Uniqueness by random perturbation: The Leray-α model of Euler equations

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[1]joint work with David Barbato & Benedetta Ferrario

Outline

The motion of incompressible inviscid fluids is described by the Euler equations

$$\begin{cases} \dfrac{\partial v}{\partial t} + (v \cdot \nabla)v + \nabla p = f \\ \text{div } v = 0 \end{cases} \tag{1}$$

For the 3D Euler system (1) neither the global existence nor the uniqueness of global solutions are known when the initial velocity is of finite energy.

We introduce a regularization in the non linear term.

$$\begin{cases} \dfrac{\partial v^\alpha}{\partial t} + (u^\alpha \cdot \nabla)v^\alpha + \nabla p = f \\ u^\alpha = G_\alpha * v^\alpha \\ \text{div } v^\alpha = 0 \end{cases} \tag{2}$$

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This regularization has been introduced by Leray to prove the existence of solutions to the Navier-Stokes equations in $\mathbb{R}^d$, $d = 2, 3$

$$\begin{cases} \dfrac{\partial v^\alpha}{\partial t} - \nu \Delta v^\alpha + (u^\alpha \cdot \nabla) v^\alpha + \nabla p = f \\ u^\alpha = G_\alpha * v^\alpha \\ \text{div } v^\alpha = 0 \end{cases} \tag{3}$$

where G_α is a smoothing kernel such that $u^\alpha \longrightarrow v^0$, in some sense, as $\alpha \to 0$. In particular, system (3) converges to the Navier-Stokes system as $\alpha \to 0$.

For example, one can use the Green function associated with the operator $1 - \alpha\Delta$

$$u^\alpha = G_\alpha * v^\alpha = (1 - \alpha\Delta)^{-1} v^\alpha \tag{4}$$

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Theorem (Existence for the deterministic Leray-α model)

Let $v_0 \in H$ and $T > 0$. Then there exists a global weak solution v^α of (2) such that

$$v^\alpha \in L^\infty([0,T];H) \cap C([0,T];V')$$

and

$$\langle v^\alpha(t), \phi \rangle_H - \int_0^t \langle (u^\alpha(s) \cdot \nabla)\phi, v^\alpha(s) \rangle_H ds = \langle v_0, \phi \rangle_H \qquad (5)$$

for each $t \in [0,T]$ and for all test functions $\phi : \mathbb{R}^3 \to \mathbb{R}^3$, periodic with period box $\mathbb{T}$, divergence free and of class C^1.

We prove that adding a particular random perturbation, we are able to prove the existence and uniqueness for the stochastic version. We have to choose the random perturbation to be conservative. In order to do so, we have to work on the Fourier components of the system.

We work on the 3D torus $\mathbb{T}$.
(We neglect the index α but everywhere we assume $\alpha > 0$)

Leray-α model in Fourier components

Assume $\mathbf{v}(t,\cdot)$ and $\mathbf{u}(t,\cdot)$ are in $\mathbf{L}^2(\mathbb{T},\mathbb{R}^3)$; then

$$\mathbf{v}(t,\mathbf{x}) = \sum_{\mathbf{k}\in\mathbb{Z}^3} \mathbf{v_k}(t) e_{\mathbf{k}}(\mathbf{x}) \qquad \text{and} \qquad \mathbf{u}(t,\mathbf{x}) = \sum_{\mathbf{k}\in\mathbb{Z}^3} \mathbf{u_k}(t) e_{\mathbf{k}}(\mathbf{x})$$

with $e_{\mathbf{k}}(\mathbf{x}) = e^{i\mathbf{k}\cdot\mathbf{x}}$.

We set $\mathbf{v}_0(t) = \mathbf{0}$ and $\mathbf{u}_0(t) = \mathbf{0}$ (mean value zero).

Since $\mathbf{v}$ and $\mathbf{u}$ are real vectors, we have

$$\mathbf{v}_{-\mathbf{k}}(t) = \overline{\mathbf{v_k}}(t), \qquad \mathbf{u}_{-\mathbf{k}}(t) = \overline{\mathbf{u_k}}(t).$$

We set $\|\mathbf{v}(t,\cdot)\|_{l^2}^2 = \sum_{\mathbf{k}} \|\mathbf{v_k}(t)\|^2$.

