Turbulent transport and dissipation of vorticity in the 3D NSE

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3D Navier-Stokes equations (NSE) – describing a flow of 3D incompressible viscous fluid – read

$$u_t + (u \cdot \nabla)u = -\nabla p + \nu \triangle u,$$

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taking the curl yields the vorticity formulation,

$$\omega_t + (u \cdot \nabla)\omega = (\omega \cdot \nabla)u + \nu \triangle \omega,$$

where $\omega = \operatorname{curl} u$ is the vorticity of the fluid

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$$\frac{\partial}{\partial x_i} u_j(x) = c\ P.V. \int \epsilon_{jkl} \frac{\partial^2}{\partial x_i \partial y_k} \frac{1}{|x-y|} \omega_l(y)\, dy$$

there is strong numerical evidence that the regions of intense vorticity organize in coherent vortex structures, and in particular, in elongated vortex filaments, cf.

[Siggia, 1981; Ashurst, Kerstein, Kerr and Gibson, 1987; She, Jackson and Orszag, 1991; Jimenez, Wray, Saffman and Rogallo, 1993; Vincent and Meneguzzi, 1994] there is strong numerical evidence that the regions of intense vorticity organize in coherent vortex structures, and in particular, in elongated vortex filaments, cf.

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an in-depth analysis of creation and dynamics of vortex tubes in 3D turbulent incompressible flows was presented in [Constantin, Procaccia and Segel, 1995]; see also

[Galanti, Gibbon and Heritage, 1997; Gibbon, Fokas and Doering, 1999; Ohkitani, 2009; Hou, 2009]

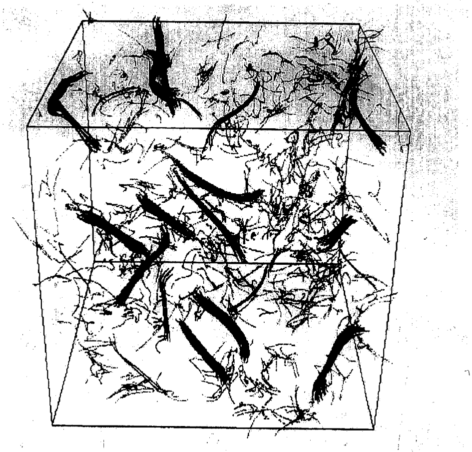

Figure 11: Intermittent vortex filaments in a 3D computational simulation of equilibrium homogeneous turbulence. (She *et al.*, 1991)

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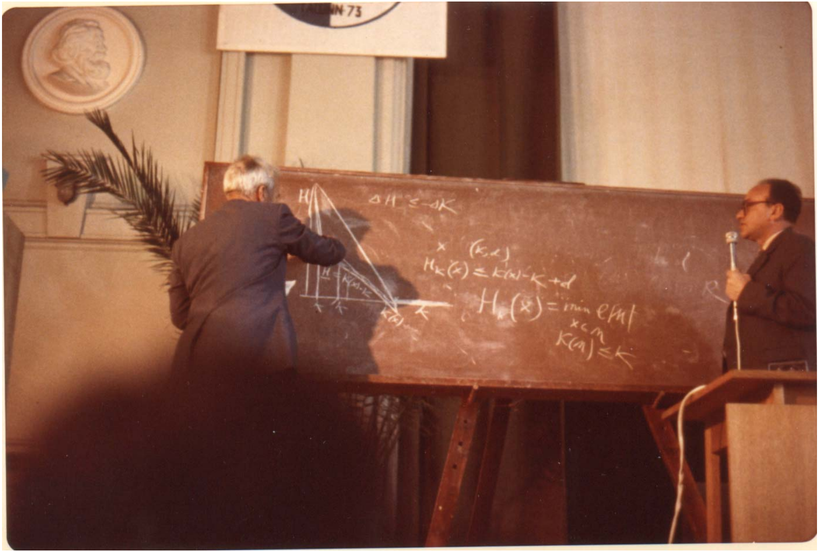
$\longrightarrow$ find a mathematical framework suitable for *encoding geometric information on the flow* in the theory of turbulent cascades; work in the *physical space*



"Half a century after Kolmogorov's work on the statistical theory of fully developed turbulence, we still wonder how his work can be reconciled with Leonardo's half a millennium old drawings of eddy motion in the study for the elimination of rapids in the river Arno."

