Production of dissipative vortices by solid bodies in incompressible fluid flows: comparison between Prandtl, Navier-Stokes and Euler solutions

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IPAM, UCLA, October 1st 2014

What is turbulence?

Turbulence is a state that **fluid flows** reach
when they become **unstable** and highly **fluctuating**.

Hypotheses *:*

-The **fluid** is supposed to be a **continuous medium**
when the observation scale is much larger
than the mean free path of molecules,

- The **fluid flow** is supposed to be **incompressible**, *i.e.*, non-divergent.

Etymology of the word 'turbulence' *:*

turba-ae, crowd, mob
turbo-inis, vortex

A **mob of vortices interacting together**
on a wide range of temporal and spatial scales.

Fluid flows reach the **fully-developed turbulent regime**
when they become **highly mixing.**

1.

History of d’Alembert’s paradox

Jean Le Rond d'Alembert
(1717-1783)

Leonhard Euler
(1707-1783)

Mathematical Prize 1750

On 16th May 1748 the Prussian Academy of Sciences, presided by Euler, offered a prize to the mathematician who could propose a:

'Theoria resistentiae quam patitur corpus in fluido motum, ex principiis omnino novis et simplissimis deducta, habita ratione tum velocitatis, figurae, et massae corporis moti, tum densitatis & compressionis partium fluidi'.

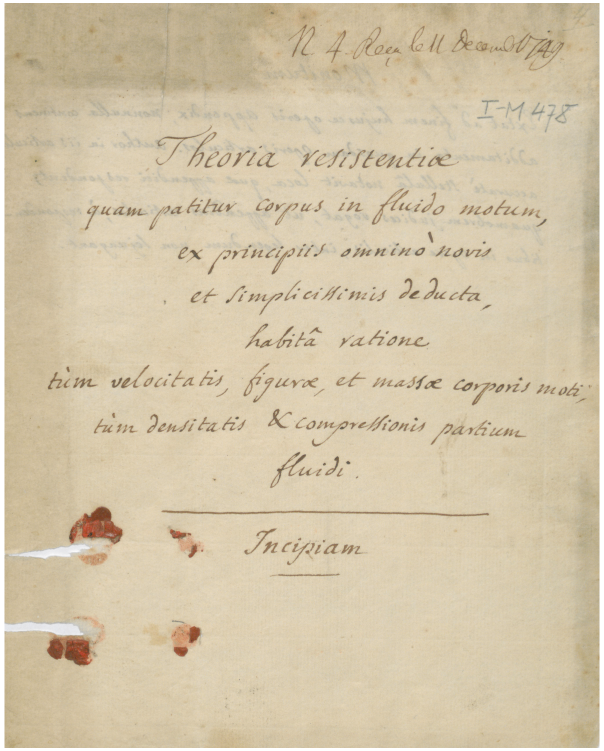
On 25th November 1749 d'Alembert sent a 137 pages manuscript, but Euler decided to postpone the prize to 1752.

***Grimberg**, D'Alembert et les équations aux dérivées partielles en hydrodynamique, Thèse de Doctorat, Université de Paris VII, 1998*

D'Alembert's theory of fluid resistance

D'Alembert was upset and took back his manuscript of 1749.
He translated it into French and published it in 1752 under the title
'Essai d'une nouvelle théorie de la résistance des fluides'.

1749

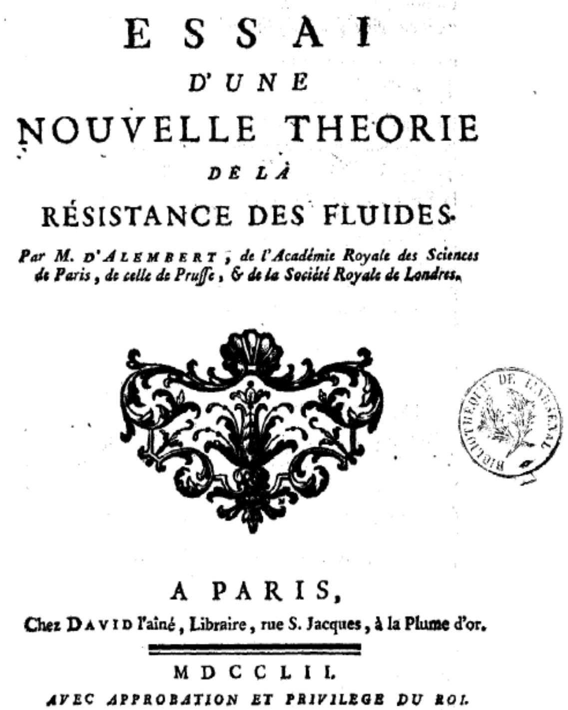


N 4 Reçu le 11 Decembre 1749

I-M 478

Theoria resistentiae
quam patitur corpus in fluido motum,
ex principiis omnino novis
et simplicissimis deducta,
habitâ ratione
tum velocitatis, figurae, et massae corporis moti,
tum densitatis & compressionis partium
fluidi.

Incipiam



ESSAI
D'UNE
NOUVELLE THEORIE
DE LA
RÉSISTANCE DES FLUIDES.
Par M. D'ALEMBERT, de l'Académie Royale des Sciences de Paris, de celle de Prusse, & de la Société Royale de Londres.

A PARIS,
Chez DAVID l'aîné, Libraire, rue S. Jacques, à la Plume d'or.
MDCCLII.
AVEC APPROBATION ET PRIVILEGE DU ROI.

1752

The prize was finally given in 1752 to Jacob Adami,
a friend of Euler, and published by the Prussian Academy.

D’Alembert’s paradox

Euler had already noticed the fact that potential flow exerts no drag on moving bodies in a work he published in 1745 on ‘New principles of gunnery’.

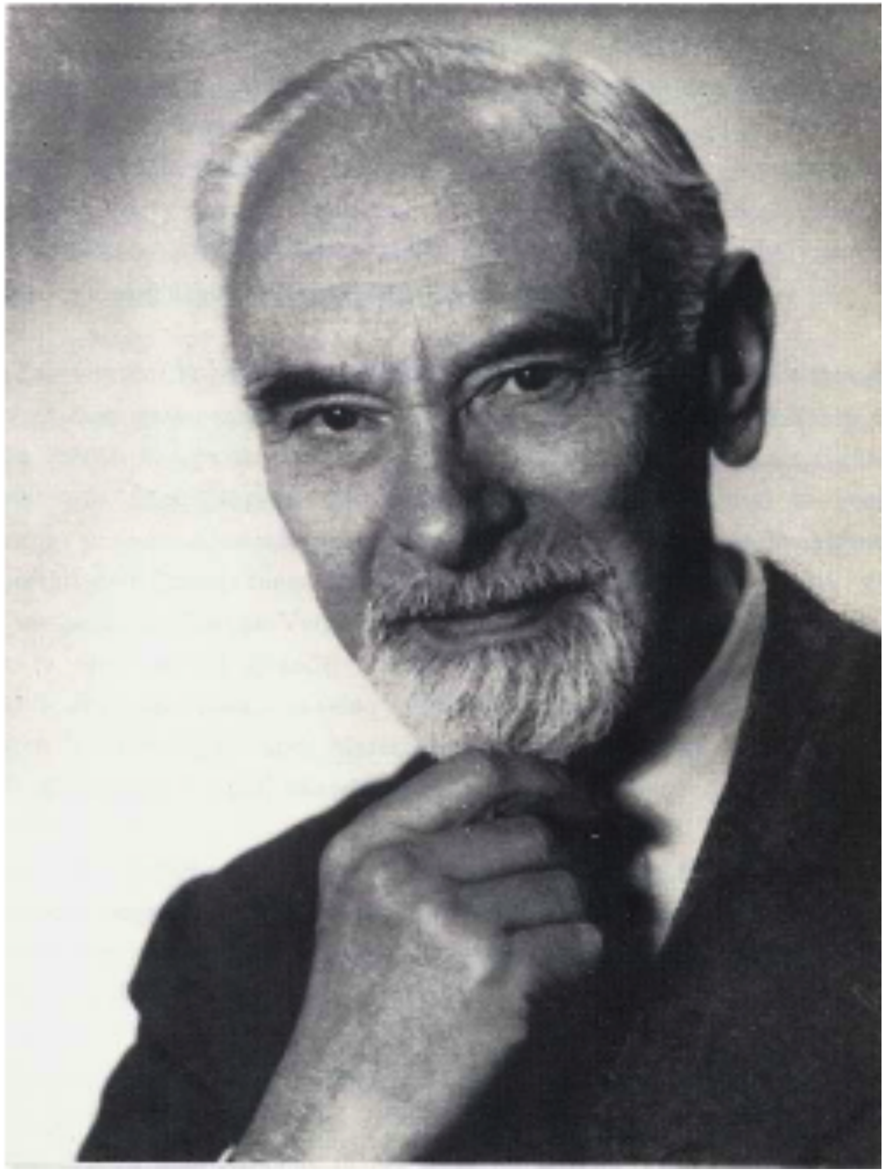
While working on the Berlin Academy Prize, d’Alembert was also conscious of that problem and wrote:

’It seems to me that the theory, developed in all possible rigor, gives, at least in several cases, a strictly vanishing resistance, a singular paradox which I leave to future geometers to elucidate.’

Darrigol, *World of flows: a history of hydrodynamics from Bernoulli to Prandtl, Oxford university Press, 2005*

Adhémar Jean-Claude Barré de Saint-Venant (1797-1886)

Ludwig Prandtl (1875-1953)



Resolution proposed by Saint-Venant

In 1846 he wrote a note to the 'Société Philomatique', published later by the 'Académie des Sciences', stating that:

'But one finds another result if, instead of an inviscid fluid, object of the calculation of the geometers of the last century, one uses a real fluid, composed of a finite number of molecules and exerting in its state of motion unequal pressure forces having components tangential to the surface elements through which they act; components to which we refer as the friction of the fluid, a name which has been given to them since Descartes and Newton until Venturi.'

***Saint-Venant**, Résistance des fluides: considérations historiques, physique et pratiques relatives au problème de l'action dynamique mutuelle d'un fluide à un solide, dans l'état de permanence supposé acquis par leurs mouvements, Mémoires de l'Académie des sciences, 44, 1-280, 1888*

Resolution proposed by Prandtl

At the 3rd ICM conference held in 1904 in Heidelberg,
Prandtl proposed a theory based on the hypothesis:

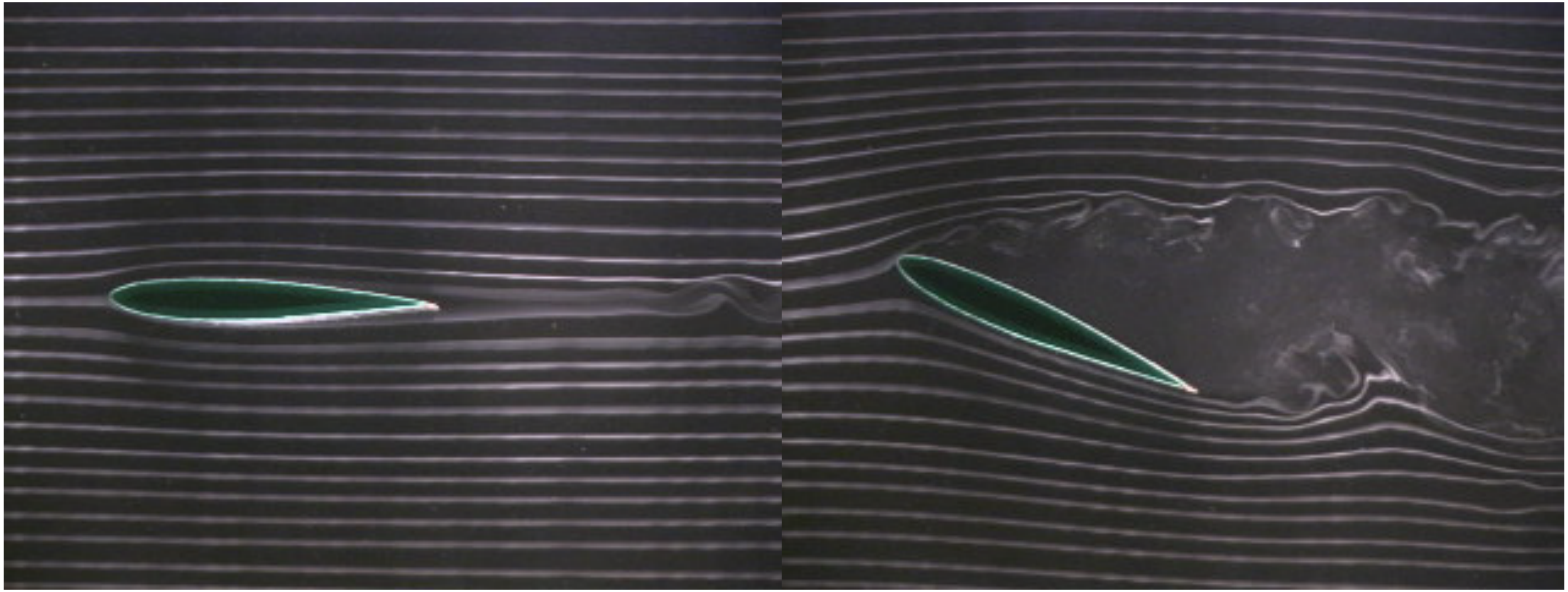
'The viscosity is supposed to be so small that it can be disregarded wherever there are no great velocity differences. […] The most important aspect of the problem is the behavior of the fluid on the surface of the solid body. […] In the thin transition layer, the great velocity differences will […] produce noticeable effects in spite of the small viscosity constants. […] It is therefore possible to pass to the limit $\nu = 0$ and still retain the same flow figure.'

Inviscid limit = Euler equation + Prandtl viscous equation

***Prandtl**, NACA YM-342,*
English translation, 1927

Boundary layer theory

- Prandtl was aware that his approach is only valid if the boundary layer remains attached to the wall *(left)*, *i.e.*, away from separation points.
- Separated flow regions, *i.e.*, where the boundary layer detaches *(right)*, have to be included « by hand » since Prandtl's theory doesn't predict their behavior.



2.

Inviscid limit of the Navier-Stokes equations

What is the inviscid limit of Navier-Stokes?

Navier-Stokes equations with no-slip boundary conditions:

$$\text{(NS)} \begin{cases} \partial_t \mathbf{u} + (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} = -\boldsymbol{\nabla} p + \frac{1}{\mathrm{Re}}\boldsymbol{\nabla}^2\mathbf{u} \\ \boldsymbol{\nabla}\cdot\mathbf{u} = 0 \\ \mathbf{u}_{|\partial\Omega} = \mathbf{0}, \qquad \mathbf{u}(0,\cdot) = \mathbf{v} \end{cases}$$

→ ? Navier-Stokes solutions $\mathbf{u}_{\mathrm{Re}}(t,\mathbf{x})$ for $\nu \to 0$, $\mathrm{Re} \to +\infty$

The Reynolds number Re = U $/\nu$ appears when non dimensional quantities are introduced.

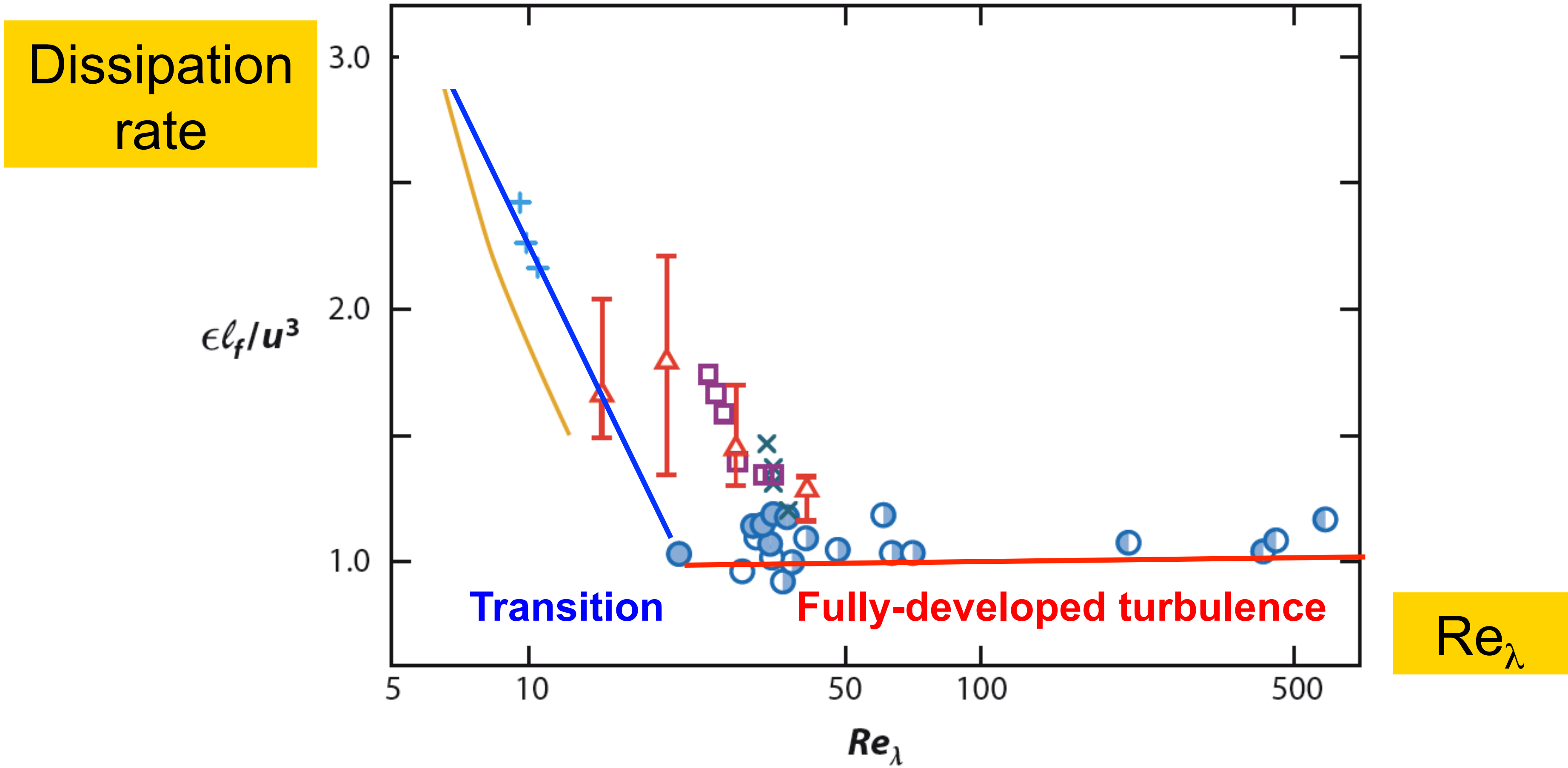
Euler equations with slip b.c.:

$$\text{(E)} \begin{cases} \partial_t \mathbf{u} + (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} = -\boldsymbol{\nabla} p \\ \boldsymbol{\nabla}\cdot\mathbf{u} = 0 \\ \mathbf{u}_{|\partial\Omega}\cdot\mathbf{n} = \mathbf{0}, \qquad \mathbf{u}(0,\cdot) = \mathbf{v} \end{cases}$$

→ ? Euler's solutions $\mathbf{u}(t,\mathbf{x})$ for $\nu = 0$, $\mathrm{Re} = +\infty$

↕ ?