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Substituting in

$$\begin{cases} \dfrac{\partial \mathbf{v}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{v} + \nabla p = \mathbf{0} \\ \mathbf{v} = (1 - \alpha\Delta)\mathbf{u} \\ \text{div } \mathbf{v} = 0 \end{cases} \tag{6}$$

we obtain the following system for the Fourier components

$$\begin{cases} \dfrac{d\mathbf{v_k}}{dt}(t) = -i \displaystyle\sum_{\mathbf{h} \in \mathbb{Z}^3} \dfrac{\langle \mathbf{v_h}(t), \mathbf{k} \rangle}{1 + \alpha \|\mathbf{h}\|^2} P_{\mathbf{k}}(\mathbf{v_{k-h}}(t)) \\ \langle \mathbf{v_k}(t), \mathbf{k} \rangle = 0 \\ \mathbf{v_{-k}}(t) = \overline{\mathbf{v_k}}(t) \end{cases} \tag{7}$$

for any $\mathbf{k}$, where $P_{\mathbf{k}}(\mathbf{v}) := \mathbf{v} - \dfrac{\langle \mathbf{v}, \mathbf{k} \rangle}{\langle \mathbf{k}, \mathbf{k} \rangle}\, \mathbf{k}$.

Remark

Given $\alpha \geq 0$, if $\sum_k \|\mathbf{v_k}(t)\|^2 < \infty$, then the series in the r.h.s. of $(7)_1$ is convergent and $\left\| \frac{d\mathbf{v_k}}{dt}(t) \right\| \leq \|\mathbf{v}(t)\|^2_{l^2} \|\mathbf{k}\|$.

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For system (7), like for the Euler equations, the energy

$$E(t) := \frac{1}{2} \sum_{\mathbf{k}} \|\mathbf{v}_{\mathbf{k}}(t)\|^2$$

is formally conserved.

Indeed,

$$\frac{d}{dt}\|\mathbf{v}_{\mathbf{k}}(t)\|^2 = \sum_{\mathbf{h}} 2\Re \left\{ -i \frac{\langle \mathbf{v}_{\mathbf{h}}(t), \mathbf{k} \rangle}{1+\alpha\|\mathbf{h}\|^2} \langle P_{\mathbf{k}}(\mathbf{v}_{\mathbf{k}-\mathbf{h}}(t)), \mathbf{v}_{\mathbf{k}}(t) \rangle \right\}$$

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Summing over all components, we formally obtain

$$\frac{dE}{dt}(t) = \sum_{\mathbf{k}} \sum_{\mathbf{h}} \Im \left\{ \frac{\langle \mathbf{v}_{\mathbf{h}}(t), \mathbf{k} \rangle}{1 + \alpha \|\mathbf{h}\|^2} \langle \mathbf{v}_{\mathbf{k}-\mathbf{h}}(t), \mathbf{v}_{\mathbf{k}}(t) \rangle \right\}$$

which vanishes, since the sum contains terms which cancel each other according to the following equality

$$\frac{\langle \mathbf{v}_{\mathbf{h}'}, \mathbf{k}' \rangle}{1 + \alpha \|\mathbf{h}'\|^2} \langle \mathbf{v}_{\mathbf{k}'-\mathbf{h}'}, \mathbf{v}_{\mathbf{k}'} \rangle = \overline{\frac{\langle \mathbf{v}_{\mathbf{h}}, \mathbf{k} \rangle}{1 + \alpha \|\mathbf{h}\|^2} \langle \mathbf{v}_{\mathbf{k}-\mathbf{h}}, \mathbf{v}_{\mathbf{k}} \rangle} \tag{8}$$

for $\mathbf{h}' = -\mathbf{h}$ and $\mathbf{k}' = \mathbf{k} - \mathbf{h}$.

We are interested in a stochastic version of system (2), that is the following stochastic Leray-α model of Euler equations

$$\begin{cases} dv^\alpha + [(u^\alpha \cdot \nabla)v^\alpha + \nabla p]\, dt = ((\sigma \circ dW) \cdot \nabla)v^\alpha \\ v^\alpha = (1 - \alpha\triangle)u^\alpha \\ \text{div } v^\alpha = 0 \end{cases} \tag{9}$$

W is a Brownian motion (in time) and $\circ dW$ referes to the Stratonovitch differential

$\alpha > 0$

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Adding this random perturbation, we prove global existence and uniqueness for the stochastic model. We have to choose the random perturbation to be conservative.
The choice of this noise has been introduced previously for some inviscid models.

