GAFD Tutorial I

Keith Julien

Department of Applied Mathematics, University of Colorado Boulder

Mathematics of Turbulence Long Program

IPAM, UCLA
Sept 8th-12th

Geophysical Fluid Dynamics (GFD)

- GFD: Turbulent dynamics of (stably and unstably) stratified fluids on a giant rotating sphere
  - ~ planetary atmospheres, oceans, interiors
  - ~ stellar interiors
  - ~ other geometries also occur, .e.g. cylindrical geometries observed in astrophysical accretion disks.

- Motivation
  - ~ understand complex phenomena occurring in GFD object
  - ~ gain predictive understanding with perhaps the ability to forecast (e.g. climate, solar weather)

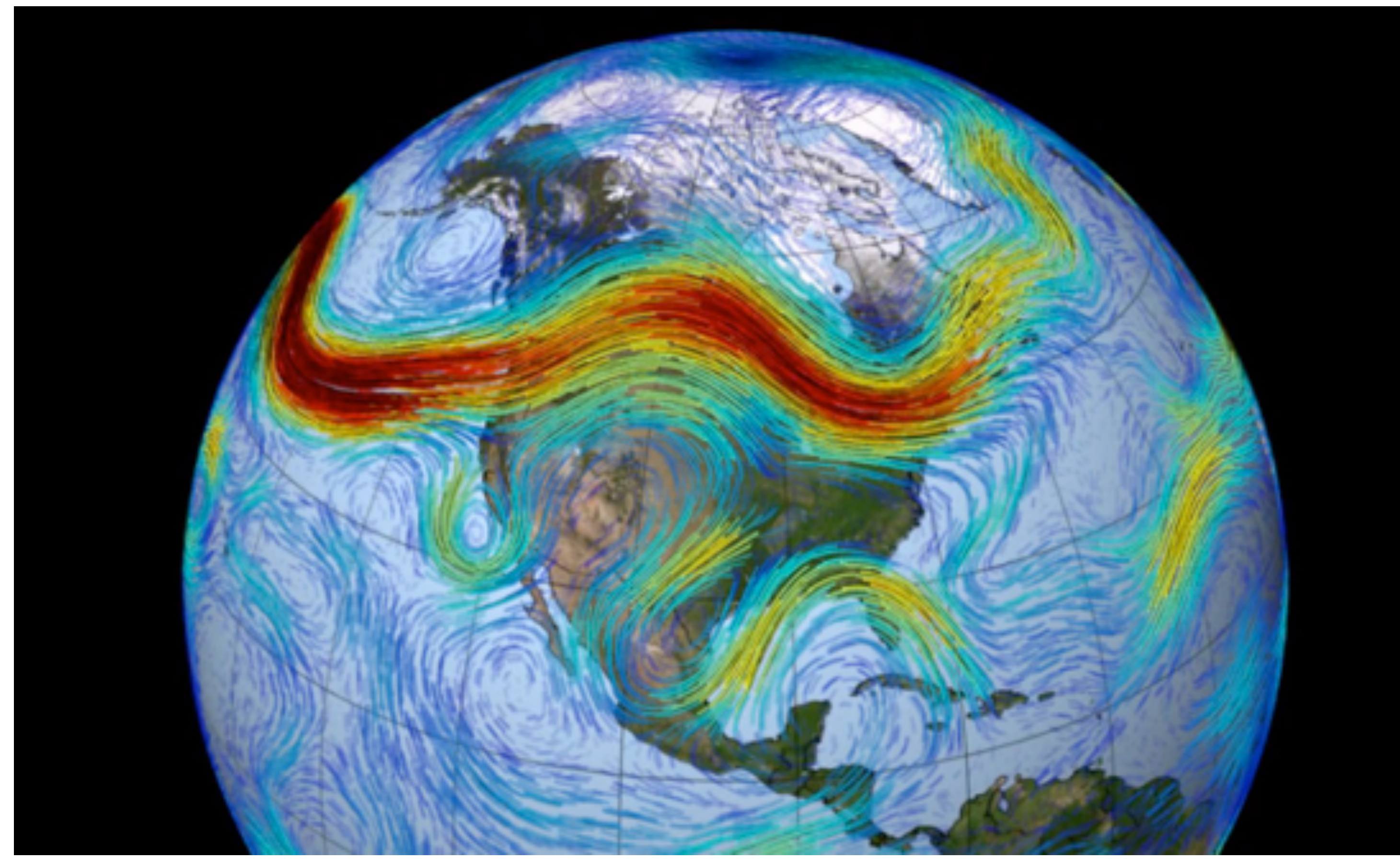
- Methodology (challenge: spatio-temporal complexity)
  - ~ finding good mathematical, experimental representations that provide good physical interpretations of GFD systems => mix of (applied) mathematics and theoretical physics
  - ~ Paucity of data from observation/laboratory ==> GFD heavily computational

Some examples (an incomplete sample)

A mix of observations and simulations

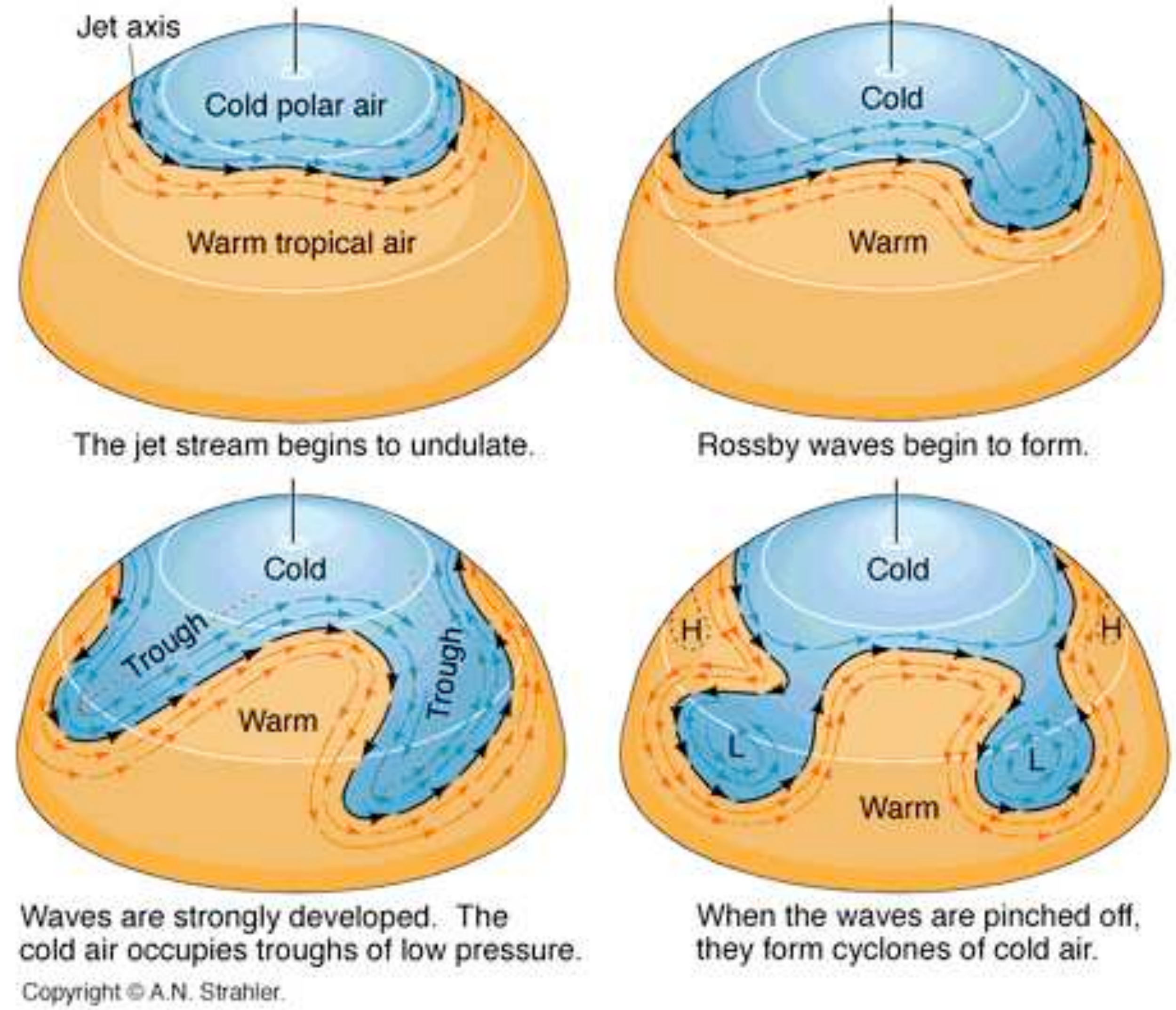
Earth: Atmosphere & Ocean System

Earth: Atmosphere & Ocean System



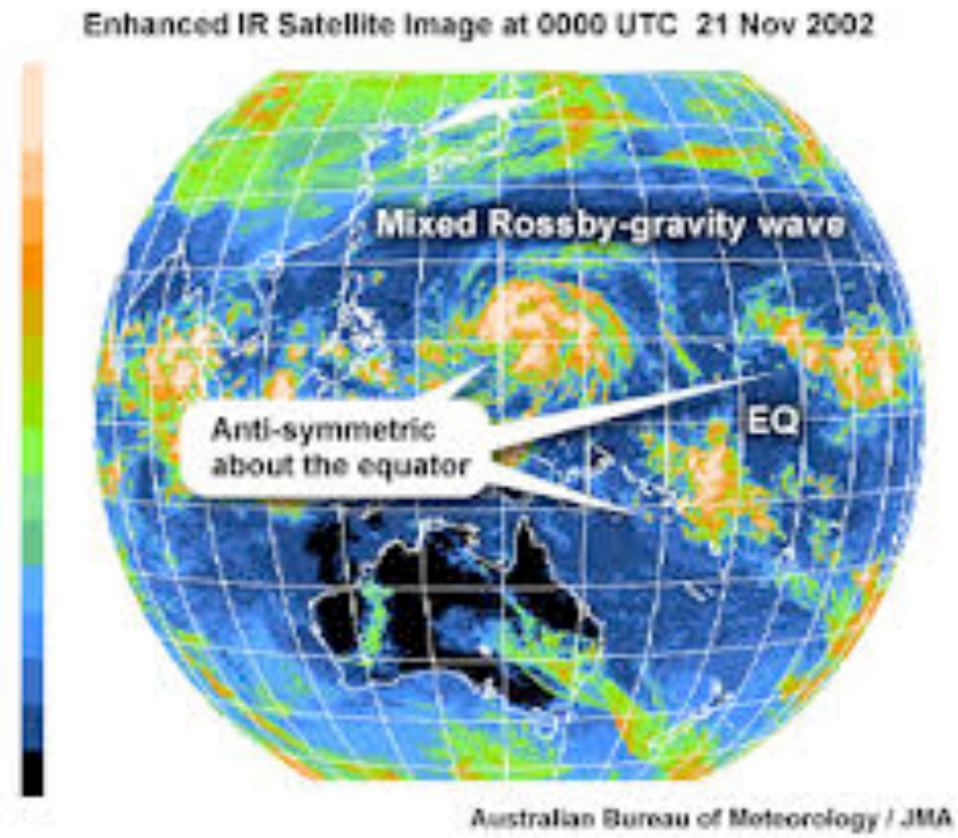
General circulation, rotation and stratification gives rise to jets: e.g. polar & subtropical. Instability - Rossby waves

Earth: Atmosphere & Ocean System



General circulation, rotation and stratification gives rise to jets: e.g. polar & subtropical.
Instability - Rossby waves

Earth: Atmosphere & Ocean System

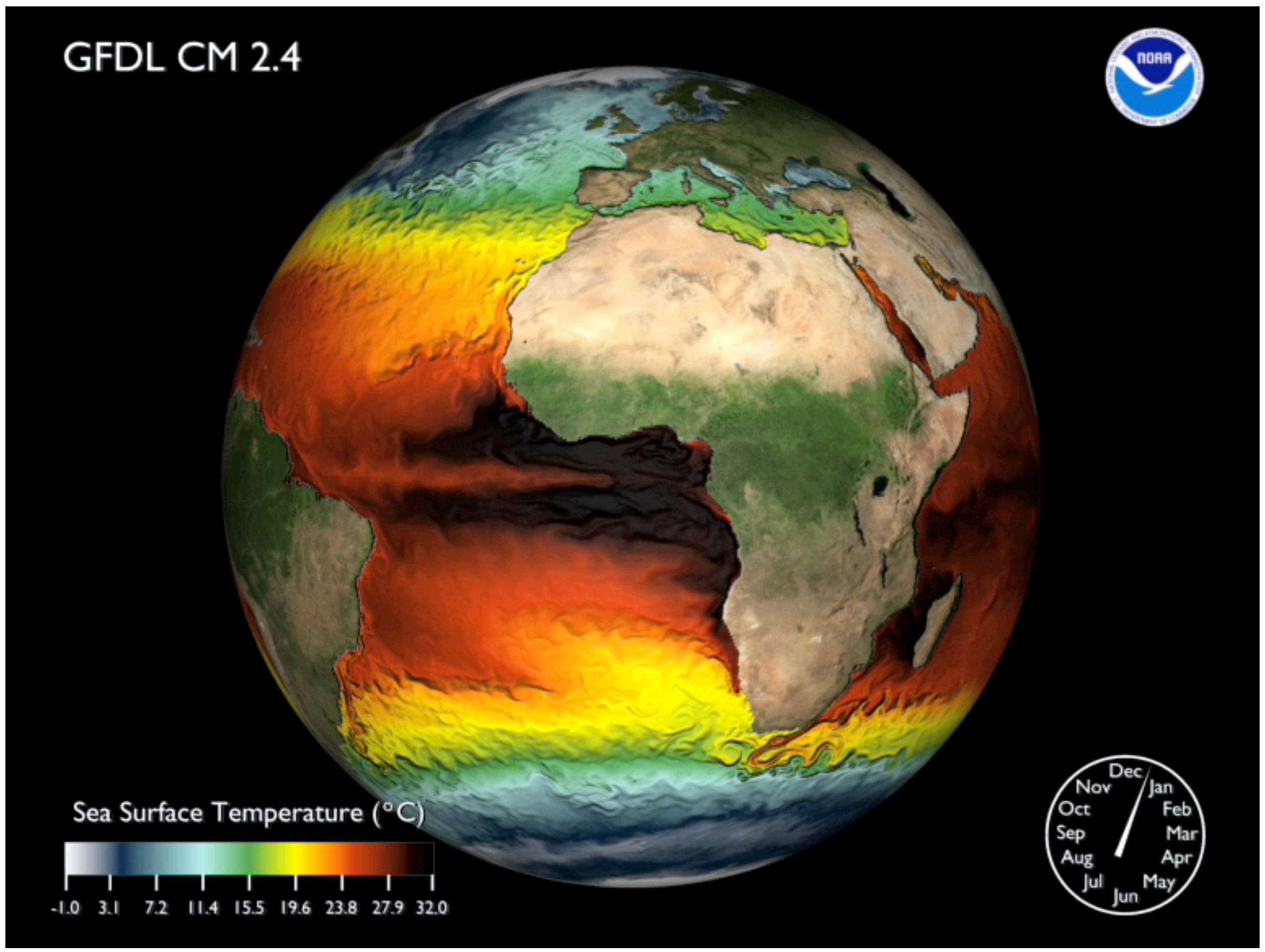


Equatorial region is a wave-guide rich in trapped waves.
Coupling with moisture in the form of deep convection leads to persistent oscillatory features on seasonal and intraseasonal timescale.

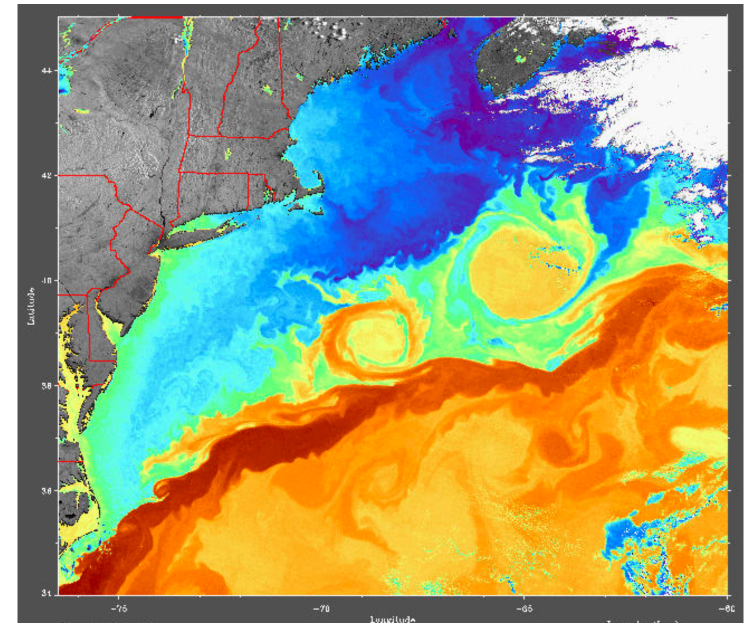
Madden-Julian-Oscillation (eastward propagation of anomalous rainfall)

Ocean dynamics

planetary - gyre scale ~ O(1000) km

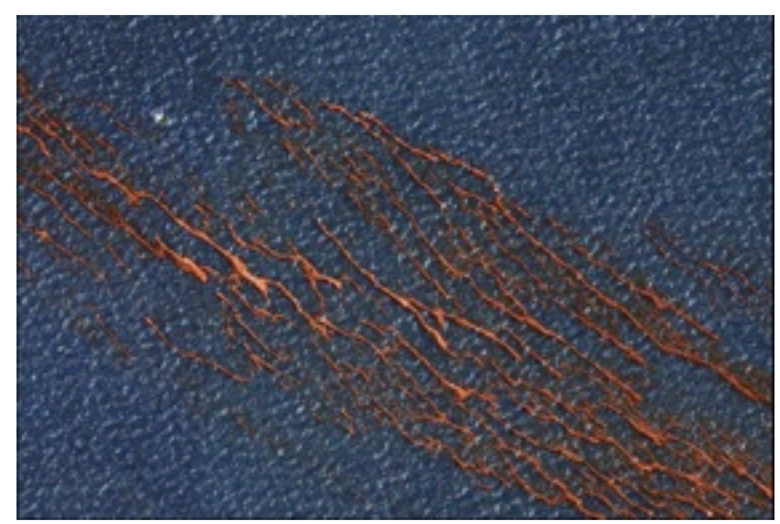


mesoscale ~ O(100) km



submesoscale ≤ O(1) km

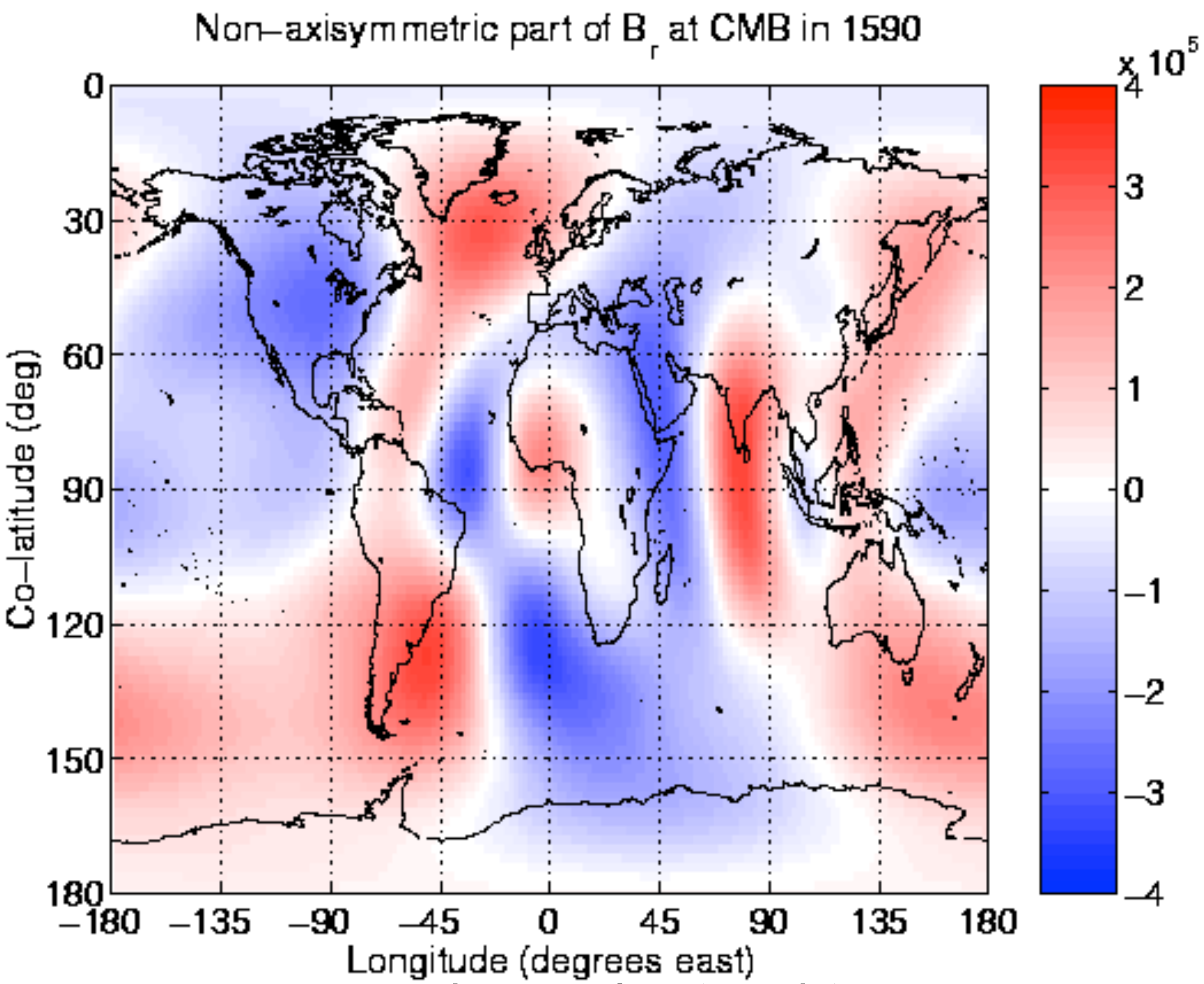
Langmuir Turbulence



Earth's Interior

$$Ro \sim 10^{-7}$$
$$U \sim 3 \times 10^{-4}\ m/s$$
$$\Omega \sim 7 \times 10^{-5}\ rad/s$$
$$L \sim 2260\ km$$

Complex wave phenomena



Movie: Chris Finlay (Leeds)

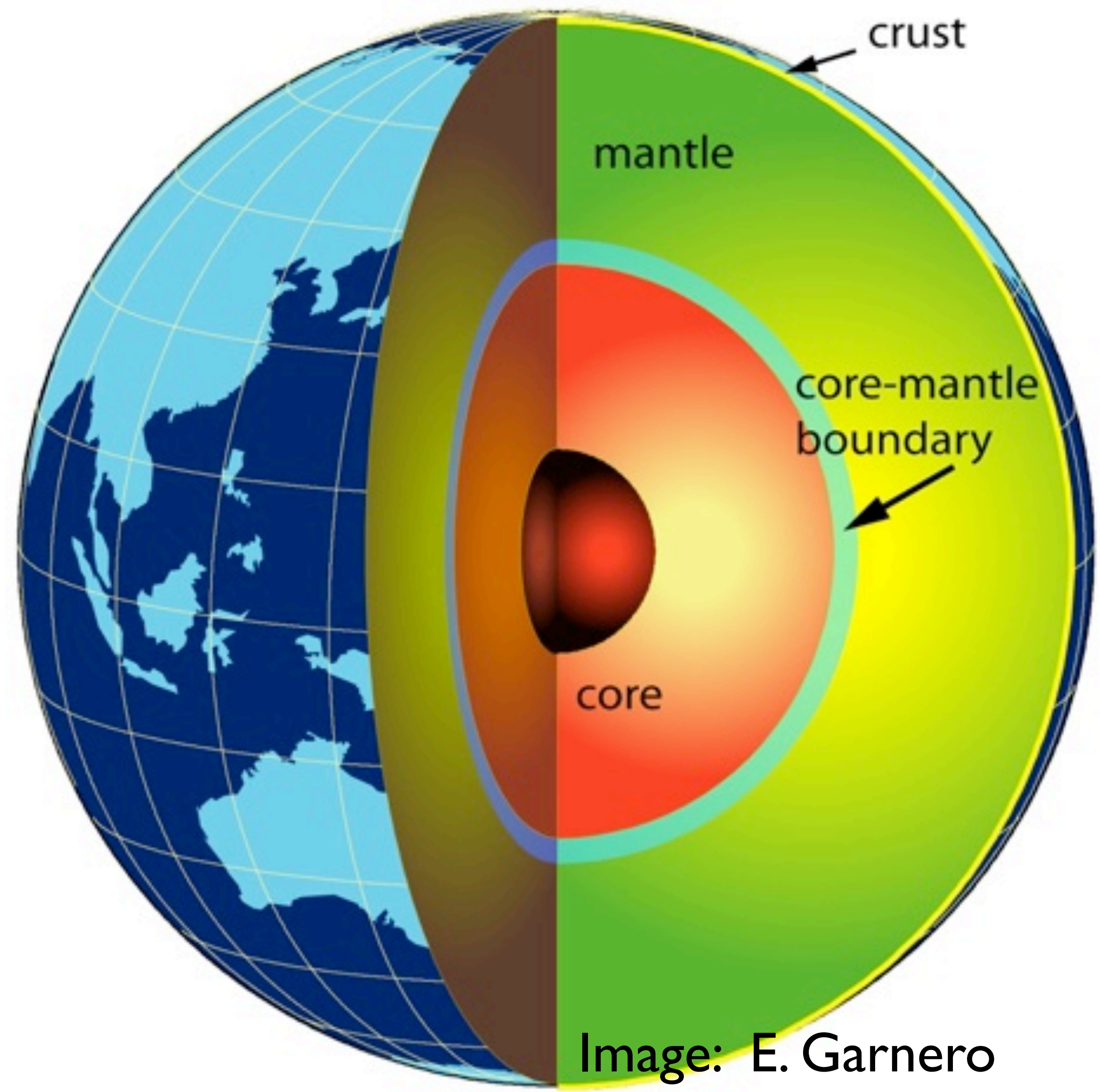


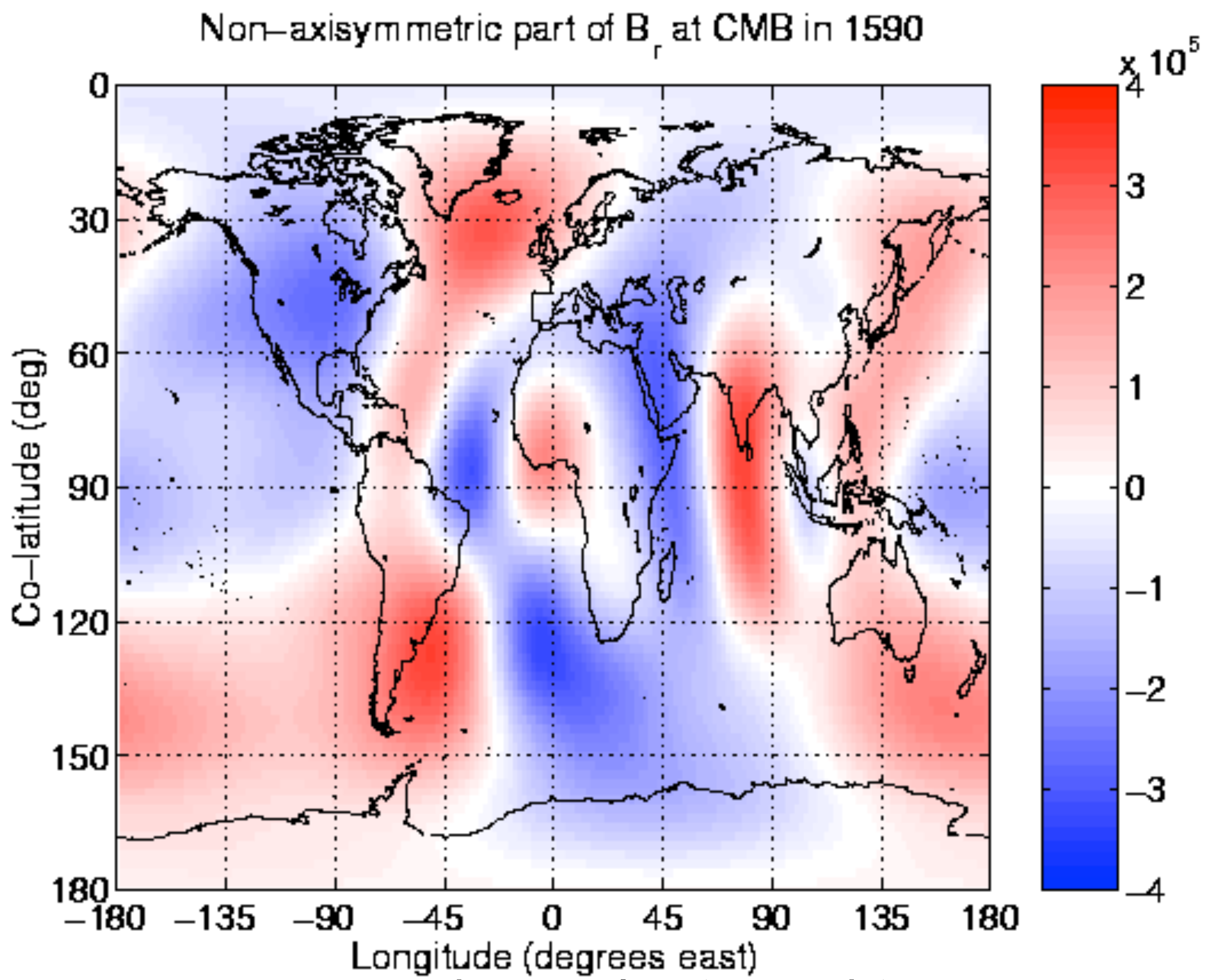
Image: E. Garnero

Liquid iron outer core is the seat of the geodynamo.