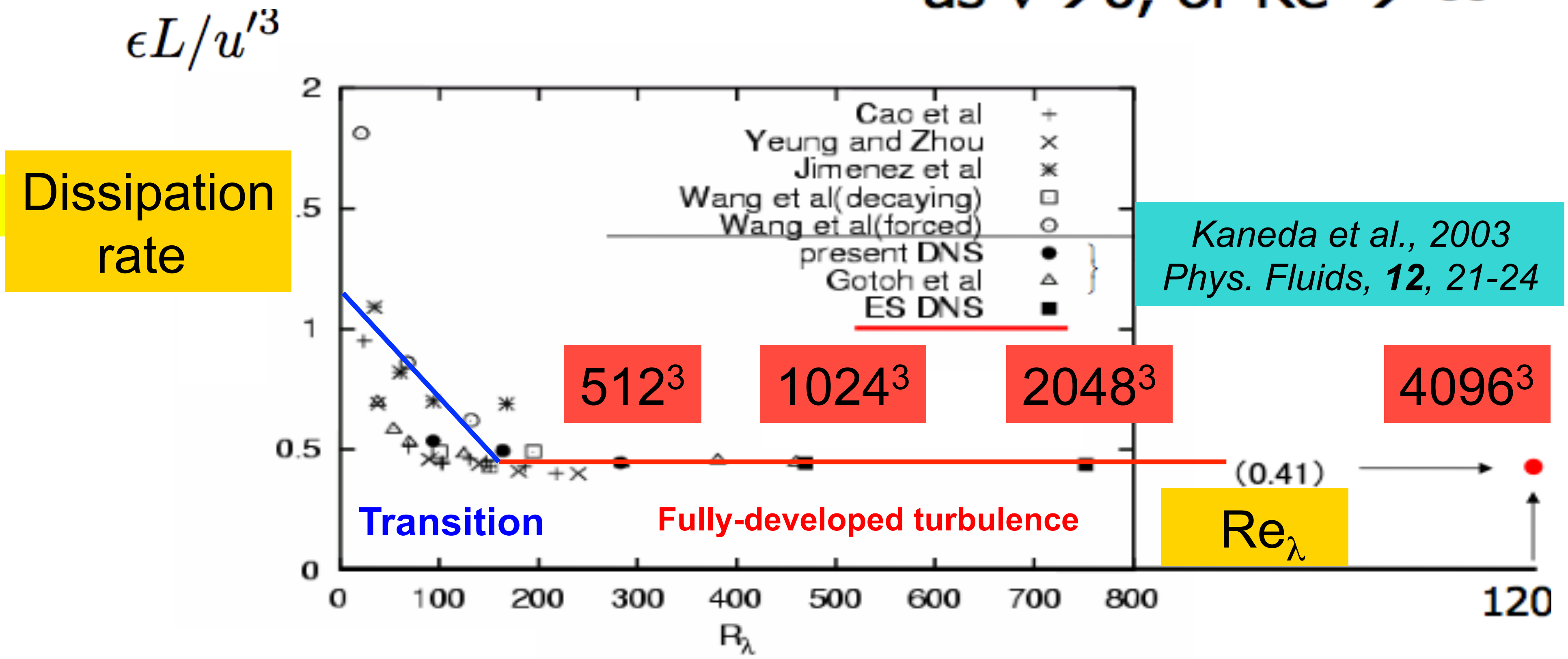
Dissipation rate: laboratory experiments



*Vassilicos, Ann. Rev. Fluid Mech.,**47**, 2015*

Dissipation rate : numerical experiments

Normalized energy dissipation → ?
as ν→0, or Re → ∞



Both laboratory experiments and
numerical experiments of turbulent flows
show that the dissipation rate
becomes independent of the fluid viscosity

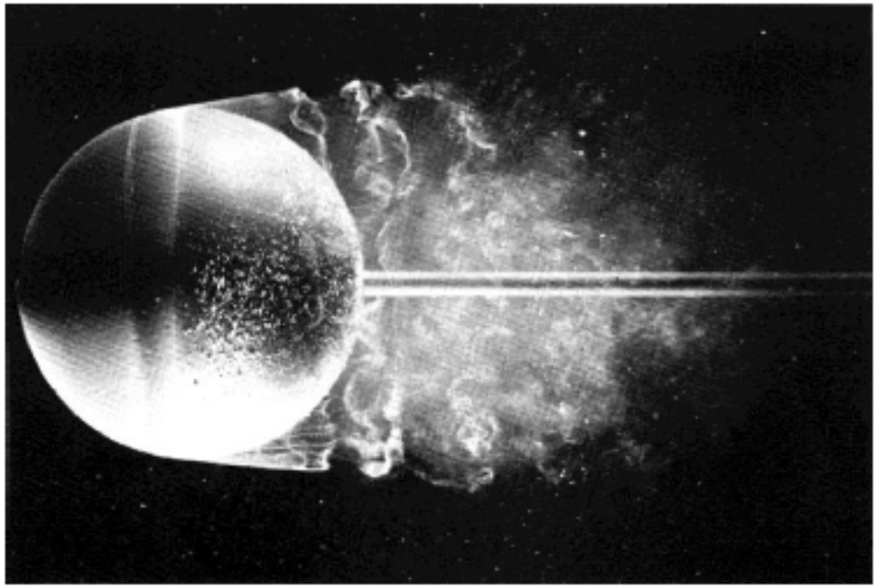
Dissipation of energy in the inviscid limit

- In an incompressible flow ($\rho = 1$)

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\int \frac{\mathbf{u}^2}{2} = -\nu \int \omega^2 = -2\nu Z$$

- To dissipate energy, vorticity needs to be created and/or amplified, in such a way that $Z \sim \nu^{-1}$.

Possible vorticity distributions:
$\omega \sim \nu^{-1/2}$ over $O(1)$ area,
$\omega \sim \nu^{-1}$ over $O(\nu)$ area.



E energy, Z enstrophy,
ν fluid kinematic viscosity
ω flow vorticity.

Why is dissipation of energy so essential ?

- Kato (1984) proved (roughly stated):

The NS solution converges towards the Euler solution in L^2:

$$\forall t \in [0,T], \left\| u_{\mathrm{Re}}(t) - u(t) \right\|_{L^2(\Omega)} \underset{\mathrm{Re}\to\infty}{\longrightarrow} 0,$$

if and and only if

the energy dissipation during this interval vanishes:

$$\Delta E_{\mathrm{Re}}(0,T) = \mathrm{Re}^{-1} \int_0^T \mathrm{d}t \int_\Omega \mathrm{d}\mathbf{x} \left| \nabla \mathbf{u}(t,\mathbf{x}) \right|^2 \underset{\mathrm{Re}\to\infty}{\longrightarrow} 0,$$

and even if and only if

it vanishes in a strip of width prop to **$\mathbf{Re}^{-1}$** around the solid:

$$\mathrm{Re}^{-1} \int_0^T \mathrm{d}t \int_{\Gamma_{c\,\mathrm{Re}^{-1}}} \mathrm{d}\mathbf{x} \left| \nabla \mathbf{u}(t,\mathbf{x}) \right|^2 \underset{\mathrm{Re}\to\infty}{\longrightarrow} 0, \qquad \Gamma_{c\,\mathrm{Re}^{-1}} = \left\{ \mathbf{x} \,\middle|\, d(\mathbf{x},\partial\Omega) < c\,\mathrm{Re}^{-1} \right\}.$$

An important practical consequence

- To have any chance of observing energy dissipation (i.e. default of convergence towards the Euler solution), we need a smaller grid than Prandtl's (1904) prediction for attached boundary layers:

~~$\delta x \propto \mathrm{Re}^{-\frac{1}{2}}$~~

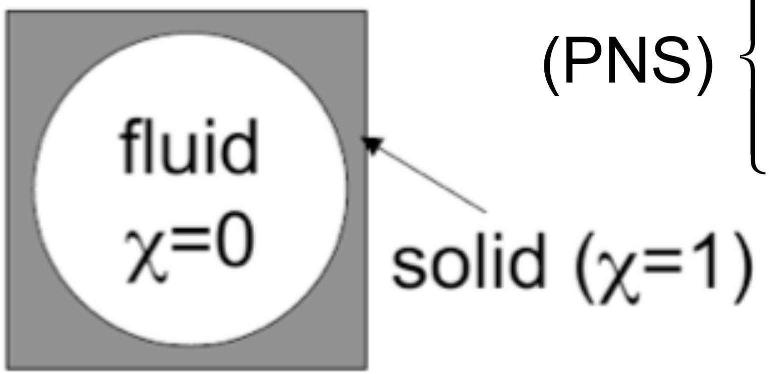
$$\delta x \propto \mathrm{Re}^{-1}$$

3.

Numerical experiments

Volume penalization method to compute NS

- For efficiency and simplicity, we would like to stick to a spectral solver in periodic, cartesian coordinates.
- as a counterpart, we need to add an additional term in the equations to approximate the effect of the boundaries,
- the geometry is encoded in a mask function χ ,



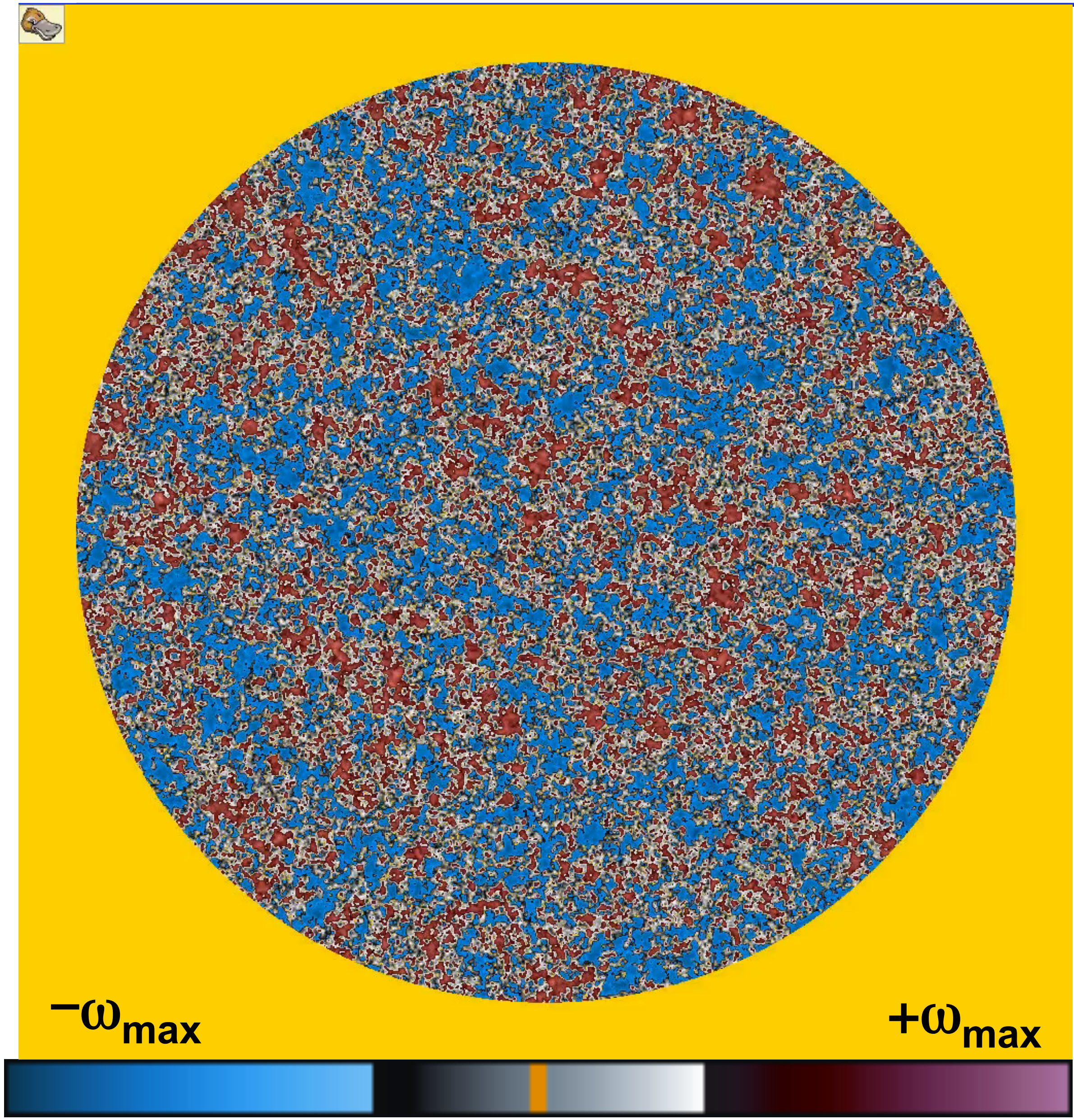
$$\text{(PNS)} \begin{cases} \partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla p + \frac{1}{\mathrm{Re}}\Delta\mathbf{u} - \frac{1}{\eta}\chi_0\mathbf{u} \\ \nabla\cdot\mathbf{u} = 0, \quad \mathbf{u}(0,\mathbf{x}) = \mathbf{v} \end{cases}$$