1. S. Attanasio, F. Flandoli (2011), *Renormalized solutions for stochastic transport equations and the regularization by bilinear multiplication noise*. Comm. Partial Differential Equations **36**, no. 8, 1455–1474.
2. D. Barbato, F. Flandoli, F. Morandin (2010), *Uniqueness for a stochastic inviscid dyadic model*. Proc. Amer. Math. Soc. **138** no. 7, 2607–2617.
3. D. Barbato, F. Morandin, *Stochastic inviscid shell models: Well-posedness and anomalous dissipation*, Nonlinearity, **26** (2013), no. 7, 1919–1943.

A brief review on solutions for SDEs and Girsanov

Let W be a Wiener process on $(\Omega, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ a probability space and let us define a process Z with paths a.e. in $C([0,T];\mathbb{R})$ which is adapted and solves

$$dZ(t) = a(t,Z)dt + \sigma(t,Z)dW, \quad z(0) = z \tag{10}$$

where a, σ are adapted measurable functionals. It is necessary that

$$\mathbb{P}\left\{\int_0^T |a(s,Z)|^2 ds < \infty\right\} = \mathbb{P}\left\{\int_0^T |\sigma(s,Z)|^2 ds < \infty\right\} = 1.$$

For simplicity, we fix the initial data z.

Definition

(Weak solution) We say there exists a weak solution to (10) if there exist a stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, $\mathcal{F}_t$-Wiener process W and an $\mathcal{F}_t$-adapted process Z defined in it such that Z solves (10) $\mathbb{P}$-a.s. We denote the solution by the triplet

$$(Z, (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}), W).$$

Definition

(Strong solution) We say there exists a strong solution to (10) if given any stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ and $\mathcal{F}_t$-Wiener process W, there exists and $\mathcal{F}_t$-adapted process Z defined in it such that Z solves (10) $\mathbb{P}$-a.s.

Moreover, we have two kind of uniqueness

Definition

(Uniqueness in law) We say that uniqueness in law holds for equation (10) if any two processes Z_1 and Z_2 solving equation (10) with the same initial data have the same law, that is $\mathbb{P} \circ Z_1^{-1} = \mathbb{P} \circ Z_2^{-1}$.

Definition

(Pathwise uniqueness) We say that the pathwise uniqueness holds for equation (10) if given two processes Z_1 and Z_2 solving equation (10) with the same initial data and defined on the same probability basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ and Wiener process, we have that

$$\mathbb{P}\left\{Z_1(t) = Z_2(t) \, \text{for all}\, t\right\} = 1.$$

Now, let us define another process X on the same probability basis which solves

$$\begin{aligned} dX(t) &= a(t,X)dt + g(t,X)dt + \sigma(t,X)dW, \quad X(0) = x \\ &= a(t,X)dt + \sigma(t,X)(b(t,X)dt + dW) \\ &= a(t,X)dt + \sigma(t,X)dW^b \end{aligned} \tag{11}$$

where $b(t,X) = b^+(t,X)g(t,X)$

$$b^+ = \begin{cases} \frac{1}{\sigma}, & \sigma \neq 0 \\ 0 & \sigma = 0. \end{cases}$$

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If $\mathbb{E}e^{\frac{1}{2}\int_0^t\int_0^t b^2(s,Z)ds} < \infty$, then Girsanov theorem gives that

$$W^b(t) = W(t) - \int_0^t b(s,Z)ds, \quad t \in [0,T],$$

is a Wiener process with respect to $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\le t\le T}, \mathbb{P}^b)$, where

$$d\mathbb{P}^b = e^{\int_0^t b(s,Z)dW(s) - \frac{1}{2}\int_0^t b^2(s,Z)ds} d\mathbb{P}.$$

The properties of equation (10) transfers to (11) and viceversa.

