Local vs. Nonlocal Diffusions
— A Tale of Two Laplacians

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Outline

1. Einstein & Wiener: The Local diffusion
2. Lévy: A nonlocal diffusion
3. Effects of Nonlocal Laplacian
4. Summary
Normal distribution (or Gaussian distribution):
\[ X \sim \mathcal{N}(0, 1) \]
Probability density function for a Gaussian random variable
There is only one normal distribution. All others: Non-normal (i.e., anomalous) distributions

\[ f(x) \quad \alpha = 0.6, \quad \beta = -0.25 \]
Normal vs. non-normal distributions

Gaussian vs. non-Gaussian random variables

Light vs. heavy tails

Local vs. nonlocal diffusions ?

Local vs. nonlocal Laplacians ?
Brownian motion: Einstein’s theory

Brownian motion: Particles randomly moves in a liquid

Einstein 1905: Macroscopic theory (probability density $p$ for particles)

**Assumptions:**
Particles spreading area grows linearly in time (i.e., variance grows linearly in time)

Particle paths on non-overlapping time intervals are independent

L. C. Evans: An Intro to Stochastic Diff Eqns, 2013
Brownian motion: Einstein’s theory

A particle randomly walks on 1D lattice: Space step \( \Delta x \), time step \( \Delta t \), location \((m, n)\)

\[
p(m, n + 1) = \frac{1}{2} [p(m - 1, n) + p(m + 1, n)]
\]

Rewrite:

\[
p(m, n + 1) - p(m, n) = \frac{1}{2} [p(m - 1, n) - 2p(m, n) + p(m + 1, n)]
\]

Assumption: Particles spreading area growing linearly in time

\[
\frac{(\Delta x)^2}{\Delta t} = D
\]

\[
\frac{p(m, n + 1) - p(m, n)}{\Delta t} = \frac{D}{2} \frac{p(m - 1, n) - 2p(m, n) + p(m + 1, n)}{(\Delta x)^2}
\]

Letting \( \Delta x \to 0 \) and \( \Delta t \to 0 \): \( p_t = \frac{D}{2} p_{xx} \)
Diffusion equation (Fokker-Planck eqn for Brownian motion):

\[ p_t = \frac{D}{2} p_{xx} \]

Local Laplacian: \( \Delta = \partial_{xx} \)

For \( p(x, 0) = \delta(0) \) and \( D = 1 \):

\[ p(x, t) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} \]

**Estimate:** \( 0 < p(x, t) \leq \frac{1}{\sqrt{2\pi}} t^{-\frac{1}{2}} \)

Diffusion!

Particle paths: Normal distribution \( \mathcal{N}(0, t) \)

**Guess:** Brownian motion \( B_t \sim \mathcal{N}(0, t) \)
Related works around the time

Macroscopic equations for microscopic motions

**Bachelier** 1900

**Smoluchowski** 1906

**G. I. Taylor** 1921

**Uhlenbeck-Ornstein** 1930
Brownian motion $B_t$: Wiener’s theory

Wiener’s theory 1923: Microscopic theory (Paths of a particle)

- Independent increments: $B_{t_2} - B_{t_1}$ and $B_{t_3} - B_{t_2}$ independent
- Stationary increments with $B_t - B_s \sim \mathcal{N}(0, t - s)$
- Continuous sample paths (but nowhere differentiable in time)

Remarks:

$B_t \sim \mathcal{N}(0, t)$

$\text{Var}(B_t) = t$

Variance linear in time; spreading area linear in time

I. Karatzas and S. E. Shreve,

Brownian Motion and Stochastic Calculus
White noise

White noise: $\frac{dB_t}{dt}$

Generalized time derivative
Brownian particles in a moving liquid with velocity “b”

Brownian motion with an ambient or underlying velocity field “b”:
\[
\frac{dX_t}{dt} = b + \frac{dB_t}{dt} \quad \text{or}
\]
\[
dX_t = b \, dt + dB_t
\]
\[
X_t = b \, t + B_t \sim \mathcal{N}(bt, t)
\]
For \( p(x, 0) = \delta(0) \) and \( D = 1 \):
\[
p(x, t) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-bt)^2}{2t}}
\]
This satisfies: Diffusion-Advection equation
\[
p_t = \frac{D}{2} p_{xx} - bp_x
\]
\( b \): Drift, or convection
Fokker-Planck eqn for Brownian motion with a drift “b”
For Brownian motion with a constant drift "b"

\[ dX_t = b \, dt + dB_t, \]

the Fokker-Planck eqn is:

\[ p_t = \frac{D}{2} p_{xx} - b p_x \]