- *convectively unstable and strongly influence by rotation*

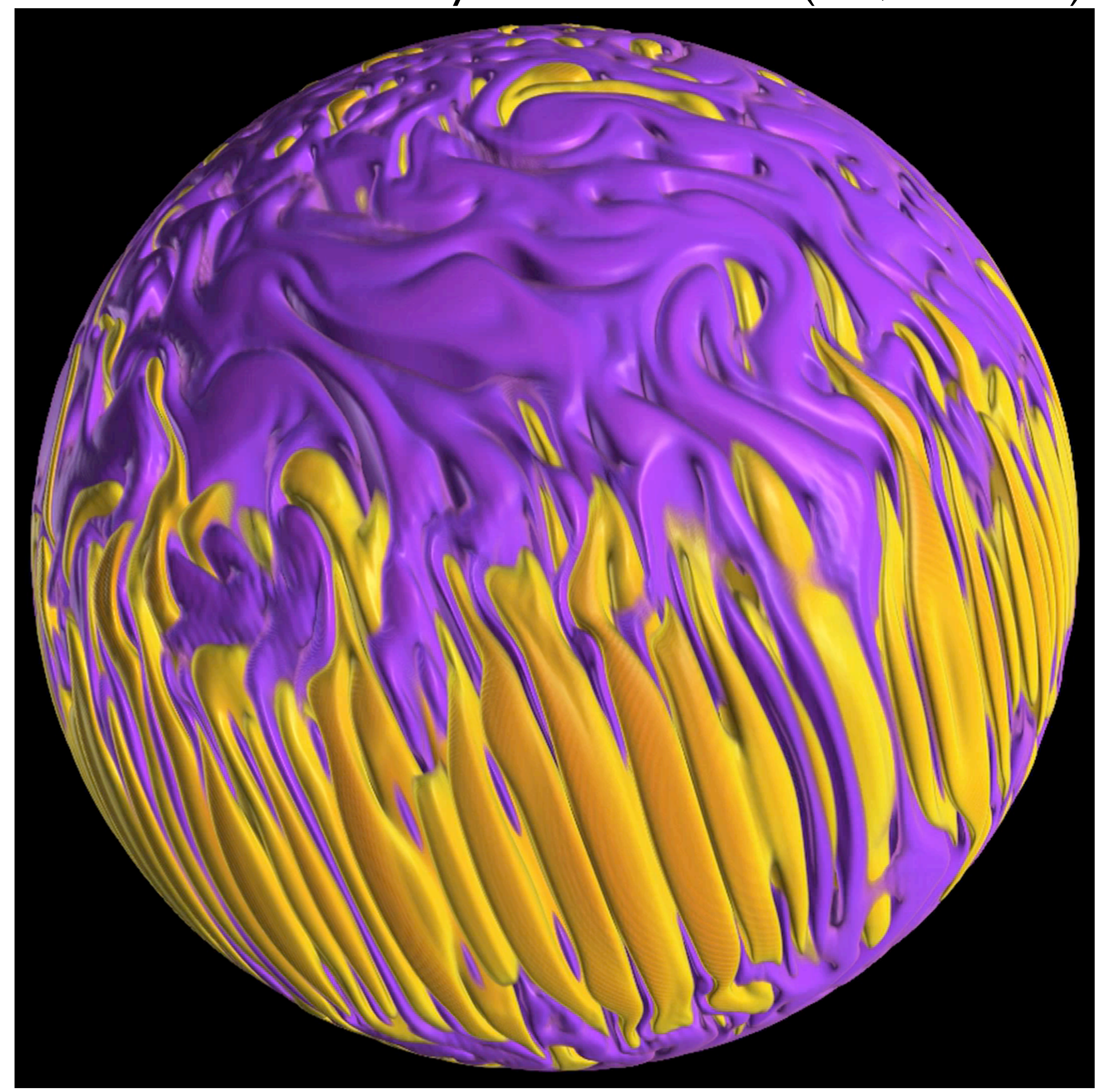
Earth's Interior

Complex wave phenomena



Movie: Chris Finlay (U Leeds)

courtesy N. Featherstone (CU, NSF CIG)



$$Ra = 2.5 \times 10^7. \quad Pr = 1, \quad Ek = 10^{-5}$$

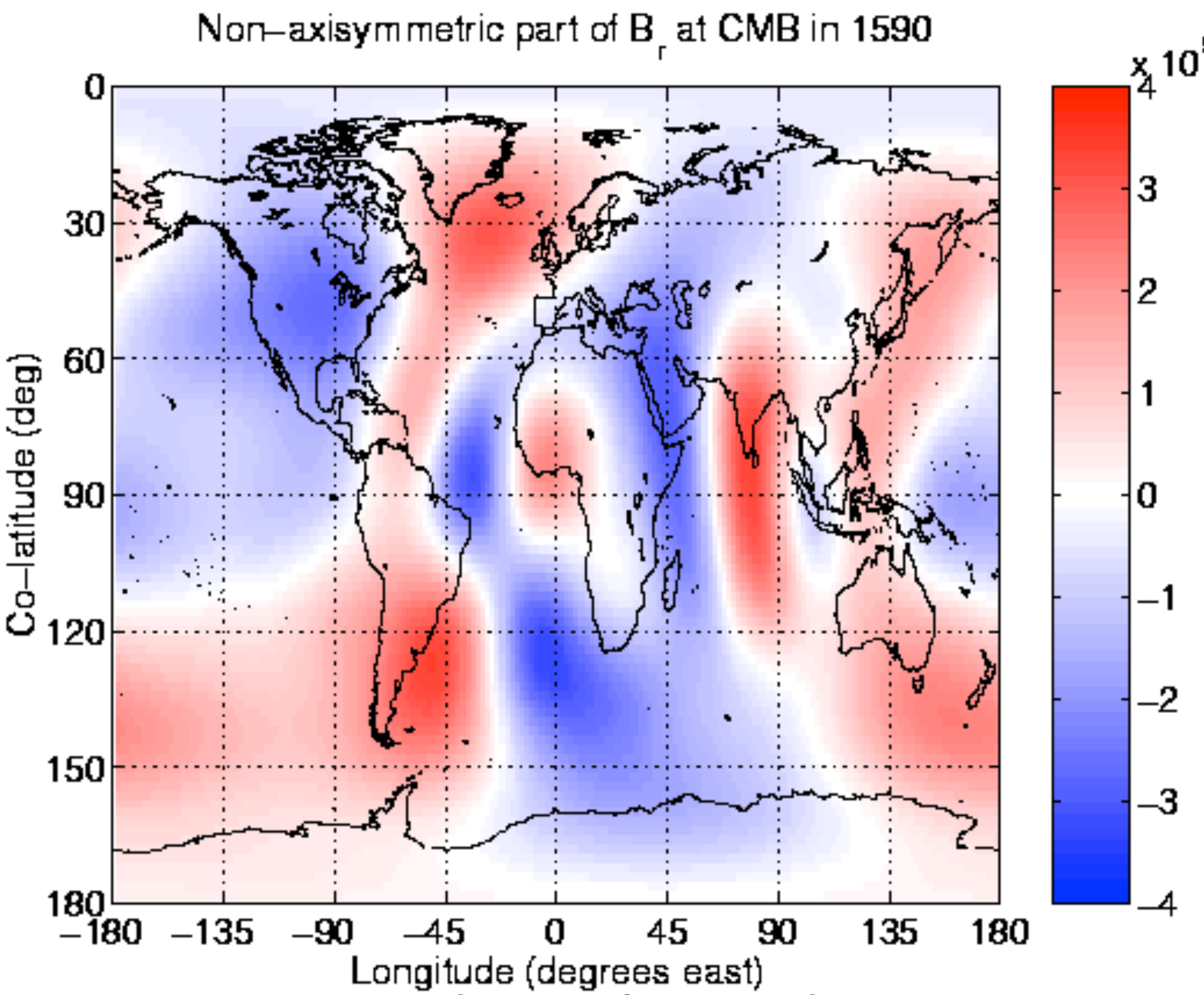
Electrically conducting fluid heavily influenced by Earth's rotation s.t. motions are highly anisotropic.

B-field free decay time: 20,000 yrs

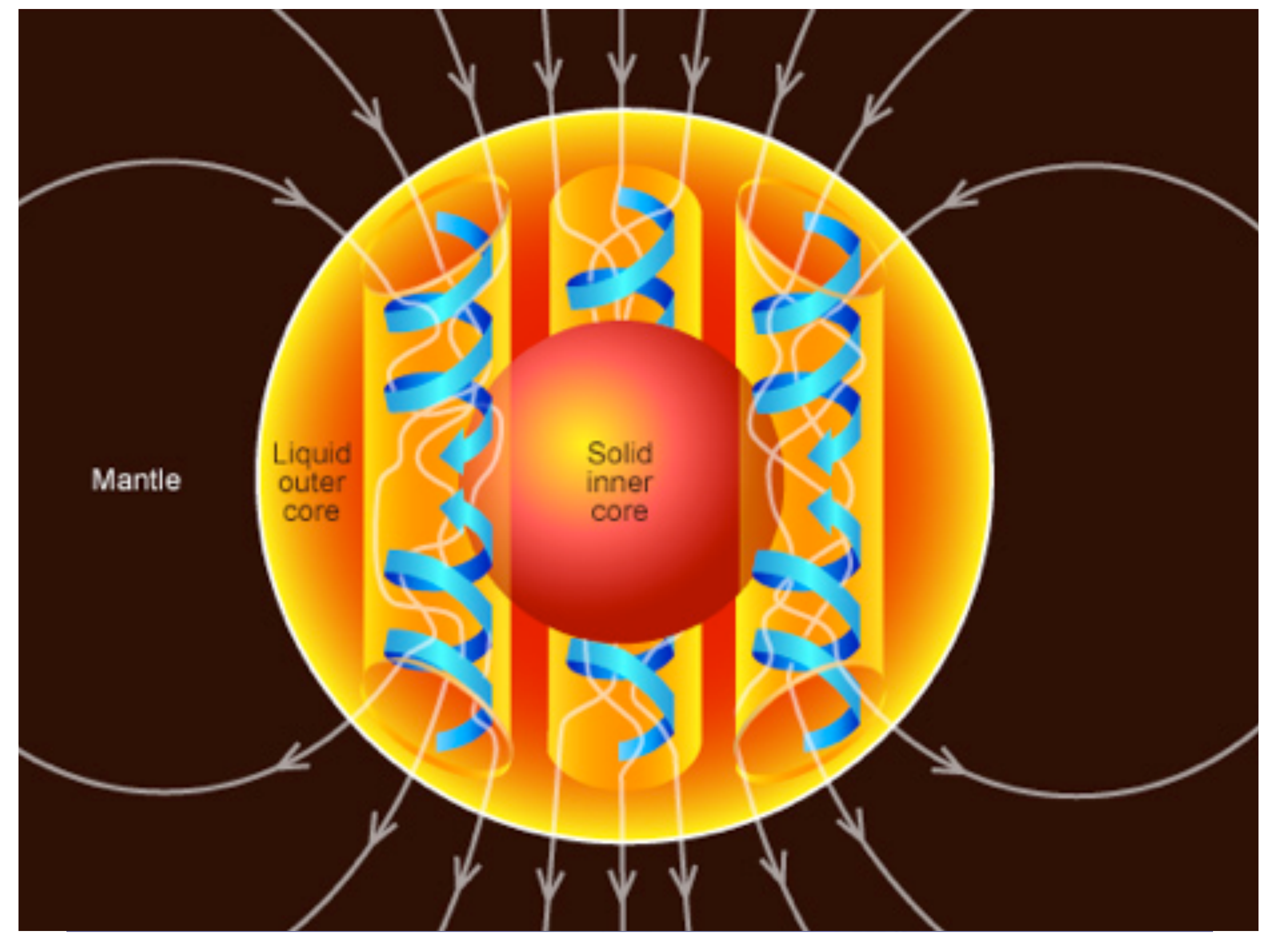
Existence: 4 billion years

Earth's Interior

Complex wave phenomena



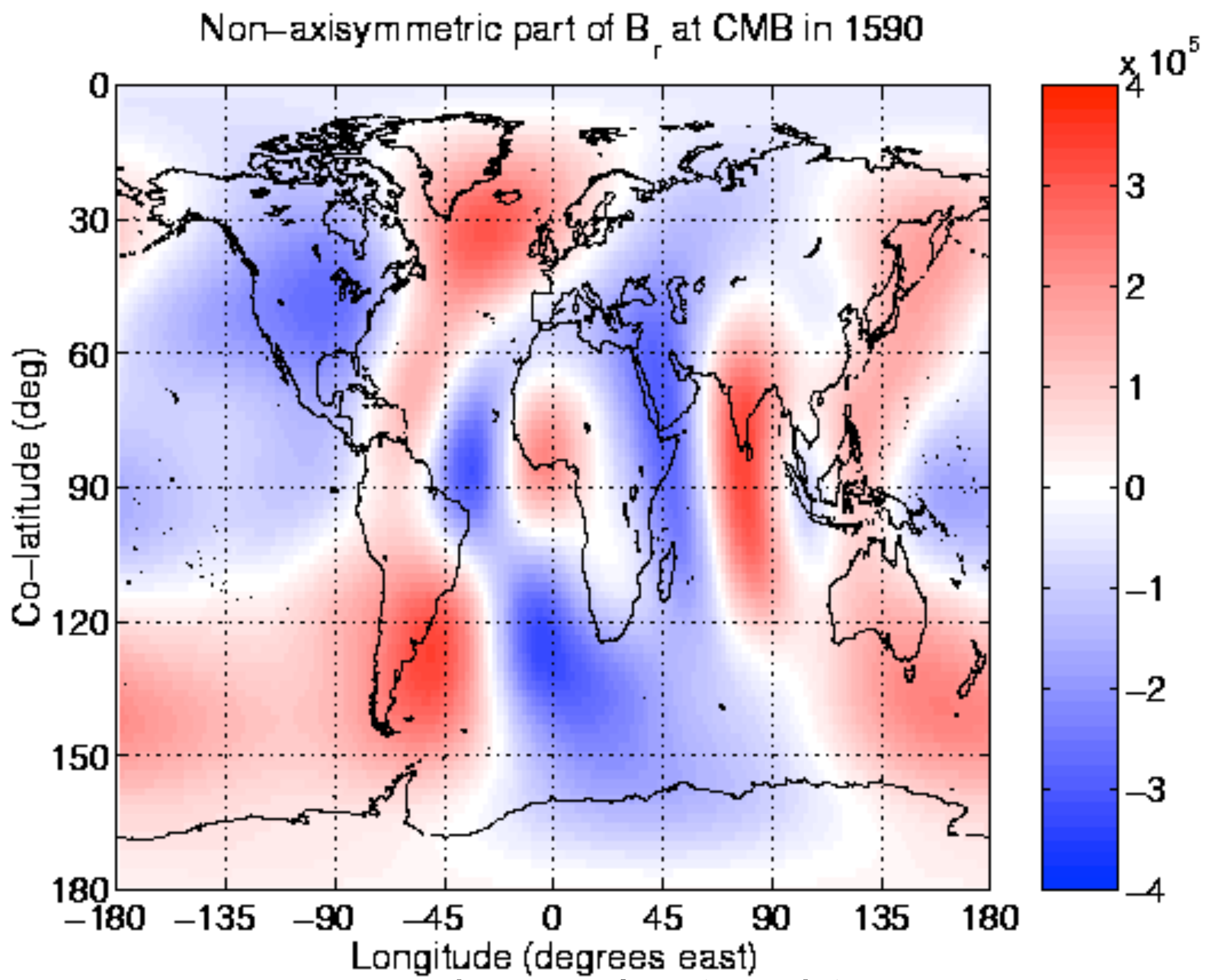
Movie: Chris Finlay (Leeds)



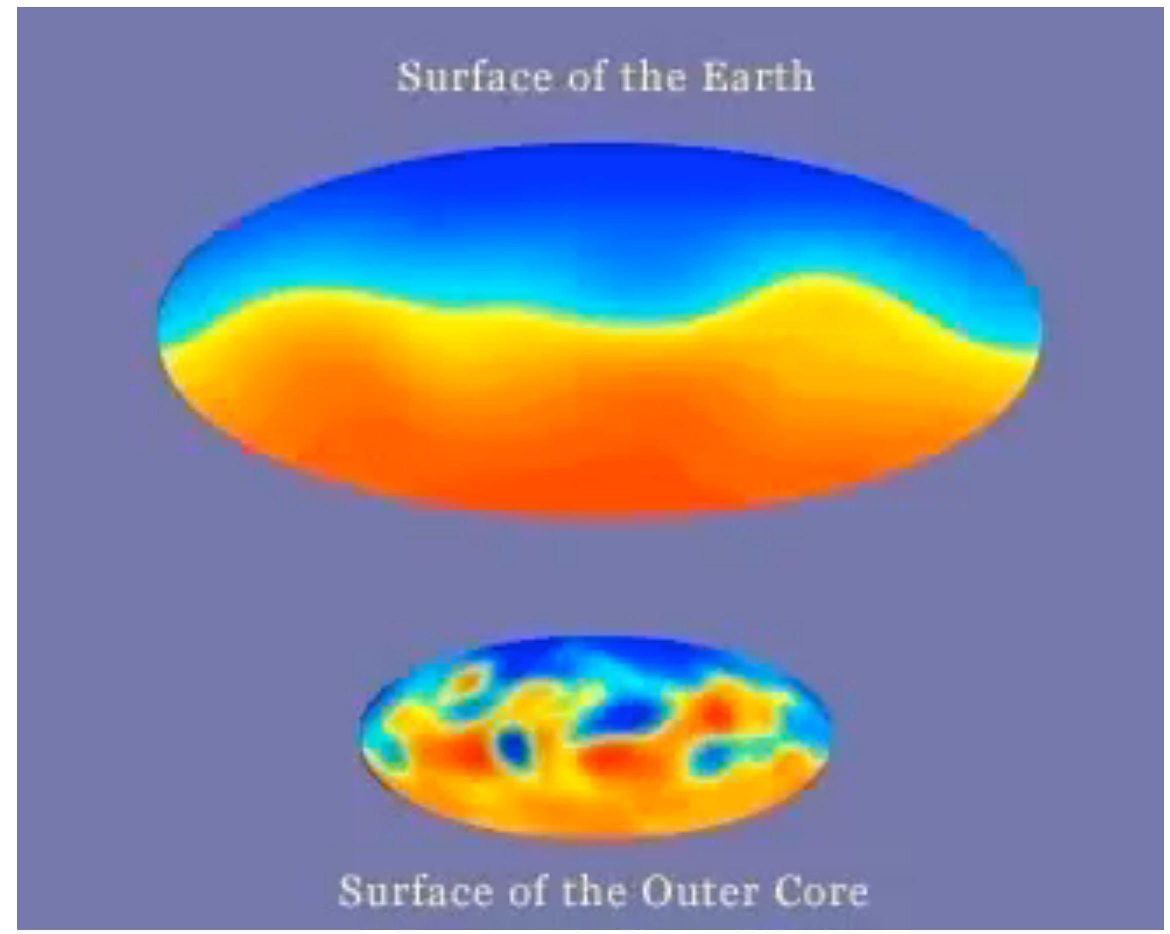
Electrically conducting fluid heavily influenced by Earth's rotation s.t. motions are highly anisotropic.
B-field free decay time: 20,000 yrs
Existence: 4 billion years

Earth's Interior

Complex wave phenomena



Movie: Chris Finlay (Leeds)

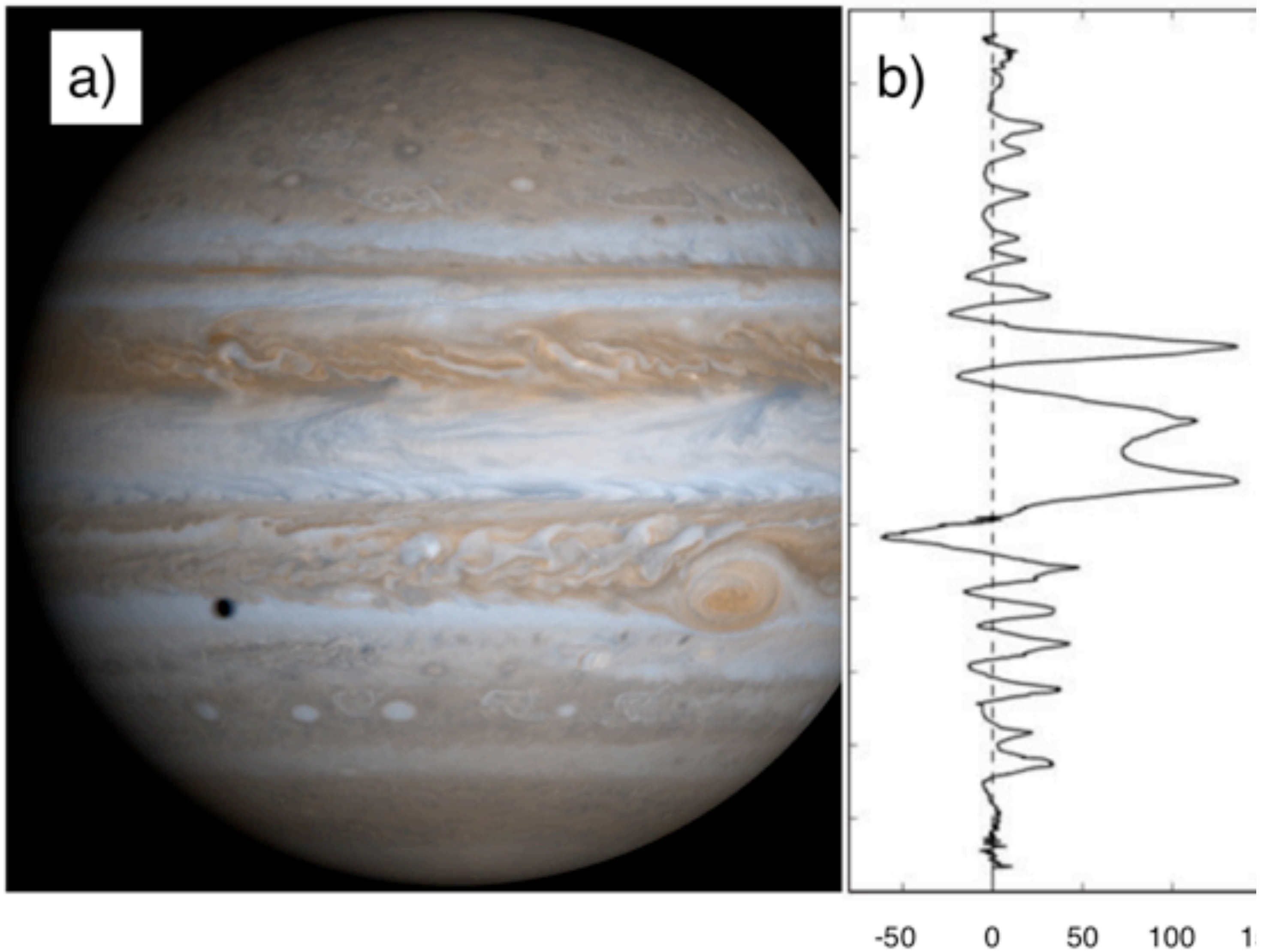


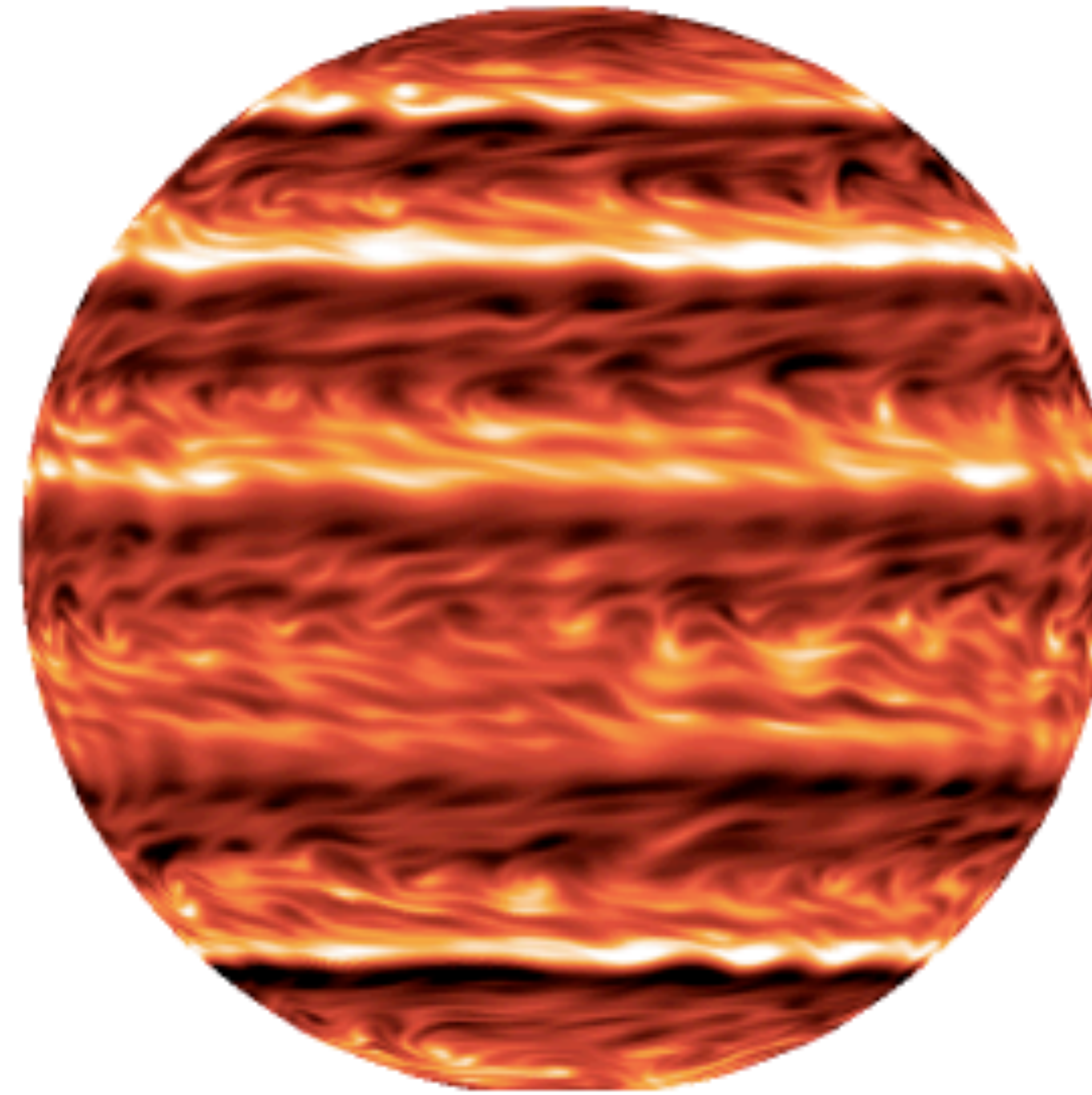
Movie: Glatzmaier & Roberts Model.