$$\xrightarrow{\text{solution}} \mathbf{u}_{\mathrm{Re},\eta}$$

E. Arquis and J.P.Caltagirone,CRAS, 1984

*M. F. and K. Schneider, PRL, **95**, 2005*

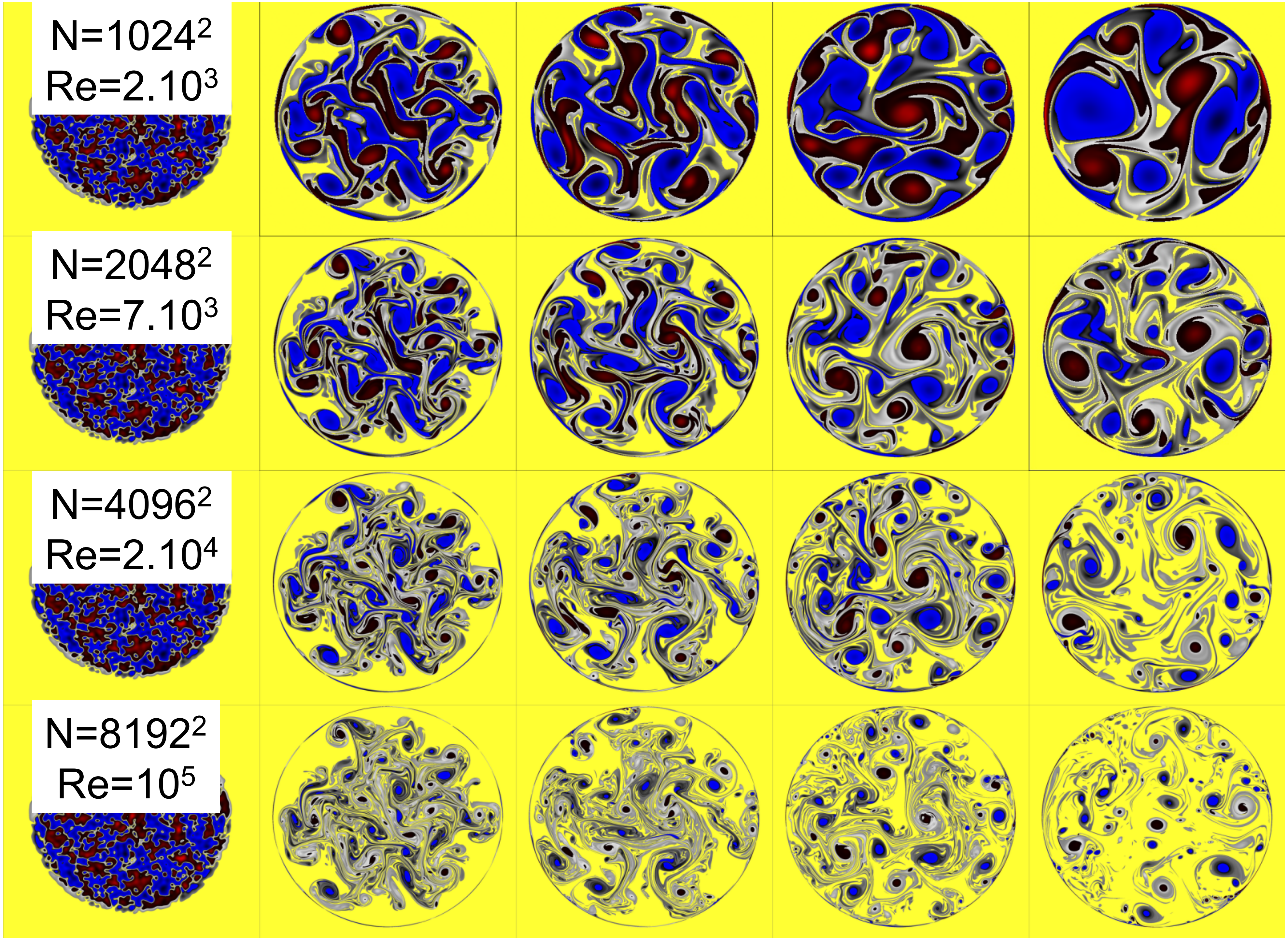
31. Wall-bounded 2D turbulent flow



DNS
Resolution
N=1024^2

Dealiased pseudo-spectral In space and 3rd order Runge-Kutta In time

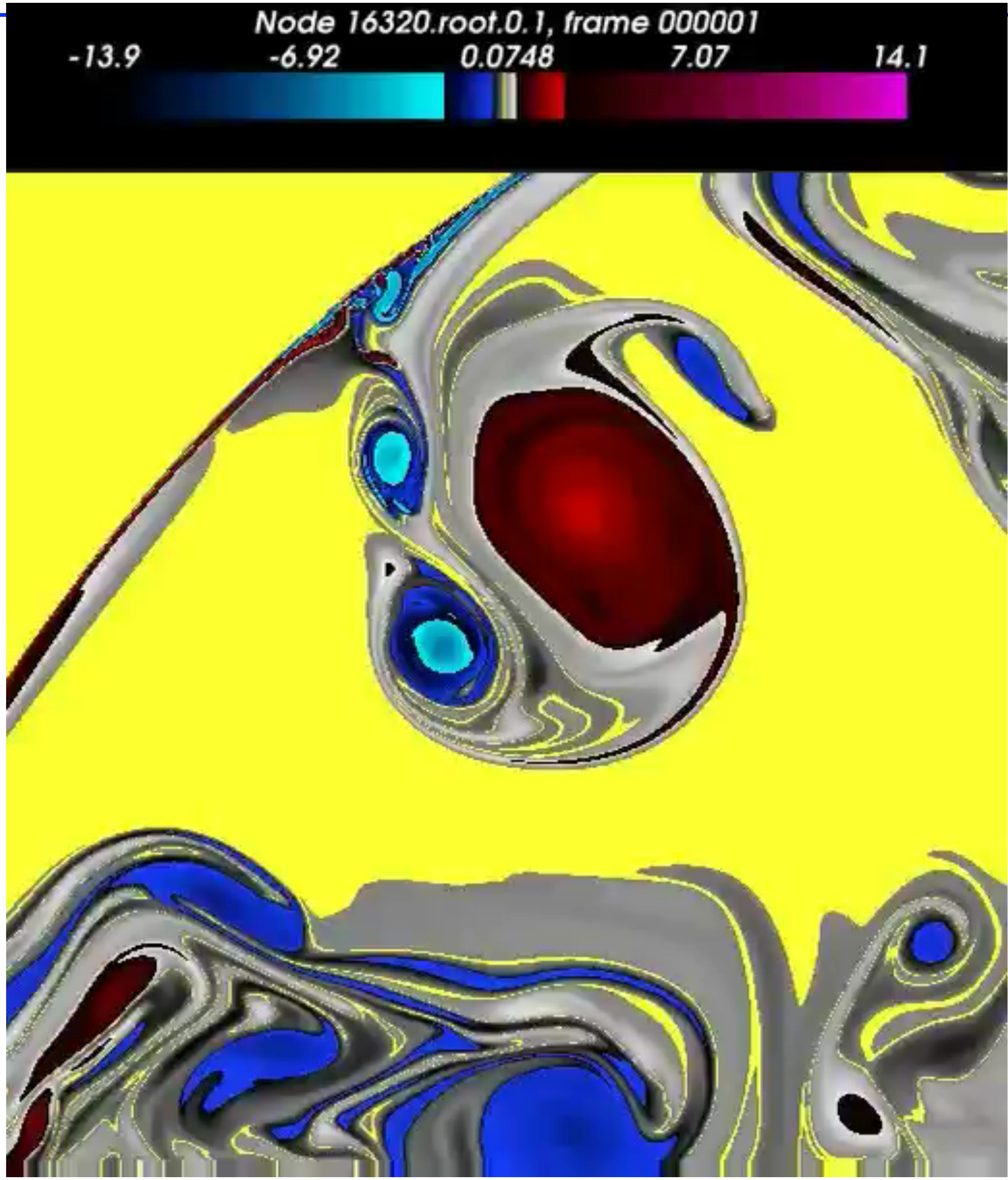
*K. Schneider and M. F., Phys. Rev. Lett., **95**, 244502 (2005)*



N=1024²
Re=2.10³
N=2048²
Re=7.10³
N=4096²
Re=2.10⁴
N=8192²
Re=10⁵

Flow interaction with the wall

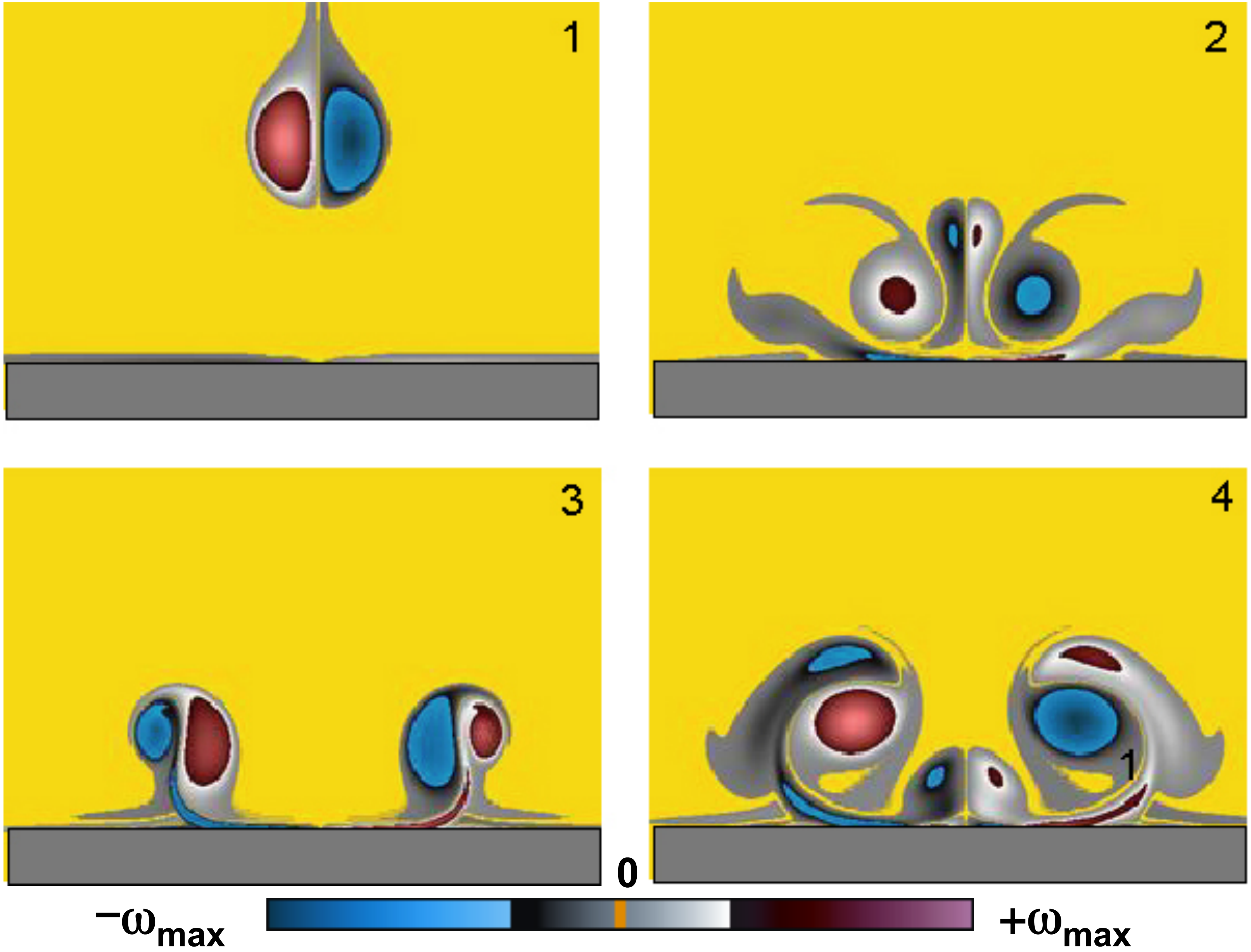
Resolution
$N=8192^2$



Time evolution of vorticity at the wall computed on IBM Blue-Gene, IDRIS, 2010 (100 Tflops)

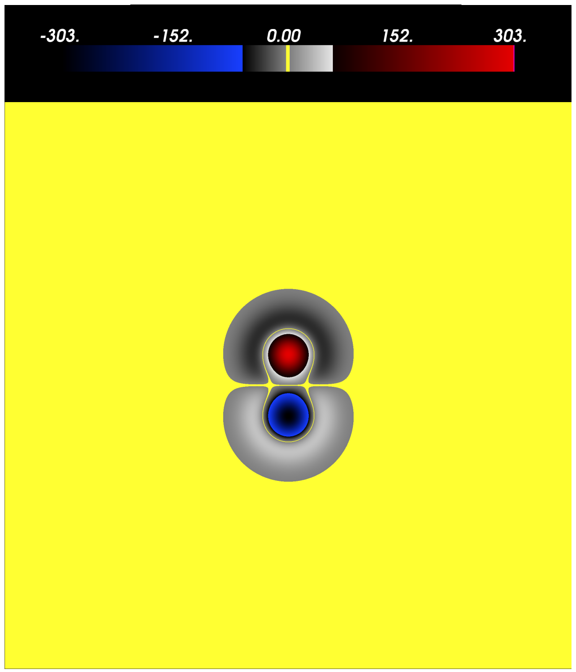
Nguyen van yen, M. F. and Schneider, 2010

