Incompressible Navier-Stokes Well-Posedness Explored through Special Solutions

C. David Levermore
Department of Mathematics *and*
Institute for Physical Science and Technology
University of Maryland, College Park, MD
lvrmr@math.umd.edu

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Introduction

Incompressible (low Mach number) fluid flows are commonly modeled by initial-value problems for a Navier-Stokes system that take the form

$$\nabla_x \cdot u = 0\,, \tag{1a}$$

$$\partial_t u + u \cdot \nabla_x u + \nabla_x p = \nu \Delta_x u\,, \tag{1b}$$

$$u(x, 0) = u_o(x)\,. \tag{1c}$$

Of course, the spatial domain and appropriate boundary conditions must also be specified. This system was introduced by Navier (1823). For many such problems a unique classical solution is known to exist for a finite time whenever the initial data u_o is sufficiently regular — see Constantin and Foias (1988). Questions of whether or not such problems are globally well-posed remain open beyond two spatial dimensions. Answering these questions are key to understanding turbulence.

Models of turbulence strive to capture the macroscopic behavior of flows without resolving molecular dissipative scales. To do this right requires a good understanding of how the unresolved small scales and the large scales might interact. Said another way, it requires a statistical theory of the possible small scale dynamics given an approximately known large scale dynamics.

Engineers have largely turned to Large-scale Eddy Simulation (LES) as a practical tool for simulating turbulent flows. Theoretical physicists and mathematicians have largely turned to theories of either time-averaged or ensemble-averaged quantities as a practical tool for building a statistical understanding of small scale dynamics. These efforts are being informed by new results from physical experiments and from direct numerical simulations (DNS).

Theorems about the existence of global *weak solutions* to INS initial-value problems (1) for any divergence-free L^2 initial data go back to Leray (1934). *Suitable weak solutions* were introduced by Scheffer (1976) and were used by Cafferelli, Kohn, and Nirenberg (1982) in partial regularity theorems. These are global weak solutions of (1) that also satisfy (in a weak sense) the inequality

$$\partial_t \left(\tfrac{1}{2}|u|^2\right) + \nabla_x \cdot \left(\tfrac{1}{2}|u|^2 u + p\, u\right) \leq \nu \Delta_x \left(\tfrac{1}{2}|u|^2\right) - \nu |\nabla_x u|^2 \,. \qquad (2)$$

This more restrictive class of weak solutions is more natural physically than that of Leray because (2) expresses the second law of thermodynamics. But even for these weak solutions the uniqueness question remains open!

So-called *weak-strong uniqueness* theorems state that there is a unique weak solution of (1) for so long as a sufficiently strong solution exists.

De Lellis and Székelyhidi (2010) showed that weak solutions of the incompressible Euler (IE) system (1 with $\nu = 0$) are not unique, even within the class of suitible weak solutions that satisfy the second law of thermodynamics (inequality 2 with $\nu = 0$). Buckmaster, De Lellis, and Székelyhidi (2014) have constructed spatially periodic solutions in $L^1(C^{\frac{1}{3}-\epsilon})$ where this inequality is strict. There are weak-strong uniqueness theorems for the IE system within the class of suitible weak solutions.

In light of this work it is now easier to believe that uniqueness might fail for solutions of the INS system that lose enough regularity. We want to explore the question of how such a loss of regularity might arise. More generally, we want to explore the uniqueness question for the INS system. We do this with special solutions.

Outline

- Illustrative Example: The Burgers Equation
- Linear Solutions of the Incompressible Navier-Stokes (INS) System
- Vieillefosse Model of Velocity Gradient Dynamics
- Other Models of Velocity Gradient Dynamics
- Concluding Remarks

Illustrative Example: The Burgers Equation

Before studying the INS system, let us consider the initial-value problem for the one-dimensional Burgers equation, about which more is known,

$$\partial_t u + u\partial_x u = \nu \partial_{xx} u \,, \qquad u(x, 0) = u_o(x) \,. \tag{3}$$