Therefore we consider the system of Stratonovich equations (for the Fourier components)

$$\begin{cases} d\mathbf{Y}_{\mathbf{k}}(t) = -i\sum_{\mathbf{h}\in\mathbb{Z}^3} P_{\mathbf{k}}(\mathbf{Y}_{\mathbf{k}-\mathbf{h}}(t))\dfrac{\langle \mathbf{Y}_{\mathbf{h}}(t)dt + \sigma\circ d\mathbf{W}_{\mathbf{h}}(t), \mathbf{k}\rangle}{1+\alpha\|\mathbf{h}\|^2} \\ \langle \mathbf{Y}_{\mathbf{k}}(t), \mathbf{k}\rangle = 0 \\ \mathbf{Y}_{-\mathbf{k}}(t) = \overline{\mathbf{Y}_{\mathbf{k}}}(t) \end{cases} \tag{12}$$

Set $J = \{(k_1, k_2, k_3) \in \mathbb{Z}^3 : k_1 > 0 \text{ or } (k_1 = 0, k_2 > 0) \text{ or } (k_1 = 0, k_2 = 0, k_3 > 0)\}$.
$\{\mathbf{W}_{\mathbf{h}}\}_{\mathbf{h}\in J}$ is a family of independent $\mathbb{C}^3$-valued Brownian motions on a filtered probability space $(\Omega, (\mathcal{F}_t)_{t\geq 0}, P)$,

$$\langle \mathbf{W}_{\mathbf{h}}(t), \mathbf{h}\rangle = 0, \qquad \mathbf{W}_{-\mathbf{h}}(t) = \overline{\mathbf{W}_{\mathbf{h}}}(t)$$

and $\sigma \neq 0$.

According to the properties of Stratonovich integral, we have formally that energy is conserved:

$$dE(t) = 0$$

i.e. for any $t \in [0, T]$

$$E(t) = E(0) \qquad a.s.$$

The computations are similar to the previous ones, using (8) and

$$\frac{\langle \mathbf{Y}_{\mathbf{k}'-\mathbf{h}'}, \mathbf{Y}_{\mathbf{k}'} \rangle \circ \langle d\mathbf{W}_{\mathbf{h}'}, \mathbf{k}' \rangle}{1 + \alpha \|\mathbf{h}'\|^2} = \overline{\frac{\langle \mathbf{Y}_{\mathbf{k}-\mathbf{h}}, \mathbf{Y}_{\mathbf{k}} \rangle \circ \langle d\mathbf{W}_{\mathbf{h}}, \mathbf{k} \rangle}{1 + \alpha \|\mathbf{h}\|^2}}.$$

for $\mathbf{h}' = -\mathbf{h}$ and $\mathbf{k}' = \mathbf{k} - \mathbf{h}$.

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for $\mathbf{h}' = -\mathbf{h}$ and $\mathbf{k}' = \mathbf{k} - \mathbf{h}$.

We will assume $\sigma = 1$.
Let

$$\sigma_{\mathbf{h}} := \frac{1}{1 + \alpha \|\mathbf{h}\|^2}. \tag{13}$$

Therefore (12) is

$$\begin{aligned} d\mathbf{Y}_{\mathbf{k}}(t) =& -i \sum_{\mathbf{h} \in \mathbb{Z}^3} \sigma_{\mathbf{h}} \langle \mathbf{Y}_{\mathbf{h}}(t), \mathbf{k} \rangle P_{\mathbf{k}}(\mathbf{Y}_{\mathbf{k}-\mathbf{h}}(t)) dt \\ & -i \sum_{\mathbf{h} \in \mathbb{Z}^3} \sigma_{\mathbf{h}} \| P_{\mathbf{h}}(\mathbf{k}) \| P_{\mathbf{k}}(\mathbf{Y}_{\mathbf{k}-\mathbf{h}}(t)) \circ d\widetilde{W}_{\mathbf{h},\mathbf{k}}(t) \end{aligned} \tag{14}$$

with

$$\widetilde{W}_{\mathbf{h},\mathbf{k}}(t) \| P_{\mathbf{h}}(\mathbf{k}) \| := \widetilde{\widetilde{W}}_{\mathbf{h},\mathbf{k}}(t) = \langle P_{\mathbf{h}}(W_{\mathbf{h}}(t)), \mathbf{k} \rangle \tag{15}$$

so $\Re \widetilde{W}_{\mathbf{h},\mathbf{k}}$ and $\Im \widetilde{W}_{\mathbf{h},\mathbf{k}}$ are standard real Brownian motions.
This is the system to be analyzed.

Now we transform the equations from the Stratonovich form into the Itô form, which is more convenient for computations.