**Guess:** For Brownian motion with a state-dependent drift "b",
the Fokker-Planck eqn is:

\[ p_t = \frac{D}{2} p_{xx} - (bp)_x \]
Related works

Fokker 1914

Planck 1918

Smoluchowski 1915

Kolmogorov forward equation 1931
So, local Laplacian $\Delta$:
Macroscopic description of Brownian particles

In fact, it is also the **generator** for Brownian motion
Generator for Brownian motion $B_t$

Brownian motion starting at $x$: $X_t = x + B_t$

**Generator**: Time derivative of ‘mean observation of a stochastic process’

$$Af(x) \triangleq \frac{d}{dt} \bigg|_{t=0} \mathbb{E} f(X_t)$$

It is a linear operator.
Connecting **stochastics** with **deterministics**.

**Generator $A$ carries info about process $X_t$**

Generator for Brownian motion: Local Laplacian!

$$\Delta = \partial_{xx}$$
Let us verifying this

For \( X_t = x + B_t \),

\[
\mathbb{E}f(X_t) = \frac{1}{\sqrt{2\pi t}} \int f(y) \, e^{-\frac{(y-x)^2}{2t}} \, dy
\]

\[
= \frac{1}{\sqrt{2\pi}} \int f(x + z\sqrt{t}) \, e^{-\frac{z^2}{2}} \, dz,
\]

where we have changed variables via \( z = \frac{y-x}{\sqrt{t}} \).
\[
\frac{\mathbb{E} f(X_t) - f(x)}{t} = \frac{1}{\sqrt{2\pi}} \int \frac{z\sqrt{t}f'(x) + \frac{1}{2}z^2 tf''(x + \theta z\sqrt{t})}{t} e^{-\frac{z^2}{2}} \, dz
\]

\[
= \frac{1}{\sqrt{2\pi}} \frac{f'(x)}{t} \int z \, e^{-\frac{z^2}{2}} \, dz
\]

\[
+ \frac{1}{2} \frac{1}{\sqrt{2\pi}} \int z^2 f''(x + \theta z\sqrt{t}) \, e^{-\frac{z^2}{2}} \, dz
\]

\[
= 0 + \frac{1}{2} \frac{1}{\sqrt{2\pi}} \int z^2 f''(x + \theta z\sqrt{t}) \, e^{-\frac{z^2}{2}} \, dz.
\]

Finally: the generator \( A \) is local Laplacian —

\[
Af(x) = \frac{d}{dt} \bigg|_{t=0} \mathbb{E} f(X_t) = \lim_{t \downarrow 0} \frac{\mathbb{E} f(X_t) - f(x)}{t}
\]

\[
= \frac{1}{2} f''(x),
\]
So, local Laplacian $\Delta$:

**Macroscopic description** of Brownian motion

**Generator** for Brownian motion

Same in the context of Fokker-Planck eqns

How about the **nonlocal Laplacian**: $(-\Delta)^{\frac{\alpha}{2}}$, $0 < \alpha < 2$?
Nonlocal Laplacian: Macroscopic description of Lévy motion

Nonlocal Laplacian: $(-\Delta)^{\alpha/2}, \ 0 < \alpha < 2$:

Macroscopic description or generator for symmetric $\alpha$–stable motion $L^\alpha_t$
Central Limit Theorem

$X_1, X_2, \cdots, X_n$ are independent, identically distributed (iid) random variables (i.e., ‘measurements’) and then “averaging”:

Central Limit Theorem
A stable random variable $X$ comes from “averaging the measurements”:

$$\lim_{n \to \infty} \frac{X_1 + \cdots + X_n - b_n}{a_n} = X \quad \text{in distribution}$$

for some constants $a_n, b_n$ ($a_n \neq 0$)

**Notation:** $X \sim S_\alpha, \quad 0 < \alpha \leq 2$

$\alpha$—stable random variable

$\alpha$: Non-Gaussianity index
A special case: $\alpha = 2$

Well-known normal random variable emerges when $\alpha = 2$

Well-known normal random variable emerges when $\alpha = 2$

$\mathbb{E} X_i = \mu$, $\text{Var}(X_i) = \sigma^2$

Central limit theorem: A normal random variable comes from “averaging the measurements”

$$\lim_{n \to \infty} \frac{X_1 + \cdots + X_n - n\mu}{\sigma \sqrt{n}} = X \sim \mathcal{N}(0, 1) \quad \text{in distribution}$$