Geomagnetic reversal in dipole field occur on O(100K yrs) with transient on O(10Kyrs)
Non-axisymmetric component temporal changes on O(1Kyr) including westward drift

Giant Planets

Scott & Polvani JAS 2007





Characterized by alternating bands of zonal jets w vortical eddy generation in region of strong shear

Jet formation is an open question - observed dynamics are likely a combination of strongly-stably stratified weather layer and rotationally constrained convective turbulence in the deep interior.

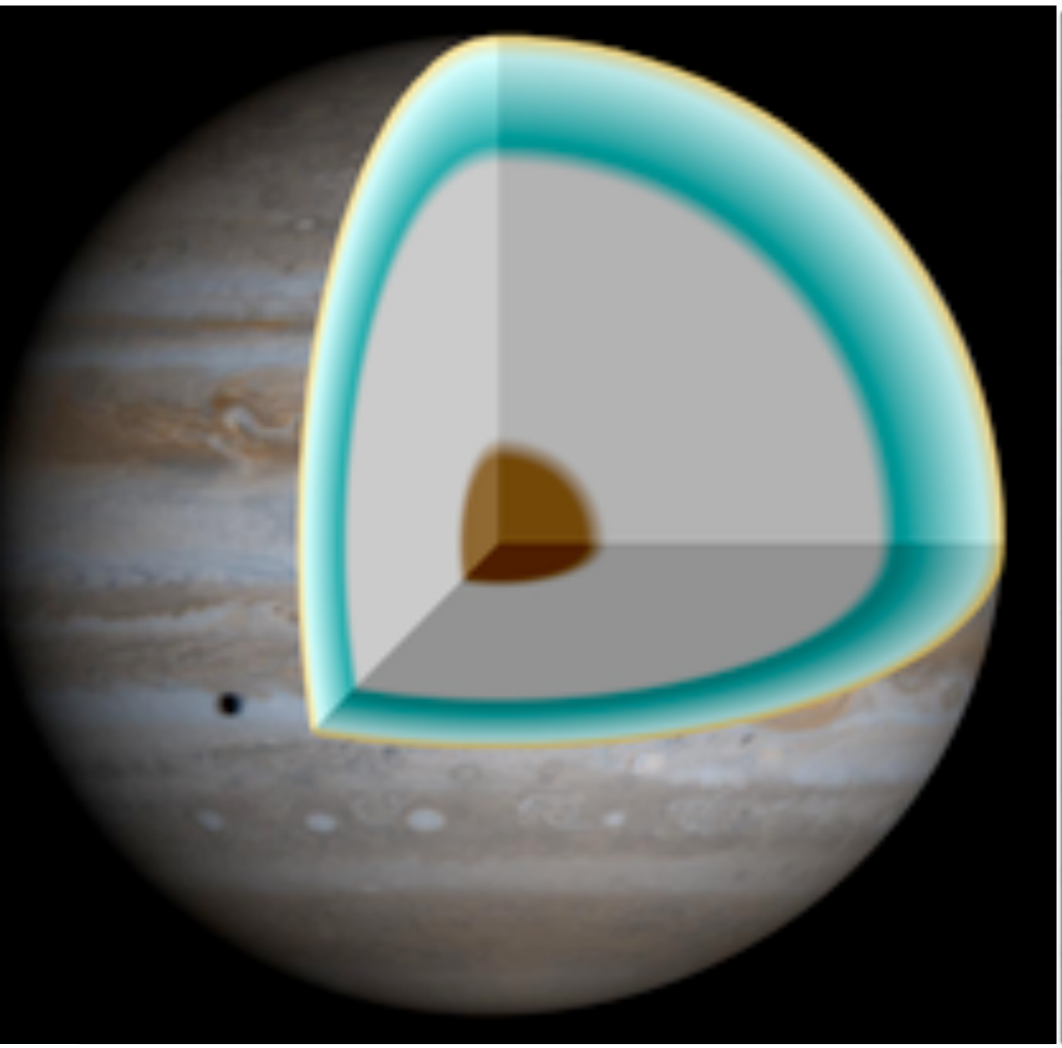
$Ro \sim 10^{-2}$

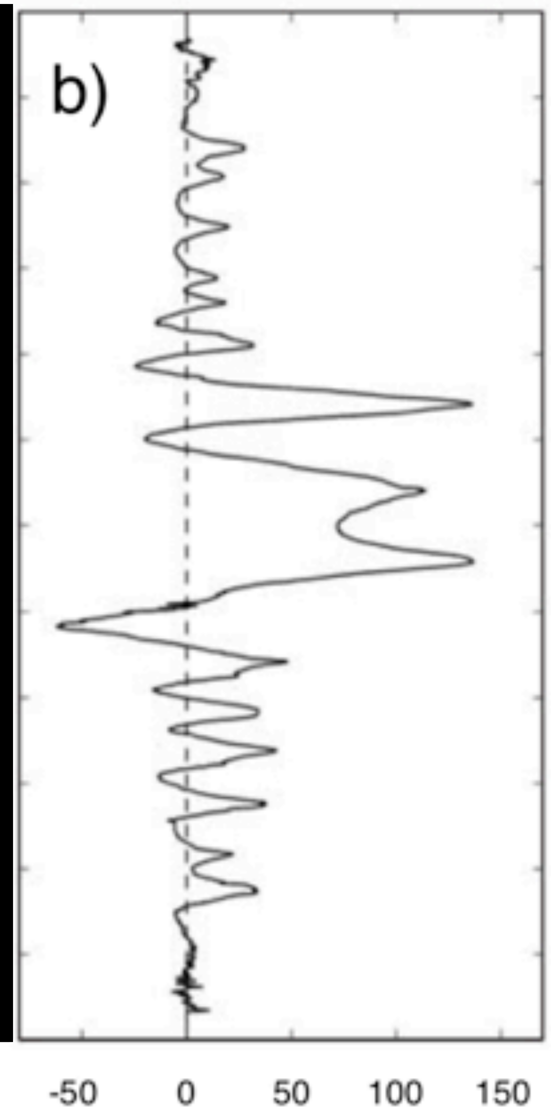
$U \sim 100 m/s$

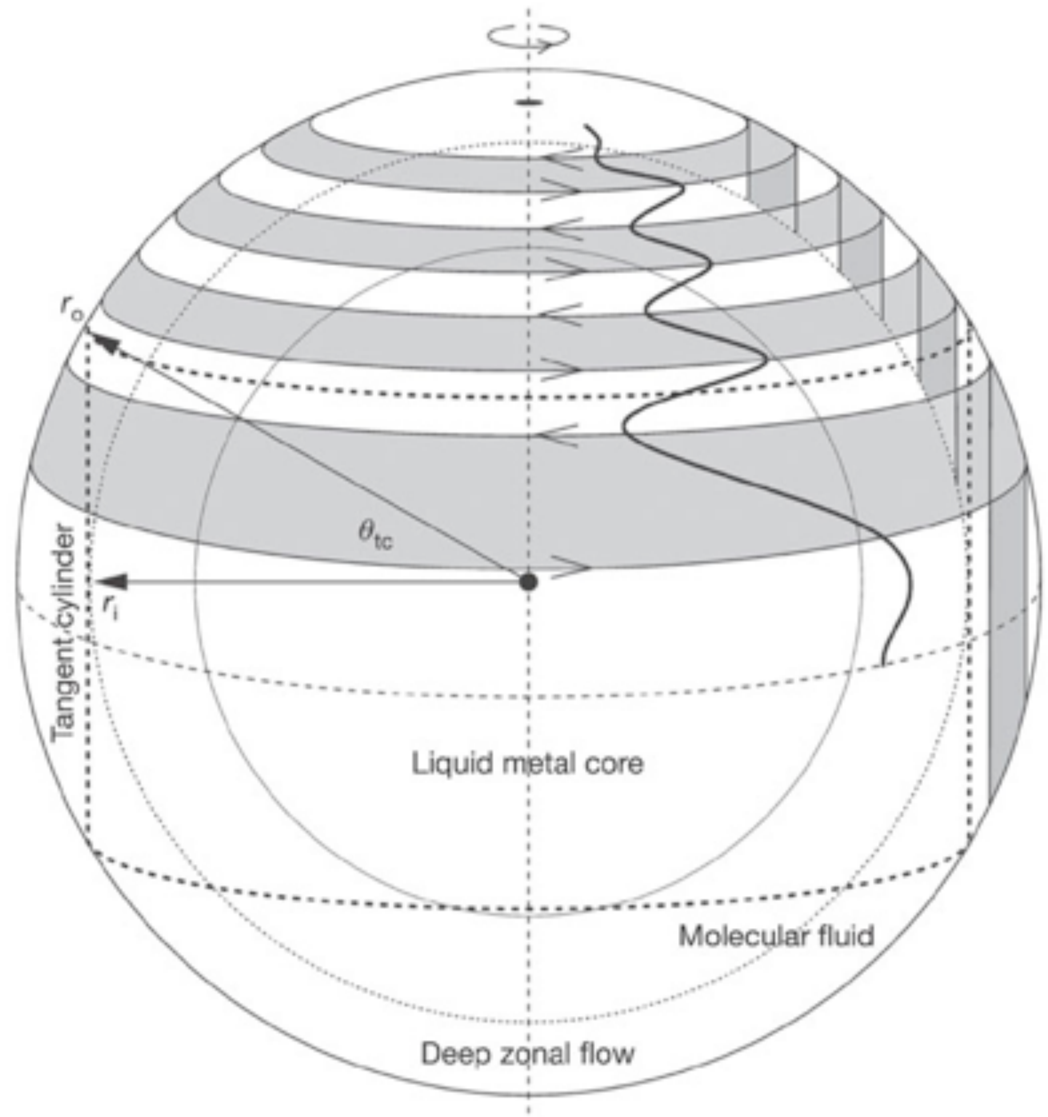
$\Omega \sim 2 \times 10^{-4} \; rad/s$

$L \sim 15 \; Mm$

Giant Planets







Characterized by alternating bands of zonal jets w vortical eddy generation in region of strong shear

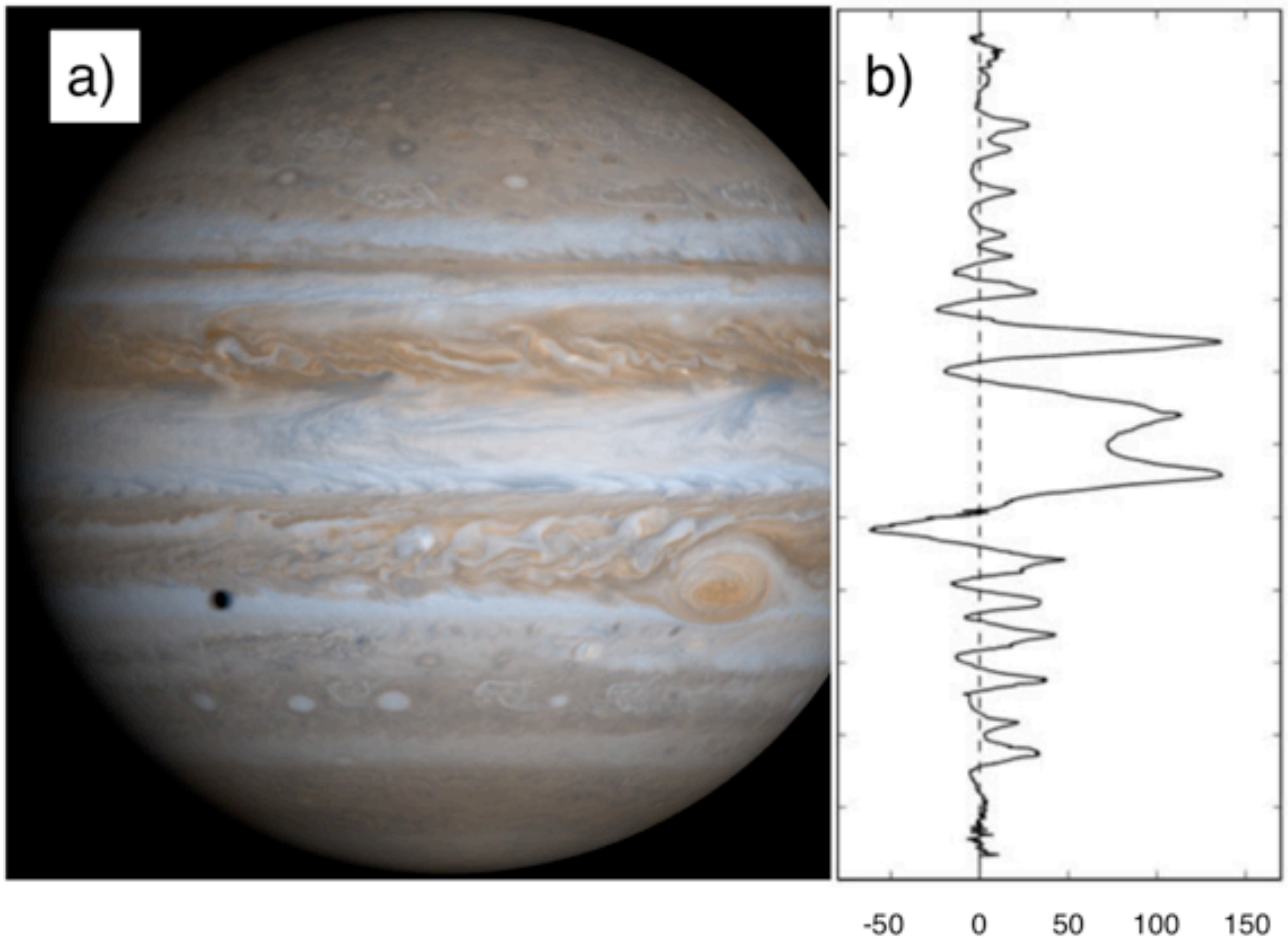
Open question - observed dynamics are likely a combination of strongly-strably stratified weather layer and rotationally constrained convective turbulence in the deep interior.

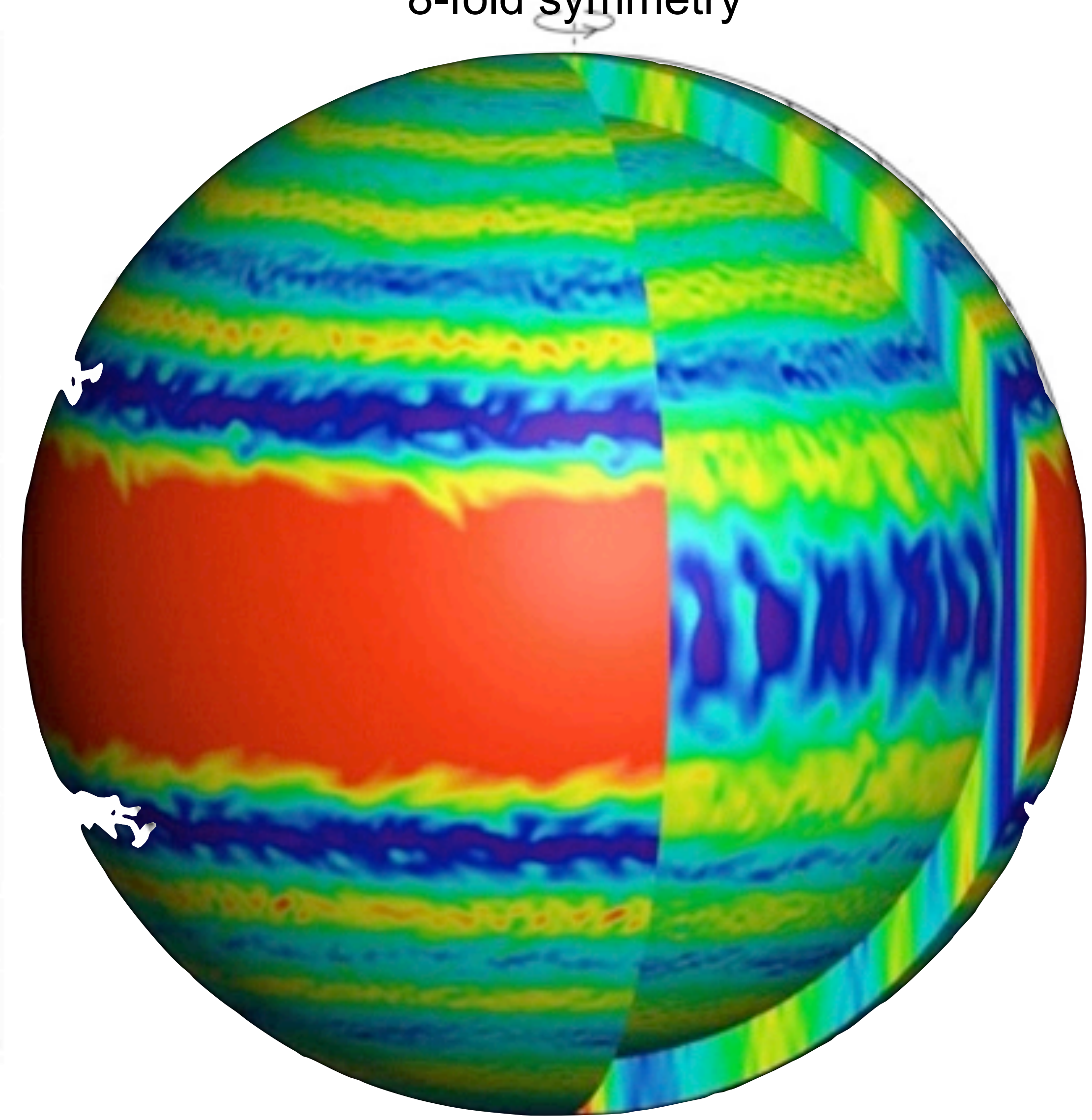
$Ro \sim 10^{-2}$
$U \sim 100 m/s$
$\Omega \sim 2 \times 10^{-4}\ rad/s$
$L \sim 15\ Mm$

Giant Planets

Heimpel, Aurnou & Wicht, Nature 2005

E~3e-6, Ra=5.6e8, Pr=0.1, r_i/r_o=0.9
8-fold symmetry

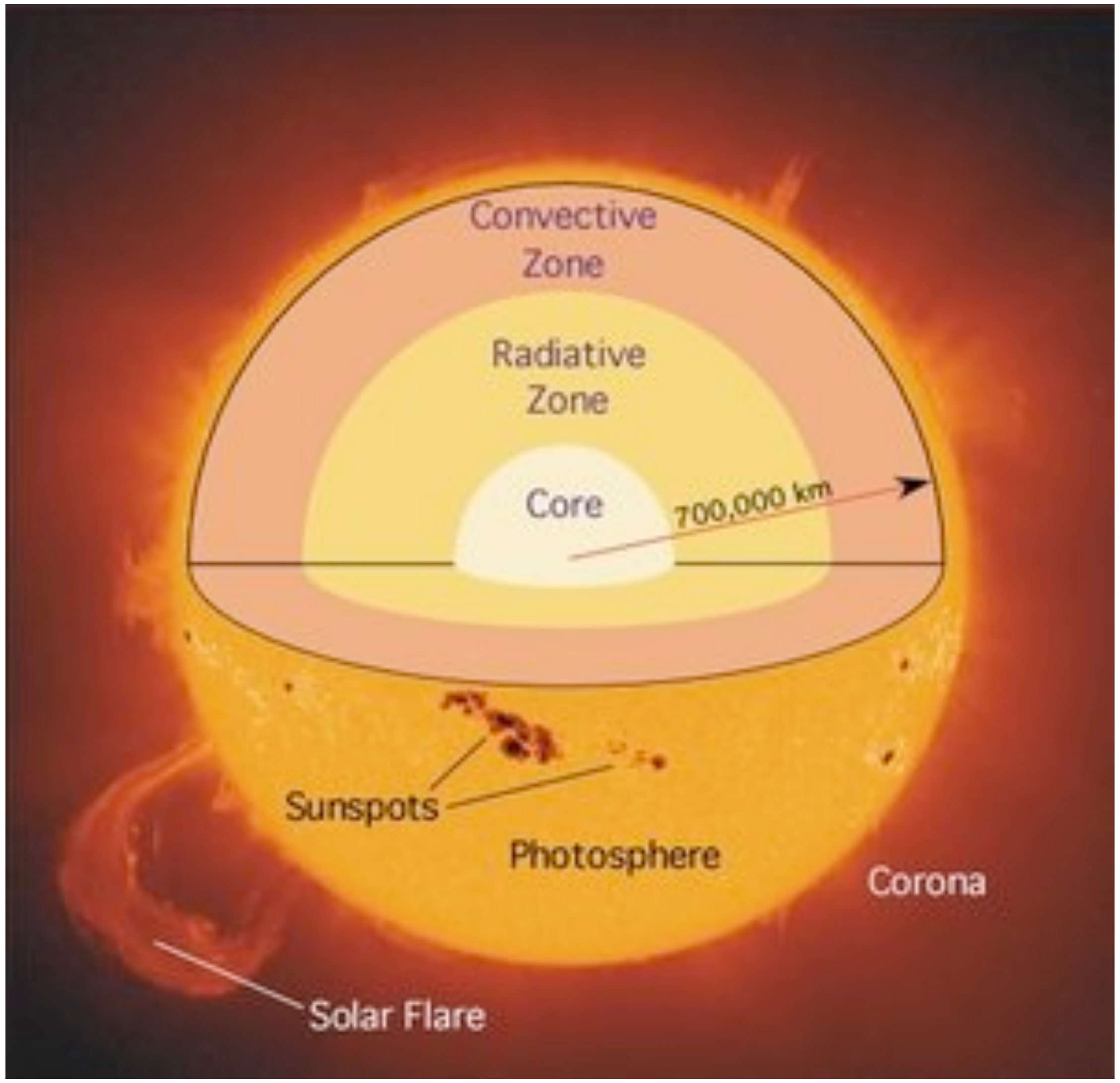




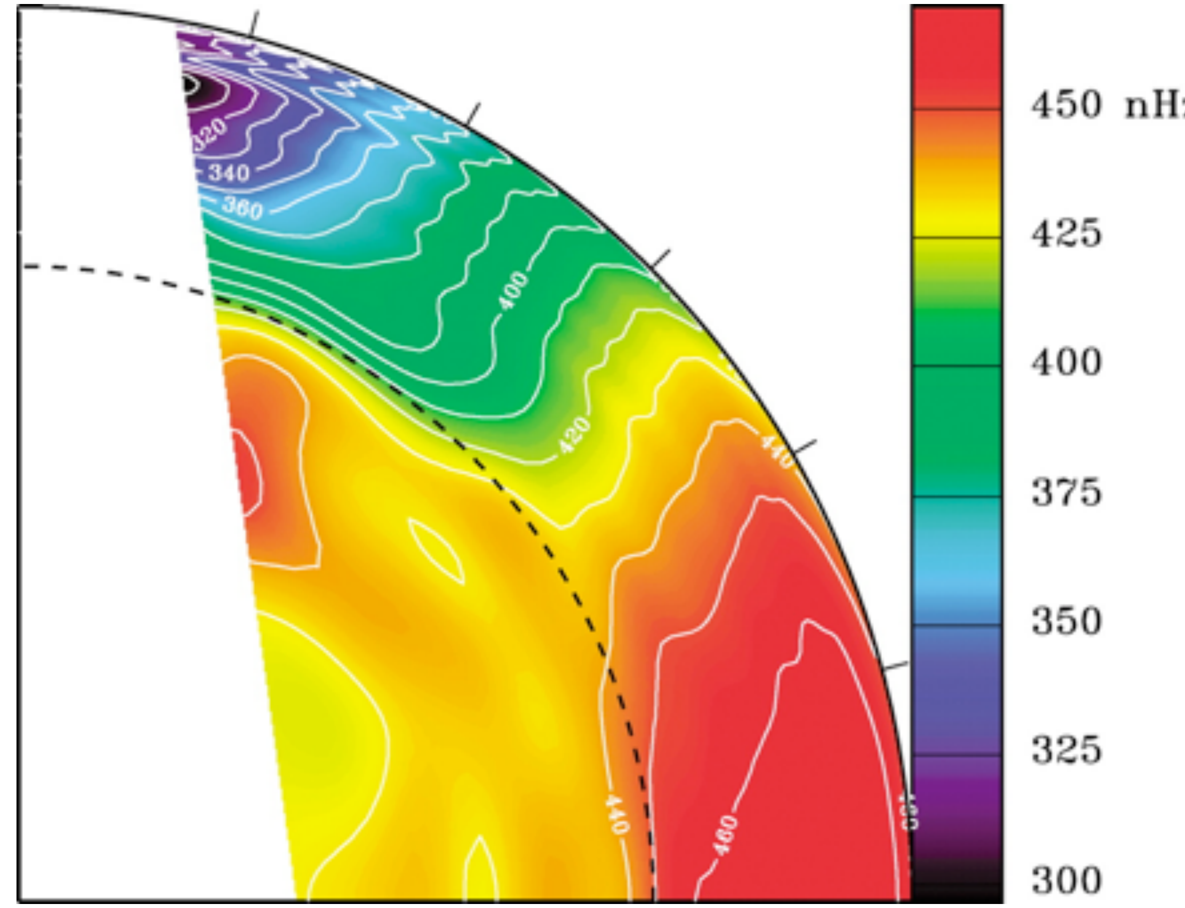
Characterized by alternating bands of zonal jets w vortical eddy generation in region of strong shear

Open question - observed dynamics are likely a combination of strongly-stably stratified weather layer and rotationally constrained convective turbulence in the deep interior.