Proposition (Itô formulation)

Let $\{\mathbf{Y_k}\}_{\mathbf{k}\in\mathbb{Z}^3,\mathbf{k}\neq\mathbf{0}}$ *be a sequence of continuous and adapted processes defined on a given filtered probability space such that* $\sum_{\mathbf{k}} \|\mathbf{Y_k}(t)\|^2 < \infty$ *a.s.. If the sequence solves the following system*

$$\begin{aligned} d\mathbf{Y_k}(t) = & -i\sum_{\mathbf{h}\in\mathbb{Z}^3} \sigma_{\mathbf{h}}\langle \mathbf{Y_h}(t),\mathbf{k}\rangle P_{\mathbf{k}}(\mathbf{Y_{k-h}}(t))dt \\ & -i\sum_{\mathbf{h}\in\mathbb{Z}^3} \sigma_{\mathbf{h}}\|P_{\mathbf{h}}(\mathbf{k})\| P_{\mathbf{k}}(\mathbf{Y_{k-h}}(t))d\widetilde{W}_{\mathbf{h},\mathbf{k}}(t) \\ & -\sum_{\mathbf{h}\in\mathbb{Z}^3} \sigma_{\mathbf{h}}^2\|P_{\mathbf{h}}(\mathbf{k})\|^2 P_{\mathbf{k}}(P_{\mathbf{k-h}}(\mathbf{Y_k}(t)))dt, \qquad \forall \mathbf{k} \end{aligned} \tag{16}$$

then it solves system (14)*.*

Let us consider the linear system obtained by neglecting the nonlinear terms in (16):

$$\begin{cases} d\mathbf{Y_k}(t) & = & -i\sum_{\mathbf{h}\in\mathbb{Z}^3}\sigma_{\mathbf{h}}\|P_{\mathbf{h}}(\mathbf{k})\|P_{\mathbf{k}}(\mathbf{Y_{k-h}}(t))d\widetilde{B}_{\mathbf{h},\mathbf{k}}(t) \\ & & -\sum_{\mathbf{h}\in\mathbb{Z}^3}\sigma_{\mathbf{h}}^2\|P_{\mathbf{h}}(\mathbf{k})\|^2 P_{\mathbf{k}}(P_{\mathbf{k-h}}(\mathbf{Y_k}(t)))dt \\ \langle \mathbf{Y_k}(t),\mathbf{k}\rangle & = & 0 \\ \mathbf{Y_{-k}}(t) & = & \overline{\mathbf{Y_k}(t)} \\ \mathbf{Y_k}(0) & = & \mathbf{y_k} \end{cases} \tag{17}$$

for each $\mathbf{k}\neq\mathbf{0}$. Here, $\{\widetilde{B}_{\mathbf{h},\mathbf{k}}\}$ is a family of $\mathbb{C}$-valued BM defined on $(\Omega,\{\mathcal{F}_t\}_{t\geq 0},Q)$ as before.

Remark

The deterministic integral in the r.h.s. of $(17)_1$ *is well defined if* $\sum_{\mathbf{h}}\sigma_{\mathbf{h}}^2 \equiv \sum_{\mathbf{h}}\frac{1}{(1+\alpha\|\mathbf{h}\|^2)^2} < \infty$.

Theorem (Existence of a strong solution)

Given $\mathbf{y} \in l^2$, *there exists a family of continuous and adapted* $\mathbb{C}^3$*-valued stochastic processes* $\{\mathbf{Y_k}\}_{\mathbf{k}\in\mathbb{Z}^3}$ *such that for all* $t \geq 0$

$$\begin{cases} \mathbf{Y_k}(t) & = & \mathbf{y_k} - i\sum_{\mathbf{h}\in\mathbb{Z}^3} \sigma_{\mathbf{h}} \| P_{\mathbf{h}}(\mathbf{k})\| \int_0^t P_{\mathbf{k}}(\mathbf{Y_{k-h}}(s)) d\widetilde{B}_{\mathbf{h},\mathbf{k}}(s) \\ & & - \sum_{\mathbf{h}\in\mathbb{Z}^3} \sigma_{\mathbf{h}}^2 \| P_{\mathbf{h}}(\mathbf{k})\|^2 \int_0^t P_{\mathbf{k}}(P_{\mathbf{k-h}}(\mathbf{Y_k}(s)))ds \\ \langle \mathbf{Y_k}(t), \mathbf{k}\rangle & = & 0 \\ \mathbf{Y_{-k}}(t) & = & \overline{\mathbf{Y_k}(t)} \end{cases}$$

Q-a.s. for all $\mathbf{k}$, *and for any* $t > 0$

$$\sum_{\mathbf{k}\in\mathbb{Z}^3} \|\mathbf{Y_k}(t)\|^2 \leq \sum_{\mathbf{k}\in\mathbb{Z}^3} \|\mathbf{y_k}\|^2 \qquad Q - a.s.$$

The solution $\mathbf{Y} = \{\mathbf{Y_k}\}_{\mathbf{k}\in\mathbb{Z}^3}$ is called an energy controlled strong solution.