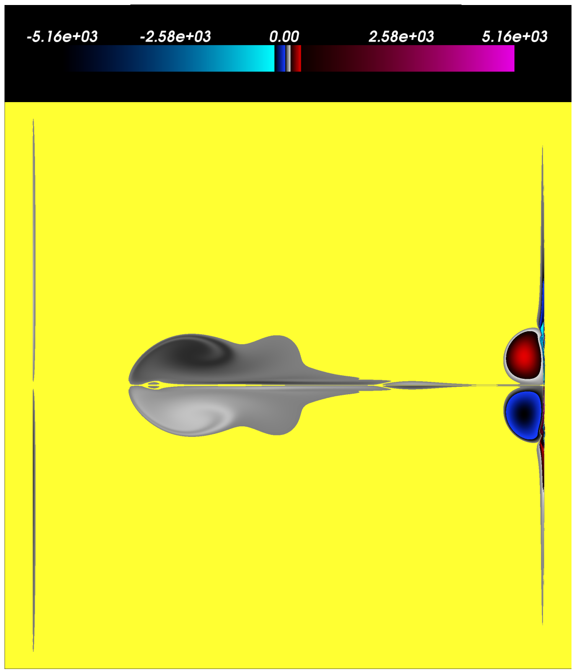
32. Dipole crashing onto a wall

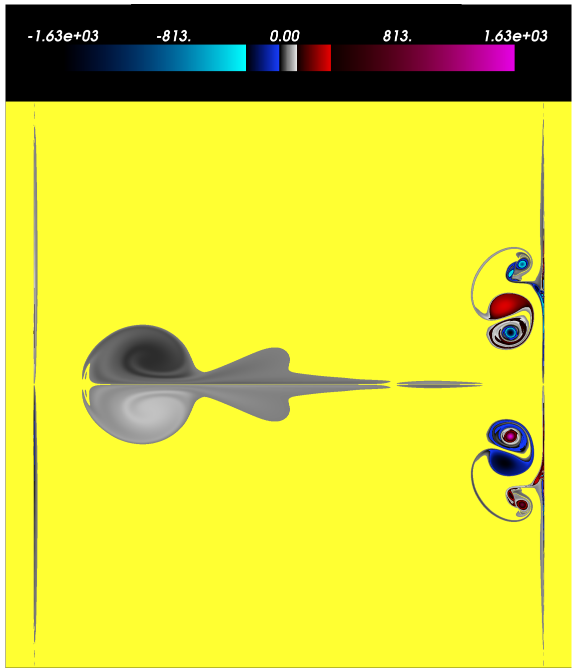


Dipole crashing onto a wall at Re=8000

DNS
Resolution
$N=8192^2$

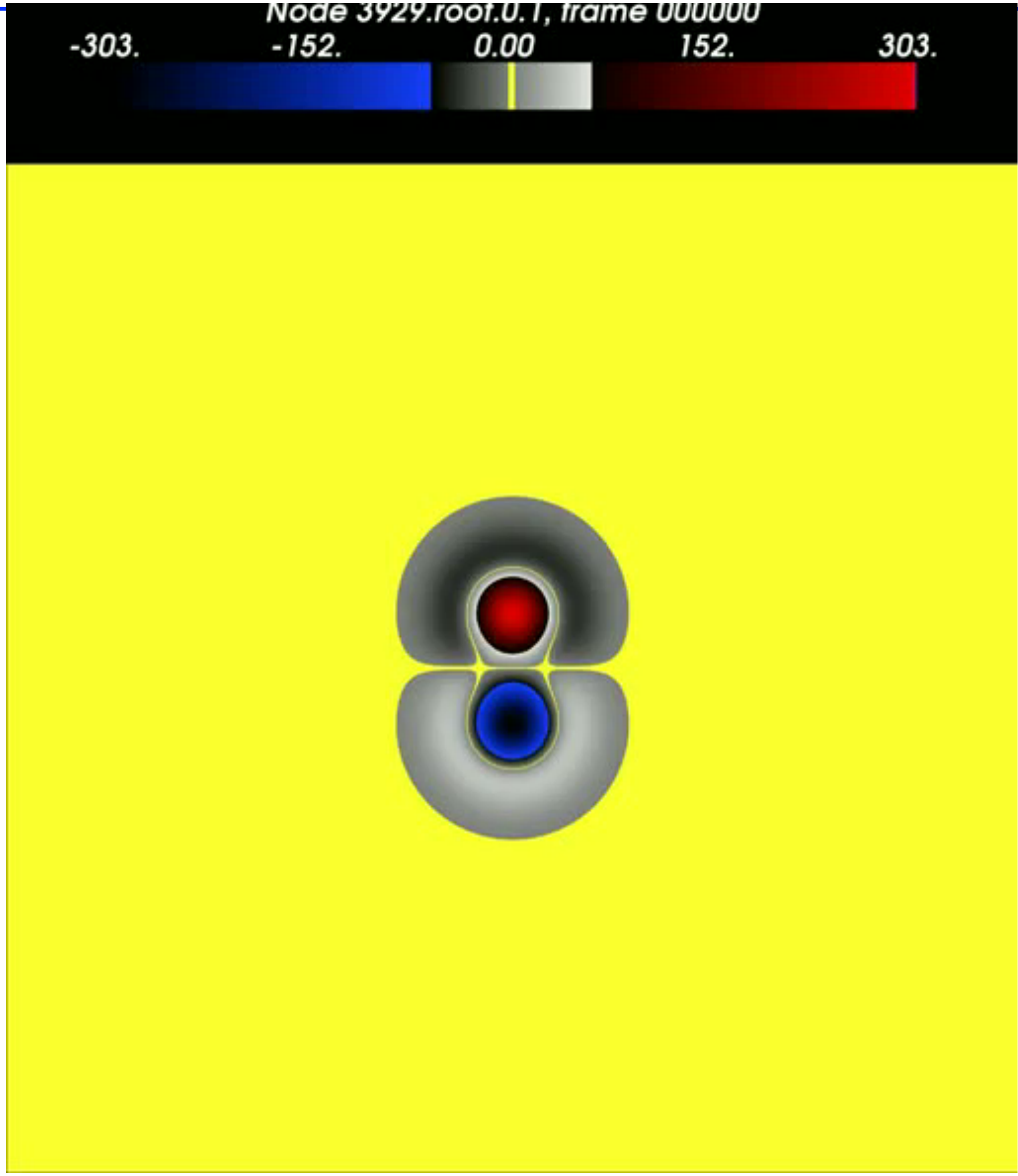






Dipole crashing onto a wall at Re=8000

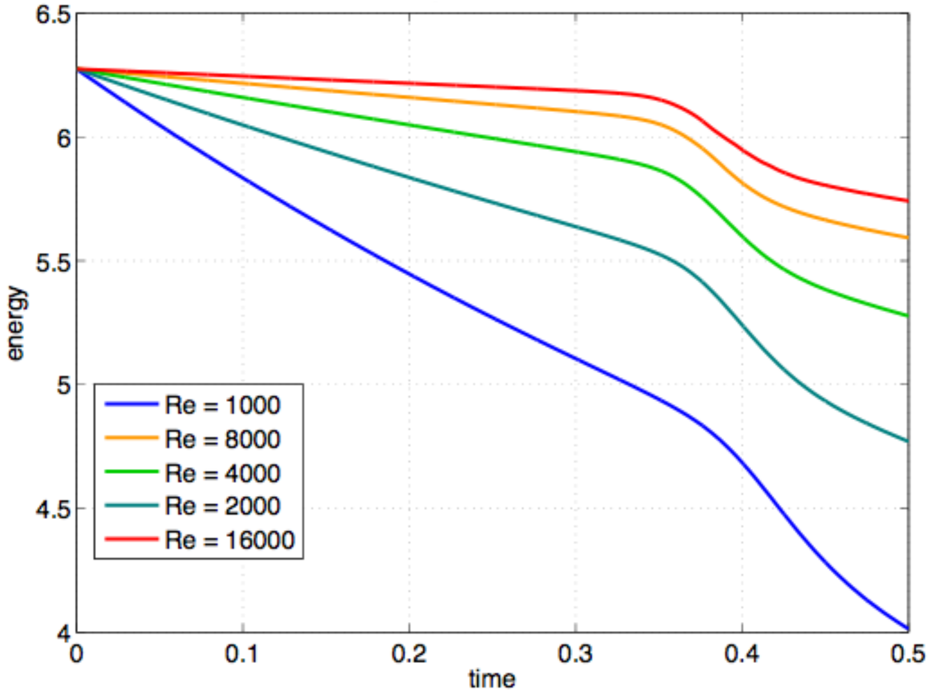
DNS
Resolution
N=8192^2



Energy evolution

DNS
Resolution
$N=8192^2$

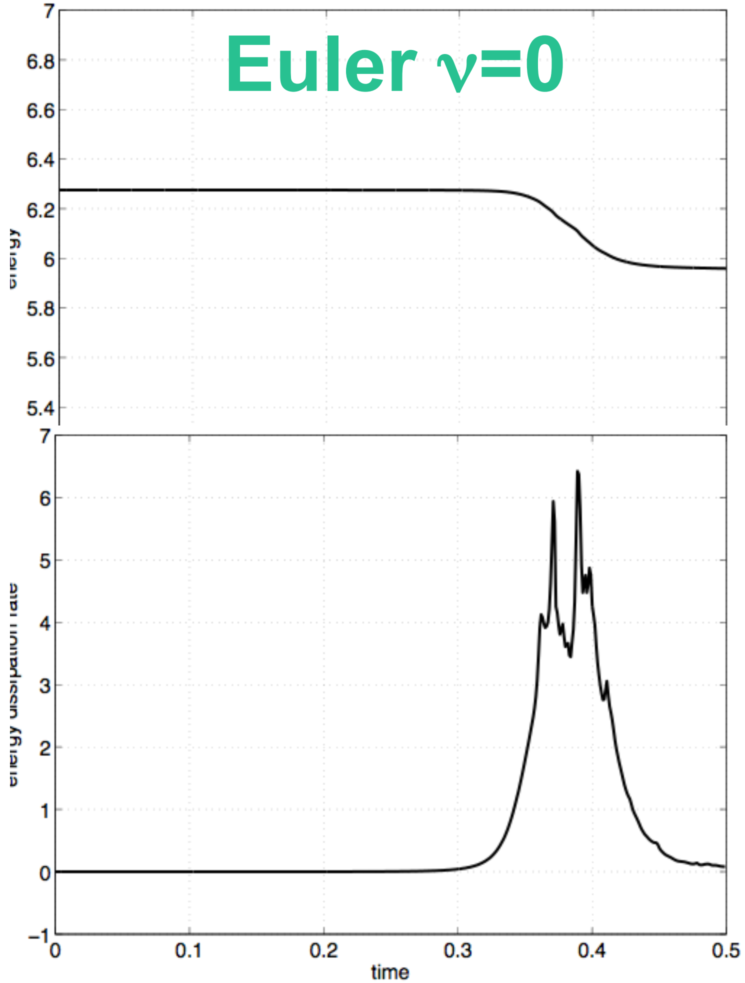
Navier-Stokes
$\nu>0$



Time evolution of energy

Time evolution of energy

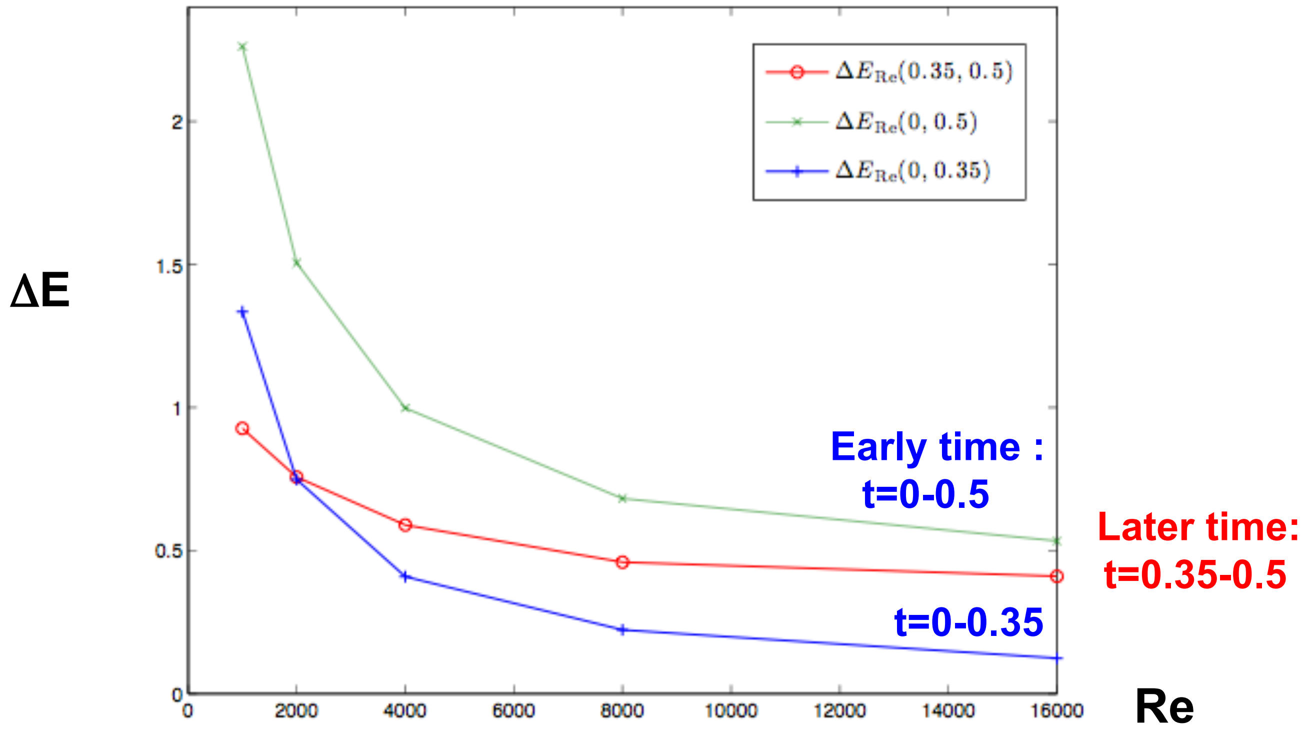
Euler $\nu=0$



Evolution of energy dissipation rate

Energy dissipation

Energy dissipated
when the dipole crashes onto the wall
at increasing Reynolds numbers



Dissipative structures

- Our experiments with the dipole crashing onto a wall suggest that the flow remains dissipative
  in the inviscid limit,
- it is tempting to relate these structures
  to energy dissipation,
- the kinetic energy density $e = \frac{|\mathbf{u}|^2}{2}$ obeys:

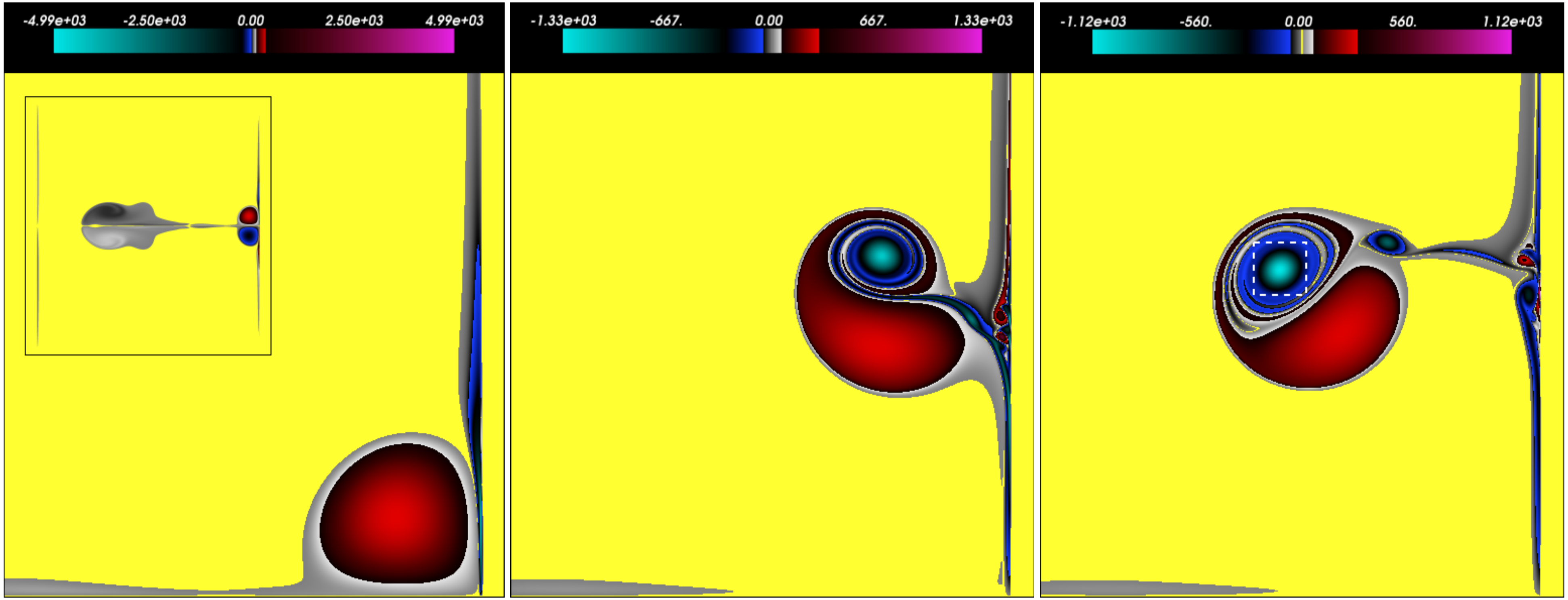
$$\partial_t e + \mathbf{u} \cdot \nabla (e + p) = \nu \Delta e - \nu |\nabla \mathbf{u}|^2$$

Local dissipation rate

DNS of dipole crashing onto a wall

Resolution
N=16384^2

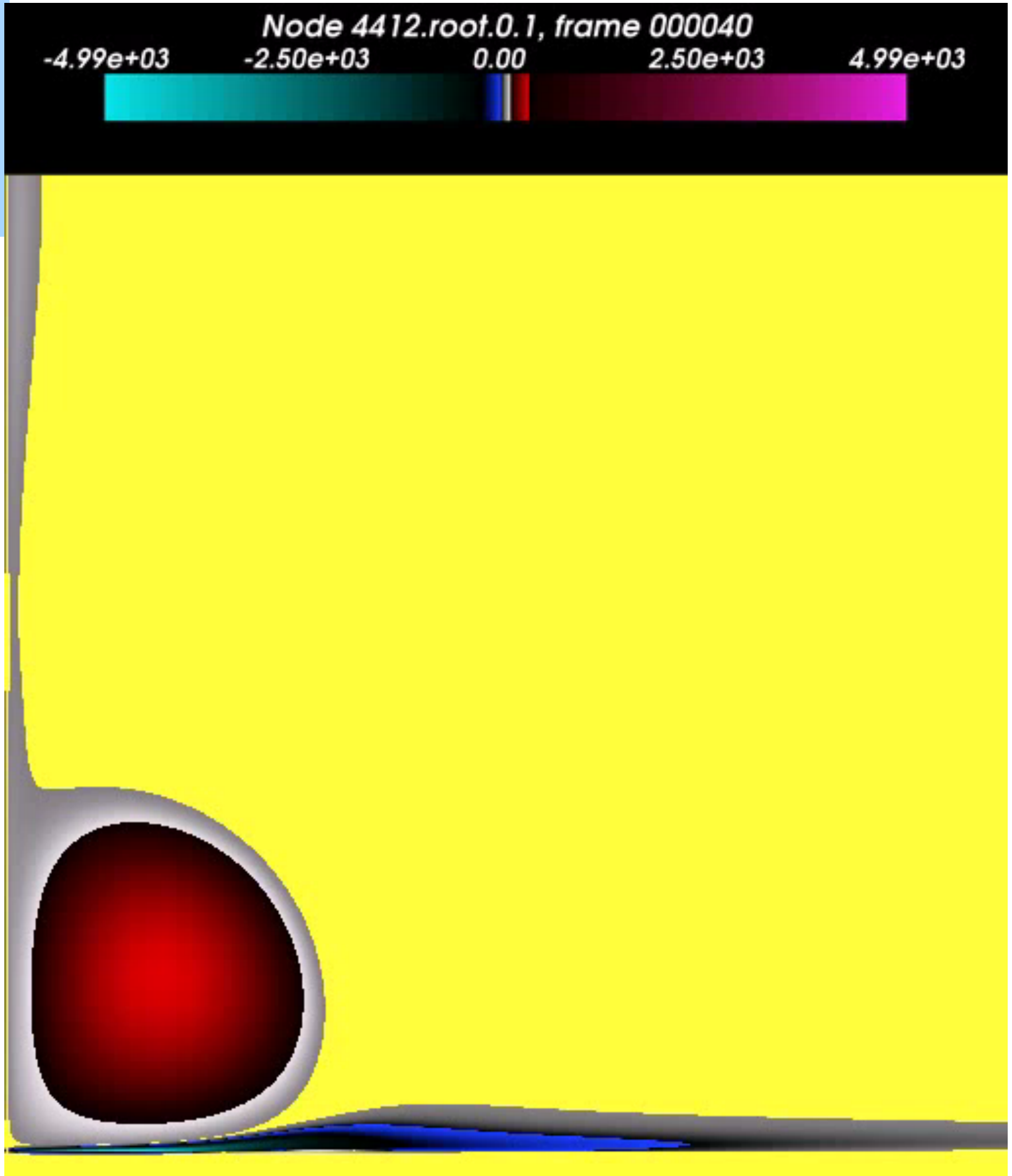
*Nguyen van yen, M. F. and Schneider, PRL, **106**(18)*



t=0.3 *t=0.4* *t=0.5*

Dipole-wall collision at Re=8000

DNS
Resolution
$N=8192^2$

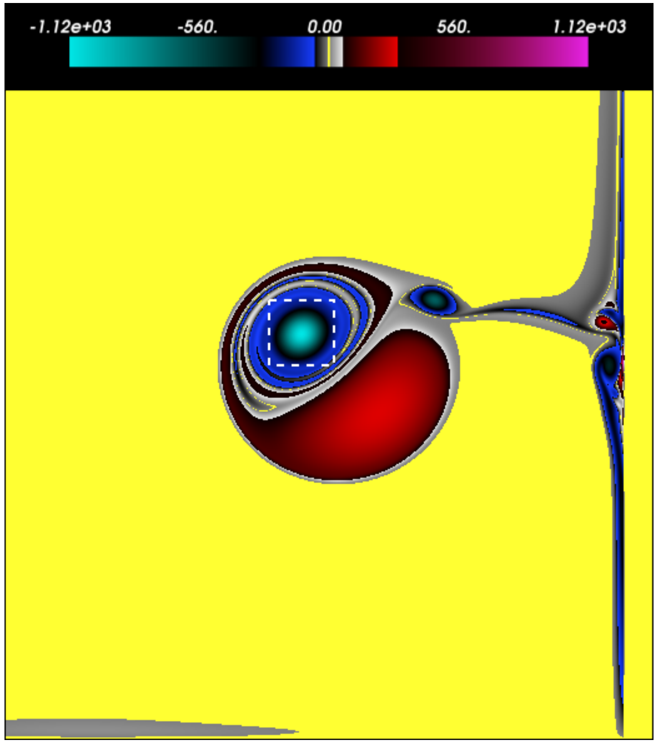


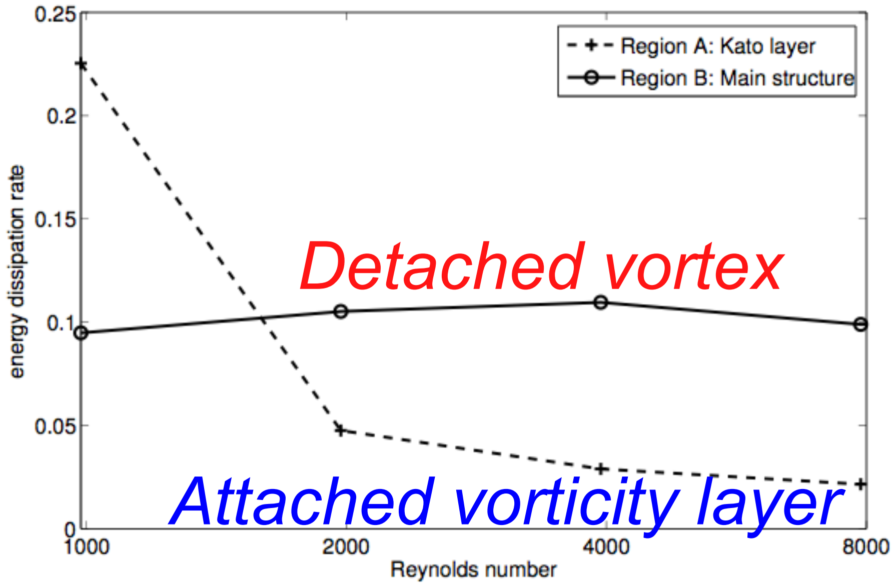
Production of vortices where boundary layer detaches