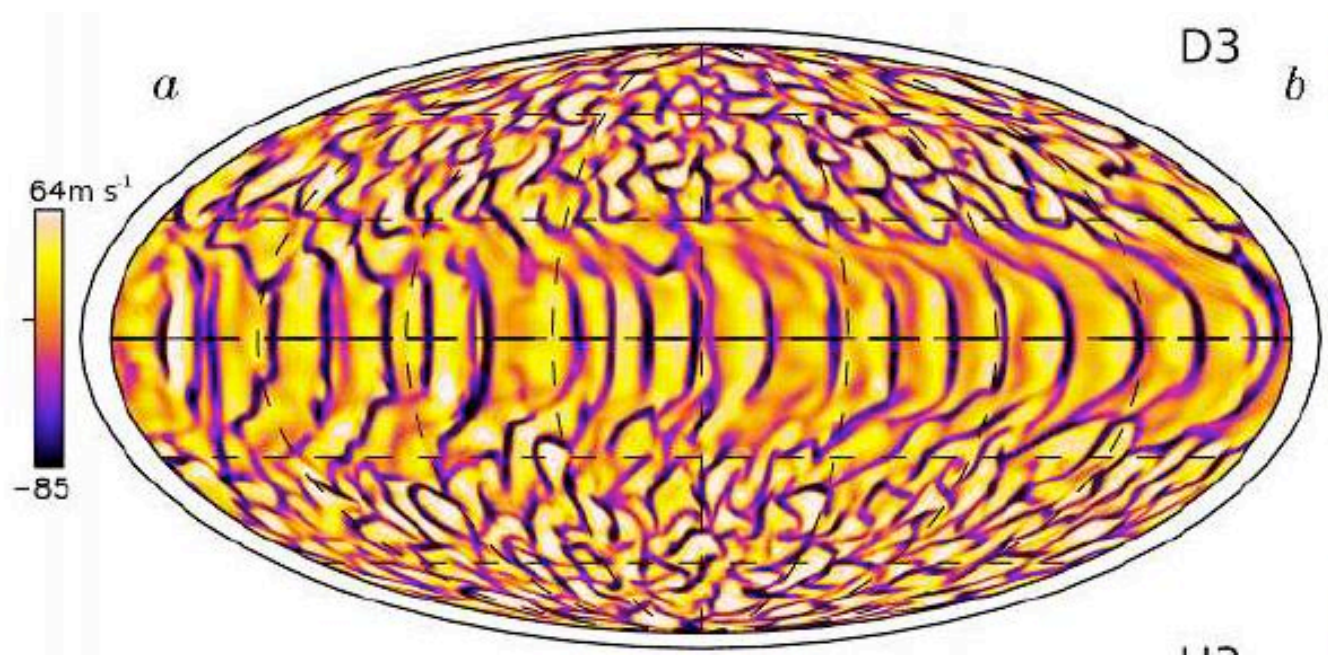
Solar Dynamics



Solar Structure



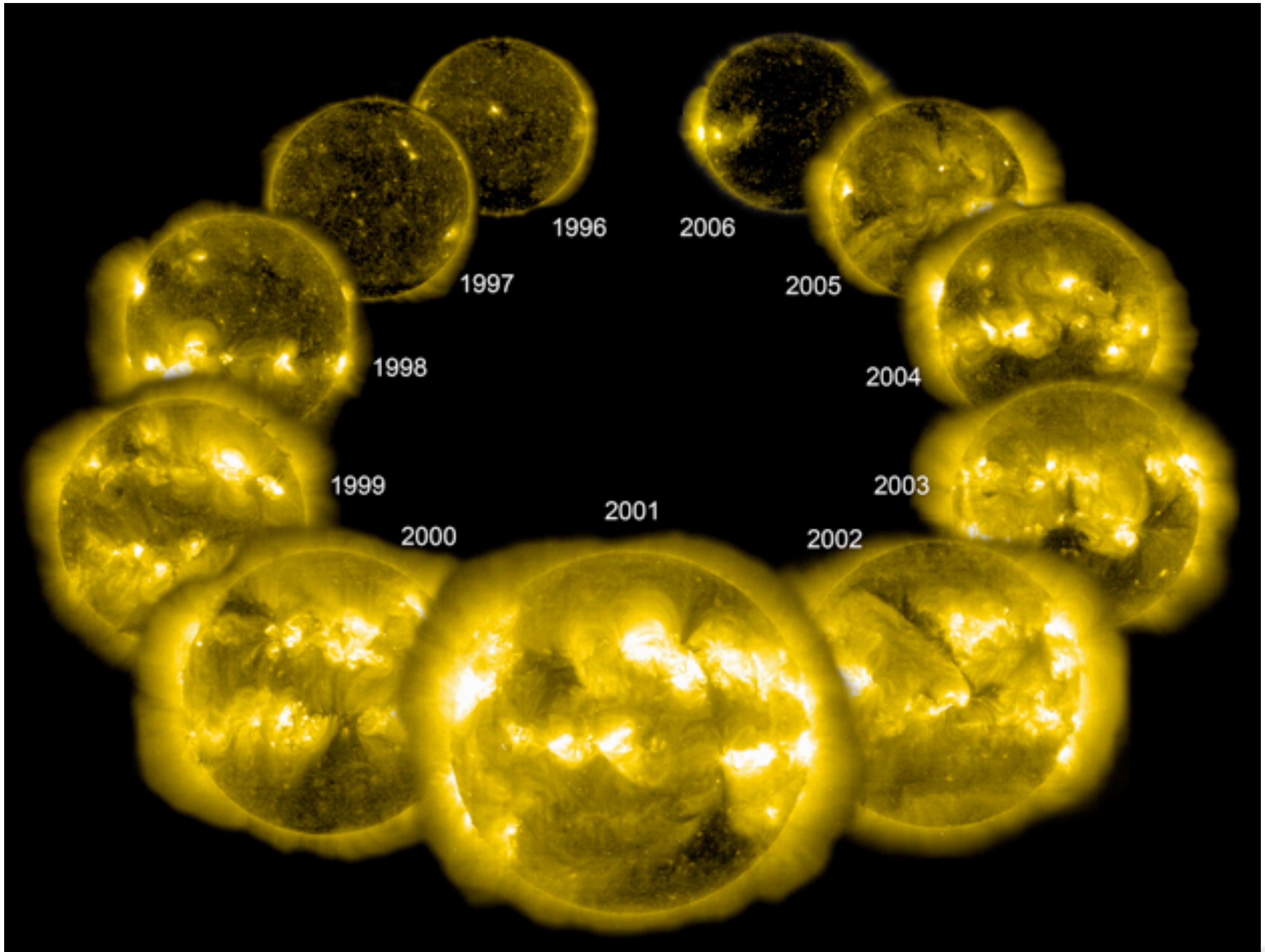
Solar Differential Rotation

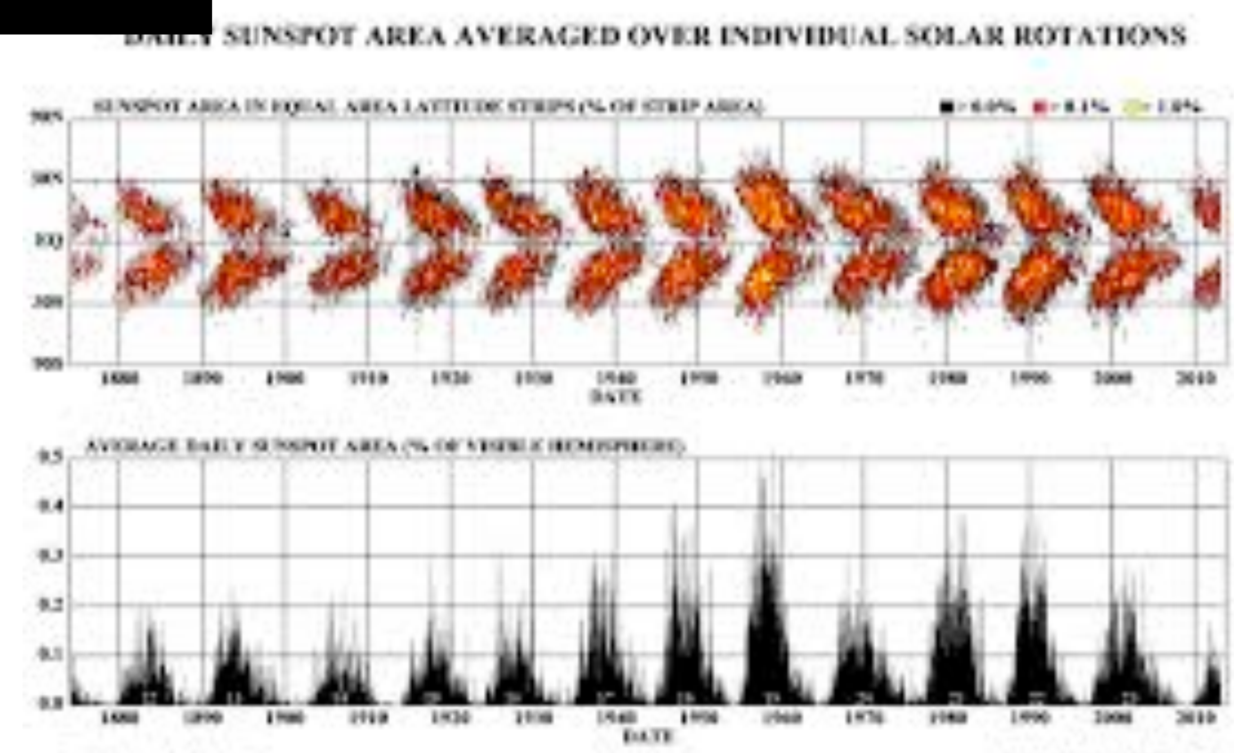


Brown et al ApJ 2010

Rotating convective turbulence is the prime driver of the interior structure.

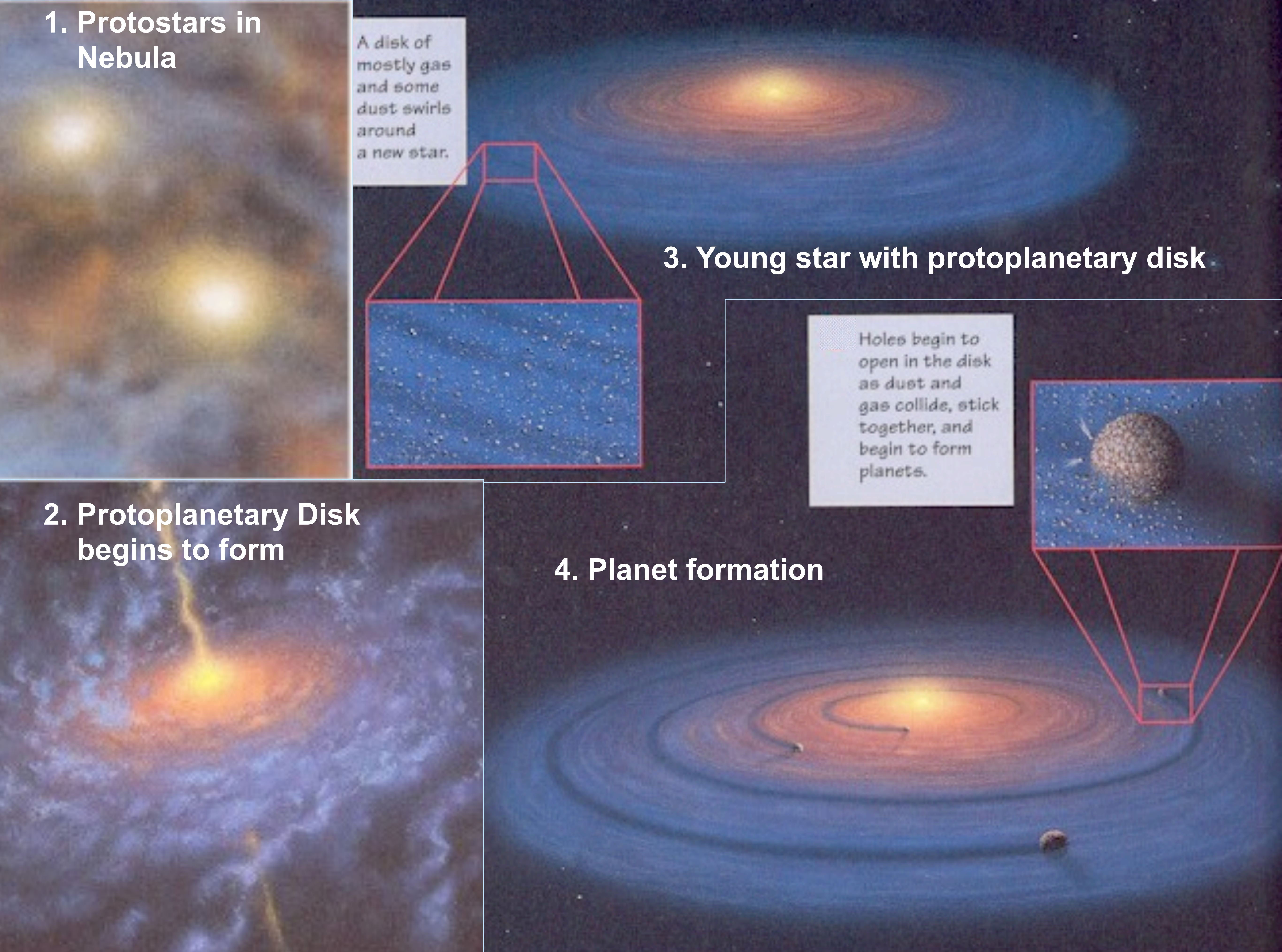
Open question - model demonstrations of differential rotation pattern, solar tachocline, self consistent dynamo mechanism





Rotating convective turbulence is the prime driver of interior structure.

Open question - self consistent model demonstrations of differential rotation pattern, solar tachocline, self consistent dynamo mechanism


1. Protostars in Nebula
A disk of mostly gas and some dust swirls around a new star.
3. Young star with protoplanetary disk
Holes begin to open in the disk as dust and gas collide, stick together, and begin to form planets.
2. Protoplanetary Disk begins to form
4. Planet formation

Courtesy H. Khlar

Open problem - generation of turbulence, efficiency of turbulence in disk
generation and longevity of vortical eddies (? self-sustaining baroclinic instability)

Navier-Stokes Equation

Strouhal, Str $\frac{L}{UT}$

Euler, Eu $\frac{p}{\rho_0 U^2}$

Reynolds, Re $\left(\frac{UL}{\nu}\right)^{-1}$

$$\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\frac{1}{\rho_0}\nabla p + \nu \nabla^2 \boldsymbol{u} + \boldsymbol{F}$$

inertia pressure viscous body force

$$\nabla \cdot \boldsymbol{u} = 0$$

incompressibility

Generic nondimensionalization $U,\ L,\ T,\ P$

Nonlinear Cascade

$$\partial_t \boldsymbol{u} + \boxed{\boldsymbol{u} \cdot \nabla \boldsymbol{u}} = -Eu \nabla p + \frac{1}{Re} \nabla^2 \boldsymbol{u} + \boldsymbol{F}$$

$$\nabla \cdot \boldsymbol{u} = 0$$

Big whorls have little whorls
That feed on their velocity,
And little whorls have lesser whorls
And so on to viscosity.

Lewis F. Richardson, 1922

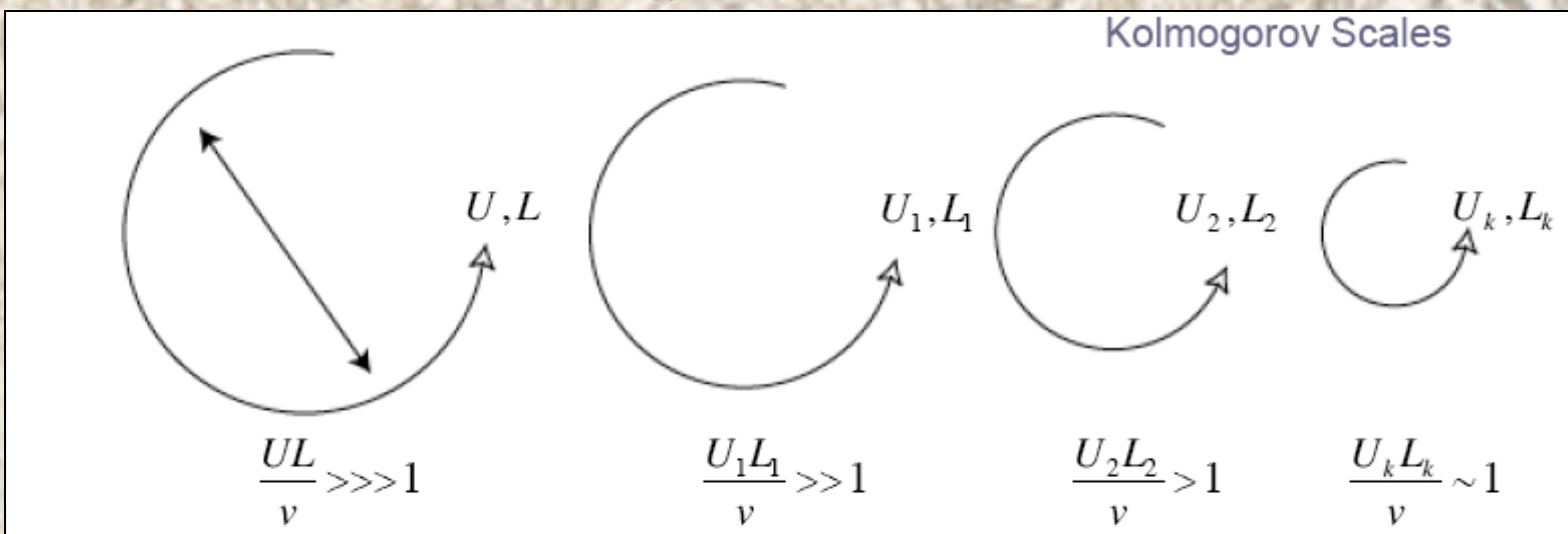
Nonlinear Cascade

$$\partial_t \boldsymbol{u} + \boxed{\boldsymbol{u} \cdot \nabla \boldsymbol{u}} = -Eu\nabla p + \frac{1}{Re}\nabla^2 \boldsymbol{u} + \boldsymbol{F}$$

$$\nabla \cdot \boldsymbol{u} = 0$$

ε units $L^2/T^3 \Rightarrow$ Viscosity is unimportant
Energy flux is conserved

$\varepsilon \quad l_k, u_k = f[\varepsilon, \nu]$



$$U(L) = (\varepsilon L)^{1/3}, \quad T(L) = (L^2/\varepsilon)^{1/3}$$

$\varepsilon \propto \frac{U^3}{L}$ Energy dissipation rate

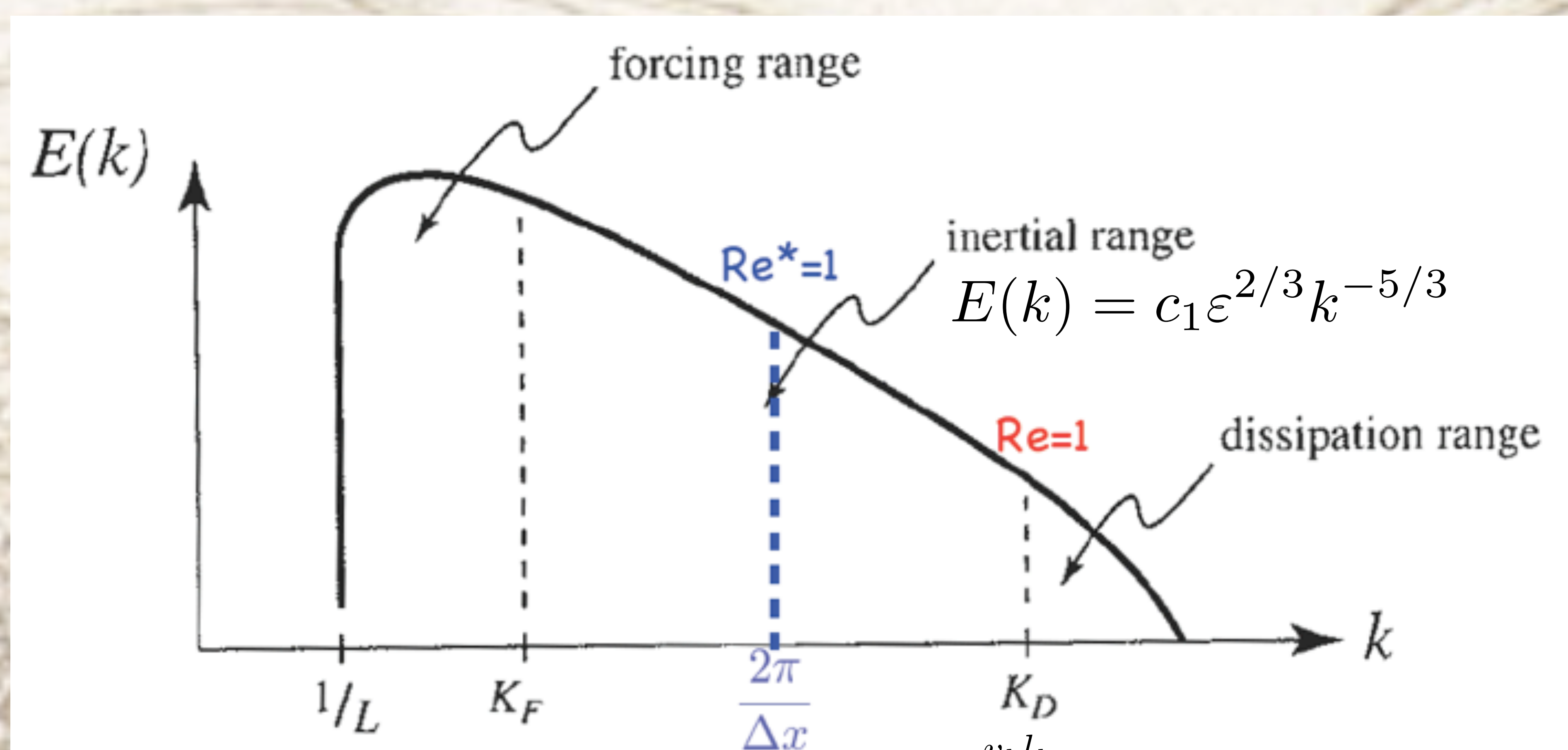
Kolmogorov Hypothesis

Kolmogorov Scales

$$l_k = \left(\frac{\nu^3}{\varepsilon}\right)^{1/4}$$ Length

$$v_k = (\nu\varepsilon)^{1/4}$$ Velocity

$$T_k = \left(\frac{\nu}{\varepsilon}\right)^{1/2}$$ Time



$$\langle E \rangle = \frac{1}{V} \iiint \frac{1}{2}(\mathbf{u} \cdot \mathbf{u})\, \mathrm{d}V = \int_0^\infty E(k) \mathrm{d}k.$$