Pathwise Uniqueness

The covariance matrices:

We have proved existence of energy controlled solutions $\mathbf{Y} = \{\mathbf{Y_k}\}_{\mathbf{k}\in\mathbb{Z}^3}$ of (17); now we want to show their uniqueness. The idea is to study the time evolution of the covariance matrices $\{A_\mathbf{k}\}_{\mathbf{k}\in\mathbb{Z}^3}$, defined as follows: for $j_1, j_2 = 1, 2, 3$,

$$A_\mathbf{k}^{j_1,j_2}(t) = \mathbb{E}^Q \left[\Re Y_\mathbf{k}^{(j_1)}(t) \Re Y_\mathbf{k}^{(j_2)}(t) + \Im Y_\mathbf{k}^{(j_1)}(t) \Im Y_\mathbf{k}^{(j_2)}(t) \right]$$

$A_\mathbf{k}(t)$ is a symmetric and semi-positive definite matrix; therefore the trace of $A_\mathbf{k}(t)$ is non negative. Moreover, we have

$$\sum_{\mathbf{k}\in\mathbb{Z}^3} \mathrm{Tr}(A_\mathbf{k}(t)) \leq \|\mathbf{y}\|_{l^2}^2 \qquad (18)$$

for any $t \geq 0$. Finally, $P_\mathbf{k} A_\mathbf{k}(t) P_\mathbf{k} = A_\mathbf{k}(t)$.

With some long but easy computations we get that each $A_{\mathbf{k}}$ fulfils a linear equation.

Lemma

For each $\mathbf{k} \neq \mathbf{0}$, $A_{\mathbf{k}}$ *fulfils the differential equation*

$$\begin{aligned} \frac{dA_{\mathbf{k}}}{dt}(t) = &- \sum_{\mathbf{h}\in\mathbb{Z}^3} \sigma_{\mathbf{h}}^2 \|P_{\mathbf{h}}(\mathbf{k})\|^2 P_{\mathbf{k}} P_{\mathbf{k}-\mathbf{h}} A_{\mathbf{k}}(t) \\ &- \sum_{\mathbf{h}\in\mathbb{Z}^3} \sigma_{\mathbf{h}}^2 \|P_{\mathbf{h}}(\mathbf{k})\|^2 A_{\mathbf{k}}(t) P_{\mathbf{k}-\mathbf{h}} P_{\mathbf{k}} \\ &+ 2 \sum_{\mathbf{h}\in\mathbb{Z}^3} \sigma_{\mathbf{h}}^2 \|P_{\mathbf{h}}(\mathbf{k})\|^2 P_{\mathbf{k}} A_{\mathbf{k}-\mathbf{h}}(t) P_{\mathbf{k}} \end{aligned} \tag{19}$$

This Lemma shows the non trivial fact that the covariance matrices satisfy a closed differential system.

We prove pathwise uniqueness for system (17).

Theorem (Pathwise uniqueness)

There exists at most one energy controlled strong solution to system (17), that is given two energy controlled strong solutions $\mathbf{Y}_{[1]}$ and $\mathbf{Y}_{[2]}$ to system (17) defined on the same probability space $(\Omega, \{\mathcal{F}_t\}_{t\geq 0}, Q)$ and with the same initial data $\mathbf{y} \in l^2$ and Brownian motions, we have, Q-a.s.

$$\mathbf{Y}_{[1]}(t) = \mathbf{Y}_{[2]}(t) \qquad \forall t$$

Proof.