We start by trying to solve this problem for linear initial data

$$u_o(x) = a_o x + b_o \,. \tag{4}$$

We seek a solution of (3) in the linear form

$$u(x, t) = a(t)x + b(t) \,. \tag{5}$$

When the linear form (5) is placed into (3) we see that

$$\left(\dot{a} + a^2\right) x + \left(\dot{b} + ab\right) = 0\,,$$

which along with (4) implies that $a(t)$ and $b(t)$ satisfy

$$\dot{a} + a^2 = 0\,, \qquad a(0) = a_o\,, \tag{6a}$$

$$\dot{b} + ab = 0\,, \qquad b(0) = b_o\,. \tag{6b}$$

Upon solving (6) we obtain

$$a(t) = \frac{a_o}{1 + ta_o}\,, \qquad b(t) = \frac{b_o}{1 + ta_o}\,.$$

Hence, there is a unique solution of the Burgers initial-value problem (3, 4) in the linear form (5).

This linear solution of the Burgers equation (3) is given by

$$u(x,t) = \frac{a_o x + b_o}{1 + t a_o}\,. \tag{7}$$

- If $a_o > 0$ then this solution exists for $-1/a_o < t < \infty$.

- If $a_o = 0$ then this solution exists for all t.

- If $a_o < 0$ then this solution exists for $-\infty < t < -1/a_o$.

While these are not finite energy solutions of (3), they capture key aspects of the behavior of every finite energy solution. Specifically, the case $a_o > 0$ describes *rarefaction*, while the case $a_o < 0$ describes *shock formation*.

Finite energy solutions of the Burgers equation will not develop an infinite spatial derivative in finite time. Rather, when ν is small these solutions will develop steep smooth shock profiles where dissipation will concentrate. The development of these smooth steep shock profiles is approximated by the linear solution (7) with $a_o = \partial_x u_o(x_o) < 0$ and $b_o = u_o(x_o)$, where the point x_o is a local minimizer of $\partial_x u_o(x)$.

As $\nu \to 0$ these solutions will approach a limit that is a weak solution of the *inviscid Burgers* equation (*Hopf* equation). In this limit the shock profiles will approach jump discontinuities (shocks) along which there is *anomalous dissipation*. The large-time asymptotics of the limiting weak solution will approach a combination of shocks and rarefactions. These rarefactions are approximated by the linear solution (7) with $a_o = \partial_x u_o(x_o) > 0$ and $b_o = u_o(x_o)$, where the point x_o is a local maximizer of $\partial_x u_o(x)$.

Linear Solutions of the INS System

Now let us apply this same approach to the INS system over $\mathbb{R}^D$ for $D \geq 2$:

$$\nabla_x \cdot u = 0\,, \tag{8a}$$

$$\partial_t u + u \cdot \nabla_x u + \nabla_x p = \nu \Delta_x u\,, \qquad u(x,0) = u_o(x)\,. \tag{8b}$$

We start by trying to solve this problem for linear initial data

$$u_o(x) = A_o x + b_o\,, \tag{9}$$

where $A_o \in \mathbb{R}^{D\times D}$, $\mathrm{tr}(A_o) = 0$, and $b_o \in \mathbb{R}^D$. We seek a solution of (8) in the linear form

$$u(x,t) = A(t)x + b(t)\,. \tag{10}$$

When the linear form (10) is placed into (8) we see that

$$\begin{aligned} \mathrm{tr}(A) &= 0\,, \\ \left(\dot{A} + A^2\right) x + \left(\dot{b} + Ab\right) + \nabla_x p &= 0\,. \end{aligned}$$

This last equation can be solved for p if and only if

$$\dot{A} + A^2 \quad \text{is symmetric}\,,$$

in which case p is given by

$$p = -\tfrac{1}{2} x \cdot \left(\dot{A} + A^2\right) x - \left(\dot{b} + Ab\right) \cdot x + c\,, \tag{11}$$

where c is an arbitrary function of time. The arbitrary function c is expected because p only enters the INS system (8) through the $\nabla_x p$ term of the motion equation.

There are many choices of $A(t)$ and $b(t)$ that make $u(x,t)$ given by (10) a global solution of (8)! Indeed, this is done by every $A \in C^1([0,\infty), \mathbb{R}^{D\times D})$ and $b \in C^1([0,\infty), \mathbb{R}^D)$ such that

$$\begin{aligned} A(0) &= A_o\,, & b(0) &= b_o\,, \\ \mathrm{tr}\Big(A(t)\Big) &= 0\,, & \dot{A}(t) + A(t)^2 &\text{ is symmetric}\,. \end{aligned} \tag{12}$$

Moreover, just as easily we can construct classical solutions of (8) that blow-up at the endpoints of any given interval (t_L, t_R) containing 0!