– U. Frisch, *Turbulence*, The Legacy of A.N. Kolmogorov, 1994 geometric depletion of the nonlinearity

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and later in [Beirao da Veiga and Berselli, 2002] where the Lipschitz condition was replaced by $\frac{1}{2}$-Hölder localized vortex-stretching term can be written [G., 2009] as

$$
\begin{aligned}
(\omega\cdot\nabla)u\cdot\phi\omega\,(x) &= \phi^{\frac{1}{2}}(x)\,\frac{\partial}{\partial x_i}u_j(x)\,\phi^{\frac{1}{2}}(x)\,\omega_i(x)\,\omega_j(x)\\
&= -c\,P.V.\int_{B(x_0,2r)}\epsilon_{jkl}\,\frac{\partial^2}{\partial x_i\partial y_k}\,\frac{1}{|x-y|}\,\phi^{\frac{1}{2}}\,\omega_l\,dy\;\phi^{\frac{1}{2}}(x)\,\omega_i(x)\,\omega_j(x) + \text{ LOT}\\
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geometric cancelations in the highest order-term VST were utilized in [G., 2009] to obtain a spatiotemporal localization of $\frac{1}{2}$-Hölder coherence of the vorticity direction regularity criterion

and later in [G. and Guberović, 2010] to introduce a family of *scaling-invariant* regularity classes featuring a balance between coherence of the vorticity direction and the vorticity magnitude

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the following regularity class – a scaling-invariant improvement of $\frac{1}{2}$-Hölder coherence – is included,

$$\int_{t_0-(2R)^2}^{t_0} \int_{B(x_0,2R)} |\omega(x,t)|^2 \; \rho^2_{\frac{1}{2},2R}(x,t)\,dx\,dt < \infty; \tag{2}$$

$$\rho_{\gamma,r}(x,t) = \sup_{y\in B(x,r),\,y\neq x} \frac{|\sin\varphi\big(\xi(x,t),\xi(y,t)\big)|}{|x-y|^\gamma}$$

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a corresponding *a priori* bound had been previously obtained in [Constantin, 1990],

$$\int_0^T \int_{\mathbb{R}^3} |\omega(x,t)||\nabla\xi(x,t)|^2 \, dx\, dt \leq \frac{1}{2} \int_{\mathbb{R}^3} |u_0(x)|^2 \, dx$$

(see also [Constantin, Procaccia and Segel, 1995].)

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*   *   *

essentially, the unhappy scenario is ‘crossing of the vortex lines’ – the *vorticity direction* becomes *discontinuous* (in some sense) – as we approach the singularity physical scales

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an adaptation to the study of *forced* turbulence $\to$ Radu's talk today let $B(0, 2R_0)$ be the spatial *macro-scale* domain

x_0 in $B(0, R_0)$

$0 < R \leq R_0$

f = a locally integrable function (density) on $B(x_0, 2R)$

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spatial cut-offs

$\psi = \psi_{x_0,R}(x) \in \mathcal{D}(B(x_0,2R))$ satisfying

$$0 \le \psi \le 1, \quad \psi = 1 \text{ on } B(x_0,R), \quad \frac{|\nabla\psi|}{\psi^{\rho_2}} \le \frac{C}{R}, \quad \frac{|\triangle\psi|}{\psi^{2\rho_2-1}} \le \frac{C}{R^2}\,, \tag{3}$$

for some $\frac{1}{2} < \rho_1, \rho_2 < 1$

a *local physical scale* R – associated to the point x_0 – is realized via bounds on distributional derivatives of f,

$$|(D^\alpha f, \psi)| \le \int_{B(x_0,2R)} |f||D^\alpha \psi| \le \left(c(\alpha) \frac{1}{R^{|\alpha|}} |f|, \psi^{\delta(\alpha)} \right)$$

for some $c(\alpha) > 0$ and $\delta(\alpha)$ in $(0,1)$

a *physical scale* R, $0 < R \le R_0$ – associated to macro-scale domain $B(0, 2R_0)$ – is realized via suitable ensemble averaging of the localized quantities with respect to