Namely,

$$\lim_{n \to \infty} \mathbb{P}\left(\frac{X_1 + \cdots + X_n - n\mu}{\sigma \sqrt{n}} \leq x\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-x^2/2} dx$$
Gaussian vs. Non-Gaussian random variables

**Gaussian:** Normal random variable $X \sim \mathcal{N}(0, 1)$

**Probability density function** $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$

$$P(X \leq x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx$$

**Non-Gaussian:** $\alpha-$stable random variable $X \sim S_\alpha, \ 0 < \alpha < 2$

**Probability density function** $f_\alpha(x)$

$$P(X \leq x) = \int_{-\infty}^{x} f_\alpha(x) \, dx$$
$X \sim \mathcal{N}(0, 1)$

**Figure**: “Bell curve”: Exponential decay, light tail
Prob density function for a non-Gaussian, $\alpha$–stable random variable

$X \sim S_\alpha$

**Figure**: Polynomial power decay, heavy tail
Some references:

P. E. Protter: **Stochastic Integration and Diff Equations**, 1990

D. Applebaum: **Lévy Processes and Stochastic Calculus**, 2009
Lévy Motion $L^\alpha_t$

**Definition:** Lévy motion $L^\alpha_t$ with $0 < \alpha \leq 2$:

1. Stationary increments $L^\alpha_t - L^\alpha_s \sim S_\alpha(|t - s|^\frac{1}{\alpha}, 0, 0)$
2. Independent increments
3. Stochastically continuous sample paths (continuous in probability):
   $\mathbb{P}(|L_t - L_s| > \delta) \to 0$, as $t \to s$, for all $\delta > 0$

**Note:** Paths are stochastically continuous (i.e., right continuous with left limit; countable jumps): $L^\alpha_t \to L^\alpha_s$ in probability as $t \to s$

Countable jumps in time!
Lévy-Khintchine Theorem:
Countable jumps in time:

**Jump measure:** a Borel measure

\[ \nu_{\alpha}(dy) = C_{\alpha} \frac{dy}{|y|^{1+\alpha}}, \text{ for } 0 < \alpha < 2 \]

\[ \nu_{\alpha}(a, b) = C_{\alpha} \int_{a}^{b} \frac{dy}{|y|^{1+\alpha}}: \]

Mean number of jumps of “size” \((a, b)\) per unit time!
Brownian motion $B_t$: A **Gaussian** process
(Brownian noise: $\frac{dB_t}{dt}$)

Lévy motion $L^\alpha_t$: A **non-Gaussian** process
(Lévy noise: $\frac{dL^\alpha_t}{dt}$)
Heavy tail for $0 < \alpha < 2$: **Power law**

$$\mathbb{P}(|L_t^\alpha| > u) \sim \frac{1}{u^\alpha}$$

Light tail for $\alpha = 2$: **Exponential law**

$$\mathbb{P}(|B_t| > u) \sim \frac{e^{-u^2/2}}{\sqrt{2\pi u}}$$
Generator for Lévy Motion: a Nonlocal operator

Lévy-Khintchine Theorem:
Specifies Fourier transform (i.e., characteristic function) of $L_t^\alpha$:

$$g(k, \alpha)$$

Thus:

$$L_t^\alpha = \mathcal{F}^{-1} g(k, \alpha)$$

$$Au = \left. \frac{d}{dt} \right|_{t=0} \mathbb{E} u(x + L_t^\alpha)$$

$$= \int_{\mathbb{R}^n \setminus \{0\}} [u(x + y) - u(x)] \nu_\alpha(dy)$$

$$\triangleq -K_\alpha (-\Delta)^{\alpha/2}$$

$$\nu_\alpha(dy) = C_\alpha \frac{dy}{|y|^{n+\alpha}}$$: Jump measure for $L_t^\alpha$

$C_\alpha, K_\alpha$: Positive constants depending on $n, \alpha$

Nonlocal Laplacian: $(-\Delta)^{\alpha/2}$
Generator for Lévy Motion: a Nonlocal operator

Justify the notation for \((-\Delta)^{\frac{\alpha}{2}}\):

\[
\int_{\mathbb{R}^n \setminus \{0\}} [u(x + y) - u(x)] \nu_\alpha(dy) \triangleq -K_\alpha (-\Delta)^{\frac{\alpha}{2}}
\]

\(\mathbb{F}(\text{left hand side}) = |k|^\alpha \mathbb{F}(u)\)

Clearly, this notation is inspired by the fact that

\(\mathbb{F}(-\Delta u(x)) = |k|^2 \mathbb{F}(u)\)