Degrees of Freedom

$$L/l_k \sim Re^{3/4}$$

$$U/v_k \sim Re^{1/4}$$

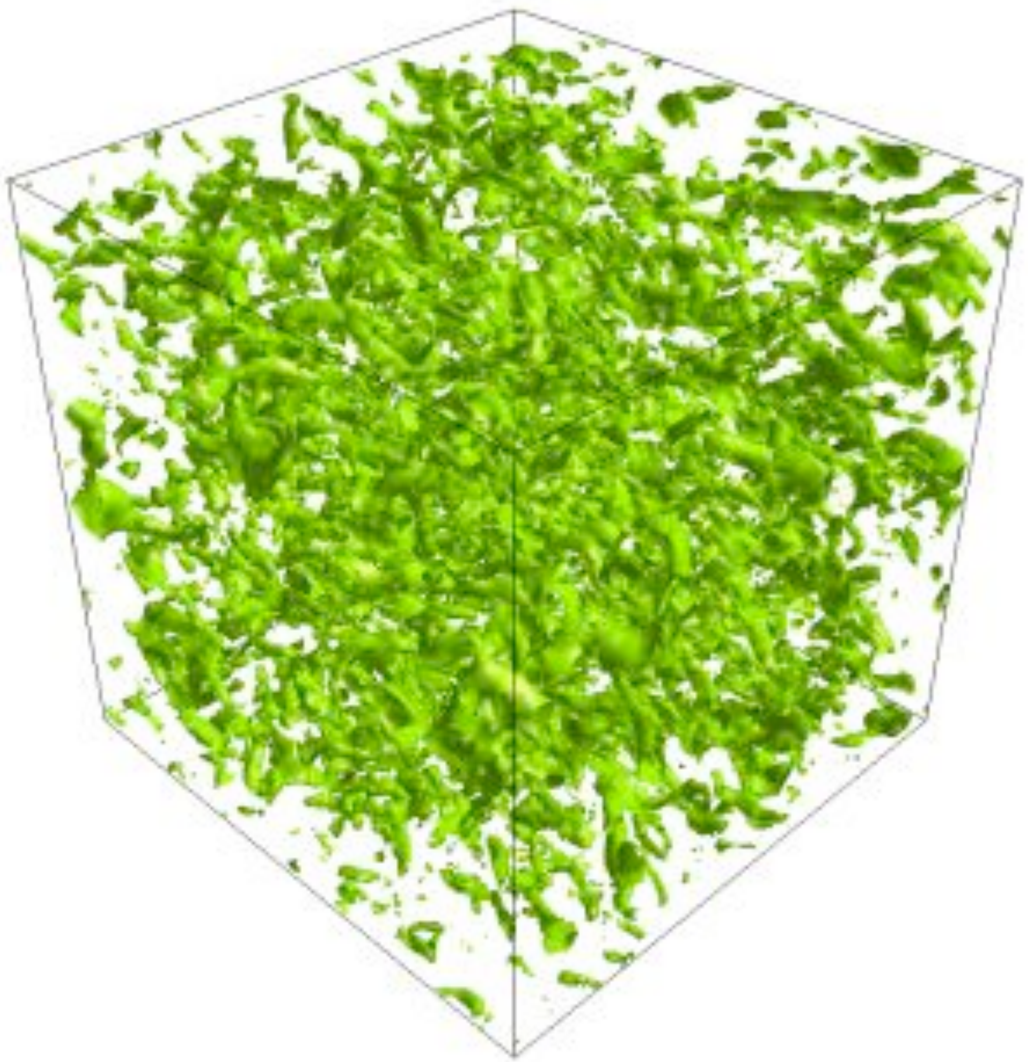
$$T/T_k \sim Re^{1/2}$$

Turbulence challenge $\Rightarrow N^3 \sim Re^{\frac{9}{4}}$

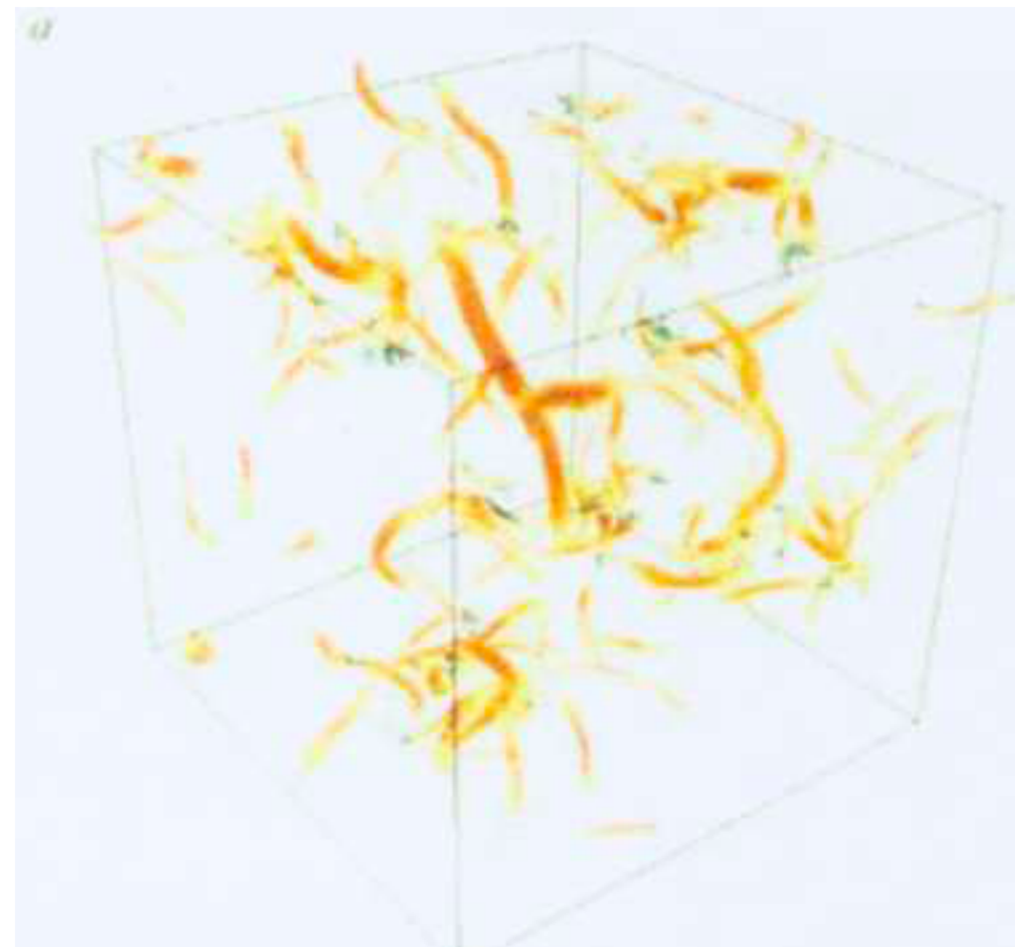
$$(10^{6+})^3 \sim (10^{8+})^{\frac{9}{4}} \Rightarrow \text{GAFD}$$

$$(10^3)^3 \sim (10^4)^{\frac{9}{4}} \Rightarrow \text{DNS}$$

Homogeneous Turbulence



Isotropic turbulence



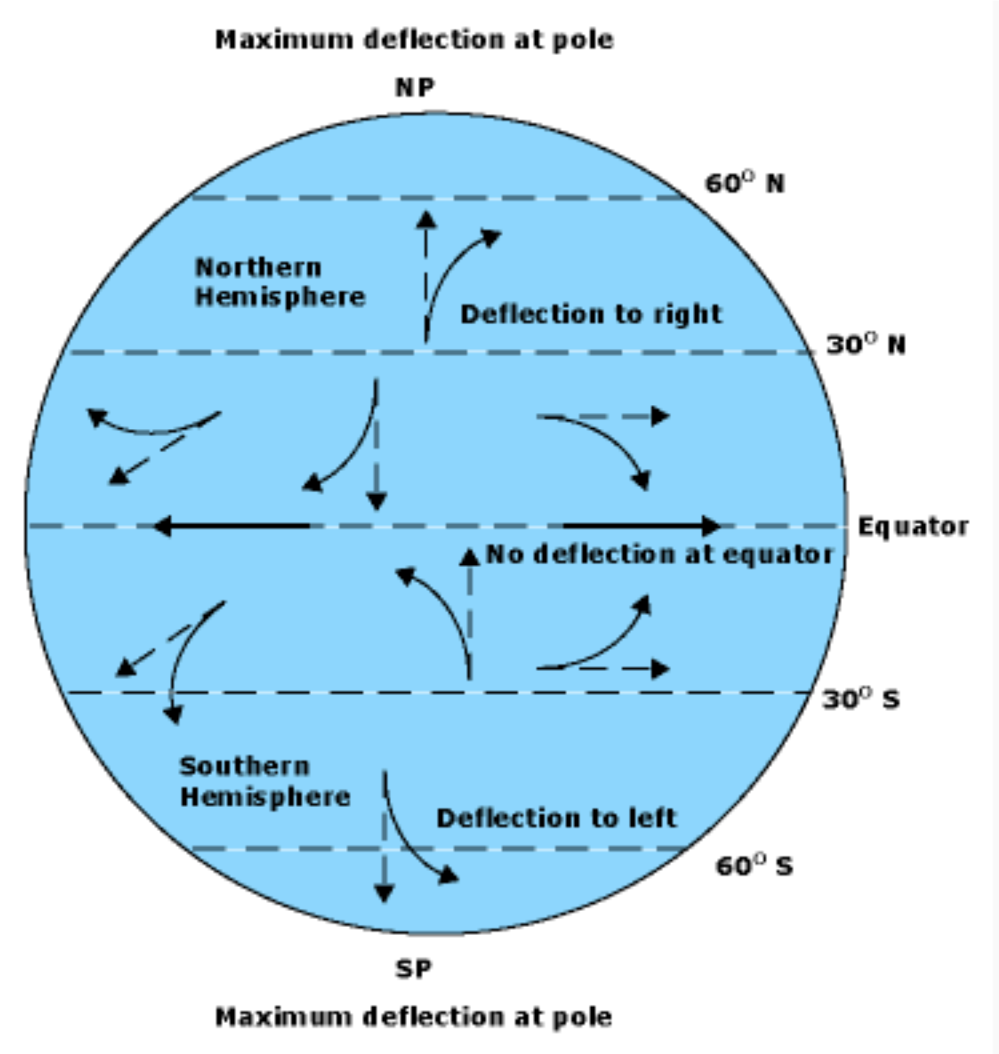
Anisotropy at dissipation scales

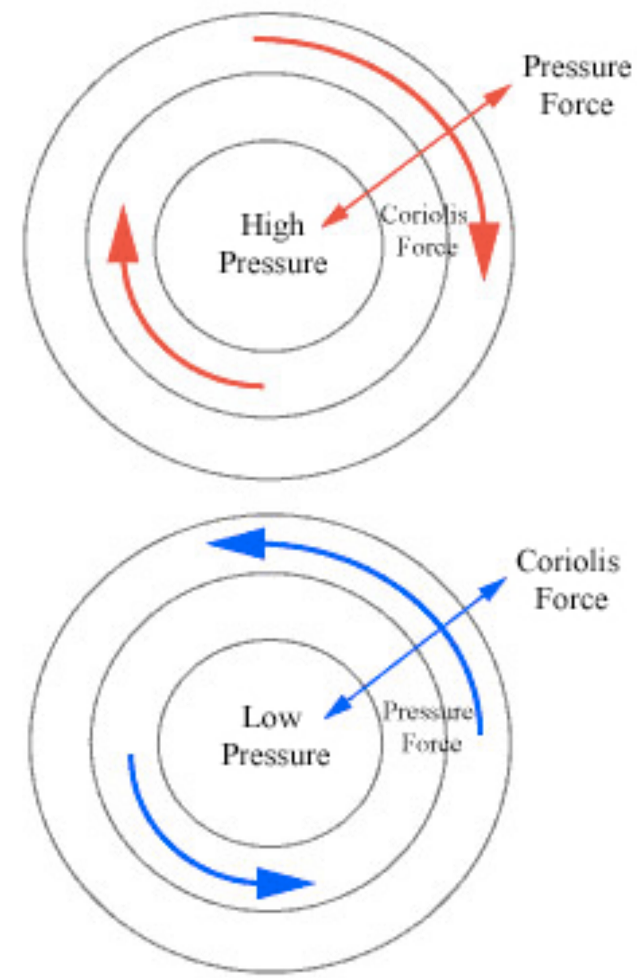
She et al., Nature, Vol. 334, 226-228, 1990

Geophysical Fluids

Inclusion of Rotation and Stratification

Rotation/Coriolis Force $F_c = 2\boldsymbol{\Omega} \times \boldsymbol{u}$





Coriolis Force: pseudo-force that appears in a rotating frame of reference w/ fixed rotation

acts to deflect fluid parcels perpendicular to their direction of motion

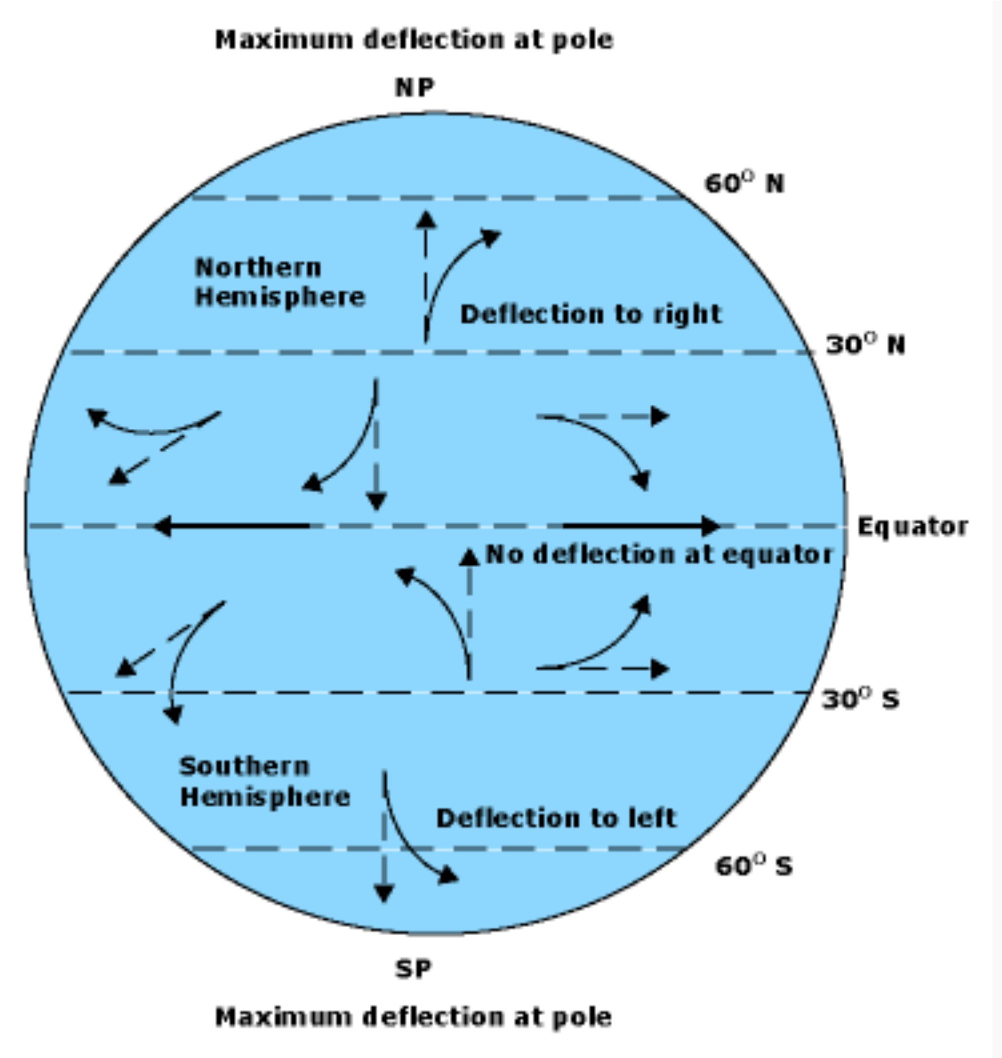
$$\left[\frac{D}{Dt}\right]_i = \left[\frac{D}{Dt}\right]_r + \boldsymbol{\Omega}\times$$

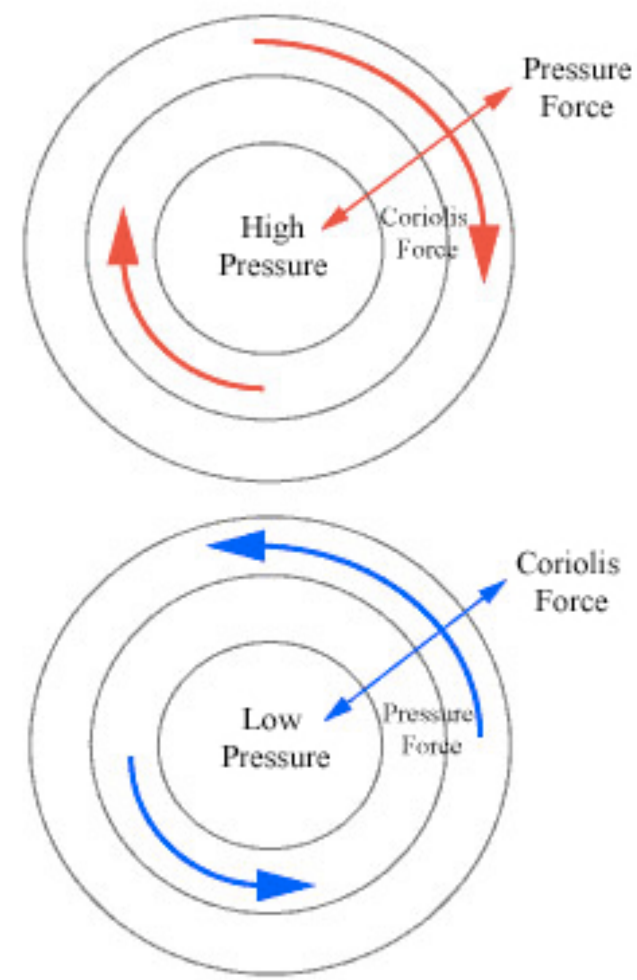
$$\boldsymbol{r}: \quad \boldsymbol{u}_i = \boldsymbol{u}_r + \boldsymbol{\Omega} \times \boldsymbol{r}$$

$$\boldsymbol{u}_r: \quad [D_t \boldsymbol{u}_r]_i = [D_t \boldsymbol{u}_r]_r + \boldsymbol{\Omega} \times \boldsymbol{u}_r$$

Rotation/Coriolis Force

$$F_c = 2\boldsymbol{\Omega} \times \boldsymbol{u}$$





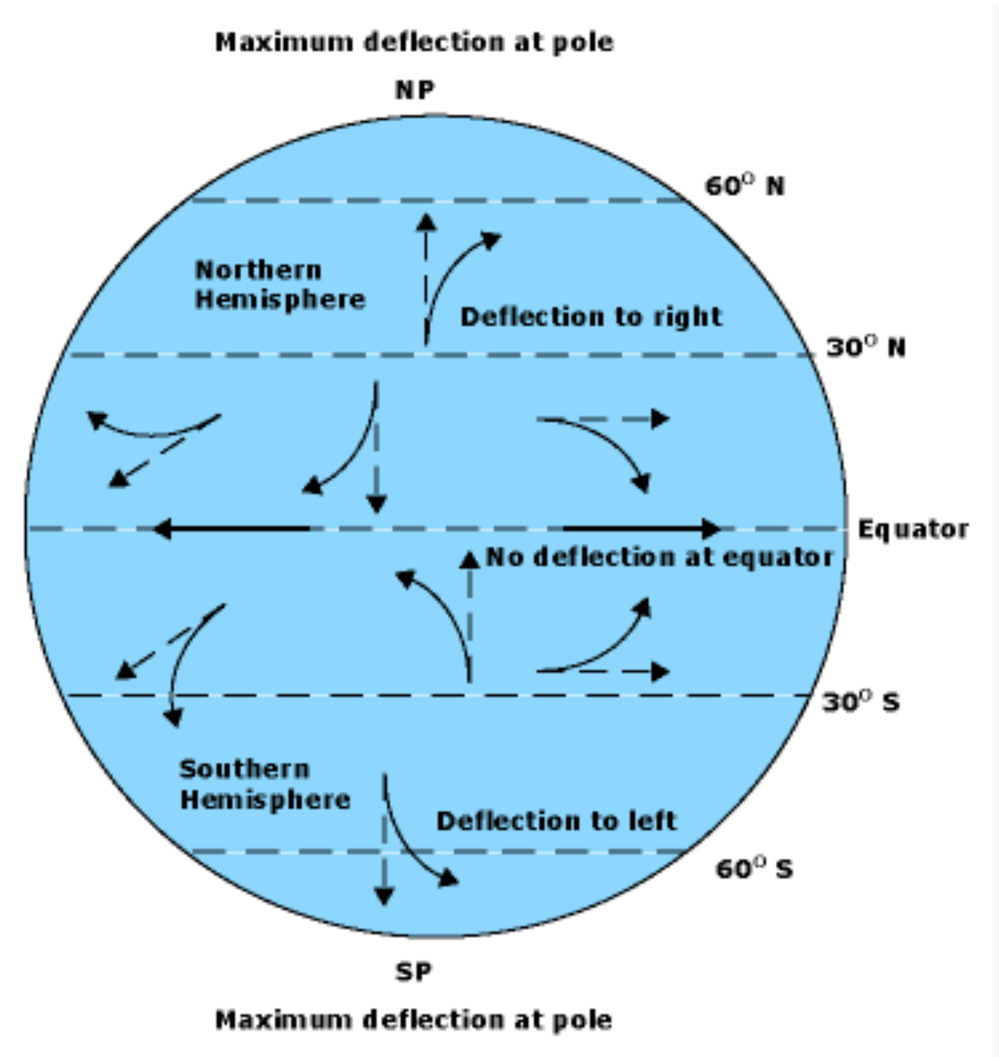
Coriolis Force: pseudo-force that appears in a rotating frame of reference w/ fixed rotation

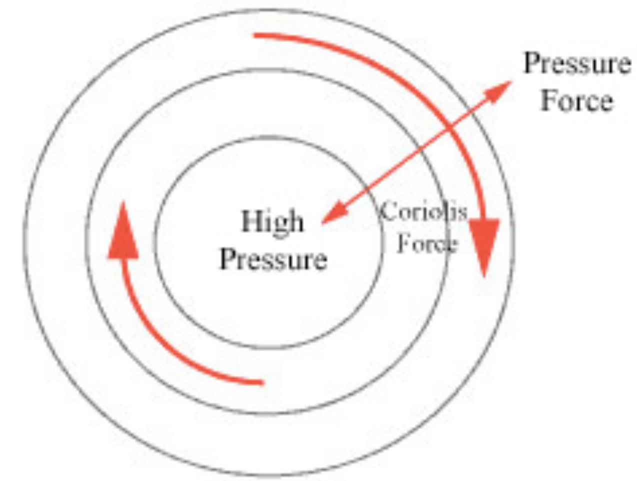
acts to deflect fluid parcels perpendicular to their direction of motion

$$\left[\frac{D}{Dt}\right]_i = \left[\frac{D}{Dt}\right]_r + \boldsymbol{\Omega}\times \qquad \boldsymbol{u}_i : \; [D_t \boldsymbol{u}_i]_i = [D_t \boldsymbol{u}_r]_r + 2\boldsymbol{\Omega} \times \boldsymbol{u}_r + \boldsymbol{\Omega} \times \boldsymbol{\Omega} \times \boldsymbol{r}$$

Rotation/Coriolis Force

$$F_c = 2\boldsymbol{\Omega} \times \boldsymbol{u}$$





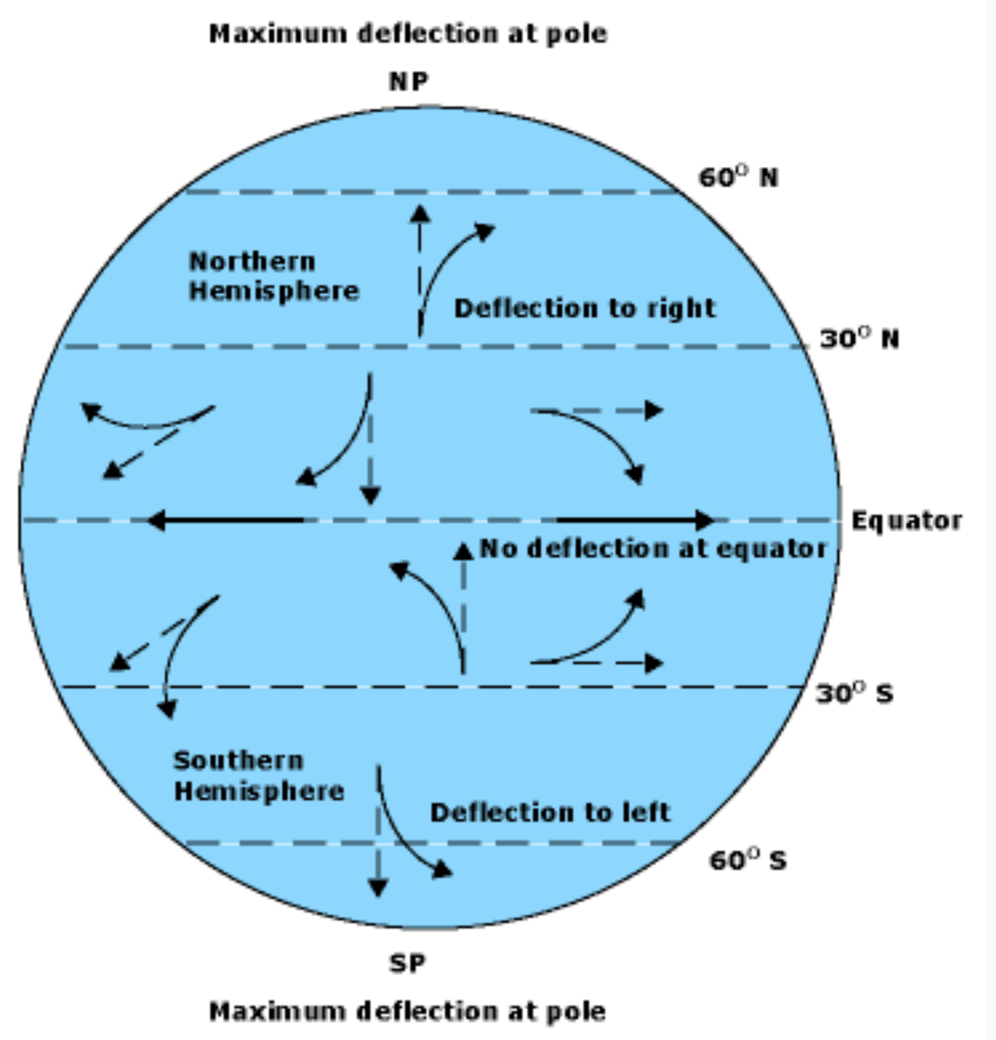
Coriolis Force: pseudo-force that appears in a rotating frame of reference w/ fixed rotation

acts to deflect fluid parcels perpendicular to their direction of motion

$$\left[\frac{D}{Dt}\right]_i = \left[\frac{D}{Dt}\right]_r + \boldsymbol{\Omega}\times \qquad \boldsymbol{u}_i : \; [D_t \boldsymbol{u}_i]_i = [D_t \boldsymbol{u}_r]_r + 2\boldsymbol{\Omega} \times \boldsymbol{u}_r + \boldsymbol{\Omega} \times \boldsymbol{\Omega} \times \boldsymbol{r}$$

Rotation/Coriolis Force

$$
\begin{aligned}
\text{Earth:} \quad & 2\Omega \sim 1.4\times 10^{-4}\mathrm{rad/s}, \quad \beta \sim 2.2\times 10^{-11}\mathrm{rad/\ m\ s}\\
\text{Jupiter:} \quad & 2\Omega \sim 3.5\times 10^{-4}\mathrm{rad/s}, \quad \beta \sim 4.9\times 10^{-12}\mathrm{rad/\ m\ s}\\
\text{sun:} \quad & 2\Omega \sim 5\times 10^{-6}\mathrm{rad/s}
\end{aligned}
$$



Local approximations of 2Ω

f-plane: $f \approx 2\Omega \sin\vartheta$

β-plane: $f \approx 2\Omega \sin\vartheta + \beta y$

eq-plane: $f \approx \beta y$

$$\beta = \frac{2\Omega}{R_p}\cos\vartheta$$

Coriolis Force: pseudo-force that appears in a rotating frame of reference w/ fixed rotation

acts to deflect fluid parcels perpendicular to their direction of motion

$$\left[\frac{D}{Dt}\right]_i = \left[\frac{D}{Dt}\right]_r + \boldsymbol{\Omega}\times$$

$$\boldsymbol{u}_i: \ [D_t\boldsymbol{u}_i]_i = [D_t\boldsymbol{u}_r]_r + 2\boldsymbol{\Omega}\times\boldsymbol{u}_r + \boldsymbol{\Omega}\times\boldsymbol{\Omega}\times\boldsymbol{r}$$

Governing Equations

$$\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \frac{1}{Ro} \widehat{\boldsymbol{z}} \times \boldsymbol{u} = -Eu \nabla \pi + \frac{1}{Re} \nabla^2 \boldsymbol{u} + \boldsymbol{F}$$

$$\nabla \cdot \boldsymbol{u} = 0$$

$$Ro = \frac{U^2/L}{2\Omega U} = \frac{inertia}{Coriolis} = \frac{U}{2\Omega L}$$

Types of motion affected by the Coriolis force:

Inherent spin is less than planetary angular velocity $Ro \leq 1 \quad \text{or} \quad U/L \leq 2\Omega$

Vary on timescales greater than a planetary day $Str\, Ro \leq 1 \quad \text{or} \quad T \geq 1/2\Omega$