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let K_1 and K_2 be two positive integers, and $0 < R \leq R_0$; a cover $\{B(x_i, R)\}_{i=1}^n$ of $B(0, R_0)$ is a (K_1, K_2)-*cover at scale* R if

$$\left(\frac{R_0}{R}\right)^3 \leq n \leq K_1 \left(\frac{R_0}{R}\right)^3,$$

and any point x in $B(0, R_0)$ is covered by at most K_2 balls $B(x_i, 2R)$

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the parameters K_1 and K_2 represent the maximal *global* and *local multiplicities*, respectively

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and, if $B(x_0, R)$ is not included in $B(0, R_0)$, then $\psi \in \mathcal{D}(B(0, 2R_0))$ with $\psi = 1$ on $B(x_0, R) \cap B(0, R_0)$ satisfying, in addition to (3), the following:

$$\psi = \psi_0 \text{ on the part of the cone centered at zero and passing through } S(0, R_0) \cap B(x_0, R) \text{ between } S(0, R_0) \text{ and } S(0, 2R_0) \tag{5}$$

and

$$\psi = 0 \text{ on } B(0, R_0) \setminus B(x_0, 2R) \text{ and outside the part of the cone centered at zero and passing through } S(0, R_0) \cap B(x_0, 2R) \text{ between } S(0, R_0) \text{ and } S(0, 2R_0) \tag{6}$$

for a physical density of interest f, consider – localized to the cover elements $B(x_i, R)$ (per unit mass) – local quantities $\hat{f}_{x_i,R}$,

$$\hat{f}_{x_i,R} = \frac{1}{R^3} \int_{B(x_i,2R)} f(x)\psi^{\delta}_{x_i,R}(x)\, dx$$

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on the other hand, if f does exhibit significant sign fluctuations at scales comparable or greater than R, suitable *rearrangements* of the cover elements up to the maximal multiplicity – emphasizing first the positive and then the negative parts of f – will result in $\langle F\rangle_R$ experiencing a wide range of values, from positive through zero to negative, respectively (the larger K_2, the finer detection..)

for a non-negative density f, the ensemble averages are all comparable to each other throughout the full range of scales, $0 < R \leq R_0$; in particular, they are all comparable to the simple average over the macro scale domain for a non-negative density f, the ensemble averages are all comparable to each other throughout the full range of scales, $0 < R \le R_0$; in particular, they are all comparable to the simple average over the macro scale domain

$$\frac{1}{K_1} F_0 \le \langle F \rangle_R \le K_2 F_0 \tag{7}$$

for all $0 < R \le R_0$, where

$$F_0 = \frac{1}{R_0^3} \int f(x) \phi_0^\delta(x)\, dx$$

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$\phi = \phi_{x_0,R,T} = \psi\,\eta$ on $B(x_0, 2R) \times (0, T)$

where $\eta = \eta_T(t) \in C^\infty(0, T)$, such that

$$0 \le \eta \le 1,\;\; \eta = 0 \text{ on } (0, T/3),\;\; \eta = 1 \text{ on } (2T/3, T),\;\; \frac{|\eta'|}{\eta^{\rho_1}} \le \frac{C}{T} \tag{8}$$

let $\mathcal{R}$ be a bounded region with smooth boundary. the inward enstrophy flux through the boundary of the region is given by

$$-\int_{\partial\mathcal{R}} \frac{1}{2}|\omega|^2(u\cdot n)\,d\sigma = -\int_{\mathcal{R}}(u\cdot\nabla)\omega\cdot\omega\,dx$$

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localization of the transport term to cylinder $B(x_0, 2R)\times(0,T)$ leads to the following version of the enstrophy flux,

$$\int \frac{1}{2}|\omega|^2(u\cdot\nabla\phi)\,dx = -\int (u\cdot\nabla)\omega\cdot\phi\omega\,dx \tag{9}$$

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since $\nabla\phi = (\nabla\psi)\eta$, and ψ can be constructed such that $\nabla\psi$ points inward, (9) represents *local, inward enstrophy flux, at scale* R (more precisely, through the layer $S(x_0, R, 2R)$) *around the point* x_0

considering a (K_1, K_2)-cover $\{B(x_i, R)\}_{i=1}^n$ at scale R, for some $0 < R \le R_0$, local inward enstrophy fluxes at scale R – associated to the cover elements $B(x_i, R)$ – are then given by

$$\int \frac{1}{2}|\omega|^2(u \cdot \nabla\phi_i)\, dx, \tag{10}$$

for $1 \le i \le n$

assuming smoothness on $(0,T)$, localizing the *enstrophy dynamics* to $B(x_i,2R)\times(0,T)$ yields the following expression for time-integrated local fluxes,