*Nguyen van yen, M. F. and Schneider, PRL, **106**(18)*

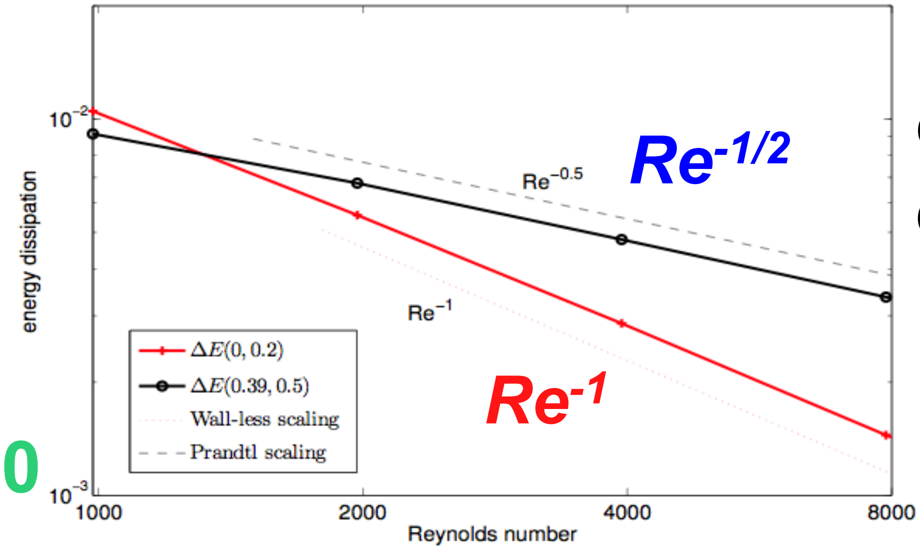
Dissipative structures

*Nguyen van yen, M. F. and Schneider, PRL, **106**(18)*





energy dissipation rate

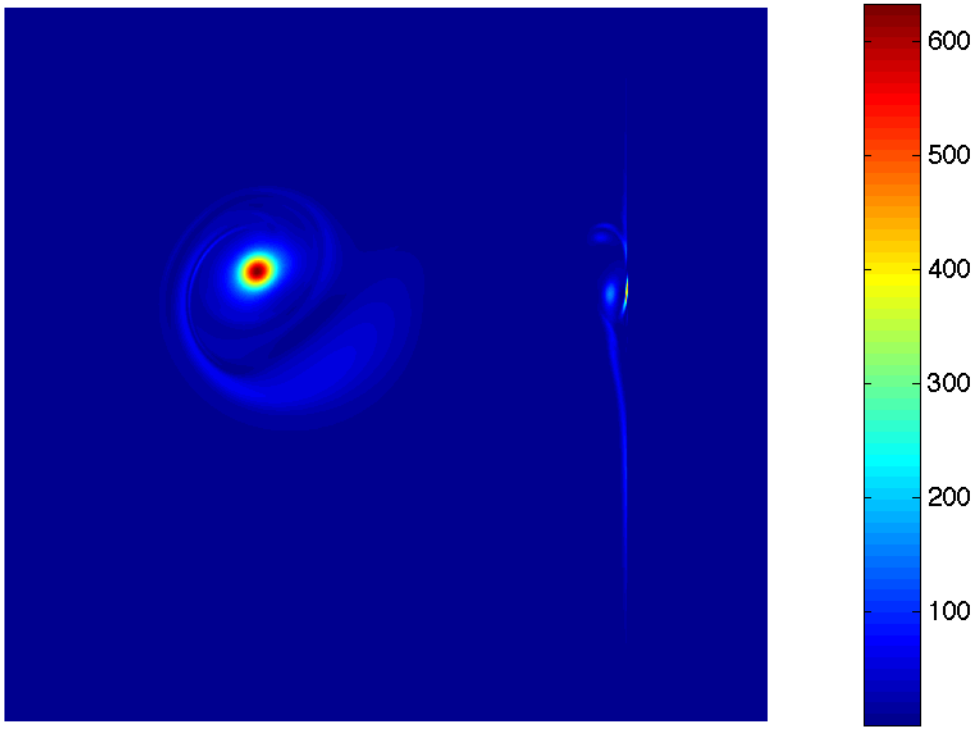


energy dissipation

Re=1000

Re=8000

Snapshot of the local dissipation rate

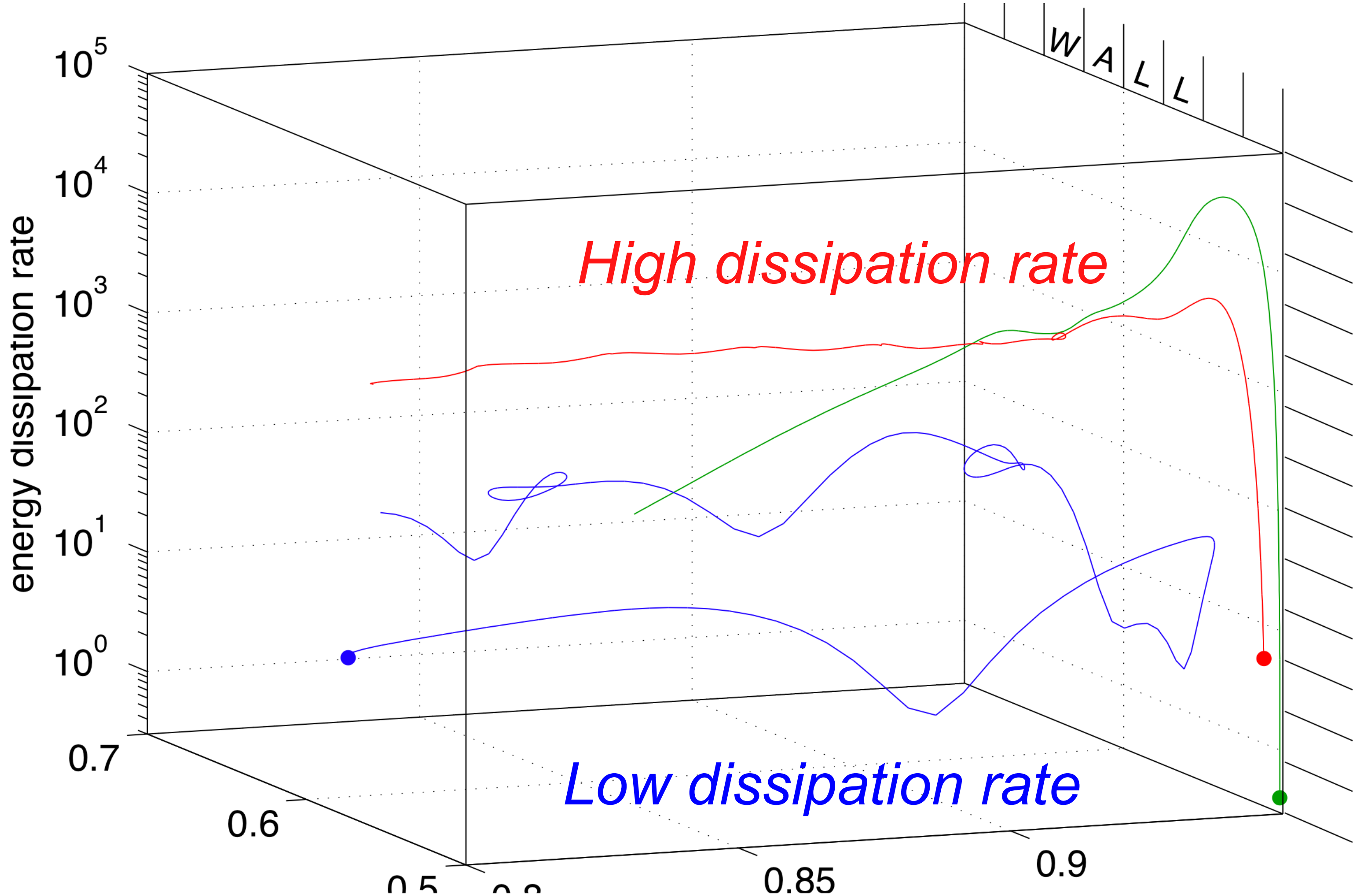


Local dissipation rate
for the dipole-wall collision
at t= 0.5

The strongest values of the energy dissipation rate is observed inside the main vortex that detached from the boundary layer,

rather than inside the boundary layer itself.

Production of dissipative structures

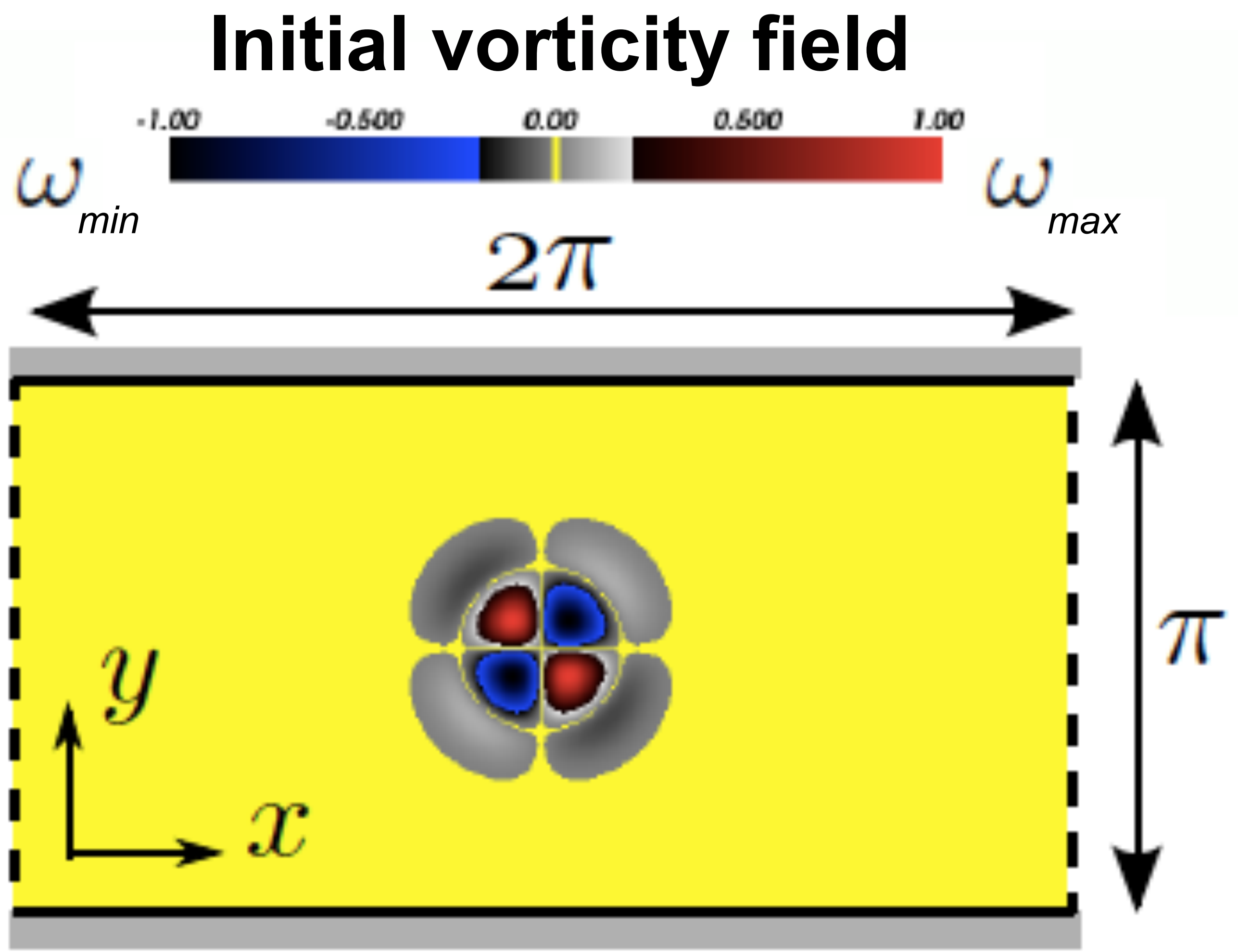


*R. Nguyen van yen, M. F. and K. Schneider, PRL, **106**(18)*

Energy Dissipating Structures Produced by Walls in Two-Dimensional Flows at Vanishing Viscosity

R. Nguyen van yen, M. F. and K. Schneider, 2011

33. Quadrupole in a channel flow



$$\psi_i(x,y) = Axy \exp\left(-\frac{(x-x_0)^2+(y-y_0)^2}{2s^2}\right)$$

Prandtl equations

Ansatz for the vorticity field as $\mathrm{Re} \to \infty$:

$$\omega(x,y) = \omega_E(x,y) + \nu^{-1/2}\omega_P(x,\nu^{-1/2}y) + \omega_R(x,y)$$

$$y_P = y / \nu^{1/2}$$

$$\partial_t \omega_P + \boldsymbol{\nabla}.(\mathbf{u}_P \omega_P) = \partial^2_{y_P}\omega_P$$

$$\omega_P(x, y_P, 0) = 0$$

$$\psi_P(x, y_P, t) = \int_0^{y_P} \mathrm{d}y'_P \int_0^{y'_P} \mathrm{d}y''_P \omega_P(x, y''_P, t)$$

$$\partial_{y_P}\omega_P(x,0,t) = -\partial_x p_E(x,0,t),$$

where p_E is the pressure calculated from ω_E which is the vorticity given by Euler equation

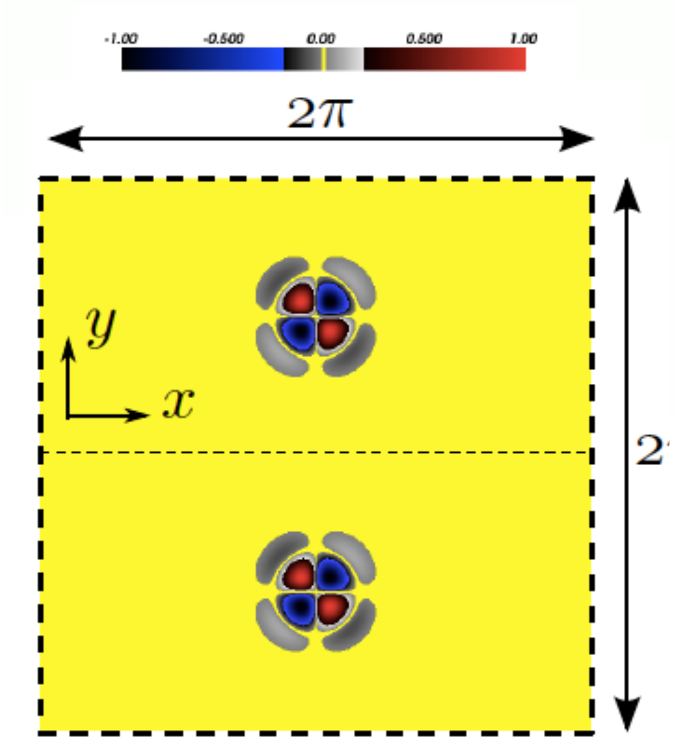
Prandtl solver

- Artificial boundary condition: $\partial_{y_P}\omega_P = 0$ at $y_P = 64$
- Spatial discretization: Fourier in x
  and compact finite differences of 5th order in y
- Time discretization: low storage third order Runge-Kutta in t

- Neumann boundary condition for vorticity:
  $$\partial_{y_P}\omega_P = -\partial_x p_E \text{ at } y_P = 0$$
  where p_E is the pressure calculated from ω_E

- To close the system we impose
  $$\partial^2_{y_P}\omega_P = 0 \text{ at } y_P = 64$$
  which is consistent with the rapid decay of ω_P

Euler solver

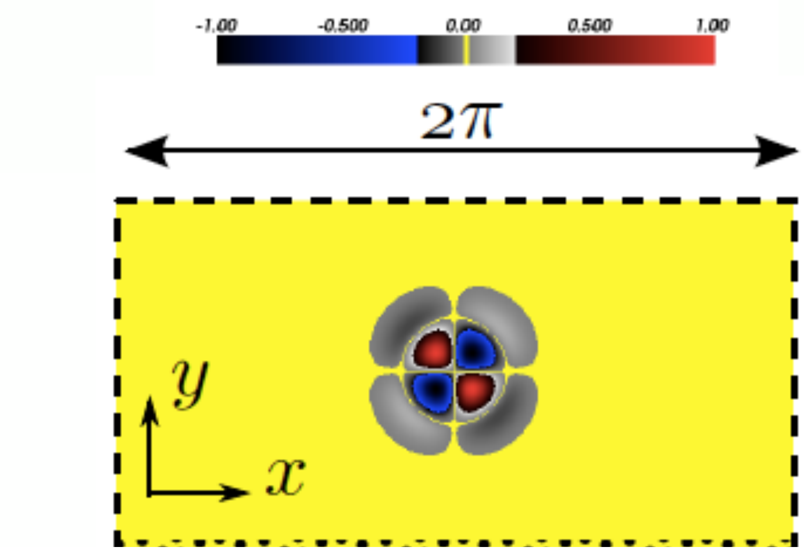


- Use mirror symmetry around $y = 0$ to impose boundary condition.
- Spatial discretization: Fourier pseudo-spectral with hyperdissipation, $k_{max} = 682$.
- Time discretization: third order low storage Runge-Kutta, with exponential propagator for the viscous term.

Navier-Stokes solver

Fourier/compact finite differences (5th order)

- Similar to the one for the Prandtl equations, except that non-uniform grids are used in the *y* direction.
- Two linear integral constraints are applied on vorticity to satisfy the no-slip boundary conditions in *y*.
-Integrating factor for the viscous term and 3rd order Runge-Kutta
-for the advection term.