Define

$$\mathbf{Y} := \mathbf{Y}_{[1]} - \mathbf{Y}_{[2]}.$$

By linearity $\mathbf{Y}$ solves (17) but with $\mathbf{Y}(0) = \mathbf{0}$.
Let $\{A_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{Z}^3}$ be the covariance matrices of $\mathbf{Y}$; these matrices satisfy (19) with zero initial condition and regularity (18). Thus, the uniqueness problem for (17) is transformed in the easier uniqueness problem for the deterministic system (19). Indeed, in order to show that for any $t > 0$ we have $\mathbf{Y}(t) = 0$ Q-a.s. it is enough to prove that system (19) with zero initial condition has the unique solution which vanishes, i.e. for any $\mathbf{k} \neq \mathbf{0}$, given $A_{\mathbf{k}}(0) = 0$ we have $A_{\mathbf{k}}(t) = 0$ for all $t > 0$. □

Consider the nonlinear system in the Itô form

$$\begin{cases} d\mathbf{Y}_{\mathbf{k}}(t) = -i \sum_{\mathbf{h}\in\mathbb{Z}^3} \sigma_{\mathbf{h}} \langle \mathbf{Y}_{\mathbf{h}}(t), \mathbf{k}\rangle P_{\mathbf{k}}(\mathbf{Y}_{\mathbf{k}-\mathbf{h}}(t))dt \\ \qquad - i \sum_{\mathbf{h}\in\mathbb{Z}^3} \sigma_{\mathbf{h}} \| P_{\mathbf{h}}(\mathbf{k})\| P_{\mathbf{k}}(\mathbf{Y}_{\mathbf{k}-\mathbf{h}}(t)) d\widetilde{B}_{\mathbf{h},\mathbf{k}}(t) \\ \qquad - \sum_{\mathbf{h}\in\mathbb{Z}^3} \sigma_{\mathbf{h}}^2 \| P_{\mathbf{h}}(\mathbf{k})\|^2 P_{\mathbf{k}}(P_{k-\mathbf{h}}(\mathbf{Y}_{\mathbf{k}}(t)))dt \\ \langle \mathbf{Y}_{\mathbf{k}}(t), \mathbf{k}\rangle = 0 \\ \mathbf{Y}_{-\mathbf{k}}(t) = \overline{\mathbf{Y}_{\mathbf{k}}}(t) \\ \mathbf{Y}_{\mathbf{k}}(0) = \mathbf{y}_{\mathbf{k}} \end{cases} \tag{20}$$

Starting from the solution of the linear system (17) we construct a solution to this nonlinear system by means of Girsanov transform. We shall deal with solutions on any fixed finite time interval $[0, T]$.

Definition

Given $\mathbf{y} \in l^2$, a weak solution of equation (20) in l^2 is a filtered probability space $(\Omega, \{\mathcal{F}_t\}_{t\in[0,T]}, P)$, a sequence of independent $\mathbb{C}^3$-valued Brownian motions $W = \{\mathbf{W_h}\}_{\mathbf{h}\in J}$ on $(\Omega, \{\mathcal{F}_t\}_{t\in[0,T]}, P)$ and an l^2-valued stochastic process $Y := (\mathbf{Y_k})_{\mathbf{k}\in\mathbb{Z}^3}$ on $(\Omega, \{\mathcal{F}_t\}_{t\in[0,T]}, P)$, with continuous and adapted components $\mathbf{Y_k}$ solving system (20) in the integral form, P-a.s.

We denote this solution by $((\Omega, \{\mathcal{F}_t\}_{t\in[0,T]}, P), Y, W)$.
Moreover, it is called an energy controlled weak solution if for all $t \in [0, T]$ this solution satisfies

$$\sum_{\mathbf{k}\in\mathbb{Z}^3} \|\mathbf{Y_k}(t)\|^2 \le \sum_{\mathbf{k}\in\mathbb{Z}^3} \|\mathbf{y_k}\|^2 \qquad P - a.s.$$

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Girsanov transform

Let $Y = \{\mathbf{Y_h}\}_\mathbf{h}$ be the strong energy controlled solution of linear model (17). Define

$$L(t) = \sum_{h \in J} \int_0^t \langle \mathbf{Y_h}(s), d\mathbf{B_h}(s) \rangle$$

Then L is a martingale: its quadratic variation $[L, L]$ is well defined and given by

$$[L, L](t) := \int_0^t \sum_{\mathbf{h} \in J} |\mathbf{Y_h}(s)|^2 ds \leq t \|\mathbf{y}\|_{l^2}^2 \qquad (21)$$

Q-a.s.