$$
\begin{aligned}
\int_0^t \int \frac{1}{2}|\omega|^2(u\cdot\nabla\phi_i)\,dx\,ds &= \int \frac{1}{2}|\omega(x,t)|^2\psi_i(x)\,dx + \int_0^t \int |\nabla\omega|^2\phi_i\,dx\,ds \\
&- \int_0^t \int \frac{1}{2}|\omega|^2\big((\phi_i)_s + \triangle\phi_i\big)\,dx\,ds \\
&- \int_0^t \int (\omega\cdot\nabla)u\cdot\phi_i\,\omega\,dx\,ds, \qquad (11)
\end{aligned}
$$

for any t in $(2T/3,T)$ and $1\le i\le n$

denoting the time-averaged local fluxes per unit mass associated to the cover element $B(x_i, R)$ by $\hat{\Phi}_{x_i,R}$,

$$\hat{\Phi}_{x_i,R} = \frac{1}{t}\int_0^t \frac{1}{R^3}\int \frac{1}{2}|\omega|^2(u\cdot\nabla\phi_i)\,dx, \tag{12}$$

the quantity of interest is the ensemble average of $\{\hat{\Phi}_{x_i,R}\}_{i=1}^n$,

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the goal is to formulate a set of physically reasonable conditions on the flow in $B(0, 2R_0)\times(0,T)$ implying the strict positivity and stability of $\langle\Phi\rangle_R$ across a suitable range of scales – *existence of the enstrophy cascade*

(A1) Coherence

assume that there exist positive constants M and C_1 such that

$$|\sin\varphi\big(\xi(x,t),\xi(y,t)\big)| \leq C_1|x-y|^{\frac{1}{2}}$$

for any (x,y,t) in $\big(B(0,2R_0)\times B(0,2R_0+R_0^{\frac{2}{3}})\times(0,T)\big)\cap\{|\nabla u|>M\}$

(A2) Modified Kraichnan scale is a small scale

denote by E_0 time-averaged enstrophy per unit mass associated with the macro-scale domain $B(0, 2R_0) \times (0, T)$,

$$E_0 = \frac{1}{T} \int \frac{1}{R_0^3} \int \frac{1}{2} |\omega|^2 \phi_0^{2\rho - 1} \, dx \, dt,$$

by P_0 a modified time-averaged palinstrophy per unit mass,

$$P_0 = \frac{1}{T} \int \frac{1}{R_0^3} \int |\nabla \omega|^2 \phi_0 \, dx \, dt + \frac{1}{T} \frac{1}{R_0^3} \int \frac{1}{2} |\omega(x, T)|^2 \psi_0(x) \, dx,$$

and by σ_0 a corresponding modified Kraichnan scale, $\sigma_0 = \left(\frac{E_0}{P_0} \right)^{\frac{1}{2}}$

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(A2) is a requirement that the modified Kraichnan scale associated with the macro-scale domain $B(0, 2R_0) \times (0, T)$ be dominated by the macro scale,

$$\sigma_0 < \beta R_0,$$

for a suitable constant $\beta = \beta\big(\rho, K_1, K_2, M, C_1, \sup_{t \in (0,T)} \|\omega(t)\|_{L^1}\big)$ $(0 < \beta < 1)$

(A3) Localization and modulation

the general set up considered is one of the Leray solutions satisfying (A1). (A1) implies smoothness – however, the control on regularity-type norms is only *local*

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the modulation assumption on the evolution of local enstrophy on $(0,T)$ – consistent with the choice of the temporal cut-off η – reads

$$\int |\omega(x,T)|^2 \psi_0(x) \, dx \geq \frac{1}{2} \sup_{t \in (0,T)} \int |\omega(x,t)|^2 \psi_0(x) \, dx$$

the following result holds [Dascaliuc and G. , *Comm. Math. Phys. 2013*]

Theorem (existence of 3D enstrophy cascade)

Let u *be a Leray solution on* $\mathbb{R}^3 \times (0,T)$ *satisfying (A1)-(A3) on the spatiotemporal macro-scale domain* $B(0, 2R_0 + R_0^{\frac{2}{3}}) \times (0,T)$*, and suppose that* ω_0 *is in* $L^1(\mathbb{R}^3)$*. Then,*

$$\frac{1}{4K_*} P_0 \le \langle \Phi \rangle_R \le 4K_* \, P_0$$

for all R, $\frac{1}{\beta}\sigma_0 \le R \le R_0$.