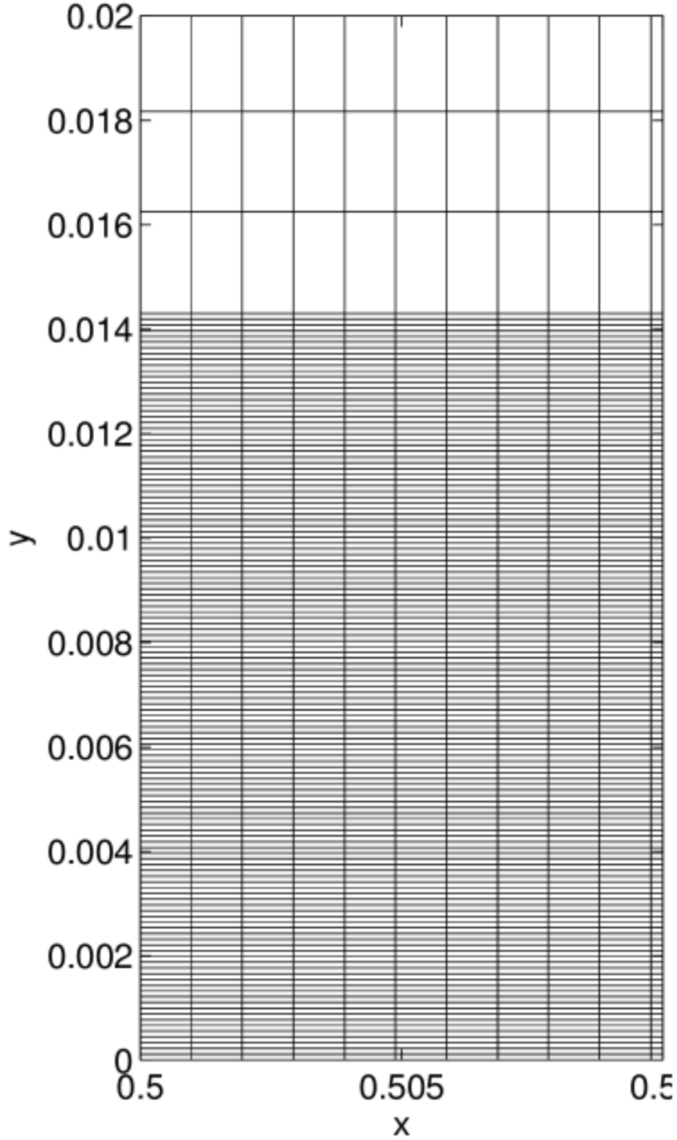


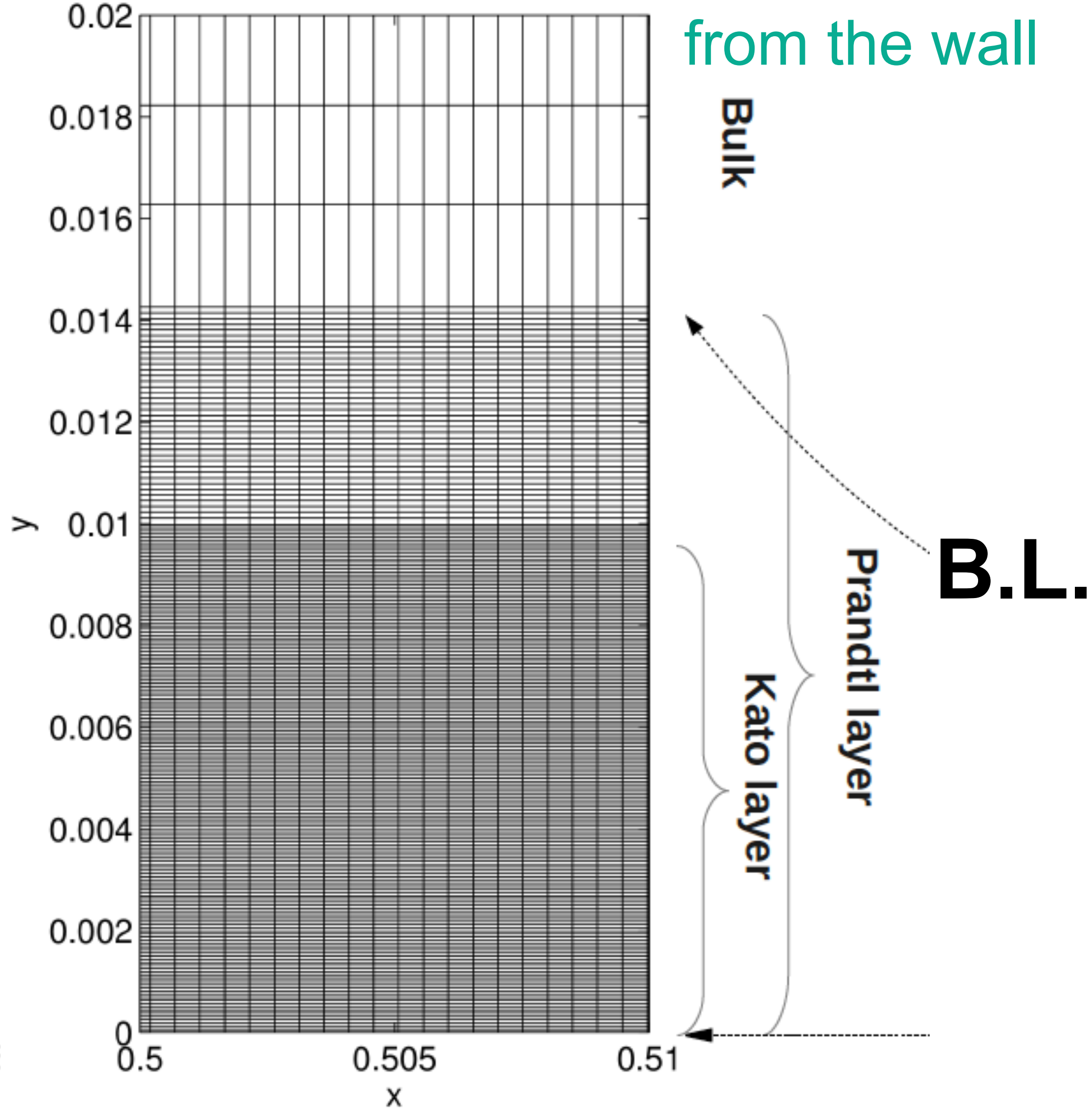
$N_x = 1024$
$N_y = 385 - 449$

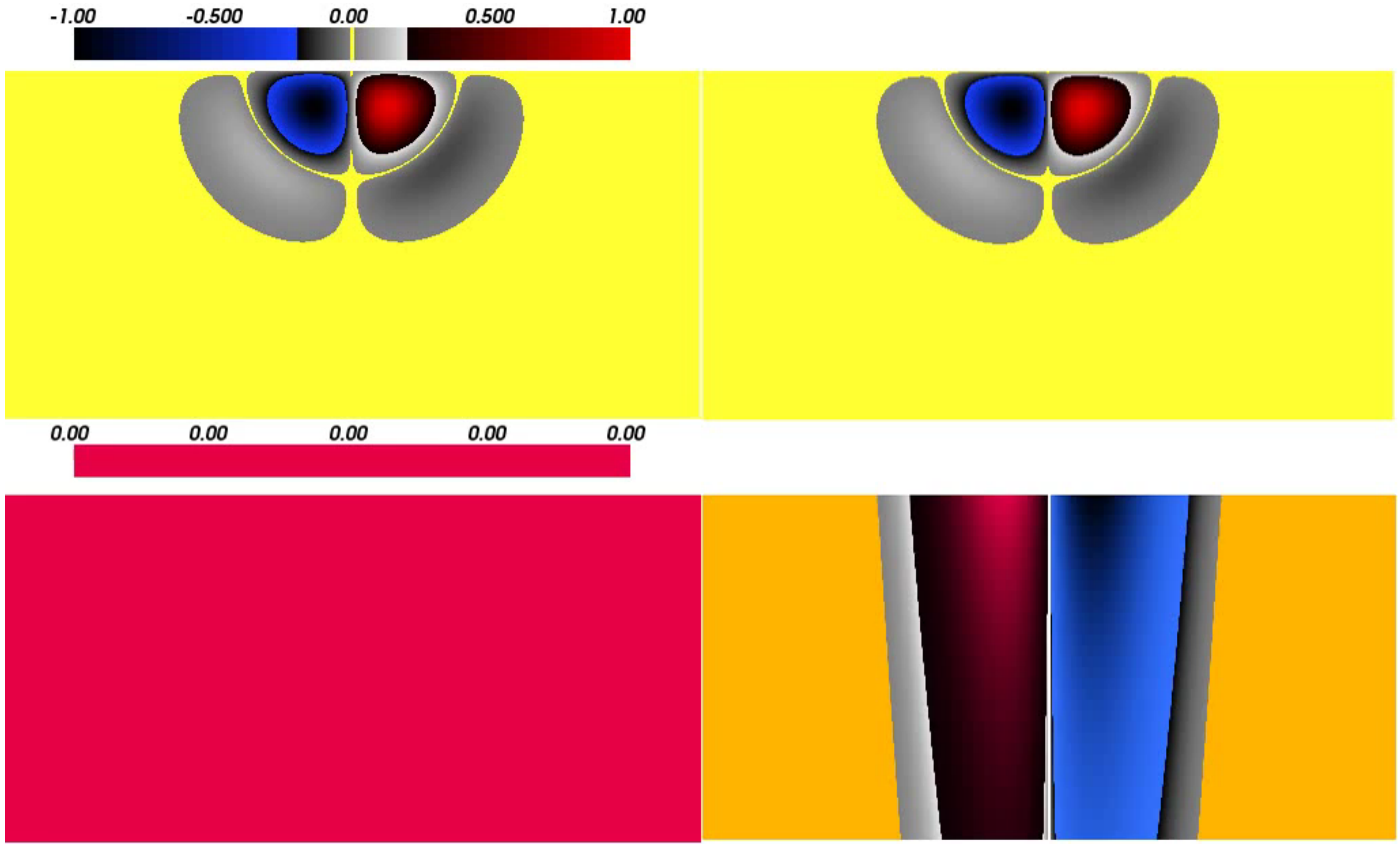
Computational grid

Before t=54

After t=54 when B.L. detachs from the wall







Euler Prandtl couplées **Navier-Stokes**

Prandtl's singularity

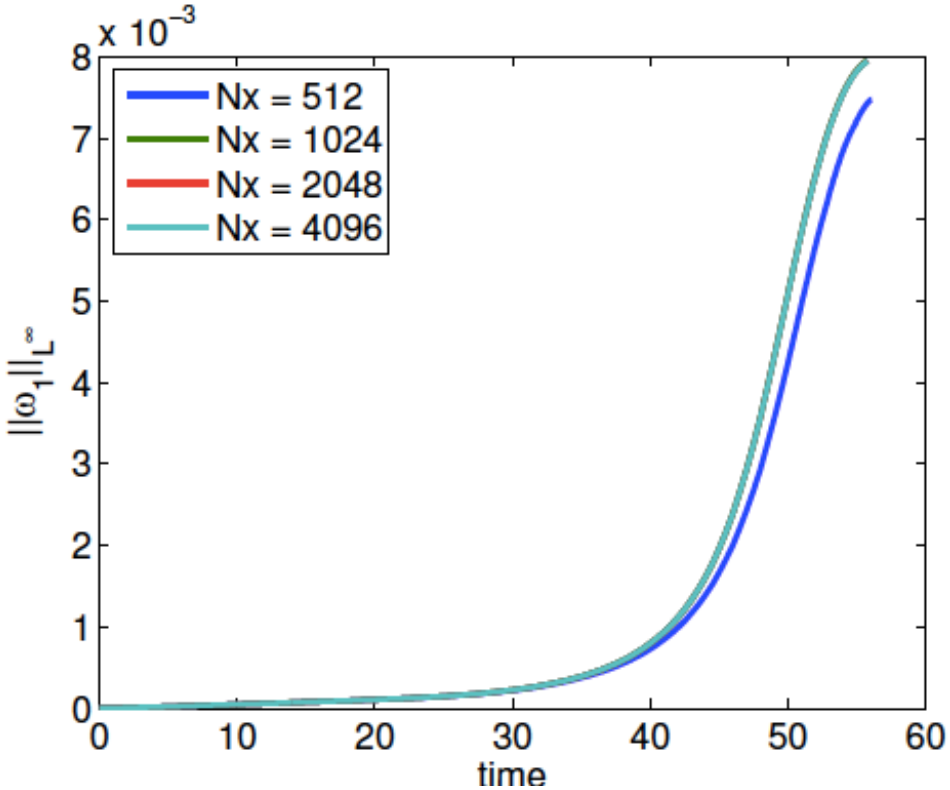
Prandtl equation has well-known finite time singularity

- $|\partial_x \omega_1|$ and $u_{1,y}$ blows up,
- ω_1 remains bounded.

L. L. van Dommelen and S. F. Shen., 1980
*J. Comp. Phys., **38**(2)*

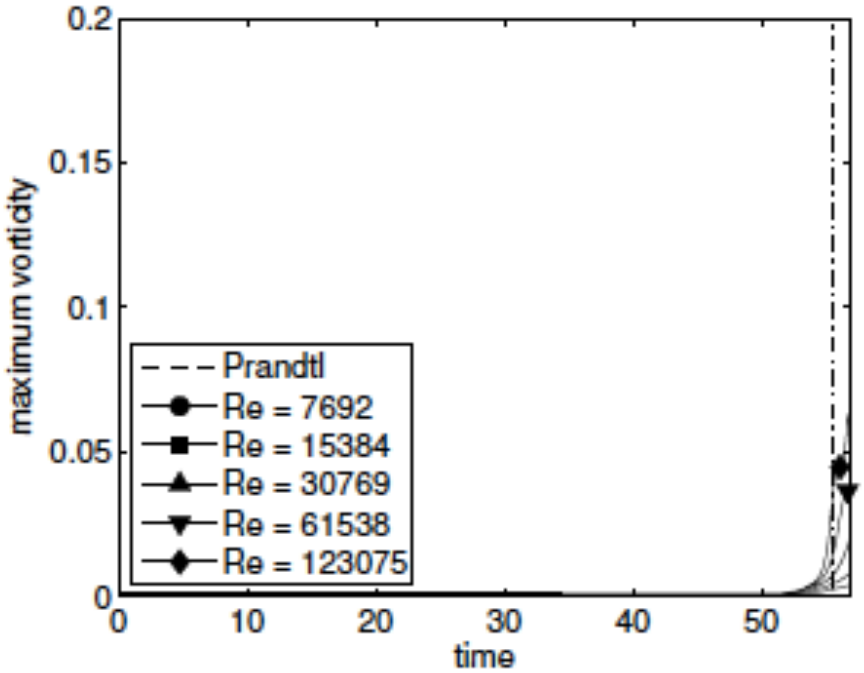
This is observed
in our computations
as expected,

for $t \to t_D \simeq 55.8$

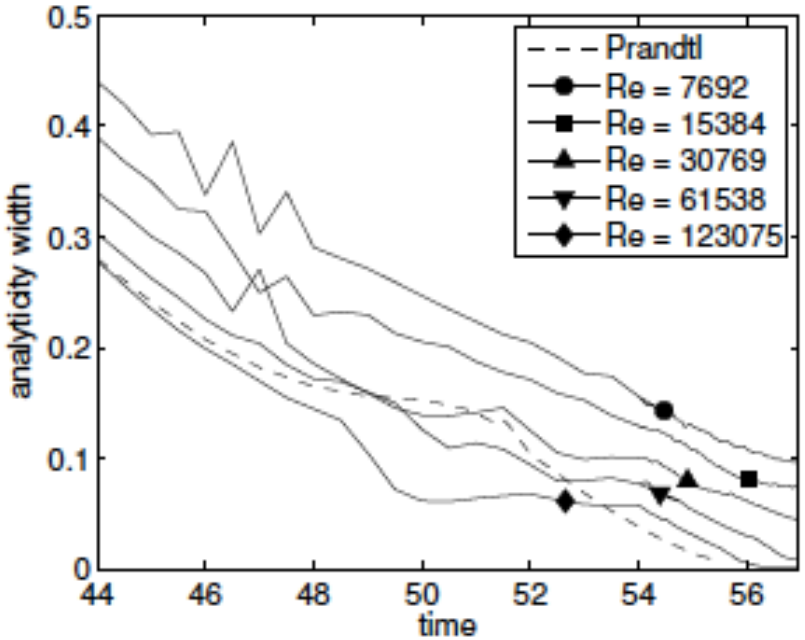


Prandtl solution's blow-up

According to Kato's theorem, and since ω_1 remains bounded uniformly until t_D, we expect that $\mathbf{u}_\nu \xrightarrow[\nu\to 0]{L^2} \mathbf{u}_0$ uniformly on $[0, t_D]$.



Evolution of vorticity max

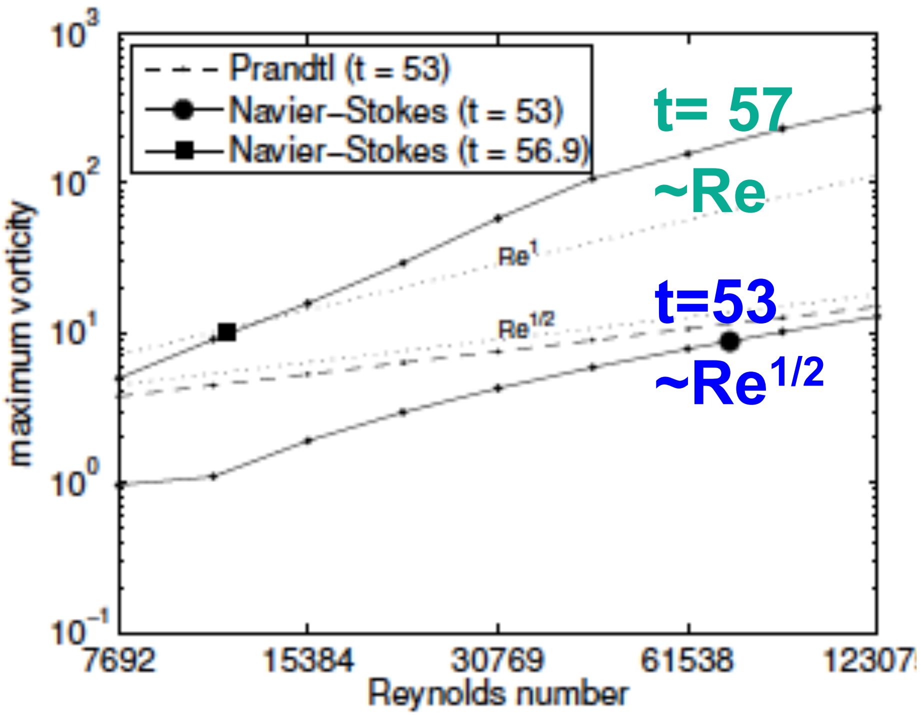


Evolution of analyticity strip

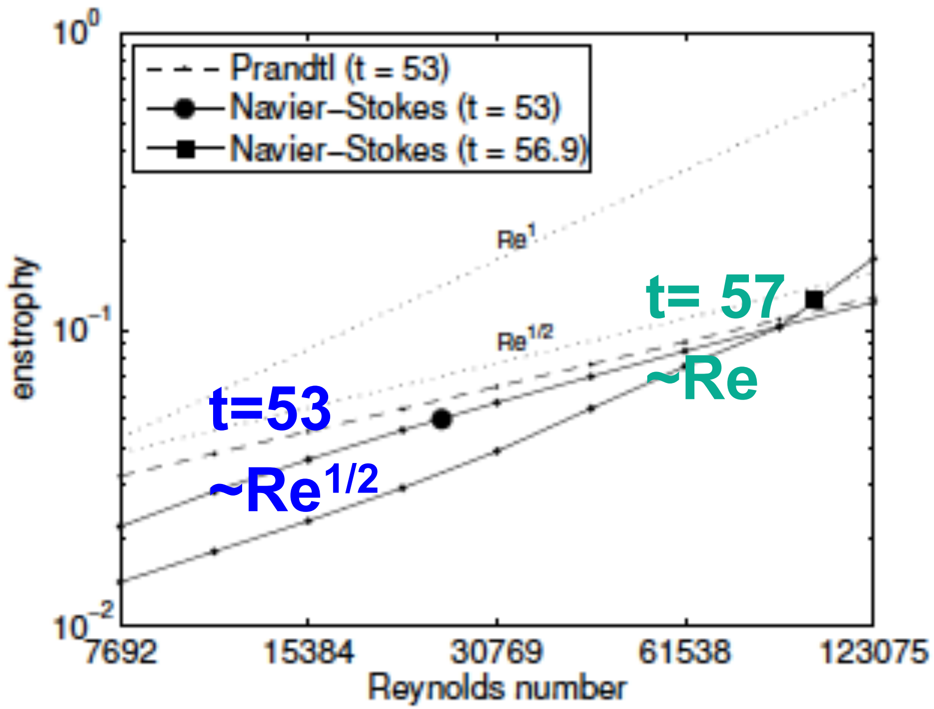
Show convergence!