Governing Equations

$$\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \frac{1}{Ro} \widehat{\boldsymbol{z}} \times \boldsymbol{u} = -Eu \nabla \pi + \frac{1}{Re} \nabla^2 \boldsymbol{u} + \boldsymbol{F}$$

$$\nabla \cdot \boldsymbol{u} = 0$$

$$Ro = \frac{U}{2\Omega L}$$

Zeman length-scale

$$Ro_{l_\Omega} \sim 1, \qquad l_\Omega = \frac{\varepsilon^{1/2}}{(2\Omega)^{3/2}} \approx L Ro^{3/2}$$

Types of motion affected by the Coriolis force:

Inherent spin is less than planetary angular velocity $Ro \leq 1 \quad \text{or} \quad U/L \leq 2\Omega$

Vary on timescales greater than a planetary day $StrRo \leq 1 \quad \text{or} \quad T \geq 1/2\Omega$

What fluid scales are influence by rotation? $l \geq l_\Omega \qquad U \sim (\varepsilon l)^{1/3}$

Inertial Oscillations

Consider case Eu << 1/Ro, inviscid motions Re >>1

$$\frac{Du}{Dt} - \frac{1}{Ro}v = 0$$

$$\frac{D^2u}{Dt^2} + \frac{1}{Ro^2}u = 0$$

$$\frac{Dv}{Dt} + \frac{1}{Ro}u = 0$$

Inertial oscillation: velocities oscillate with period $T_i = \pi/2\Omega$

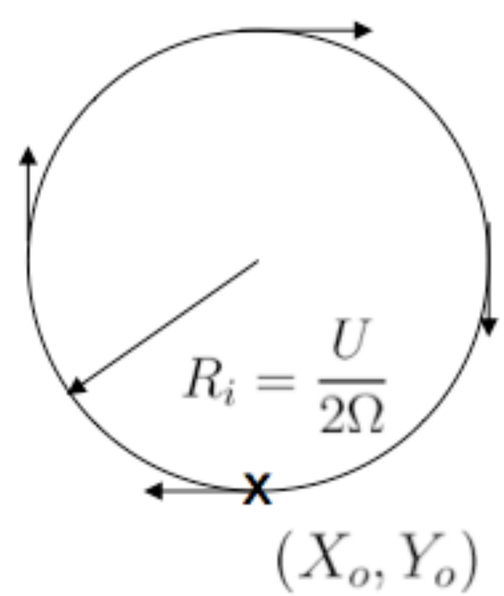
$$u = U\cos(Ro^{-1}t),\;\; v = U\sin(Ro^{-1}t) \longrightarrow u_d = U_d\cos(2\Omega t),\;\; v_d = U_d\sin(2\Omega t)$$

Inertial circles: fluid parcels trace a circle of radius R_i in one inertial period

$$\frac{D\boldsymbol{X}_\perp}{Dt} = \boldsymbol{u}_\perp, \quad \boldsymbol{X}_\perp = \boldsymbol{X}_{0\perp}$$

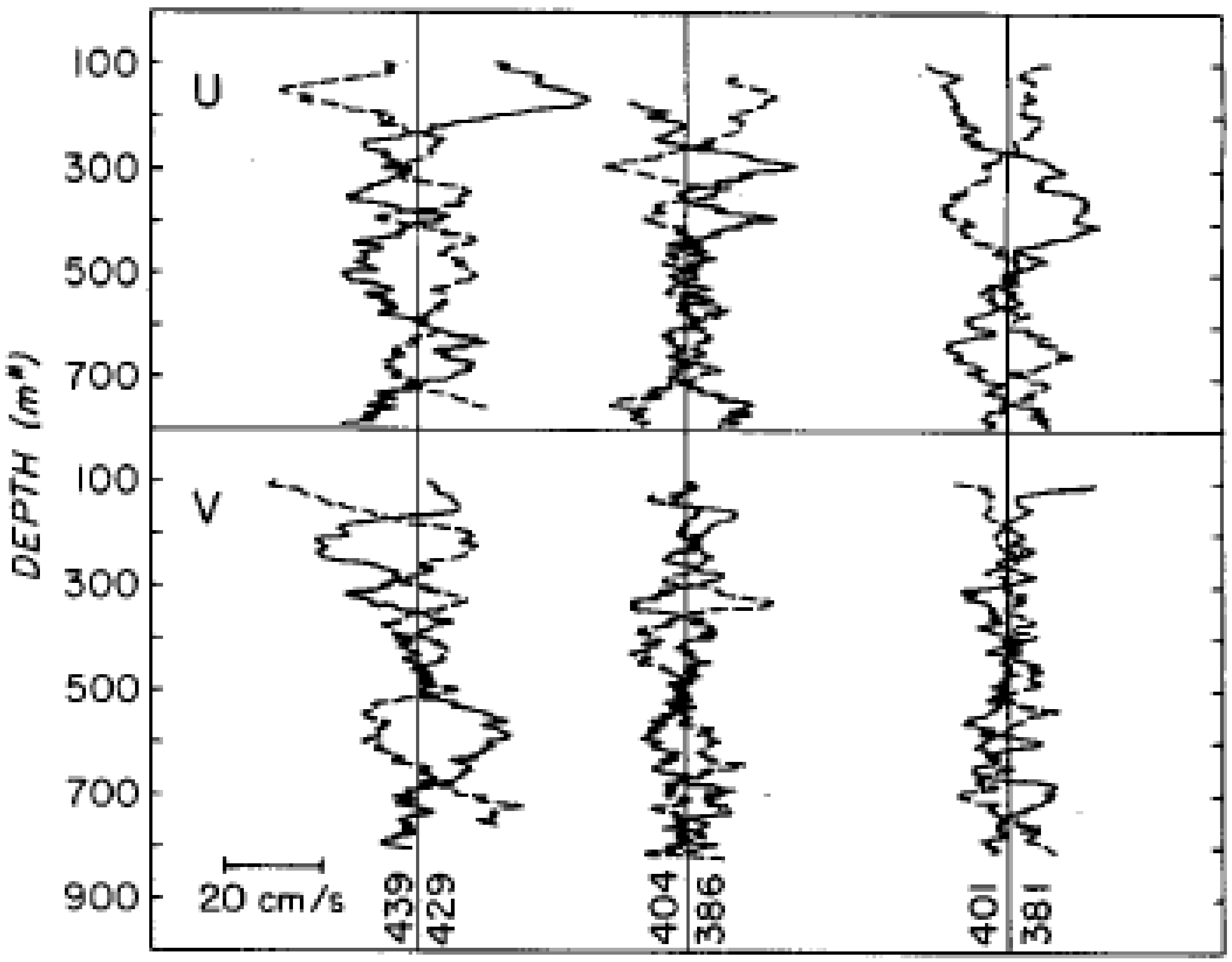
$$X = \frac{U}{2\Omega}\sin(2\Omega t) + X_0$$

$$Y = \frac{U}{2\Omega}(1 - \cos(2\Omega t)) + Y_0$$

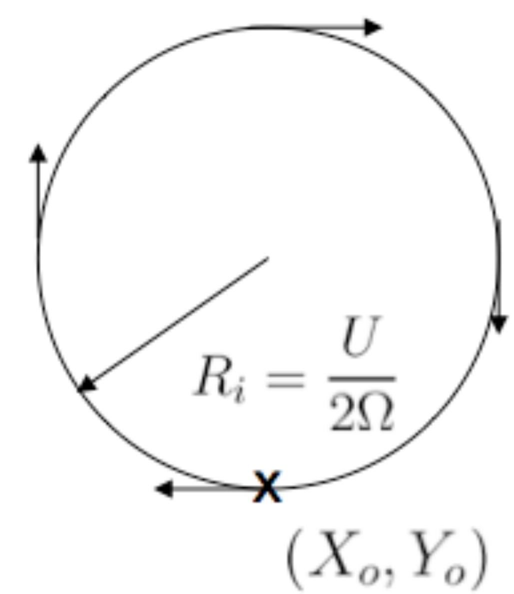


Inertial Oscillations: Oceanic example

Velocity profiles in N. Pacific Subtropical Front 30 N



Kunze & Sanford J. Phys. Ocean 1984



Horizontal velocity profiles taken a half inertial period apart are near mirror-images of each other suggesting flows are dominated by inertial motions

Half the variance in the IW band is explained by inertial motions in the upper ocean (Ferrari & Wunsch. Ann. Rev. Fluid Mech 2008).

Dispersion Relation: Waves vs Eddies

Consider case Eu ~ 1/Ro, inviscid motions Re >>1

Separation of dynamics into fast and slow motions (waves and eddies).

$$\partial_t \boldsymbol{u} + \frac{1}{Ro}\widehat{\boldsymbol{z}} \times \boldsymbol{u} \approx -Eu\nabla p$$

$$\nabla \cdot \boldsymbol{u} = 0$$

$$\boldsymbol{v} \propto e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)}, \quad \boldsymbol{k}\cdot\boldsymbol{u} = 0, \quad \boldsymbol{k} = (\boldsymbol{k}_\perp, k_z) = |\boldsymbol{k}|(\cos\phi, \sin\phi)$$

Inertial waves

$$\partial_{tt}\nabla^2 w + \frac{1}{Ro^2}\partial_{zz} w = 0$$

$$\omega^2 = \frac{1}{Ro^2}\frac{k_z^2}{|\boldsymbol{k}|^2} = \frac{1}{Ro^2}\sin^2\phi$$

Freq. only a function propagation dir'n

Vortical modes/eddies are in geostrophic balance

$$\boldsymbol{u}_\perp = EuRo\,(-\partial_y p, \partial_x p), \quad w = 0$$

$$\omega = 0, \qquad \partial_z(\boldsymbol{u}_\perp, p) = 0$$

Taylor-Proudman constraint

Dispersion Relation: Waves vs Eddies

Consider case Eu ~ 1/Ro, inviscid motions Re >>1

Separation of dynamics into fast and slow motions (waves and eddies).

$$\partial_t \boldsymbol{u} + \frac{1}{Ro}\widehat{\boldsymbol{z}} \times \boldsymbol{u} \approx -Eu\nabla p$$

$$\nabla \cdot \boldsymbol{u} = 0$$

$$\boldsymbol{v} \propto e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)}, \quad \boldsymbol{k}\cdot\boldsymbol{u} = 0, \quad \boldsymbol{k} = (\boldsymbol{k}_\perp, k_z) = |\boldsymbol{k}|(\cos\phi, \sin\phi)$$

Inertial waves:

$$\partial_{tt}\nabla^2 w + \frac{1}{Ro^2}\partial_{zz} w = 0$$

$$\omega^2 = \frac{1}{Ro^2}\frac{k_z^2}{|\boldsymbol{k}|^2} = \frac{1}{Ro^2}\sin^2\phi$$

Freq. only a function propagation dir'n

Vortical modes/eddies are in geostrophic balance

$$\boldsymbol{u}_\perp = EuRo\,(-\partial_y p, \partial_x p), \quad w = 0$$

$$\omega = 0, \qquad \partial_z(\boldsymbol{u}_\perp, p) = 0$$

Taylor-Proudman constraint

Dispersion Relation: Waves vs Eddies

Consider case Eu ~ 1/Ro >>1, inviscid motions Re >>1

Separation of dynamics into fast and slow motions (waves and eddies).

$$\cancel{\partial_t \boldsymbol{u}} + \frac{1}{Ro}\widehat{\boldsymbol{z}} \times \boldsymbol{u} \approx -Eu\nabla p$$

Geostrophy

$$\nabla \cdot \boldsymbol{u} = 0$$

$$\boldsymbol{v} \propto e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)}, \quad \boldsymbol{k}\cdot\boldsymbol{u} = 0, \quad \boldsymbol{k} = (\boldsymbol{k}_\perp, k_z) = |\boldsymbol{k}|(\cos\phi, \sin\phi)$$

Inertial waves

$$\partial_{tt}\nabla^2 w + \frac{1}{Ro^2}\partial_{zz} w = 0$$

$$\omega^2 = \frac{1}{Ro^2}\frac{k_z^2}{|\boldsymbol{k}|^2} = \frac{1}{Ro^2}\sin^2\phi$$

Freq. only a function propagation dir'n

Vortical modes/eddies are in geostrophic balance

$$\boldsymbol{u}_\perp = EuRo\,(-\partial_y p, \partial_x p), \quad w = 0$$

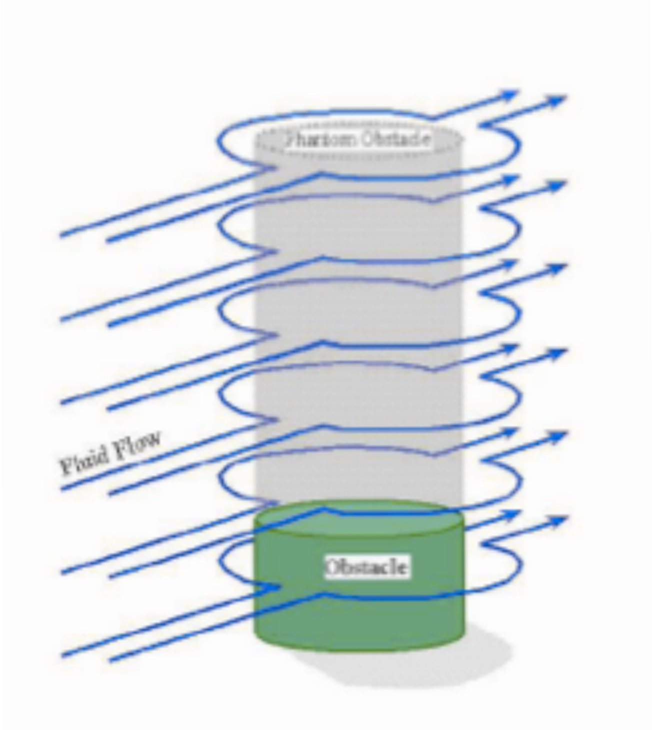
$$\omega = 0, \qquad \partial_z(\boldsymbol{u}_\perp, p) = 0$$

Taylor-Proudman constraint

Proudman-Taylor Constraint (Experiment)

Proudman 1916 Proc. Roy. Soc. Lond. A

Taylor 1923 Proc. Roy. Soc. Lond. A



Geostrophy: diagnostic balance

$$\frac{1}{Ro}\hat{\boldsymbol{z}} \times \boldsymbol{u} \approx -Eu\nabla p$$

$$\nabla \cdot \boldsymbol{u} = 0$$

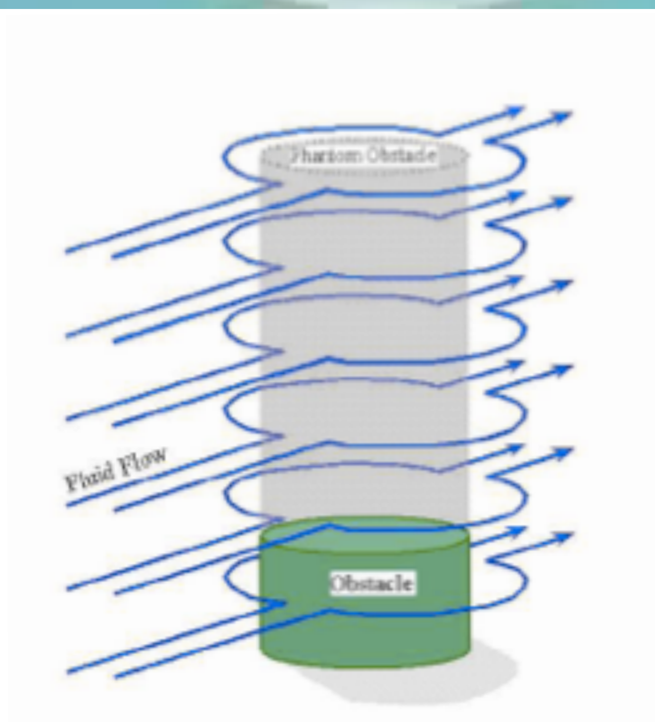
$$\boldsymbol{u}_\perp = EuRo\,(-\partial_y p, \partial_x p), \quad w = 0$$

$$\omega = 0, \qquad \partial_z(\boldsymbol{u}_\perp, p) = 0$$

Proudman-Taylor Constraint (Experiment)

Proudman 1916 Proc. Roy. Soc. Lond. A

Taylor 1923 Proc. Roy. Soc. Lond. A



Geostrophy: diagnostic balance

$$\frac{1}{Ro}\widehat{\boldsymbol{z}} \times \boldsymbol{u} \approx -Eu \nabla p$$

$$\nabla \cdot \boldsymbol{u} = 0$$

$$\boldsymbol{u}_\perp = EuRo\,(-\partial_y p, \partial_x p), \quad w = 0$$

$$\omega = 0, \qquad \partial_z(\boldsymbol{u}_\perp, p) = 0$$

Evolution ⇒

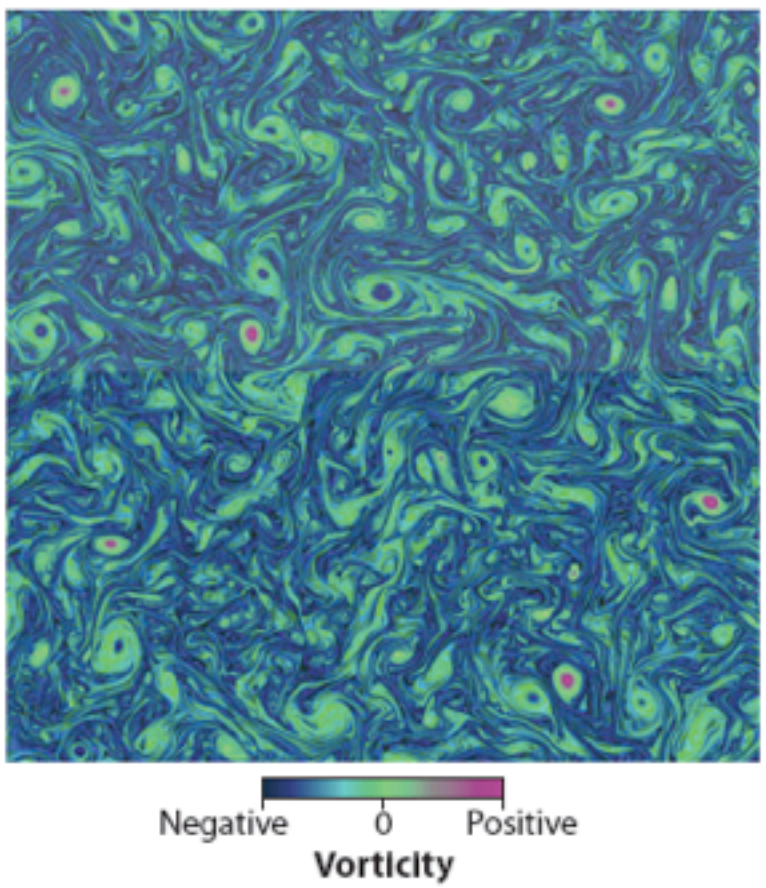
Barotropic Vorticity Equation

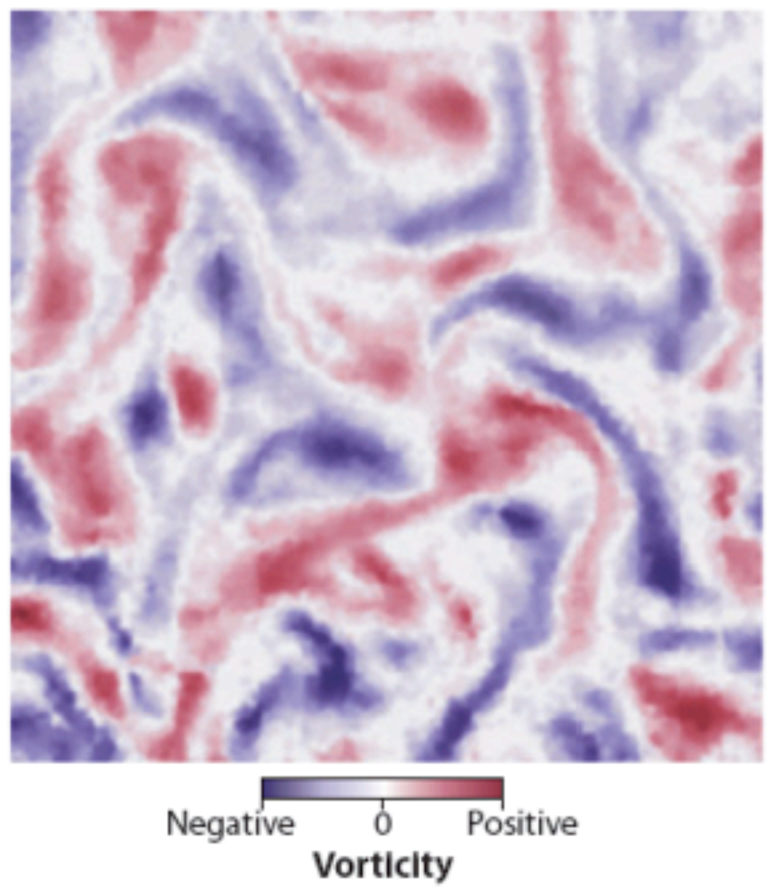
$$f \approx 2\Omega \sin\vartheta$$

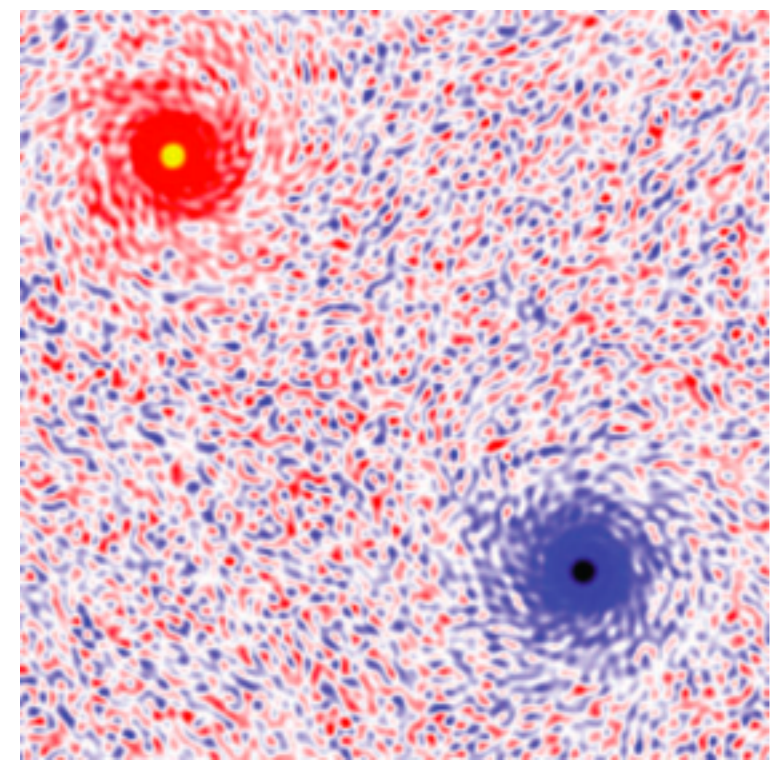
Quasigeostrophy - (Prognostic) departure from geostrophy $\quad \boldsymbol{u}_\perp = \boldsymbol{u}_\perp^G + \boldsymbol{u}_\perp^{AG}$

$$\partial_t \boldsymbol{u}_\perp + \boldsymbol{u}_\perp \cdot \nabla_\perp \boldsymbol{u}_\perp + \frac{1}{Ro}\widehat{\boldsymbol{z}} \times \boldsymbol{u}_\perp^{AG} = -Eu\nabla_\perp p^{AG} + F + D$$

$$\nabla_\perp \cdot \left(\boldsymbol{u}_\perp^{AG}\right) = 0$$







Bofetta & Ecke Ann. Rev. Fluid. Mech. 2012

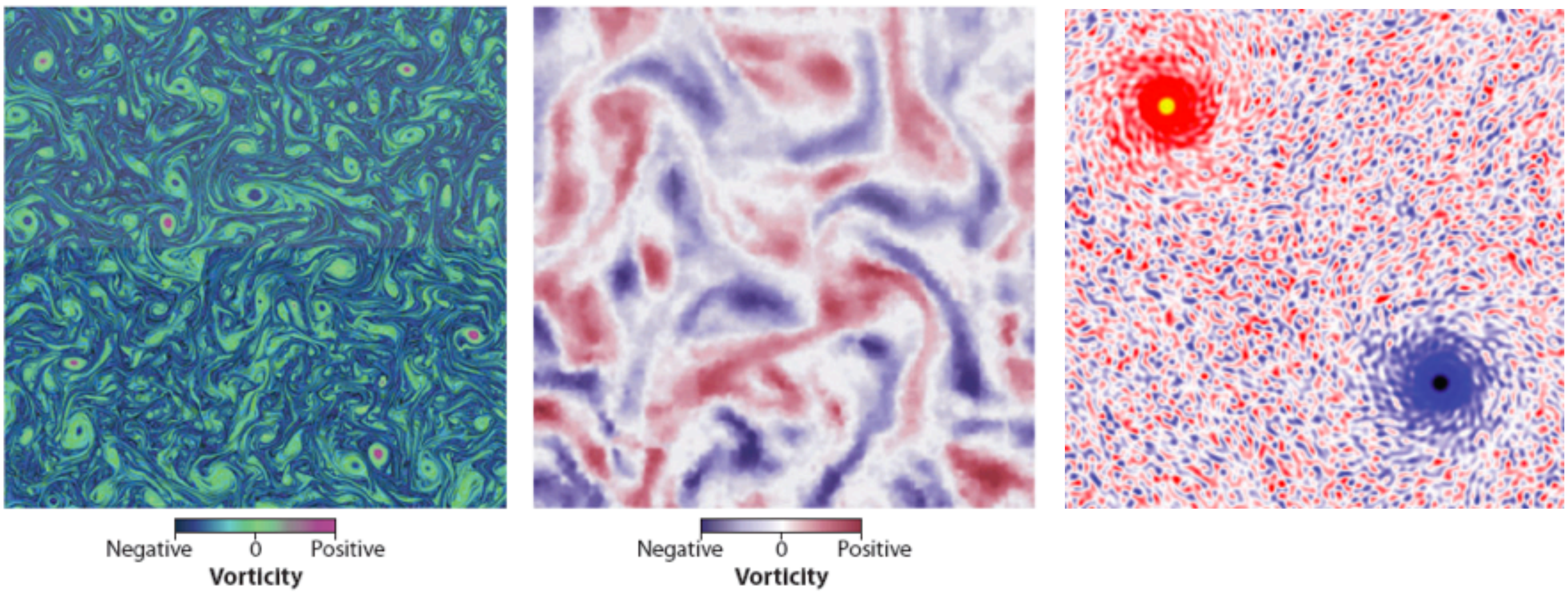
Barotropic Vorticity Equation

$f \approx 2\Omega \sin\vartheta$

Quasigeostrophy - (Prognostic) departure from geostrophy $\boldsymbol{u}_\perp = \boldsymbol{u}_\perp^G + \boldsymbol{u}_\perp^{AG}$

$$\partial_t \zeta + \boldsymbol{u}_\perp \cdot \nabla_\perp \zeta = \mathcal{F} + \mathcal{D}, \qquad \zeta_a = \widehat{\boldsymbol{z}} \cdot \nabla \times \boldsymbol{u} + Ro^{-1}$$

$$\boldsymbol{u}_\perp = -\nabla \times \psi \widehat{\boldsymbol{z}}, \quad \zeta = \nabla_\perp^2 \psi$$