it is indeed possible to remove the localization of $B(0, 2R_0) \times (0, T)$ assumption in the theorem [Leitmeyer, 2014]

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let S be a set, and denote by $M^{p \cdot q}(S)$ the restricted Morrey space of functions f such that

$$\sup_{y \in \mathbb{R}^3, R>0} \frac{1}{R^{3(1-p/q)}} \int_{B(y,R) \cap S} |f|^p \, dx < \infty$$

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then, one can replace (A3)-localization with $\omega \in L^2(0, T; M^{2 \cdot q} B(0, 2R_0))$ for some $q > 2$, and

$$\sigma_0^{1-2/q} \|\omega\|_{M^{2,q}(B(0,2R_0))} < c(\beta)$$

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denoting the time-averaged local fluxes associated to the cover element $B(x_i, R)$ by $\hat{\Psi}_{x_i,R}$,

$$\hat{\Psi}_{x_i,R} = \frac{1}{T}\int_0^T \int \frac{1}{2}|\omega|^2 (u \cdot \nabla\phi_i)\, dx, \tag{14}$$

the (time and ensemble) averaged flux is given by

$$\langle \Psi \rangle_R = \frac{1}{n}\sum_{i=1}^{n} \hat{\Psi}_{x_i,R} = R^3 \langle \Phi \rangle_R \tag{15}$$

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the following locality result is a straightforward consequence of the universality of the cascade of the time and ensemble-averaged local fluxes *per unit mass* $\langle\Phi\rangle_R$ presented in the previous result

Theorem (locality of 3D enstrophy cascade)

Let u be a Leray solution on $\mathbb{R}^3 \times (0,T)$ satisfying (A1)-(A3) on the spatiotemporal macro-scale domain $B(0, 2R_0 + R_0^{\frac{2}{3}}) \times (0,T)$, and suppose that ω_0 is in $L^1(\mathbb{R}^3)$. Let R and r be two scales within the inertial range delineated in the previous theorem. Then

$$\frac{1}{16K_*^2}\left(\frac{r}{R}\right)^3 \le \frac{\langle\Psi\rangle_r}{\langle\Psi\rangle_R} \le 16K_*^2\left(\frac{r}{R}\right)^3.$$

In particular, if $r = 2^k R$ for some integer k, i.e., through the dyadic scale,

$$\frac{1}{16K_*^2}\ 2^{3k} \le \frac{\langle\Psi\rangle_{2^k R}}{\langle\Psi\rangle_R} \le 16K_*^2\ 2^{3k}.$$

Theorem (locality of 3D enstrophy cascade)

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previous locality results include locality of the filtered flux – via coarse graining approach – presented in [Eyink, 2005] and [Eyink and Aluie, 2009], and locality of the flux in the Littlewood-Paley setting obtained in [Cheskidov, Constantin, Friedlander and Shvydkoy, 2008] an effect of coherence of the vorticity direction on the *energy cascade* $\longrightarrow$ Mike's talk on Friday another look at turbulent transport/concentration of 3D vorticity

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the production part is relatively well-understood; amplification of the vorticity via the process of vortex stretching is essentially a consequence of the conservation of the angular momentum in the incompressible flows..

let us consider a flow that is smooth on $(0, T)$, and think about T as being either the first (possible) singular time, or a regular time at which the vorticity field exhibits high magnitude and high spatial complexity (in the sense that $\|\omega(T)\|_\infty$ is 'very large' and σ_0 is 'very small') let us consider a flow that is smooth on $(0, T)$, and think about T as being either the first (possible) singular time, or a regular time at which the vorticity field exhibits high magnitude and high spatial complexity (in the sense that $\|\omega(T)\|_\infty$ is 'very large' and σ_0 is 'very small')

denote by $\Omega_\tau(M)$ the vorticity super-level set at time τ – more precisely –

$$\Omega_\tau(M) = \{x \in \mathbb{R}^3 : |\omega(x, \tau)| > M\}$$

and define *the region of intense vorticity* at time $s < T$ to be the region in which the vorticity magnitude exceeds a fraction of $\|\omega(s)\|_\infty$, i.e., the set

$$\Omega_s\left(\frac{1}{c_1}\|\omega(s)\|_\infty\right)$$

for some $c_1 > 1$

the picture painted by numerical simulations indicates that the region of intense vorticity comprises – in statistically significant significant sense/in time-average – of vortex filaments with the lengths comparable to the suitable macro-scale R_0 (e.g., the spatial period L)