Applebaum: Lévy Processes and Stochastic Calculus
Nonlocal diffusion equation (Fokker-Planck eqn for Lévy motion):

\[ p_t = -K_\alpha (-\Delta)^{\frac{\alpha}{2}} p \]

Nonlocal Laplacian: \((-\Delta)^{\frac{\alpha}{2}}\)

It is the **Generator** for Lévy motion
Two Laplacians

Local Laplacian: $\Delta$

Nonlocal Laplacian: $(-\Delta)^{\frac{\alpha}{2}}$, for $0 < \alpha < 2$

Macroscopic manifestation of corresponding microscopic descriptions:
Brownian motion and $\alpha$–stable Lévy motion
Brownian motion vs. $\alpha-$stable Lévy motion

<table>
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<td>Jump measure: $\nu_\alpha$</td>
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Fokker-Planck eqn for system with Brownian motion

For a stochastic system with Brownian motion:

\[ dX_t = b(X_t)dt + dB_t, \quad X_0 = x \]

Fokker-Planck eqn for probability density evolution \( p(x, t) \):

\[ p_t = \Delta p - \nabla \cdot (b(x)p) \]

When the vector field (drift) \( b(x) \) is divergence-free:

\[ p_t = \Delta p - b(x) \cdot \nabla p \]
Fokker-Planck eqn for system with Lévy motion:

For a stochastic system with Lévy motion:

\[ dX_t = b(X_t)dt + dL_t^\alpha, \quad X_0 = x \]

Fokker-Planck eqn for probability density evolution \( p(x, t) \):

\[ p_t = -K_\alpha (-\Delta)^{\alpha/2} p - \nabla \cdot (b(x)p) \]

When the vector field (drift) \( b(x) \) is divergence-free:

\[ p_t = -K_\alpha (-\Delta)^{\alpha/2} p - b(x) \cdot \nabla p \]

for \( 0 < \alpha < 2 \)
Fokker-Planck eqn: Nonlinear, as well as nonlocal

When the vector field $b$ depends on the distribution of system state, then we have a nonlinear, nonlocal PDE:

$$p_t = \Delta p - \nabla \cdot (\tilde{b}(p)p)$$

$$p_t = -K_\alpha (-\Delta)^{\frac{\alpha}{2}} p - \nabla \cdot (\tilde{b}(p)p)$$

for $0 < \alpha < 2$

Wellposedness & regularity of solutions? Useful for designing numerical schemes.
Effects of Nonlocal Laplacian $(-\Delta)^{\frac{\alpha}{2}}$: 

- in some partial differential equations?
- in some dynamical phenomena?
Eigenvalues of Two Laplacians on bounded domain

**Local Laplacian:** $\Delta$
One-dim, zero Dirichlet BC: $\lambda_n \sim -n^2$

**Nonlocal Laplacian:** $-\left(-\Delta\right)^{\frac{\alpha}{2}}$, for $0 < \alpha < 2$
One-dim, zero external Dirichlet BC: $\lambda_n \sim -(n - \frac{2-\alpha}{4})^\alpha + O\left(\frac{1}{n}\right)$
Kwasnicki 2010
Reducing the “diffusion power” by the “amount” $2 - \alpha$!
Effects of Nonlocal Laplacian in the Burgers eqn:

\[ u_t = -uu_x - (\Delta)^{\frac{\alpha}{2}} u \]

**Kiselev, Nazarov & Shterenberg 2008**
Under periodic boundary condition:
Blowup in finite time for \( 0 < \alpha < 1 \), but global solution for \( 1 \leq \alpha < 2 \).

**Biler, Funaki & Woyczynski 1998**
In the whole space: Global solution for \( 1.5 < \alpha < 2 \) in \( H^1(R) \)
Motion of particles under the influence of Lévy motion:

\[ dX_t = b(X_t)dt + dL_t^\alpha, \quad X_0 = x \]

• **Examine** quantities that carry dynamical information:

  **Escape probability**

  Likelihood of transition between different dynamical regimes!
Escape probability: Carrying dynamical information

- **Contaminant transport**: likelihood for contaminant to reach a specific region
- **Climate**: likelihood for temperature to be within a range
- **Tumor cell density**: likelihood for tumor density to decrease (becoming cancer-free)

How to quantify escape probability?
Escape probability from a domain $D$

Consider a SDE

$$dX_t = b(X_t)dt + dL^\alpha_t, \quad X_0 = x \in D$$

Escape probability $p(x)$:
Likelihood that a “particle x” first escapes $D$ and lands in $U$

**Figure**: Domain $D$, with a target domain $U$ in $D^c$
A surprising connection between escape probability and harmonic functions!