What happens after the singularity?

Vorticity max

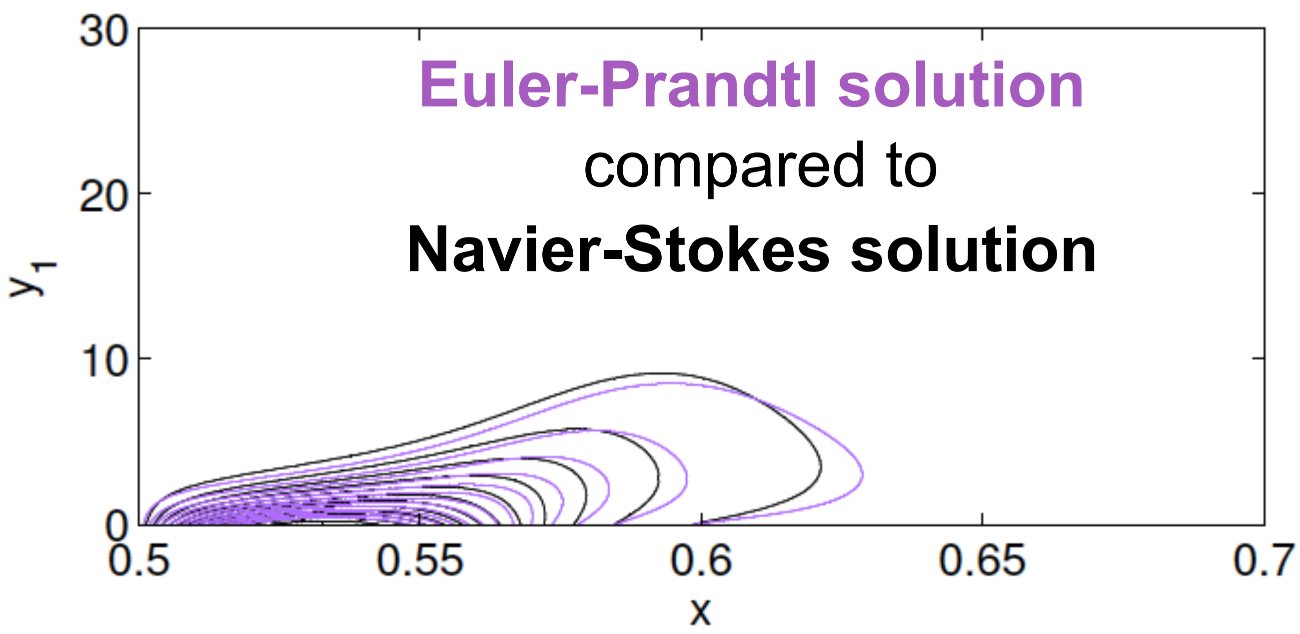


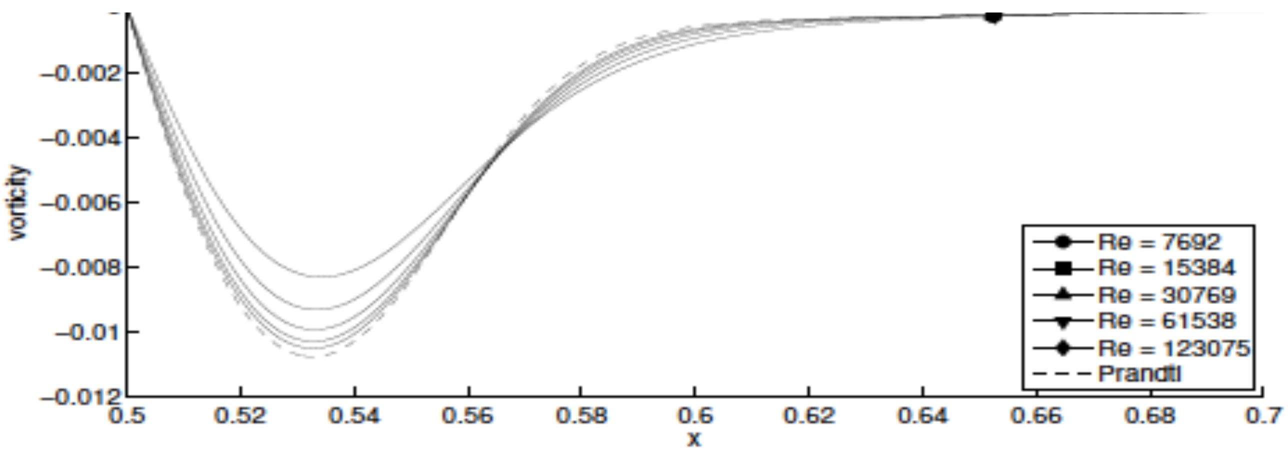
Enstrophy



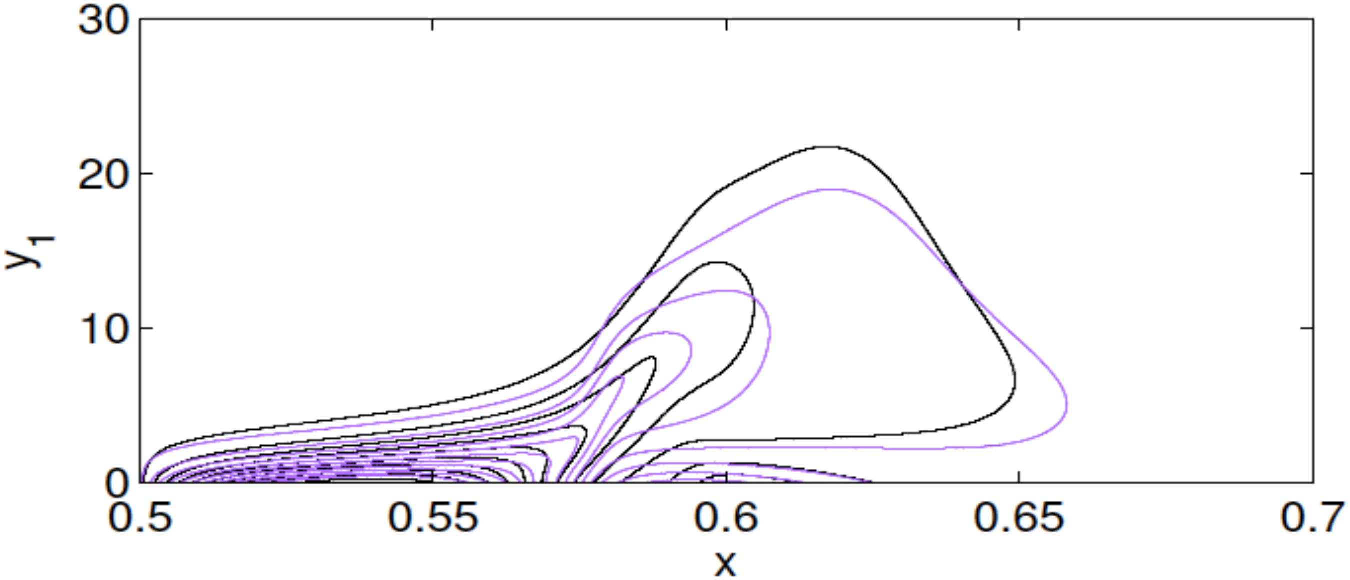
We observe Prandtl's scaling in $Re^{1/2}$ before t_D~ 55.8
and Kato's scaling in Re after