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in what follows, let us think in terms of the macro-scale long vortex filaments. a natural *micro-scale* is then the length scale associated with the diameters of the cross-sections; this can then be estimated *indirectly*, by estimating the rate of the decrease of the total volume of the region of intense vorticity $\Omega_{s(t)}\left(\frac{1}{c_1}\|\omega(t)\|_\infty\right)$

taking the initial vorticity to be in L^1, a desired estimate on the volume of the region of intense vorticity follows simply from the *a priori* L^1-bound and the Tchebyshev inequality,

$$\text{Vol}\left(\Omega_{s(t)}\left(\frac{1}{c_1}\|\omega(t)\|_\infty\right)\right) \le \frac{c_2^0}{\|\omega(t)\|_\infty} \quad (c_2^0 > 1)$$

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this implies the decrease of the diameters of the cross-section of at least

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$\longrightarrow$ 2 questions

1. is it possible to obtain a mathematical evidence of creation and persistence (in average) of R_0-long vortex filaments?

2. is λ a dissipation scale?

one way to identify the range of (axial) scales at which the dynamics of creation and persistence of vortex filaments takes place is to identify the *range of scales of positivity* of $S\omega \cdot \omega$

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denote the time-averaged localized vortex-stretching terms per unit mass associated to the cover element $B(x_i, R)$ by $VST_{x_i,R,t}$,

$$VST_{x_i,R,t} = \frac{1}{t} \int_0^t \frac{1}{R^3} \int (\omega \cdot \nabla) u \cdot \omega \ \phi_i \, dx \, ds \tag{16}$$

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the quantity of interest is the ensemble average of $\{VST_{x_i,R,t}\}_{i=1}^n$,

$$\langle VST \rangle_{R,t} = \frac{1}{n}\sum_{i=1}^n VST_{x_i,R,t} \tag{17}$$

$B(x_i, R)$-localized enstrophy level dynamics is as folows

$$\int_0^t \int (\omega \cdot \nabla) u \cdot \phi_i \, \omega \, dx \, ds = \int \frac{1}{2} |\omega(x,t)|^2 \psi_i(x) \, dx + \int_0^t \int |\nabla \omega|^2 \phi_i \, dx \, ds$$
$$- \int_0^t \int \frac{1}{2} |\omega|^2 \big((\phi_i)_s + \triangle \phi_i \big) \, dx \, ds$$
$$- \int_0^t \int \frac{1}{2} |\omega|^2 (u \cdot \nabla \phi_i) \, dx \, ds, \tag{18}$$

for any t in $(2T/3, T)$, and $1 \leq i \leq n$

denote by $E_{0,t}$ time-averaged enstrophy per unit mass associated with the macro scale domain $B(0, 2R_0) \times (0, t)$,

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and by $\sigma_{0,t}$ a corresponding modified Kraichnan scale,

$$\sigma_{0,t} = \left(\frac{E_{0,t}}{P_{0,t}} \right)^{\frac{1}{2}}$$

then the following holds [Dascaliuc and G., *J. Math. Phys.* 2012] then the following holds [Dascaliuc and G., *J. Math. Phys.* 2012]

Theorem (vortex stretching)

Let u be a global-in-time local Leray solution on $\mathbb{R}^3 \times (0, \infty)$, regular on $(0, T)$. Suppose that, for some $t \in (2T/3, T)$,

$$C \max\{M_0^{\frac{1}{2}}, R_0^{\frac{1}{2}}\} \, \sigma_{0,t}^{\frac{1}{2}} < R_0 \tag{19}$$

where $M_0 = \sup_t \int_{B(0,2R_0)} |u|^2 < \infty$, and $C > 1$ a suitable constant depending only on the cover parameters.
Then,

$$\frac{1}{C} P_{0,t} \le \langle VST \rangle_{R,t} \le C \, P_{0,t} \tag{20}$$

for all R satisfying

$$C \max\{M_0^{\frac{1}{2}}, R_0^{\frac{1}{2}}\} \, \sigma_{0,t}^{\frac{1}{2}} \le R \le R_0. \tag{21}$$