Bofetta & Ecke Ann. Rev. Fluid. Mech. 2012

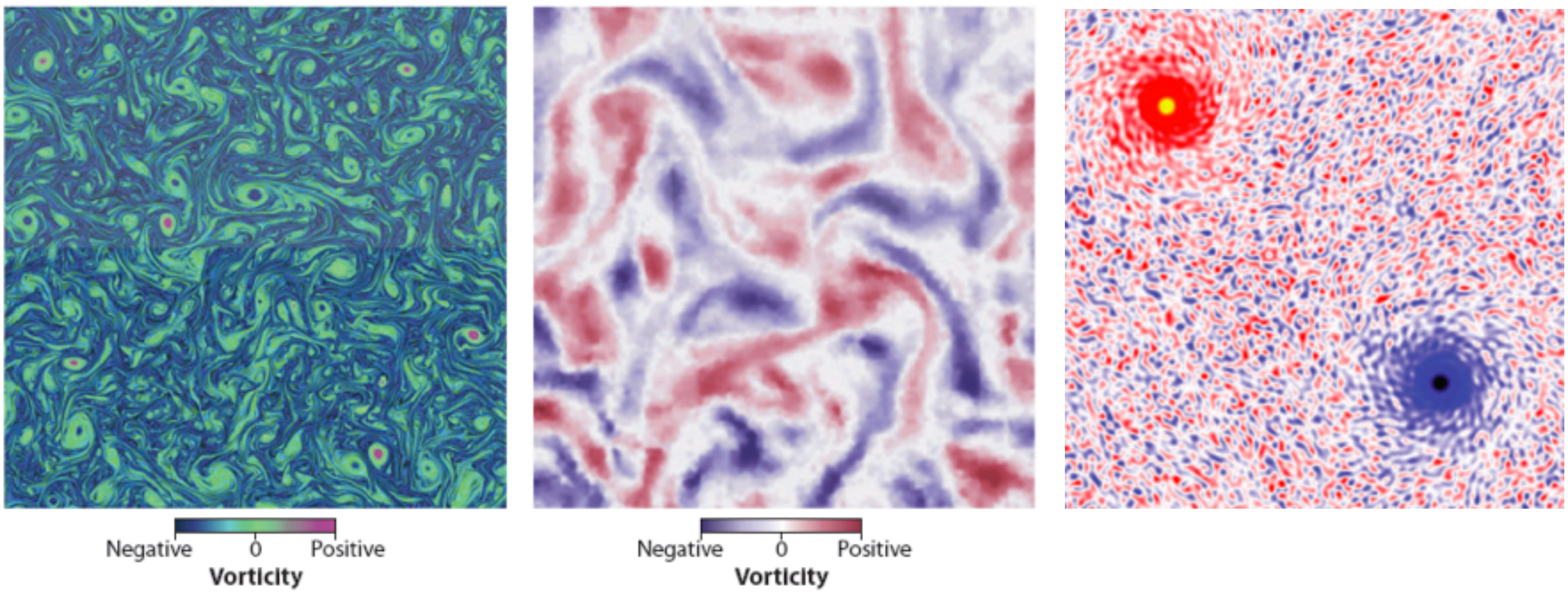
Barotropic Vorticity Equation

$f \approx 2\Omega \sin\vartheta$

Quasigeostrophy - (Prognostic) departure from geostrophy $\quad \boldsymbol{u}_\perp = \boldsymbol{u}_\perp^G + \boldsymbol{u}_\perp^{AG}$

$$\partial_t \nabla_\perp^2 \psi + J[\psi, \nabla_\perp^2 \psi] = \mathcal{F} + \mathcal{D}, \qquad \zeta_a = \widehat{\boldsymbol{z}} \cdot \nabla \times \boldsymbol{u} + Ro^{-1}$$

$$\boldsymbol{u}_\perp = -\nabla \times \psi \widehat{\boldsymbol{z}}, \quad \zeta = \nabla_\perp^2 \psi$$



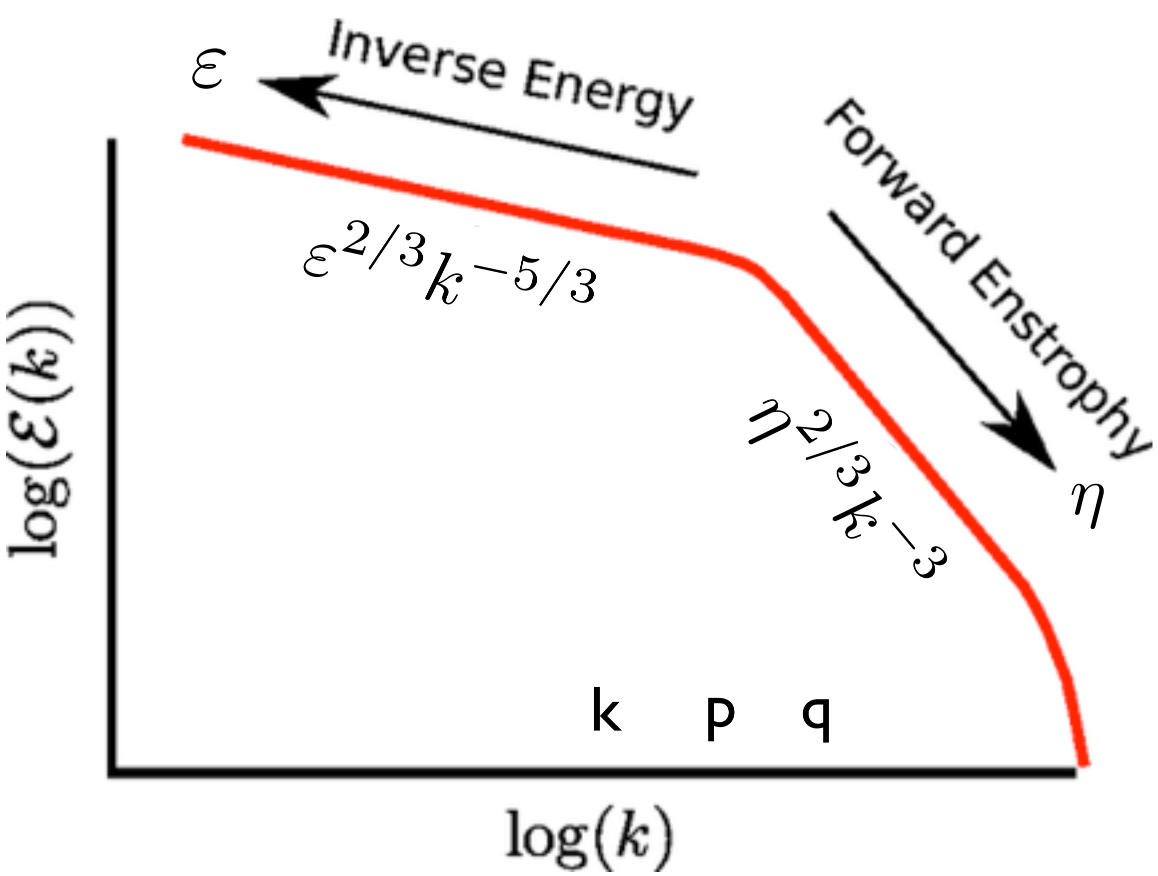
Bofetta & Ecke Ann. Rev. Fluid. Mech. 2012

Barotropic Vorticity Equation

$$\partial_t \zeta + \boldsymbol{u}_\perp \cdot \nabla_\perp \zeta = \mathcal{F} + \mathcal{D}, \qquad \zeta_a = \widehat{\boldsymbol{z}} \cdot \nabla \times \boldsymbol{u} + Ro^{-1}$$

$$\boldsymbol{u}_\perp = -\nabla \times \psi \widehat{\boldsymbol{z}}, \quad \zeta = \nabla_\perp^2 \psi$$

Two conserved quantities: Energy $\langle |\nabla_\perp \psi|^2 \rangle$ Enstrophy $\langle |\zeta_a|^2 \rangle$



Transfer between three interacting modes

$$k < p < q$$

$$\delta\mathcal{E}(k) + \delta\mathcal{E}(p) + \delta\mathcal{E}(q) = 0$$

$$k^2\delta\mathcal{E}(k) + p^2\delta\mathcal{E}(p) + q^2\delta\mathcal{E}(q) = 0$$

$$\delta\mathcal{E}(p) < 0 \Rightarrow$$

$$\delta\mathcal{E}(k) > \delta\mathcal{E}(q) > 0$$

$$\delta\mathcal{Z}(q) > \delta\mathcal{Z}(k) > 0$$

Dual cascade: Energy $\mathcal{E}(k) = \int_0^\infty E(k)dk$ Enstrophy $\mathcal{Z}(k) = \int_0^\infty k^2 E(k)dk$

Barotropic Vorticity Equation - Beta Plane

$$f \approx 2\Omega \sin\vartheta + \beta y$$

$$\partial_t \zeta + \boldsymbol{u}_\perp \cdot \nabla_\perp \zeta + \beta v = \mathcal{F} + \mathcal{D}, \qquad \zeta_a = \widehat{\boldsymbol{z}} \cdot \nabla \times \boldsymbol{u} + Ro^{-1} + \beta y$$

$$\boldsymbol{u}_\perp = -\nabla \times \psi \widehat{\boldsymbol{z}}, \quad \zeta = \nabla_\perp^2 \psi$$

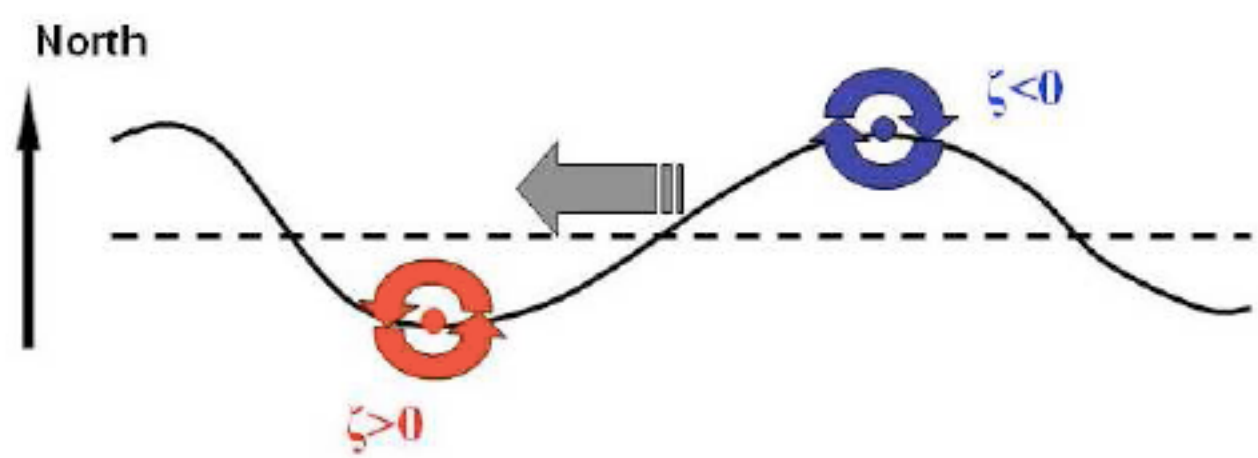
Two conserved quantities: Energy $\langle |\nabla_\perp \psi|^2 \rangle$ Enstrophy $\langle |\zeta_a|^2 \rangle$

Dispersion Rel'n

$$\omega = \frac{\beta k_x}{k_x^2 + k_y^2}$$

westward propagation

Mechanism



Rossby wave

Dual cascade: Energy $\mathcal{E}(k) = \int_0^\infty E(k)dk$ Enstrophy $\mathcal{Z}(k) = \int_0^\infty k^2 E(k)dk$

Barotropic Rossby Waves

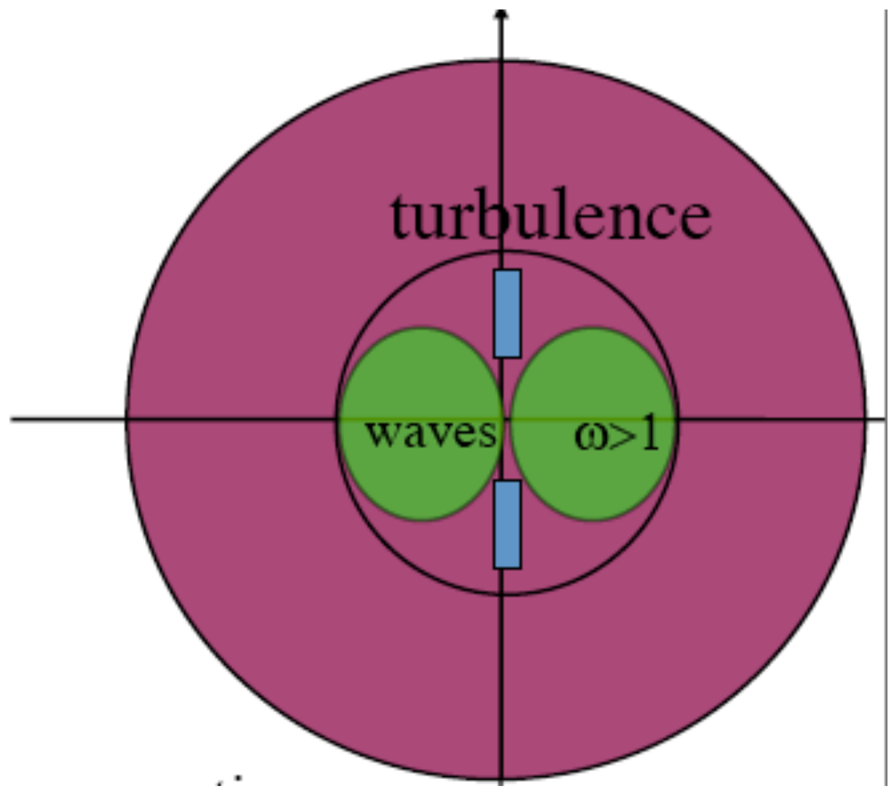
Inverse energy cascade is suppressed by low wavenumber Rossby waves (Rhines JFM 1975)

$$\omega_{turb} \approx \omega_\beta \quad \Rightarrow \quad \varepsilon^{1/3} k_\perp^{2/3} \approx \frac{\beta k_x}{k_\perp^2}$$

Inverse cascade barrier

$$k_x \approx \left(\frac{\beta^3}{\varepsilon}\right)^{1/5} \cos^{8/5}\vartheta$$

$$k_y \approx \left(\frac{\beta^3}{\varepsilon}\right)^{1/5} \sin\vartheta \cos^{3/5}\vartheta$$



Dual cascade: Energy $\mathcal{E}(k) = \int_0^\infty E(k)dk$ Enstrophy $\mathcal{Z}(k) = \int_0^\infty k^2 E(k)dk$

Barotropic Rossby Waves

Inverse energy cascade is suppressed by low wavenumber Rossby waves

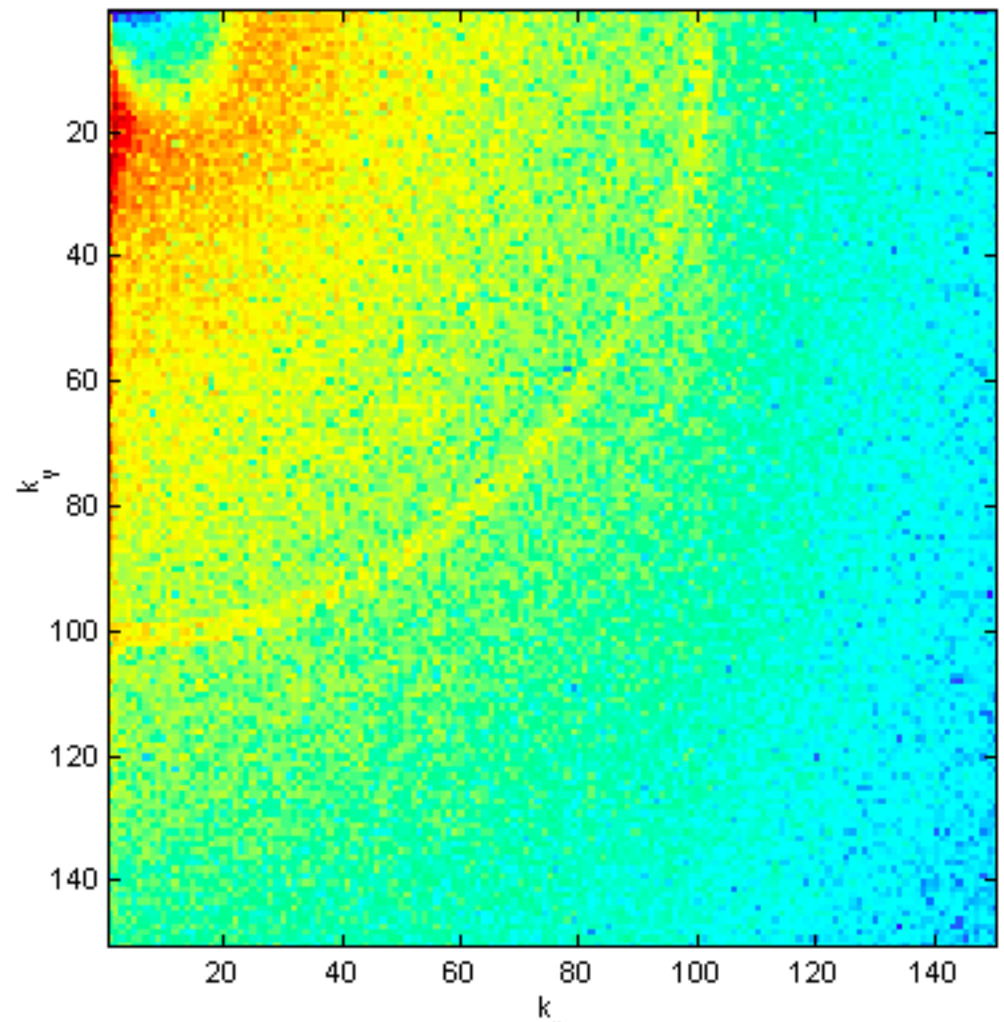
$$\omega_{turb} \approx \omega_\beta \quad \Rightarrow \quad \varepsilon^{1/3} k_\perp^{2/3} \approx \frac{\beta k_x}{k_\perp^2}$$

Inverse cascade barrier

$$k_x \approx \left(\frac{\beta^3}{\varepsilon}\right)^{1/5} \cos^{8/5}\vartheta$$

$$k_y \approx \left(\frac{\beta^3}{\varepsilon}\right)^{1/5} \sin\vartheta \cos^{3/5}\vartheta$$

Beta-plane spectrum

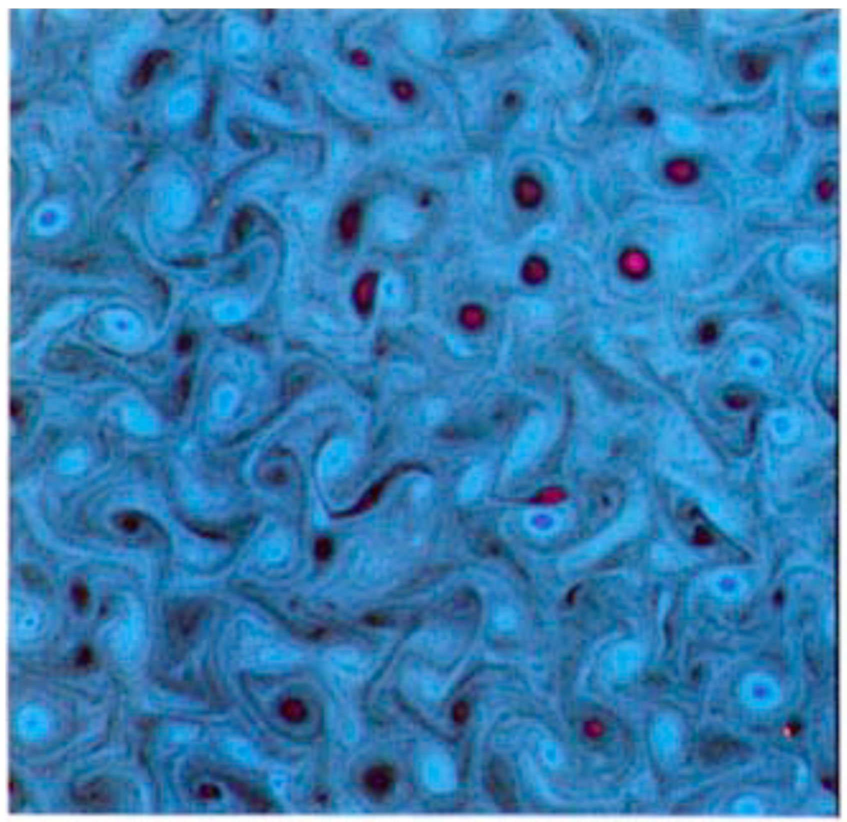


D. Gurarie (2004)

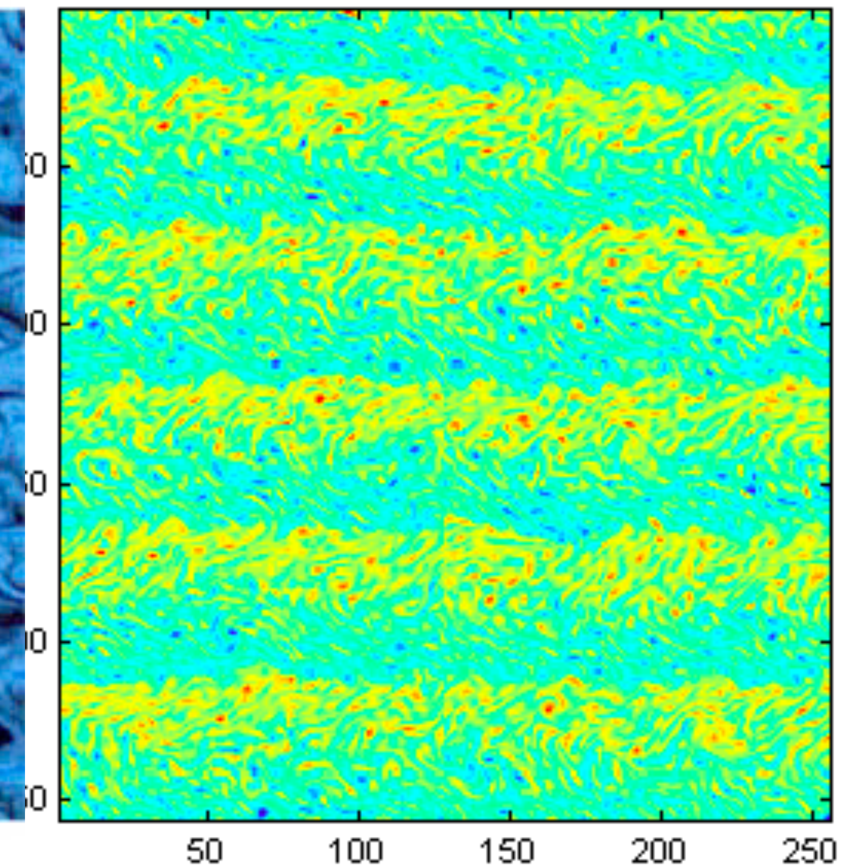
Dual cascade: Energy $\mathcal{E}(k) = \int_0^\infty E(k)dk$ Enstrophy $\mathcal{Z}(k) = \int_0^\infty k^2 E(k)dk$

Barotropic Rossby Waves

$\beta = 0$



$\beta \neq 0$



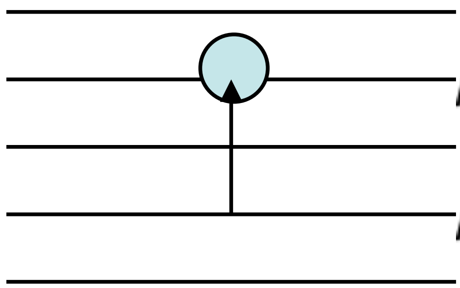
Maltrud & Vallis JFM1992

D. Gurarie (2004)

Dual cascade: Energy $\mathcal{E}(k) = \int_0^\infty E(k)dk$ Enstrophy $\mathcal{Z}(k) = \int_0^\infty k^2 E(k)dk$

Stratification/buoyancy Force

$$F_b = b\hat{z} \equiv -g\frac{\delta\rho}{\rho_0}\hat{\boldsymbol{z}}, \qquad \rho = \rho_0 + \rho(z) + \delta\rho$$



$$\frac{Dw}{Dt} \approx -b$$

$$\frac{Db}{Dt} \approx -N^2(z)w$$

$$\frac{D^2w}{Dt^2} + N^2(z)w \approx 0$$

$$N^2(z) = -\frac{g}{\rho_0}\frac{\partial\rho(z)}{\partial z}$$

Stable stratification: $N^2(z) > 0, \quad \partial_z\rho(z) < 0$ Fluid parcel oscillates about equilibrium posn