Unlike for the Burgers equation, here the balances in the INS system do not determine a unique solution in the linear form (10)! This lack of uniqueness reflects the fact that the pressure p in the INS system depends upon u in a nonlocal way.

The bottom two conditions in (12) imply that

$$\dot{A} + A^2 - \frac{1}{D}\,\mathrm{tr}\Big(A^2\Big)I \quad \text{is symmetric and traceless}\,.$$

Conversely, let $g \in C([0,\infty),\mathbb{R}^D)$ and $H \in C([0,\infty),\mathbb{R}^{D\times D})$ such that $H(t)$ is symmetric and traceless for every $t \geq 0$. Let $A(t)$ and $b(t)$ satisfy

$$\dot{A} + A^2 - \frac{1}{D}\,\mathrm{tr}\Big(A^2\Big)I = H(t)\,, \qquad A(0) = A_o\,, \tag{13a}$$

$$\dot{b} + Ab = g(t)\,, \qquad b(0) = b_o\,. \tag{13b}$$

Then $A(t)$ and $b(t)$ will satisfy (12) over the interval of definition (t_L, t_R) for the solution $A(t)$ of (13a).

For every such g and H we have constructed a solution of the INS system (8) given by

$$\begin{aligned} u(x,t) &= A(t)x + b(t)\,, \\ p(x,t) &= c(t) - g(t)\cdot x - \tfrac{1}{2D}\,\mathrm{tr}\Big(A(t)^2\Big)|x|^2 - \tfrac{1}{2}x\cdot H(t)x\,. \end{aligned} \tag{14}$$

Then the dynamical equation for A in (13a) can be recast as

$$\dot{A} + A^2 + \nabla_x^2 p = 0\,. \tag{15}$$

Here $\nabla_x^2 p$ denotes the Hessian matrix of p, not its Laplacian!

Remark. This solution will be unique if boundary conditions are imposed on the large x behavior of the pressure that select a unique g and H. It is not clear how to do this. For example, imposing a large x isotropy condition on $p(x,t)$ implies that $H(t) = 0$, but leaves the freedom to choose $g(t)$.

Vieillefosse Models of Velocity Gradient Dynamics

If we set $H(t) = 0$ in the foregoing construction of linear INS solutions then $A(t)$ is governed by the matrix Riccati initial-value problem

$$\dot{A} + A^2 - \tfrac{1}{D}\,\mathrm{tr}\Big(A^2\Big)I = 0\,, \qquad A(0) = A_o\,, \tag{16}$$

where $\mathrm{tr}(A_o) = 0$. The singularity formation in an associated linear INS solution happens whenever the solution of (16) blows up.

Patrick Vieillefosse (1982, 1984) proposed using (16) with $D = 3$ to model singularity formation in the three-dimensional incompressible Euler system — and therefore to model singularity or near singularity formation in the three-dimensional INS system. The hope is that these systems are related to (16) the same way the Burgers equation is related to the scalar Riccati initial-value problem

$$\dot{a} + a^2 = 0\,, \qquad a(0) = a_o\,.$$

The motivation for this hope is seen by considering the dynamics of the gradient tensor $A(x,t) = \nabla_{\!x} u(x,t)$ for solutions $u(x,t)$ of the INS system (8). Indeed, taking the gradient of (8b) yields

$$\partial_t A + u \cdot \nabla_{\!x} A + A^2 + \nabla_{\!x}^2 p = \nu \Delta_x A \,, \qquad A(x,0) = \nabla_{\!x} u_o(x) \,. \tag{17}$$

(Again $\nabla_{\!x}^2 p$ is the Hessian matrix of p, not its Laplacian!) Equation (8a) implies that $\mathrm{tr}(A(x,t)) = 0$, whereby taking the trace of (17) yields

$$-\Delta_x p = -\,\mathrm{tr}\Big(\nabla_{\!x}^2 p\Big) = \,\mathrm{tr}\Big(A^2\Big) \,. \tag{18}$$