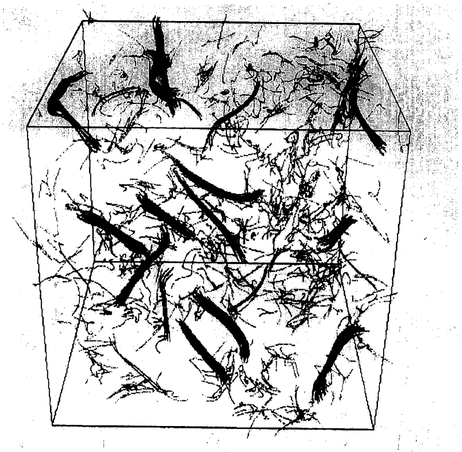

Figure 11: Intermittent vortex filaments in a 3D computational simulation of equilibrium homogeneous turbulence. (She *et al.*, 1991)

local anisotropic diffusion

Definition

Let x_0 be a point in $\mathbb{R}^3$, $r > 0$, S an open subset of $\mathbb{R}^3$ and δ in $(0, 1)$.

The set S is linearly δ-sparse around x_0 at scale r in weak sense if there exists a unit vector d in S^2 such that

$$\frac{|S \cap (x_0 - rd, x_0 + rd)|}{2r} \leq \delta.$$

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recall that $\Omega_t(M)$ denotes the vorticity super-level set at time t,

$$\Omega_t(M) = \{x \in \mathbb{R}^3 : |\omega(x, t)| > M\}$$

the following result holds [G., *Nonlinearity* 2013]

Theorem (local anisotropic diffusion)

Suppose that a solution u is regular on an interval $(0, T^*)$.

Assume that either

(i) *there exists t in $(0, T^*)$ such that $t + \dfrac{1}{d_0^2 \|\omega(t)\|} \geq T^*$, or*

(ii) $t + \dfrac{1}{d_0^2 \|\omega(t)\|} < T^*$ *for all t in $(0, T^*)$, and there exists ϵ in $(0, T^*)$ such that for any t in $(T^* - \epsilon, T^*)$, there exists $s = s(t)$ in $\left[t + \frac{1}{4 d_0^2 \|\omega(t)\|}, t + \frac{1}{d_0^2 \|\omega(t)\|}\right]$ with the property that for any spatial point x_0, there exists a scale $r = r(x_0)$, $0 < r \leq \frac{1}{2 d_0^2 \|\omega(t)\|_\infty^{\frac{1}{2}}}$, such that the super-level set $\Omega_s(M)$ is linearly δ-sparse around x_0 at scale r in weak sense; here, $\delta = \delta(x_0)$ is an arbitrary value in $(0,1)$, $h = h(\delta) = \frac{2}{\pi} \arcsin \frac{1-\delta^2}{1+\delta^2}$, $\alpha = \alpha(\delta) \geq \frac{1-h}{h}$, and $M = M(\delta) = \frac{1}{d_0^\alpha} \|\omega(t)\|_\infty$.*

Then, there exists $\gamma > 0$ such that ω is in $L^\infty\Big((T^* - \epsilon, T^* + \gamma); L^\infty\Big)$, i.e., T^* is not a singular time. *(d_0 is a suitable absolute constant.)*

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(i) a local-in-time lower bound on the radius of spatial analyticity in L^∞

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(iii) a consequence of the general harmonic measure majorization principle:

let D be open and K closed in $\mathbb{C}$, f analytic in $D \setminus K$, $|f| \leq M$, and $|f| \leq m$ on K.

then

$$|f(z)| \leq m^\theta \, M^{1-\theta}$$

for any z in $D \setminus K$, where $\theta = h(z, D, K)$ is the harmonic measure of K with respect to D evaluated at z

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(iv) a result on extremal properties of the harmonic measure in the unit disk $\mathbb{D}$ [Solynin, 1999]