What is a harmonic function?
Recall: What is a harmonic function?

It is a solution of the Laplace equation:

\[ \Delta h(x) = 0 \]

But \( \Delta \) is the generator of Brownian motion \( B_t \)

So we say:
\( h(x) \) is a harmonic function with respect to Brownian motion
An analogy:

Harmonic function with respect to Lévy motion $L_t^\alpha$:

$$(-\Delta)^{\frac{\alpha}{2}} h(x) = 0$$

where $(-\Delta)^{\frac{\alpha}{2}}$ is the generator of $L_t^\alpha$

**Note:** Feedback of Probability Theory to Analysis!
Consider a stochastic system

\[ dX_t = b(X_t)dt + dL_t^\alpha \]

Generator for solution process \( X_t \):

\[ A_\alpha h(x) = b(x) \cdot \nabla h(x) - K_\alpha (-\Delta)^{\frac{\alpha}{2}} h(x) \]

Harmonic function with respect to \( X_t \): \( A_\alpha h(x) = 0 \)

Nonlocal deterministic partial differential equation
What is the connection between escape probability & harmonic functions?
Escape probability from a domain $D$

Escape probability $p(x)$:
Likelihood that a “particle $x$” first escapes $D$ and lands in $U$

Exit time: $\tau_{D^c}(x)$ is the first time for $X_t$ to escape $D$

Figure: Domain $D$, with a target domain $U$ in $D^c$
Connection: Escape probability & harmonic function

\[ dX_t = b(X_t)dt + dL_t^\alpha, \quad X_0 = x \in D \]

For

\[ \varphi(x) = \begin{cases} 
1, & x \in U, \\
0, & x \in D^c \setminus U,
\end{cases} \]

\[
E[\varphi(X_{\tau_{D^c}}(x))] = \int_{\{\omega: X_{\tau_{D^c}} \in U\}} \varphi(X_{\tau_{D^c}}) d\mathbb{P}(\omega) \\
+ \int_{\{\omega: X_{\tau_{D^c}} \in D^c \setminus U\}} \varphi(X_{\tau_{D^c}}) d\mathbb{P}(\omega) \\
= \mathbb{P}\{\omega: X_{\tau_{D^c}} \in U\} \\
= p(x)
\]

But, left hand side is a harmonic function with respect to \( X_t \)

Liao 1989
Escape probability from a domain $D$

$$dX_t = b(X_t)dt + dL^\alpha_t, \quad X_0 = x \in D$$

Escape probability $p(x)$: Likelihood that a “particle $x$” first escapes $D$ and lands in $U$

**Theorem**

*Escape probability $p$ is solution of Balayage-Dirichlet problem*

$$\begin{cases} 
A_\alpha p = 0, \\
p|_U = 1, \\
p|_{D^c \setminus U} = 0,
\end{cases}$$

with $A_\alpha = b(x) \cdot \nabla - K_\alpha (-\Delta)^{\frac{\alpha}{2}}$.

Qiao-Kan-Duan 2013
Δρ = 0, escape from \( D = (-2, 2) \) to \( U = (2, +\infty) \):

**Figure**: Escape probability: The case of Brownian motion
Escape probability to the right: under Lévy motion, no drift

\((-\Delta)^{\frac{\alpha}{2}} p = 0\), escape from \(D = (-2, 2)\) to \(U = (2, +\infty)\):

\[\alpha = 0.25\]

\[\alpha = 1\]

\[\alpha = 1.25\]

\[\alpha = 1.99\]

**Figure**: Escape probability: The case of Lévy motion
Impact of local & nonlocal diffusions

Under Brownian fluctuations (i.e., local diffusion):
— Escape probability $p(x)$ is linear in location

Under Lévy fluctuations (i.e., nonlocal diffusion):
— Escape probability $p(x)$ is nonlinear in location
When velocity field (drift) is present:

**Escape probability** under interactions between nonlinearity and fluctuations

Gao-Duan-Li-Song 2014
Fokker-Planck eqn:
Numerical simulations

Wang-Duan-Li-Lou 2014

Wellposedness under realistic conditions?
Behavior of solutions?
Impact of nonlocal Laplacian?
Summary

\[ \Delta \text{ and } (-\Delta)^{\frac{\alpha}{2}} \]

- Microscopic origins of two Laplacians:
  Macroscopic descriptions of Brownian & Lévy motions

- Comparing Local & Nonlocal Diffusions:
  Escape probability: Quantifying particle dynamics under non-Gaussian fluctuations
  Fokker-Planck eqn: Quantifying probability density evolution