By combining this with the dynamical equation in (17) we get

$$\begin{aligned} \partial_t A + u \cdot \nabla_{\!x} A + A^2 - \tfrac{1}{D}\,\mathrm{tr}\Big(A^2\Big) I & \\ + \nabla_{\!x}^2 p - \tfrac{1}{D} \Delta_x p \, I &= \nu \Delta_x A \,. \end{aligned} \tag{19}$$

We now argue that we can make the approximations

$$\nabla_x^2 p - \tfrac{1}{D}\Delta_x p\, I = 0\,, \qquad \nu \Delta_x u = 0\,. \tag{20}$$

The first states that the Hessian of p is nearly isotropic compared to the anisotropies that arise from A. The second states that the viscosity will have little effect on singularity formation.

The result is the so-called *Restricted Euler* (RE) equation

$$\partial_t A + u \cdot \nabla_x A + A^2 - \tfrac{1}{D}\,\mathrm{tr}\Big(A^2\Big) I = 0\,. \tag{21}$$

For any given u this equation has the property that along the characteristics $\dot{x} = u(x,t)$ the matrix $A(x,t)$ satisfies the initial-value problem (16). But this equation does not preserve the relationship that the three-tensor $\nabla_x A$ is symmetric in its last two indices, which is required for A to be given as $A = \nabla_x u$ for some u. Hence, the approximation that $\nabla_x^2 p$ is isotropic will generally not be valid globally.

Now let's turn our attention to finding the solution of the matrix Riccati initial-value problem (16). Because $\mathrm{tr}(A_o) = 0$, and because

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{tr}(A) = \mathrm{tr}\Big(\dot{A}\Big) = -\,\mathrm{tr}\Big(A^2 - \tfrac{1}{D}\,\mathrm{tr}\Big(A^2\Big)I\Big) = 0\,,$$

we conclude that $\mathrm{tr}(A) = 0$ for so long as the solution of (16) exists.

For $D = 2$ the Cayley-Hamilton theorem states that

$$A^2 - \mathrm{tr}(A)A + \mathrm{det}(A)I = 0\,.$$

Moreover, because $\mathrm{det}(A) = \frac{1}{2}\Big(\mathrm{tr}(A)^2 - \mathrm{tr}\Big(A^2\Big)\Big)$ when $D = 2$, we see that every 2×2 matrix that satisfies $\mathrm{tr}(A) = 0$ also satisfies

$$A^2 - \tfrac{1}{2}\,\mathrm{tr}\Big(A^2\Big)I = 0\,.$$

Therefore (16) reduces to $\dot{A} = 0$, $A(0) = A_o$, whereby $A(t) = A_o$.

For $D = 3$ the solution of (16) was analyzed by Vieillefosse (1982, 1984) and an explicit formula for it was found by Cantwell (1992). For $D > 3$ it was analyzed by Liu and Tadmor (2002), by Liu, Tadmor, and Wei (2010), and by Wei (2011). Here we give a fresh original analysis.

For $D > 2$ the solution of (16) is given by

$$\begin{aligned} A(t) = &\det(I + \tau A_o)^{\frac{2}{D}} \\ &\left[(I + \tau A_o)^{-1} A_o - \frac{1}{D} \operatorname{tr}\Big((I + \tau A_o)^{-1} A_o \Big) \, I \right] , \end{aligned} \tag{22a}$$

where $\tau(t)$ is the solution of the scalar initial-value problem

$$\dot{\tau} = \det(I + \tau A_o)^{\frac{2}{D}} \,, \qquad \tau(0) = 0 \,. \tag{22b}$$

This can be checked by direct calculation, but we will not do so here.

The stationary points of the scalar differential equation in (22b) are given by $-1/\lambda$, where λ is a nonzero eigenvalue of A_o. A phase-line analysis shows that as t increases the value of $\tau(t)$ will run from τ_{min} to τ_{max} where

$$\begin{aligned} \tau_{\text{min}} &= \sup\left\{-\frac{1}{\lambda} \,:\, \lambda \in \mathrm{Sp}(A_o)\,,\, \lambda > 0\right\}\,, \\ \tau_{\text{max}} &= \inf\left\{-\frac{1}{\lambda} \,:\, \lambda \in \mathrm{Sp}(A_o)\,,\, \lambda < 0\right\}\,. \end{aligned} \tag{23}$$

Recall that $\sup\{\emptyset\} = -\infty$ and $\inf\{\emptyset\} = \infty$, so these formulas cover the cases when A_o either has no postive eigenvalues or has no negative eigenvalues. Note that in all cases $\tau_{\text{min}} < 0 < \tau_{\text{max}}$.