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there is a recent work on localization of spatial analyticity properties of solutions to the 3D NSE $\longrightarrow$ Igor's talk tomorrow

recall the upper bound on the diameters of the vortex filaments,

$$\lambda(t) = \frac{c_3^0}{\|\omega(t)\|_\infty^{\frac{1}{2}}}$$

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$(c_3^0, c_4^0 > 1)$

$\longrightarrow$ try to get a uniform-in-time estimate on

$$\int \psi\, w \log w \, dx$$

where $w = \sqrt{1 + |\omega|^2}$; this would – in turn – yield extra log-decay on the distribution function of the vorticity for any τ in $[0, T)$,

$$I(\tau) \equiv \int \psi(x)\, w(x,\tau) \log w(x,\tau)\, dx \le I(0) + c \int_0^\tau \int_x \omega \cdot \nabla u \cdot \psi\, \xi\ \log w\, dx\, dt$$

$$+ \ \textit{a priori} \ \ \text{bounded}$$

for any τ in $[0, T)$,

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$+$ *a priori* bounded

in order to take the advantage of the Coifman-Rochberg's estimate –

$$\| \log \mathcal{M} f \|_{BMO} \leq c(n),$$

for any locally integrable function f – we decompose the logarithmic factor as

$$\log w = \log \frac{w}{\mathcal{M} w} + \log \mathcal{M} w$$

for the second term, we have the following string of inequalities

$$
\begin{aligned}
J_2 &\le c \int_0^{\tau} \|\omega \cdot \nabla u\|_{\mathfrak{h}^1} \|\psi\, \xi \log \mathcal{M} w\|_{bmo}\, dt \\
&\le c \int_0^{\tau} \|\omega \cdot \nabla u\|_{\mathcal{H}^1} \|\psi\, \xi \log \mathcal{M} w\|_{\widetilde{bmo}}\, dt \\
&\le c \int_0^{\tau} \|\omega\|_2 \|\nabla u\|_2 \Big(\|\psi\, \xi\|_\infty + \|\psi\, \xi\|_{\widetilde{bmo}_{\frac{1}{|\log r|}}} \Big) \Big(\|\log \mathcal{M} w\|_{BMO} + \|\log \mathcal{M} w\|_1 \Big)\, dt \\
&\le c \sup_{t \in (0,T)} \left\{ \Big(1 + \|\psi\, \xi\|_{\widetilde{bmo}_{\frac{1}{|\log r|}}} \Big) \Big(\|\log \mathcal{M} w\|_{BMO} + \|\log \mathcal{M} w\|_1 \Big) \right\} \int_t \int_x |\nabla u|^2 \\
&\le c \left(1 + \sup_{t \in (0,T)} \|\psi\, \xi\|_{\widetilde{bmo}_{\frac{1}{|\log r|}}} \right) \left(1 + \sup_{t \in (0,T)} \|\omega\|_1 \right) \int_t \int_x |\nabla u|^2
\end{aligned}
$$

by $\mathfrak{h}^1 - bmo$ duality, the Div-Curl Lemma, the pointwise $\widetilde{bmo}$-multiplier theorem, the Coifman-Rochberg's estimate, and the L^1-bound on the modified maximal operator $\mathcal{M}$

this implies the following result [Bradshaw and G., Indiana Univ. Math. J. 2014]

Theorem (broken ξ can still generate dissipation)

Let u be a Leray solution to the 3D NSE, smooth on $(0,T)$. Assume that the initial vorticity ω_0 is in $L^1 \cap L^2$. Suppose that

$$\sup_{t\in(0,T)} \|(\psi\xi)(\cdot,t)\|_{\widetilde{bmo}_{\frac{1}{|\log r|}}} < \infty.$$

Then,

$$\sup_{t\in(0,T)} \int \psi(x)\, w(x,t) \log w(x,t)\, dx < \infty.$$

$\widetilde{bmo}_\phi$ contains discontinuous functions if and only if $\displaystyle\int_0^{\frac{1}{2}} \frac{\phi(r)}{r}\, dr = \infty$

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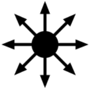
$\widetilde{bmo}_\phi$ contains discontinuous functions if and only if $\displaystyle\int_0^{\frac{1}{2}} \frac{\phi(r)}{r}\, dr = \infty$

in particular, $\widetilde{bmo}_{\frac{1}{|\log r|}}$ contains bounded functions with the singularities of, say,

$\sin\log|\log($ something algebraic $)|$-type

ξ can (as it approaches T) oscillate among infinitely many points on the unit sphere –

$$\xi(\mathsf{sing}_x, T) \;\sim$$



– and still yield extra log-decay of the distribution function of ω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