Then Novikov condition holds and we can apply Girsanov theorem:

Theorem

Let Y be the strong solution of system (17) with the family of independent $\mathbb{C}^3$-valued standard Brownian motions $\{B_h\}_{h\in J}$ on $(\Omega, \{\mathcal{F}_t\}_{t\geq 0}, Q)$. Then

$$W_h(t) = B_h(t) - \int_0^t Y_h(s)ds, \qquad 0 \leq t \leq T, \tag{22}$$

defines a family of independent $\mathbb{C}^3$-valued standard Brownian motions on $(\Omega, \{\mathcal{F}_t\}_{0\leq t\leq T}, P)$ with the measure P, defined on $(\Omega, \mathcal{F}_T)$, which is absolutely continuous with respect to the measure Q and

$$\frac{dP}{dQ} = e^{L(T)-\frac{1}{2}[L,L](T)}$$

We can use Girsanov transform to pass from the strong solution $\mathbf{Y}$ (defined on any $(\Omega, \{\mathcal{F}_t\}, Q)$) of the linear system

$$d\mathbf{Y}_\mathbf{k}(t) = -\, i \sum_{\mathbf{h}\in\mathbb{Z}^3} \sigma_\mathbf{h} \|P_\mathbf{h}(\mathbf{k})\| P_\mathbf{k}(\mathbf{Y}_{\mathbf{k}-\mathbf{h}}(t)) d\widetilde{B}_{\mathbf{h},\mathbf{k}}(t) - \sum_{\mathbf{h}\in\mathbb{Z}^3} \sigma_\mathbf{h}^2 \|P_\mathbf{h}(\mathbf{k})\|^2 P_\mathbf{k}(P_{k-\mathbf{h}}(\mathbf{Y}_\mathbf{k}(t)))dt$$

to the weak solution (defined on $(\Omega, \{\mathcal{F}_t\}, P)$) of the non linear system

$$d\mathbf{Y}_\mathbf{k}(t) = -i \sum_{\mathbf{h}\in\mathbb{Z}^3} \sigma_\mathbf{h} \|P_\mathbf{h}(\mathbf{k})\| P_\mathbf{k}(\mathbf{Y}_{\mathbf{k}-\mathbf{h}}(t))(\langle \mathbf{Y}_\mathbf{h}(t), \mathbf{k}\rangle dt + d\widetilde{W}_{\mathbf{h},\mathbf{k}}(t)) - \sum_{\mathbf{h}\in\mathbb{Z}^3} \sigma_\mathbf{h}^2 \|P_\mathbf{h}(\mathbf{k})\|^2 P_\mathbf{k}(P_{k-\mathbf{h}}(\mathbf{Y}_\mathbf{k}(t)))dt$$

Our main result is

Theorem (Weak existence and uniqueness)

For any initial data of finite energy, the nonlinear system (20) has an energy controlled weak solution. Moreover, this solution is unique in law.

Remark

i) Our technique can be applied for any $\alpha > 0$ to more general models, that is we can deal with a noise defined by means of $\sigma_h = \frac{1}{(1+\alpha\|h\|^2)^p}$ and with the smoothing term given by $u^\alpha = (1-\alpha\Delta)^{-p}v^\alpha$, for any $p > 3/4$.
Indeed, all what we need is $\sum_{\mathbf{h}\in\mathbb{Z}^3}\sigma_{\mathbf{h}}^2 < \infty$.
ii) In the 2-dimensional case, all our computations can be extended for $p > 1/2$.

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ii) In the 2-dimensional case, all our computations can be extended for $p > 1/2$.

The formulation in SPDE

The stochastic model considered so far in Fourier components can be written as a stochastic partial differential equation, as follows

$$\begin{cases} dv + (u \cdot \nabla)v \ dt \ + \nabla p \ dt \ = \sum_{h\in\mathbb{Z}^3} \sum_{j=1}^{3} \sigma_h e_h \frac{\partial v}{\partial x^{(j)}} \circ dW_h^{(j)} \\ v = (1-\alpha\triangle)u \\ \text{div } v = 0 \\ v(0) = v_0 \end{cases} \tag{23}$$

In a more compact form as:

$$dv + \sum_{j=1}^{3} \frac{\partial v}{\partial x^{(j)}} \; u^{(j)} \; dt \; + \nabla p \; dt \; = \sum_{j=1}^{3} \frac{\partial v}{\partial x^{(j)}} \circ dW^{(j)}, \qquad (24)$$

and

$$W(t,x) = 2 \sum_{h \in J} \frac{\cos(\langle h, x \rangle) \Re W_h(t) - \sin(\langle h, x \rangle) \Im W_h(t)}{1 + \alpha \|h\|^2}, \qquad (25)$$

THANK YOU!