The orbit of $A(t)$ is simply the range of the right-hand side of (22a) over all τ in $(\tau_{\text{min}}, \tau_{\text{max}})$. This can be analyzed without explicitly solving (22b).

If $-\infty < \tau_{\text{min}}$ then $\tau(t)$ will reach τ_{min} in finite time as t decreases from 0 unless the eigenvalue of A_o associated with τ_{min} has multiplicity $\geq \frac{D}{2}$.

If $-\infty = \tau_{\text{min}}$ then $\tau(t)$ will reach $-\infty$ in finite time as t decreases from 0 unless zero is an eigenvalue of A_o with multiplicity $\geq \frac{D}{2}$.

If $\tau_{\text{max}} < \infty$ then $\tau(t)$ will reach τ_{max} in finite time as t increases from 0 unless the eigenvalue of A_o associated with τ_{max} has multiplicity $\geq \frac{D}{2}$.

If $\tau_{\text{max}} = \infty$ then $\tau(t)$ will reach ∞ in finite time as t increases from 0 unless zero is an eigenvalue of A_o with multiplicity $\geq \frac{D}{2}$.

The solution $A(t)$ is eternal if and only if either:

- zero is an eigenvalue of A_o with multiplicity $\geq \frac{D}{2}$ and there are no other real eigenvalues;

- A_o has exactly two real eigenvalues, each with multiplicity $= \frac{D}{2}$.

The second case can only occur when D is even and the eigenvalues are negatives of each other. When $D = 3$ the first case can only occur when $\mathrm{Sp}(A_o) = \{0, 0, 0\}$. When $D = 4$ the first case can only occur when $\mathrm{Sp}(A_o) = \{0, 0, i\xi, -i\xi\}$ for some $\xi \in \mathbb{R}$ while the second case can only occur when $\mathrm{Sp}(A_o) = \{\xi, \xi, -\xi, -\xi\}$ for some $\xi > 0$.

Other Models of Velocity Gradient Dynamics

Rather than set $H(t) = 0$ in our construction of linear INS solutions, we can seek alternatives. For example, we can set

$$H = -\gamma \left(A^T A + A A^T - \tfrac{2}{D} \operatorname{tr}\!\left(A^T A\right) I \right) . \tag{24}$$

This choice of H has the virtue of being quadratic in A, and so shares the scaling of the term it is trying to model. It is not the only such choice, but it has other virtues that will become apparent. Here γ is a unitless scalar that can depend upon A in principle, but that we will assume is constant.

With this choice of H, the pressure p of the associated INS solution (14) becomes

$$\begin{aligned} p = c - g \cdot x - \tfrac{1}{2D} \operatorname{tr}\!\left(A^2\right) |x|^2 \\ + \tfrac{1}{2}\gamma\, x \cdot \left(A^T A + A A^T - \tfrac{2}{D} \operatorname{tr}\!\left(A^T A\right) I \right) x \,. \end{aligned} \tag{25}$$

Define the strain-rate matrix S and the rotation matrix Ω by

$$S = A + A^T\,, \qquad \Omega = A - A^T\,. \tag{26}$$

Clearly,

$$S^T = S\,, \qquad \mathrm{tr}(S) = 0\,, \qquad \Omega^T = -\Omega\,.$$

Because $A = \frac{1}{2}(S + \Omega)$ and $A^T = \frac{1}{2}(S - \Omega)$, we find that

$$\mathrm{tr}\Big(A^2\Big) = \tfrac{1}{4}\,\mathrm{tr}\Big(S^2 + \Omega^2\Big)\;, \qquad A^T A + A A^T = \tfrac{1}{2}\left(S^2 - \Omega^2\right)\;.$$

We then see from (25) that the Hessian of the pressure is

$$\begin{aligned} \nabla_x^2 p &= \gamma\;\Big(A^T A + A A^T - \tfrac{2}{D}\,\mathrm{tr}\Big(A^T A\Big)\;I\Big) - \tfrac{1}{D}\,\mathrm{tr}\Big(A^2\Big)I \\ &= \tfrac{\gamma}{2}S^2 - \tfrac{2\gamma+1}{4}\tfrac{1}{D}\,\mathrm{tr}\Big(S^2\Big)I - \tfrac{\gamma}{2}\Omega^2 + \tfrac{2\gamma-1}{4}\tfrac{1}{D}\,\mathrm{tr}\Big(\Omega^2\Big)I\;. \end{aligned} \tag{27}$$

Because the pressure should have a local minimum inside a strengthening vortex, the pressure Hessian (27) should be nonnegative definite in a good model for vortex concentration.