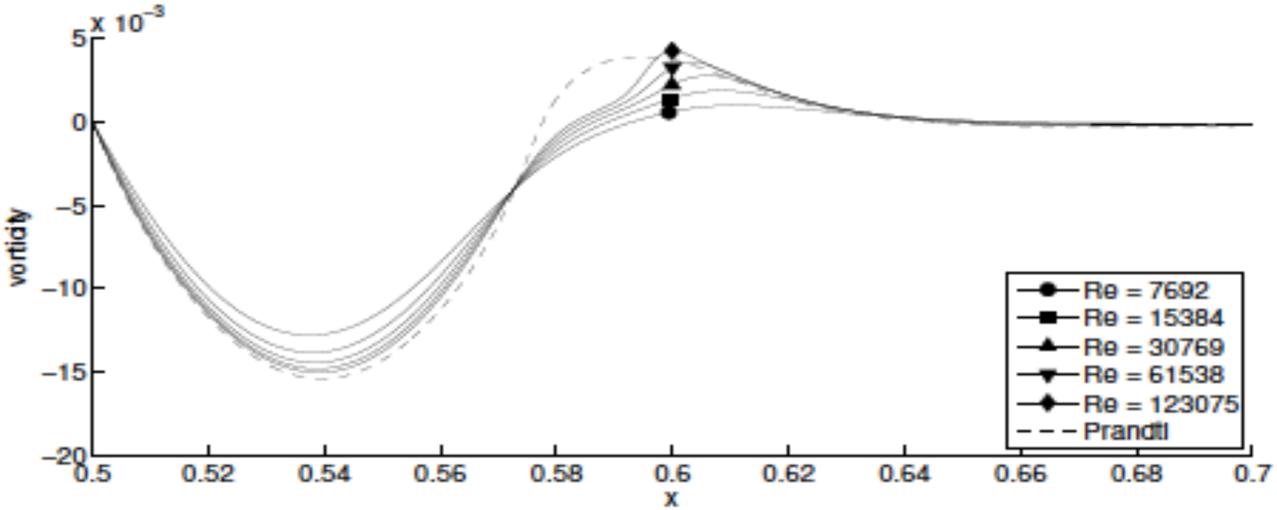
Vorticity along the wall at t=50



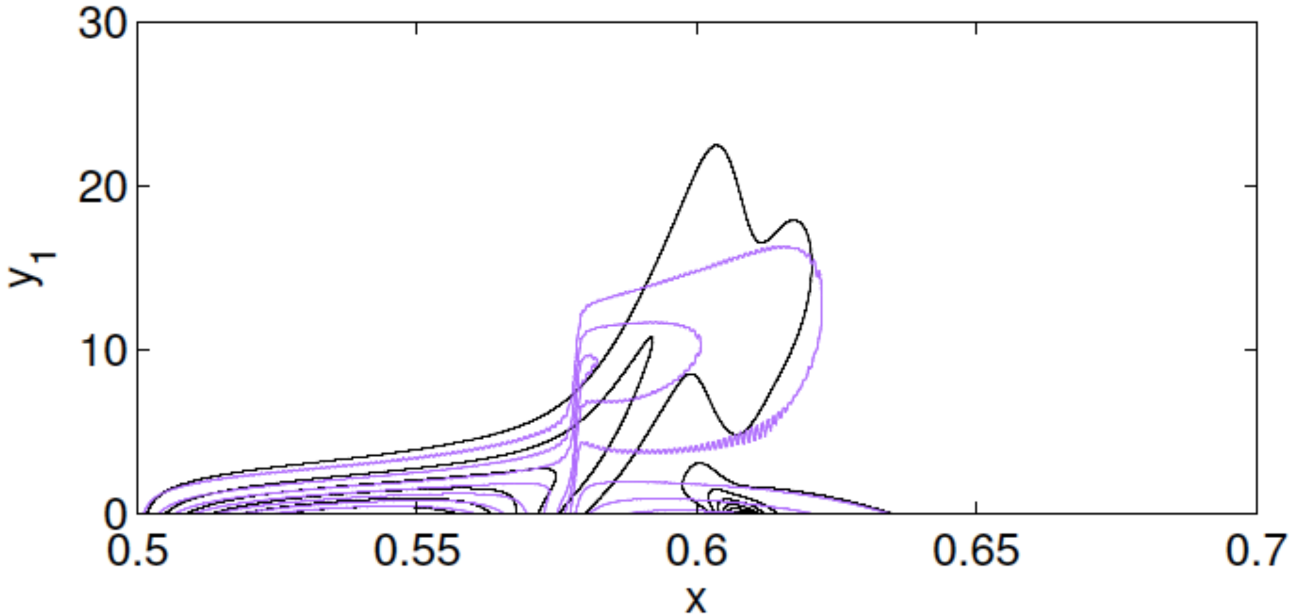


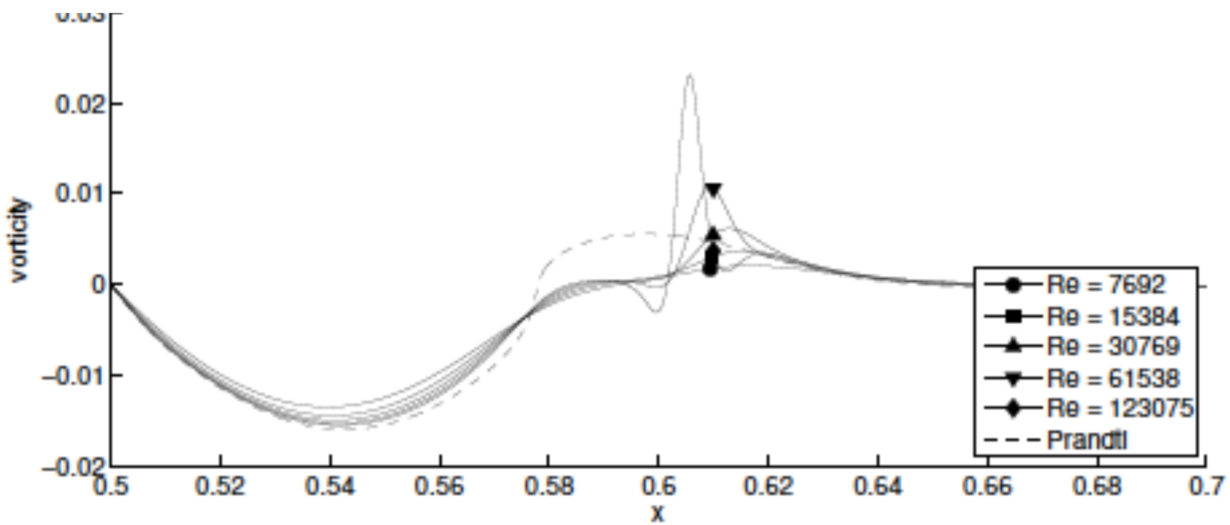
Vorticity along the wall at t=54



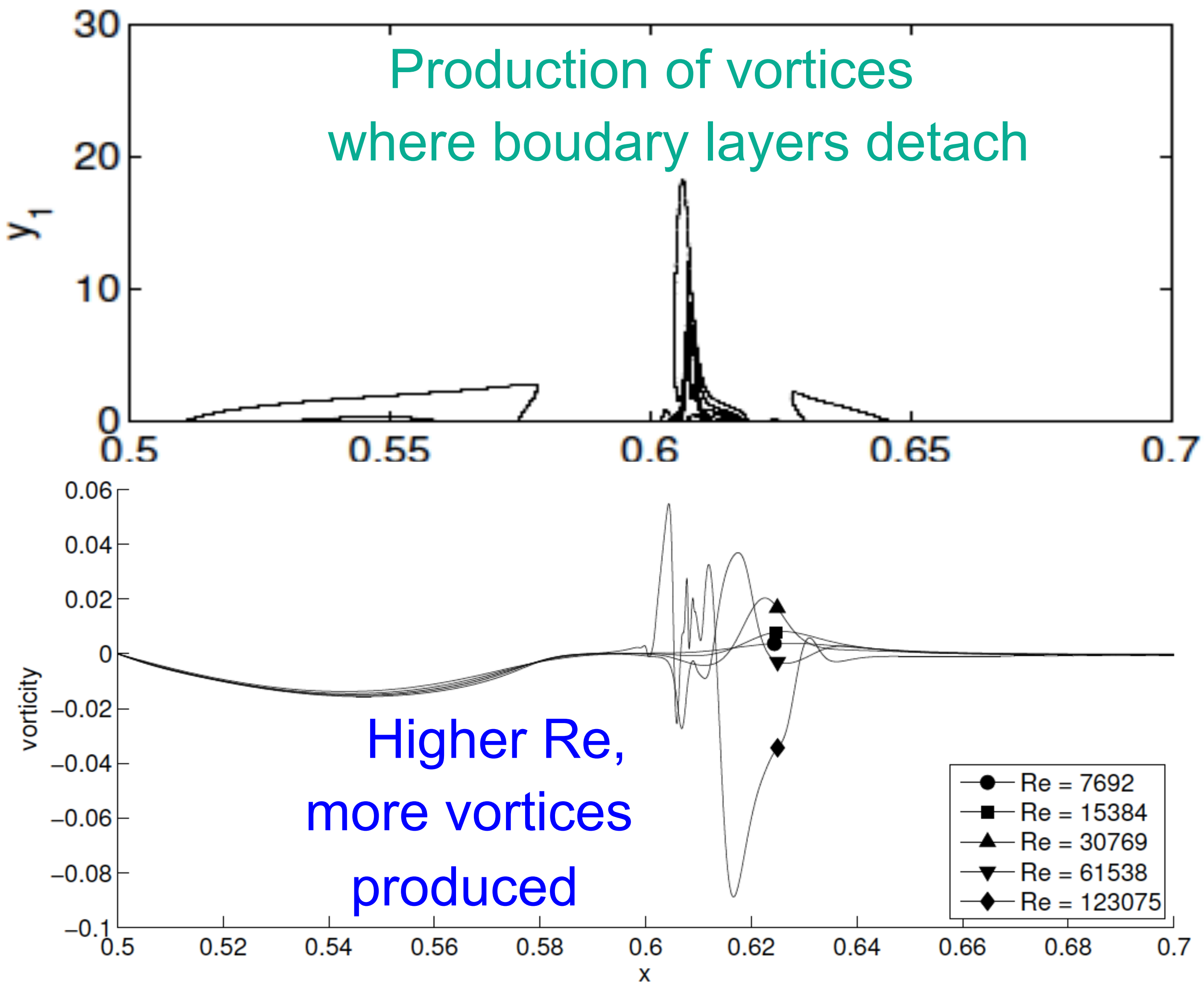


Vorticity along the wall at t=55

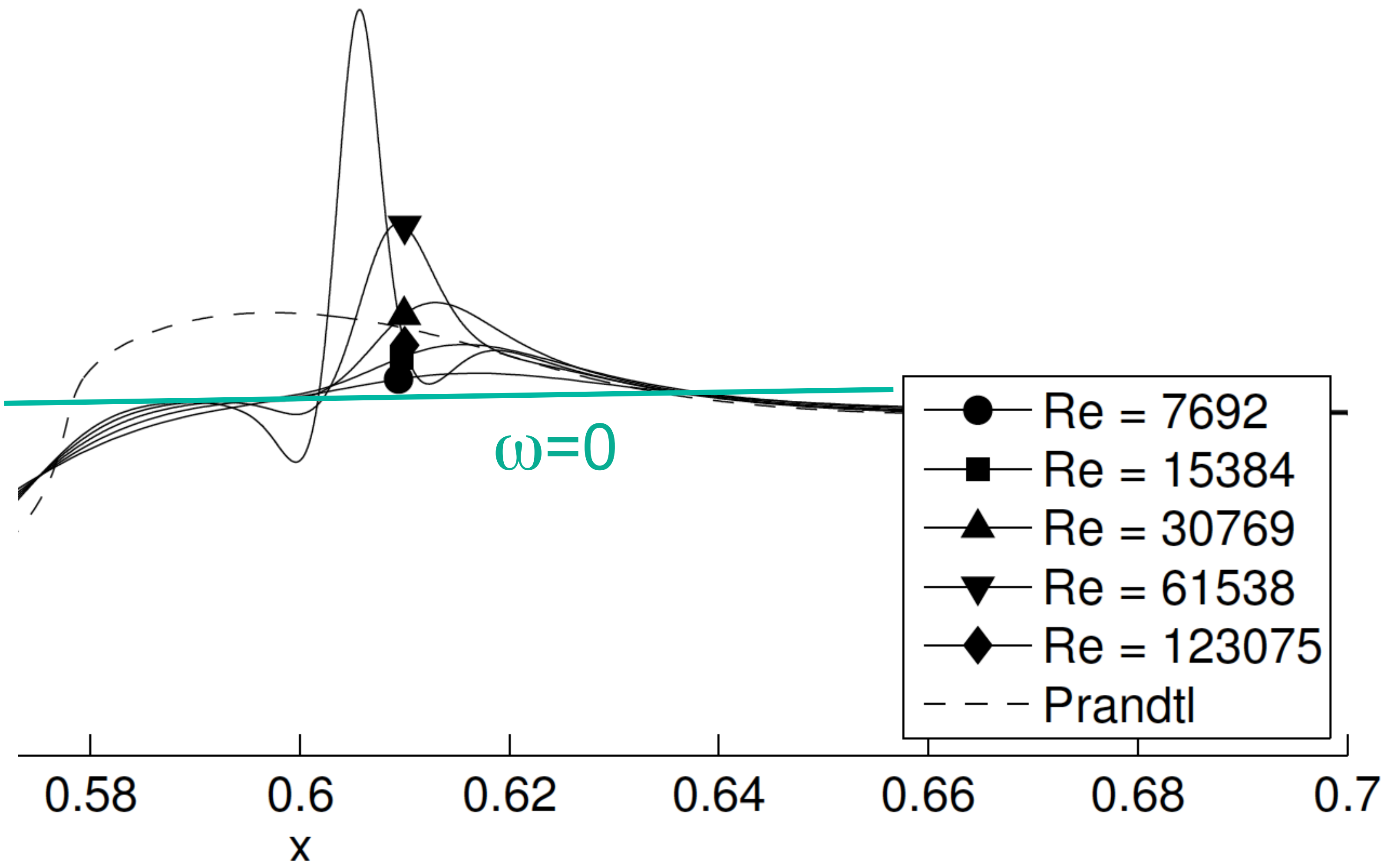




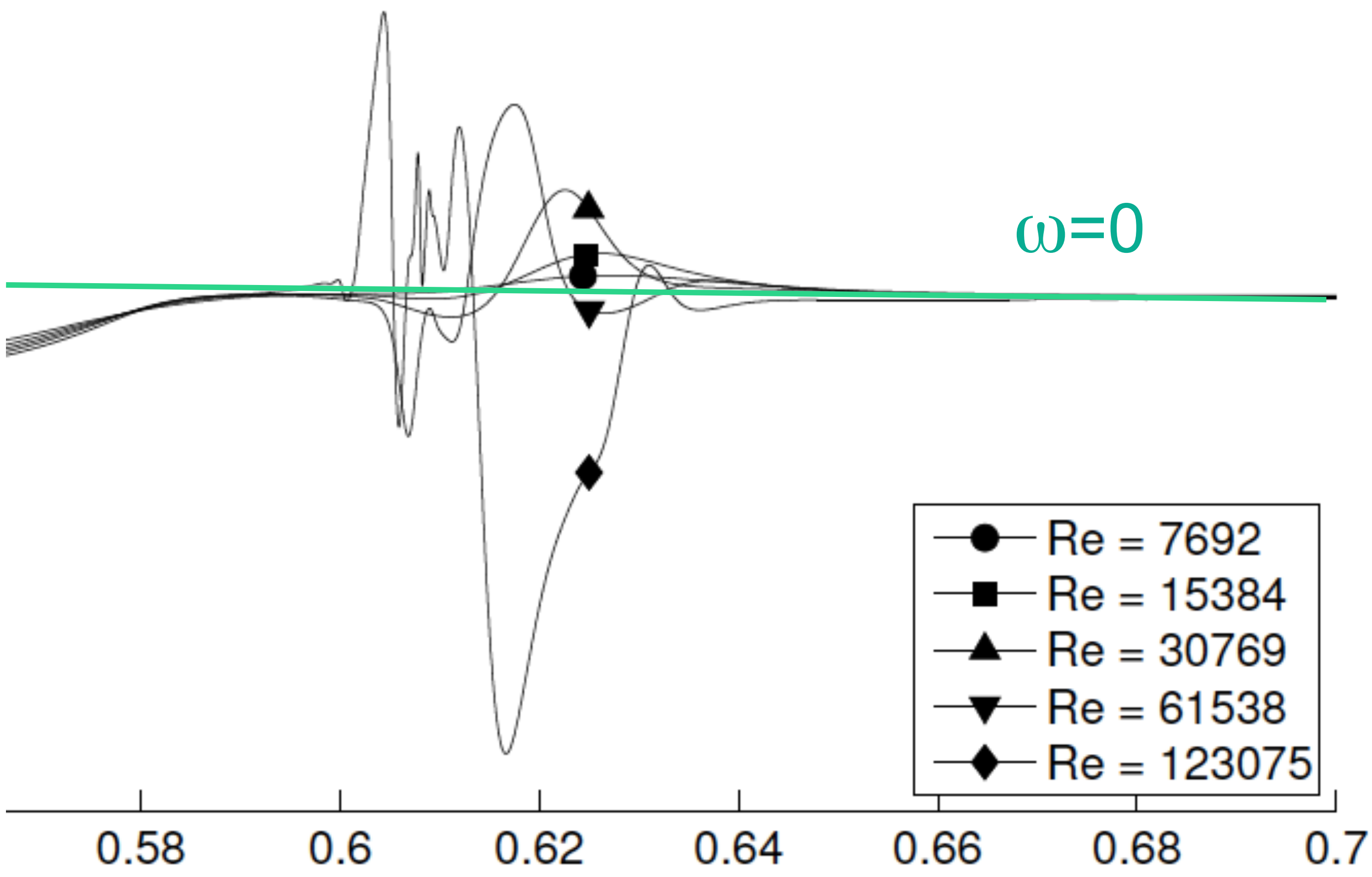
Vorticity along the wall at t=57>t_D



Vorticity along the wall at t=55.3



Vorticity along the wall at t=57.5



Conclusion

The production of dissipative structures
is the key feature of boundary layer (BL) detachment
at vanishing viscosity limit of incompressible flows.

The viscous Prandtl solution becomes singular at t_D
That corresponds to the instant when BL detaches.

The viscous Navier-Stokes solution converges uniformly
to the inviscid Euler solution for $t<t_D$,
and ceases to converge for $t>t_D$.

The detachment process involves spatial scales
in different directions, and not only parallel to the wall,
that are as fine as Re^{-1}.

Conclusion

The Navier-Stokes boundary layer detachment dynamics are very different from the dynamics of the finite time singularity developing in Prandtl's equation with:

- non locality in the parallel direction,
- formation of small scales scaling at least as Re^{-1}, in different directions and not only in the direction parallel to the wall,
- pressure plays an essential role in the detachment process.

R. Nguyen van yen, M. F. and K. Schneider, 2011
Phys. Rev. Lett., ***106****(18), 184502*

R. Nguyen van yen, M. Waidman, R. Klein, M.F. and K. Schneider, 2014
Preprint

Open questions

Numerical results suggest that a new asymptotic description of the flow beyond the breakdown of the Prandtl regime is possible.

Studying it might help to answer the following questions:

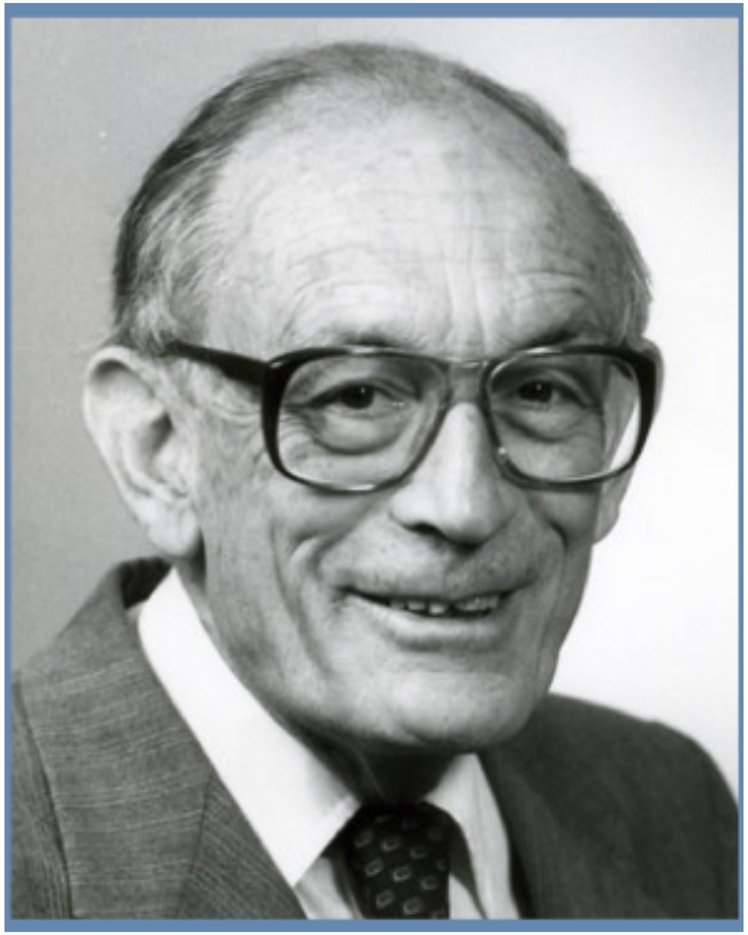
- Would Navier-Stokes solution looses smoothness after t_D?
- Would it converges to a weak singular dissipative solution of Euler's equation analog to dissipative shocks in Burgers solution?
- How can such a weak solution be approximated numerically?

This might lead to a new resolution of d'Alembert's paradox in terms of the production of weak singular dissipative structures due to the interaction of fully-developed turbulent flows with walls.

J. Leray, 1934
Sur le mouvement d'un fluide visqueux,
Acta Mathematica, ***63***

C. de Lellis and L. Székzlyhidi, 2010
Archives Rational Mechanics and Analysis,
195*(1), 221-260*

On turbulence

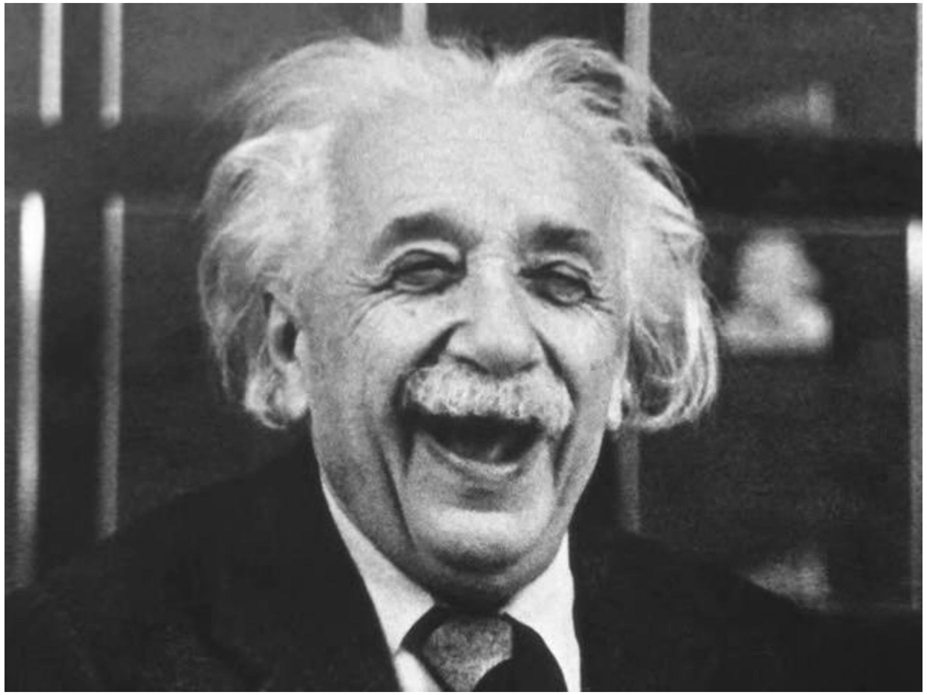


Hans Liepmann
(1914-2009)

'As long as we are not able to predict the drag on a sphere or the pressure drop in a pipe from continuous, incompressible and Newtonian assumptions without any other complications, namely from first principles, we would not have made it!'

Turbulence Workshop
UC Santa Barbara
1997

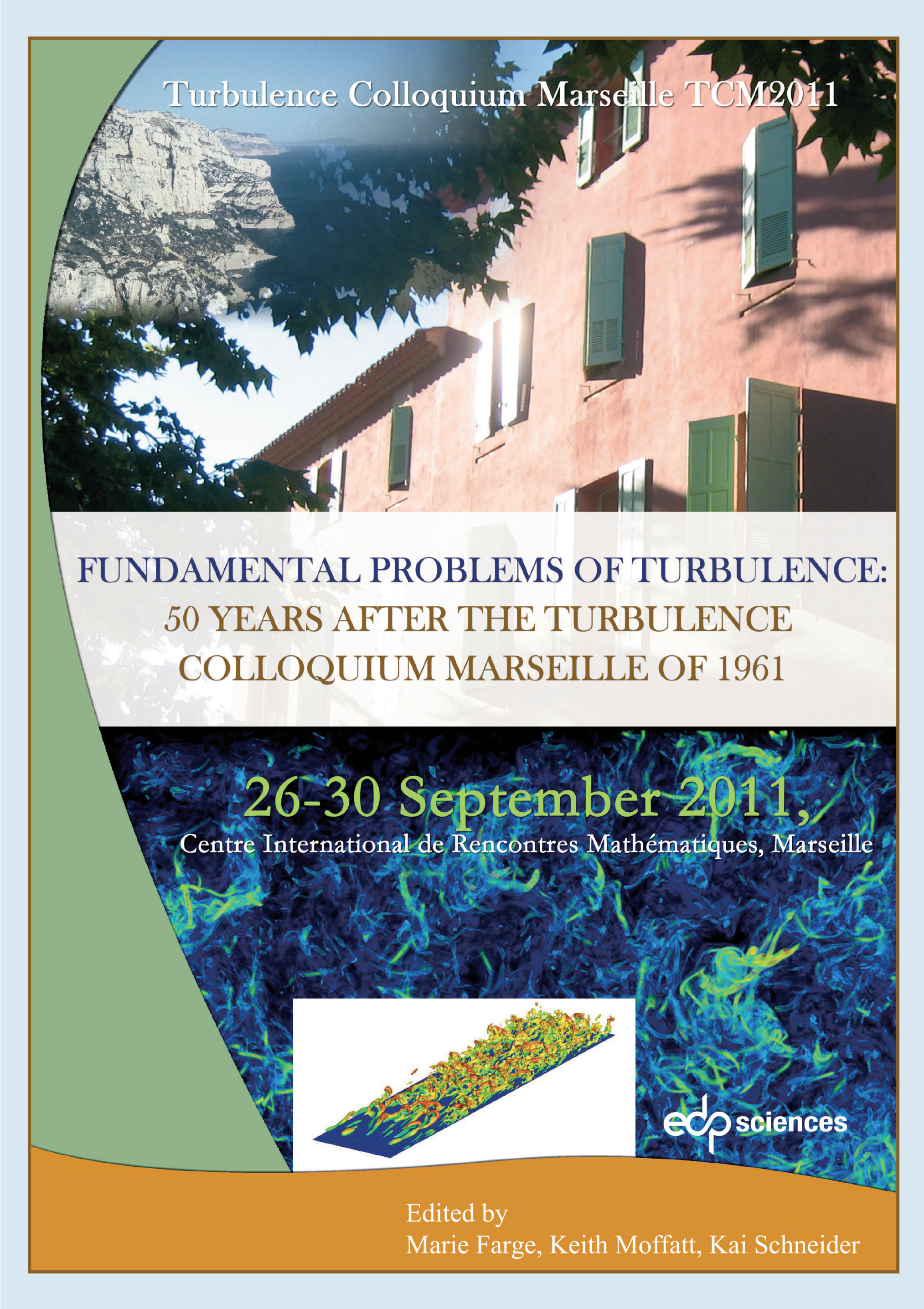
On mathematics and reality



Albert Einstein (1879-1955)

'As far as the laws of mathematics refer to reality, they are not certain, as far as they are certain, they do not refer to reality'.

'Geometry and experience', Conference given in Berlin at the Prussian Academy of Sciences on January 27th 1921



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