The contribution to the pressure Hessian (27) from the strain rate S is

$$\frac{\gamma}{2}S^2 - \frac{2\gamma+1}{4}\frac{1}{D}\,\mathrm{tr}\Big(S^2\Big)I\,.$$

Its trace is $-\frac{1}{4}\,\mathrm{tr}(S^2)$, which is negative for every nonzero S. Therefore this contribution can never be nonnegative definite when S is nonzero.

The contribution to the pressure Hessian (27) from the rotation Ω is

$$-\frac{\gamma}{2}\Omega^2 + \frac{2\gamma-1}{4}\frac{1}{D}\,\mathrm{tr}\Big(\Omega^2\Big)I\,.$$

Its trace is $-\frac{1}{4}\,\mathrm{tr}(\Omega^2)$, which is positive for every nonzero Ω. It will be nonnegative definite for every Ω if and only if γ satisfies

$$-\frac{1}{D-2} \leq \gamma \leq \frac{1}{2}\,.$$

A particularly appealing choice is $\gamma = \frac{1}{2}$ because then this contribution becomes an exact balance to the coriolis force, which will sustain rotation. When $\gamma = \frac{1}{2}$ the pressure Hessian reduces to

$$\begin{aligned} \nabla_x^2 p &= \frac{1}{2}\left(A^T A + A A^T - \frac{2}{D}\operatorname{tr}\!\left(A^T A\right) I\right) - \frac{1}{D}\operatorname{tr}\!\left(A^2\right) I \\ &= \frac{1}{4} S^2 - \frac{1}{2}\frac{1}{D}\operatorname{tr}\!\left(S^2\right) I - \frac{1}{4}\Omega^2 \,. \end{aligned} \tag{28}$$

We see that the dynamical equation for A given by (15) then becomes

$$\begin{aligned} 0 &= \dot{A} + A^2 + \nabla_x^2 p \\ &= \frac{1}{2}\left(\dot{S} + \dot{\Omega}\right) + \frac{1}{4}\left(S^2 + S\Omega + \Omega S + \Omega^2\right) \\ &\quad + \frac{1}{4} S^2 - \frac{1}{2}\frac{1}{D}\operatorname{tr}\!\left(S^2\right) I - \frac{1}{4}\Omega^2 \\ &= \frac{1}{2}\left(\dot{S} + S^2 - \frac{1}{D}\operatorname{tr}\!\left(S^2\right) I\right) + \frac{1}{2}\left(\dot{\Omega} + \frac{1}{2}(S\Omega + \Omega S)\right) . \end{aligned}$$

By separating this into its symmetric and skew-symmetric components, we arrive at the initial-value problem for a system of matrix evolution equations that is given by

$$\dot{S} + S^2 - \tfrac{1}{D}\,\mathrm{tr}\Big(S^2\Big) I = 0\,, \qquad S(0) = S_o\,, \tag{29a}$$

$$\dot{\Omega} + \tfrac{1}{2}(S\Omega + \Omega S) = 0\,, \qquad \Omega(0) = \Omega_o\,, \tag{29b}$$

where $S_o = A_o + A_o^T$ and $\Omega_o = A_o - A_o^T$. Notice that $\mathrm{tr}(S_o) = 0$.

Remark. The fact that the evolution of S given by (29a) decouples from Ω only happens for the choice $\gamma = \frac{1}{2}$.

For $D = 2$ the evolution equations in (29) reduce to $\dot{S} = 0$ and $\dot{\Omega} = 0$, whereby the solution of (29) is given by $S(t) = S_o$ and $\Omega(t) = \Omega_o$.

For $D > 2$ the initial-value problem (29a) governing S is the Vieillefosse initial-value problem (16) restricted to symmetric matrices. Therefore its solution can be read off from our solution (22) of that eariler problem. We thereby see that the solution of (29a) is given by

$$S(t) = \det(I + \tau S_o)^{\frac{2}{D}} \\ \Big[(I + \tau S_o)^{-1} S_o - \tfrac{1}{D}\,\mathrm{tr}\Big((I + \tau S_o)^{-1} S_o\Big)\, I\Big] \,, \tag{30a}$$

where $\tau(t)$ is the solution of the scalar initial-value problem

$$\dot{\tau} = \det(I + \tau S_o)^{\frac{2}{D}} \,, \qquad \tau(0) = 0 \,. \tag{30b}$$

Then the solution of (29b) is given by

$$\Omega(t) = \det(I + \tau S_o)^{\frac{1}{D}} (I + \tau S_o)^{-\frac{1}{2}} \Omega_0 (I + \tau S_o)^{-\frac{1}{2}} \,. \tag{30c}$$

We remark that $I + \tau S_o$ is positive definite, so its square root is too.

When $S_o \neq 0$, it will always have a positive and a negative eigenvalue. A phase-line analysis of (30b) shows that as t increases the value of $\tau(t)$ will run from τ_{min} to τ_{max} where

$$\begin{aligned} \tau_{\text{min}} &= \sup\left\{ -\frac{1}{\sigma} \, : \, \sigma \in \, \mathrm{Sp}(S_o)\, , \, \sigma > 0 \right\} \, , \\ \tau_{\text{max}} &= \inf\left\{ -\frac{1}{\sigma} \, : \, \sigma \in \, \mathrm{Sp}(S_o)\, , \, \sigma < 0 \right\} \, . \end{aligned} \tag{31}$$

The orbits of $S(t)$ and $\Omega(t)$ are simply the range of the right-hand sides of (30a) and (30c) over all τ in $(\tau_{\text{min}}, \tau_{\text{max}})$. The orbit of $A(t)$ is found from $A = \frac{1}{2}(S + \Omega)$. These can be analyzed without explicitly solving (30b).

When A_o is symmetric then the orbit of $A(t)$ is identical to the orbit of the solution to the Vieillefosse problem, but the dynamics along it will happen exactly twice as fast. In particular, any blow-up will happen twice as fast.

- $\tau(t)$ will reach $\tau_{\min}$ in a finite time $t_L < 0$ as t decreases from 0 unless the eigenvalue of S_o associated with $\tau_{\min}$ has multiplicity $\geq \frac{D}{2}$.

- $\tau(t)$ will reach $\tau_{\max}$ in finite time $t_R > 0$ as t increases from 0 unless the eigenvalue of S_o associated with $\tau_{\max}$ has multiplicity $\geq \frac{D}{2}$.

- Blow-ups of $S(t)$ behave like $(t_R - t)^{-1}$ as $t \to t_R$ from below and like $(t - t_L)^{-1}$ as $t \to t_L$ from above.

- $S(t)$ is eternal if and only if S_o has exactly two real eigenvalues, each with multiplicity $= \frac{D}{2}$. This can only occur when D is even and the eigenvalues are negatives of each other.

A few comments about $D = 3$.

- The vorticity will nearly align with the eigendirection associated with the most positive eigenvalue of S_o. This is consistent with what is reported by Ashurst, Kerstein, Kerr, and Gibson (1987) as being seen in simulations during vorticity stengthening.

- The blow-up of these solutions can be arrested by the inclusion of sufficient damping in the dynamical equation as a model for viscosity. There are analytic solutions in that case too.

- There is an entire family of similar models presented in a recent review by Charles Meneveau (Annu. Rev. Fluid Mech. 2011).

Concluding Remarks

- The rotation matrix $\Omega(x,t)$ of a classical solution of the INS system cannot blow-up unless its strain-rate matrix $S(x,t)$ blows up.

- The possible form of blow-up for $S(x,t)$ the IE and INS systems is very sensitive to the local behavior of the preasure, which depends nonlocally on $u(x,t)$. This dynamical amplification of small differences is a source of the chaos that makes turbulence turbulent.

- These ideas will be explored further for Gibbon-Fokas-Doering (GFD) solutions of the INS system in the Anisotropy Working Group next Monday.

Thank You!

References

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