Neutral stratification: $N^2(z) = 0, \quad \partial_z\rho(z) = 0$

Unstable stratification: $N^2(z) < 0, \quad \partial_z\rho(z) > 0$ Fluid parcel accelerates away from equilibrium posn (Convection)

$$N^2 < 0$$

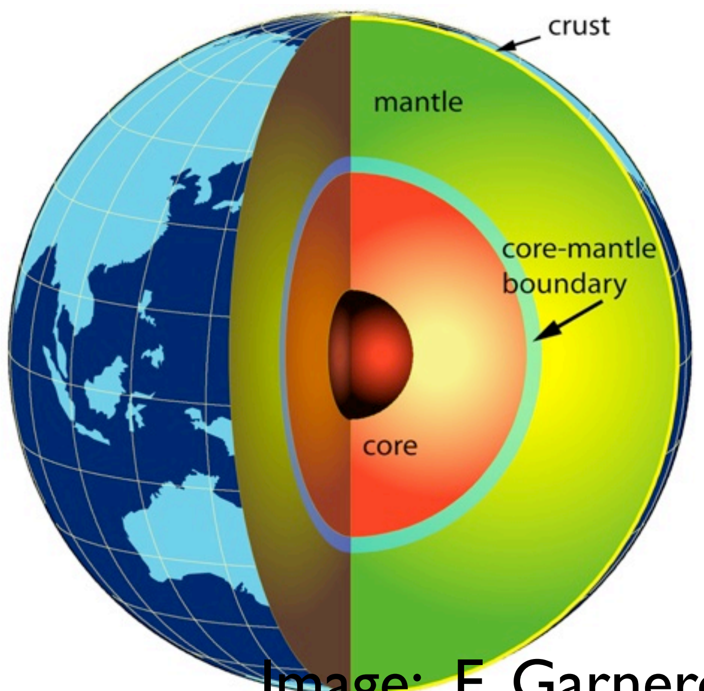
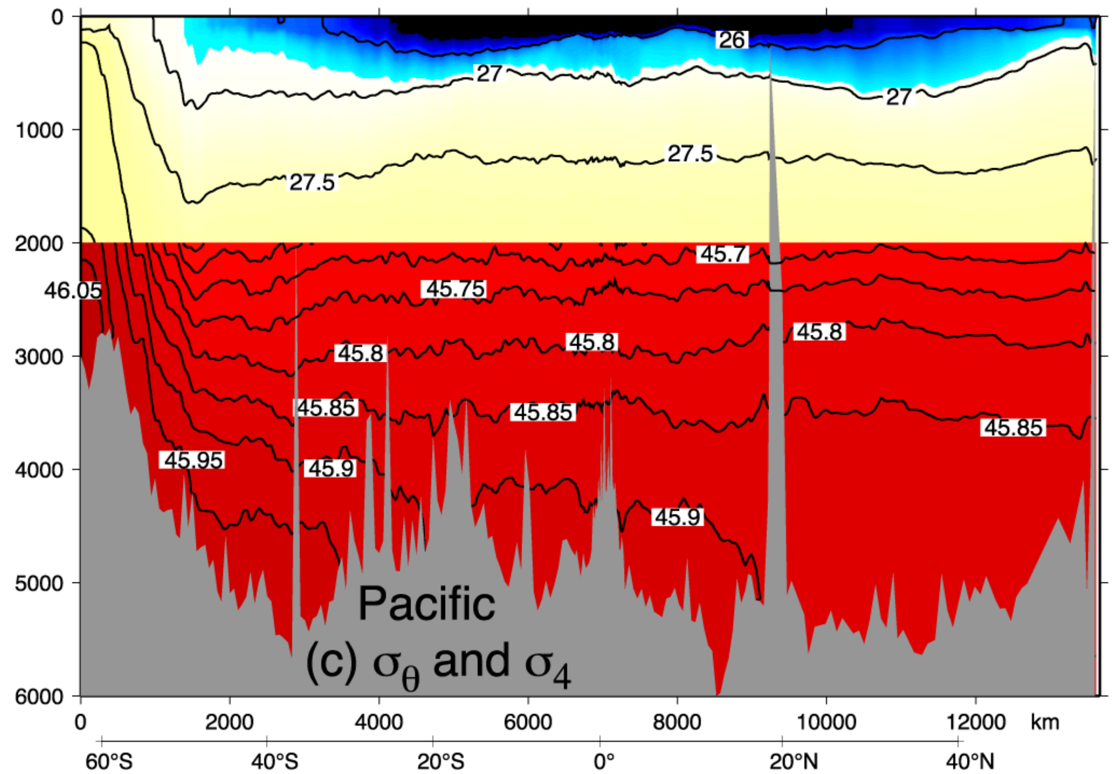


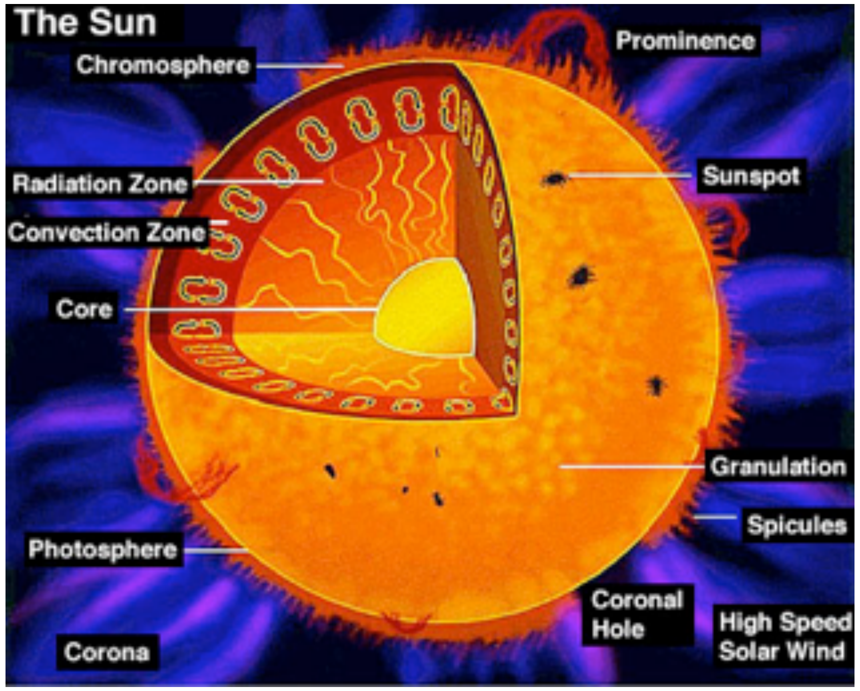
Image: E. Garnero

STRATIFICATION



The ocean is layered, with less dense water overriding denser water yielding a stratification:

$$N^2 = -\frac{g}{\rho_o}\frac{\partial \rho}{\partial z} \sim 10^{-4} - 10^{-6}\ \mathrm{s}^{-2}$$



Fluid Equations: Stratified Flows

$$\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -Eu \nabla \pi + \frac{1}{Fr} b \widehat{\boldsymbol{z}} + \frac{1}{Re} \nabla^2 \boldsymbol{u} + \boldsymbol{F}$$

$$\nabla \cdot \boldsymbol{u} = 0$$

$$\partial_t b + \boldsymbol{u} \cdot \nabla b + \frac{1}{Fr} w = \frac{1}{Pe} \nabla^2 b$$

Froude number

$$Fr = \frac{U^2/H}{NU} = \frac{\text{inertia}}{\text{buoyancy}} = \sqrt{\frac{KE}{PE}} = \frac{U}{NH}$$

Ozmidov length scale

$$Fr_{l_b} \sim 1, \qquad l_b = \frac{\varepsilon^{1/2}}{N^{3/2}} \approx HFr^{3/2}$$

Types of motions affected by stratification:

KE less than PE release $\quad Fr \leq 1, \quad \text{or} \quad U/H \leq N$

Vary on timescales greater than a buoyancy period $\quad StrFr \leq 1 \quad \text{or} \quad T \leq 1/N$

What fluid scales are influence by rotation? $\quad l \geq l_b \;\; U \sim (\varepsilon l)^{1/3}$

Dispersion Relation: Waves & Eddies

Consider case Eu ~ 1/Fr >>1, inviscid motions Re >>1

Separation of dynamics into fast and slow motions (waves and eddies).

$$\partial_t \boldsymbol{u} \approx -Eu\nabla p + \frac{1}{Fr} b\widehat{\boldsymbol{z}}$$

$$\nabla \cdot \boldsymbol{u} = 0$$

$$\partial_t b \approx -\frac{1}{Fr} w$$

Normal mode analysis:

$$\boldsymbol{v} \propto e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)}, \quad \boldsymbol{k}\cdot\boldsymbol{u} = 0, \quad \boldsymbol{k} = (\boldsymbol{k}_\perp, k_z) = |\boldsymbol{k}|(\cos\phi, \sin\phi)$$

Inertial waves (fast dynamics)

$$\partial_{tt}\nabla^2 w + \frac{1}{Fr^2}\nabla_\perp^2 w = 0$$

$$\omega^2 = \frac{1}{Fr^2}\frac{|\boldsymbol{k}_\perp|^2}{|\boldsymbol{k}|^2} = \frac{1}{Fr^2}\cos^2\phi$$

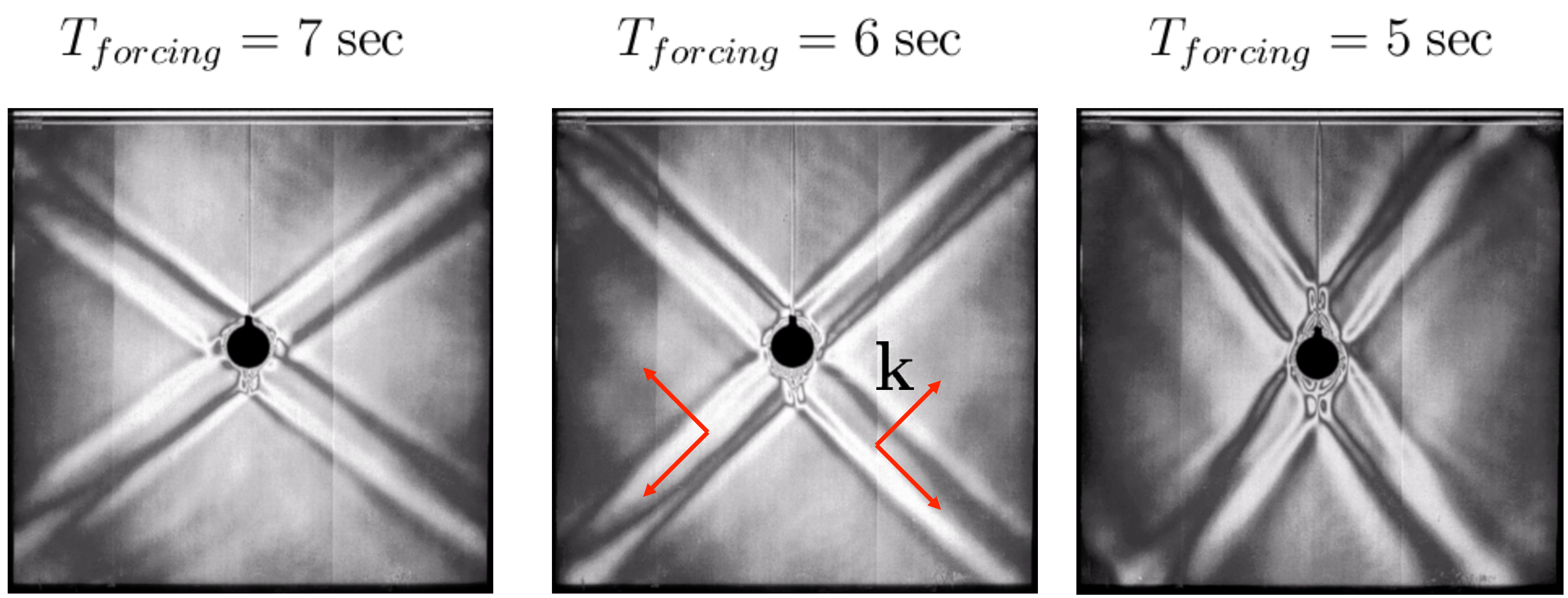
Slow dynamics (eddies) are in hydrostatic balance

$$EuFr\partial_z p = b, \quad w = 0$$

$$\omega = 0, \qquad \nabla_\perp(p, b) = 0$$

Freq. only a function propagation dir'n

Internal Gravity Waves (Experiments)



$$\frac{2\pi}{N} = 3.6 \text{ sec}$$

Experiments: Sakai's GFD lab

http://dennou.gaia.h.kyoto-u.ac.jp/library/gfd_exp/exp_e/index.htm

Dispersion Relation: Waves & Eddies

Consider case Eu ~ 1/Fr >>1, inviscid motions Re >>1

Separation of dynamics into fast and slow motions (waves and eddies).

$$0 \approx -Eu\nabla p + \frac{1}{Fr} b\widehat{\boldsymbol{z}}$$

$$\nabla \cdot \boldsymbol{u} = 0$$

$$0 \approx -\frac{1}{Fr} w$$

Normal mode analysis:

$$\boldsymbol{v} \propto e^{i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)}, \quad \boldsymbol{k}\cdot\boldsymbol{u} = 0, \quad \boldsymbol{k} = (\boldsymbol{k}_\perp, k_z) = |\boldsymbol{k}|(\cos\phi, \sin\phi)$$

Inertial waves (fast dynamics)

$$\partial_{tt}\nabla^2 w + \frac{1}{Fr^2}\nabla_\perp^2 w = 0$$

$$\omega^2 = \frac{1}{Fr^2}\frac{|\boldsymbol{k}_\perp|^2}{|\boldsymbol{k}|^2} = \frac{1}{Fr^2}\cos^2\phi$$

Slow dynamics (eddies) are in hydrostatic balance

$$EuFr\partial_z p = b, \quad w = 0$$

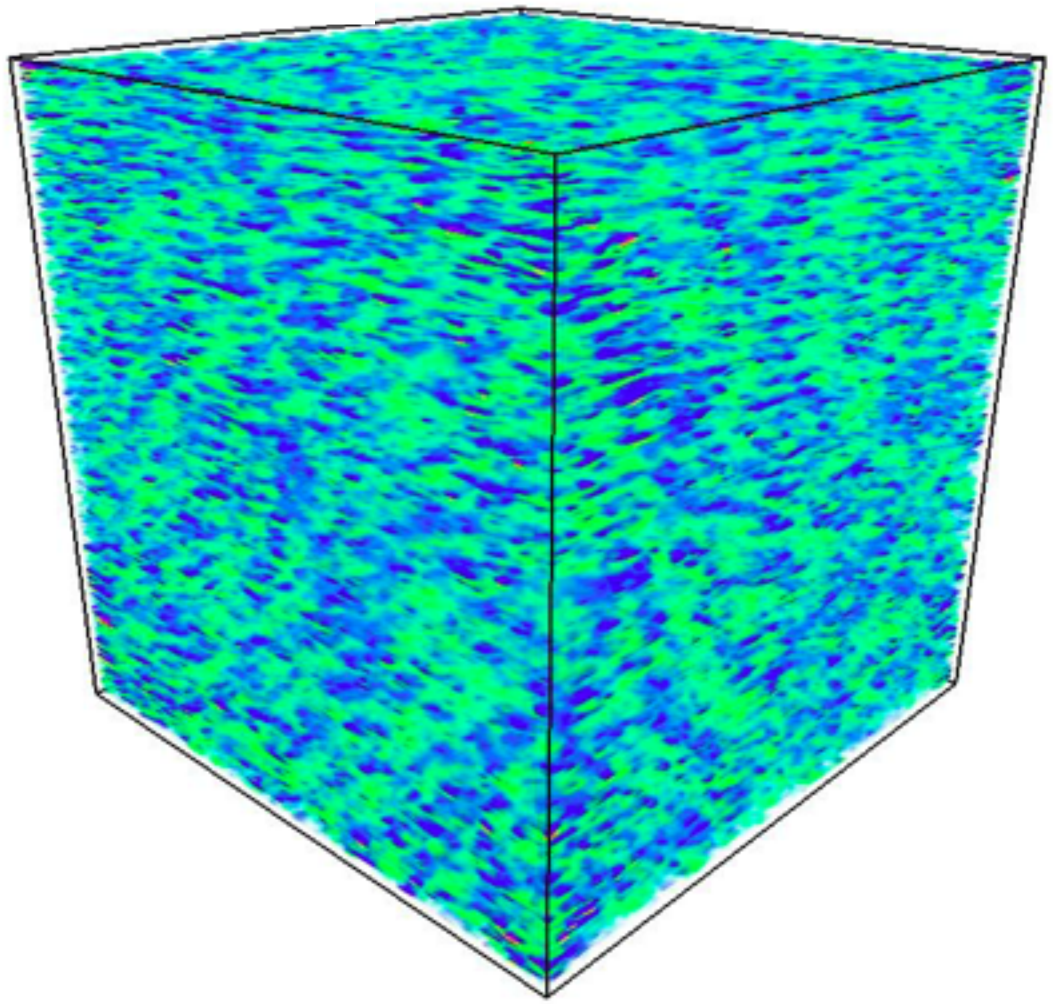
$$\omega = 0, \qquad \nabla_\perp(p, b) = 0$$

Freq. only a function propagation dir’n

Purely STRAT ($f = 0$)

$k_F = [40, 41]$

$Fr = 0.04$



1024^3 temperature
Marino et al EPL 2013

Y. Kimura and J. R. Herring

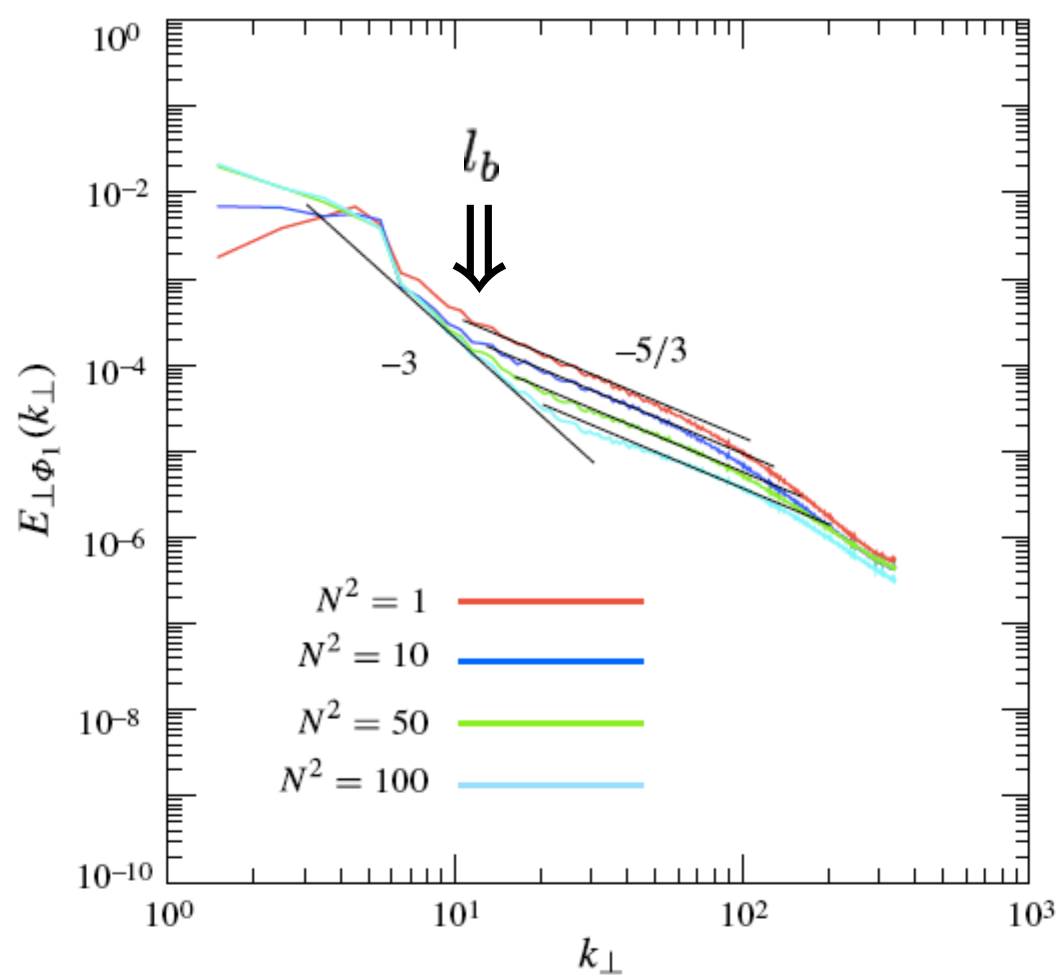
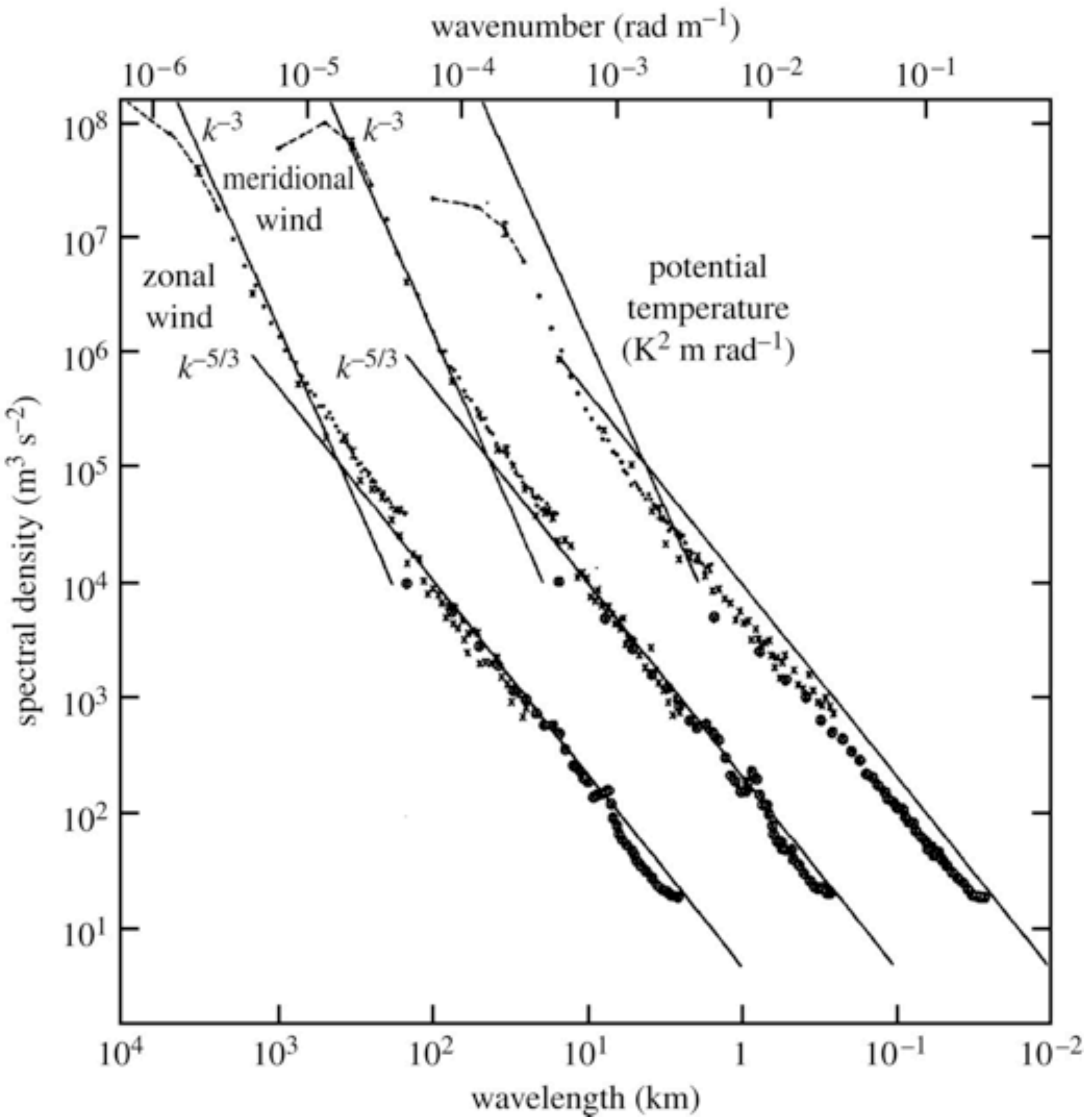


FIGURE 5. Spectra $E_{\perp\Phi_1}(k_\perp)$ for $N^2 = 1, 10, 50, 100$. The large-scale parts collapse to a single spectrum of $\sim k_\perp^{-3}$. On the other hand, the small scales form a different spectrum corresponding to the value of N, but each spectrum has the same slope of $-5/3$.

Forward cascade observed (Lindborg JFM 2004, Billant & Chomaz JFM 2000), break in exponent occurs at Ozmidov scale



The Nastrom and Gage (1983) atmospheric data

Forward or Inverse cascade?

Lindborg JFM (2006)
Billant & Chomaz JFM (2001)

GFD Fluid Equations: Rotating, Stratified Flows

$$\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \frac{1}{Ro} \widehat{\boldsymbol{z}} \times \boldsymbol{u} = -Eu \nabla \pi + \frac{1}{Fr_L} b \widehat{\boldsymbol{z}} + \frac{1}{Re} \nabla^2 \boldsymbol{u} + \boldsymbol{F}$$

$$\nabla \cdot \boldsymbol{u} = 0$$

$$\partial_t b + \boldsymbol{u} \cdot \nabla b + \frac{1}{Fr_L} w = \frac{1}{Pe} \nabla^2 b$$

Rossby, Froude numbers;

$$Ro = \frac{U}{2\Omega L}, \quad Fr_L = \frac{U}{NL}, \quad Fr_v = \frac{U}{NH}$$

GFD Fluid Equations: Rotating, Stratified Flows

$$\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \frac{1}{Ro} \widehat{\boldsymbol{z}} \times \boldsymbol{u} = -Eu \nabla \pi + \frac{1}{Fr_L} b \widehat{\boldsymbol{z}} + \frac{1}{Re} \nabla^2 \boldsymbol{u} + \boldsymbol{F}$$

$$\nabla \cdot \boldsymbol{u} = 0$$

$$\partial_t b + \boldsymbol{u} \cdot \nabla b + \frac{1}{Fr_L} w = \frac{1}{Pe} \nabla^2 b$$

Rossby, Froude numbers;

$$Ro = \frac{U}{2\Omega L}, \quad Fr_L = \frac{U}{NL}, \quad Fr_v = \frac{U}{NH}$$

GFD setting

$$\left(\frac{N}{2\Omega}\right)_{Earth} \sim 100, \;\; 2\Omega \sim 10^{-4} \mathrm{rad/s}$$

$$N > 2\Omega \;\; \Rightarrow \;\; Fr_L < Ro \;\; \Rightarrow \;\; l_b < l_\Omega$$

RST $\Rightarrow$ ST $\Rightarrow$ HT

GFD Fluid Equations: Rotating, Stratified Flows

$$\partial_t \boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \frac{1}{Ro}\widehat{\boldsymbol{z}}\times\boldsymbol{u} = -Eu\nabla\pi + \frac{1}{Fr_L}b\widehat{\boldsymbol{z}} + \frac{1}{Re}\nabla^2\boldsymbol{u} + \boldsymbol{F}$$

$$\nabla\cdot\boldsymbol{u} = 0$$

$$\partial_t b + \boldsymbol{u}\cdot\nabla b + \frac{1}{Fr_L}w = \frac{1}{Pe}\nabla^2 b$$

Rossby, Froude numbers;

$$Ro = \frac{U}{2\Omega L}, \quad Fr_L = \frac{U}{NL}, \quad Fr_v = \frac{U}{NH}$$

GFD setting

$$\left(\frac{N}{2\Omega}\right)_{Earth} \sim 100,\ \ 2\Omega \sim 10^{-4}\mathrm{rad/s}$$

$$N > 2\Omega \ \Rightarrow\ Fr_L < Ro \ \Rightarrow\ l_b < l_\Omega$$

RST $\Rightarrow$ ST $\Rightarrow$ HT

$$(i)\ \text{Hydrostatic balance:}\ \ Fr_L \ll 1 \ \Rightarrow\ Eu\partial_z \sim \frac{1}{Fr_L},\ \ w \sim Fr_L$$

$$(ii)\ \text{PGF:}\ \ Eu \sim 1 \ \Rightarrow\ \partial_z \sim \frac{1}{Fr_L}$$

$$(iii)\ \text{Rotation:}\ \ Ro \sim 1 \ \Rightarrow\ L_d = \frac{N}{2\Omega}H$$

GFD Fluid Equations: Rotating, Stratified Flows

$$\boxed{\partial_t \boldsymbol{u}_\perp + \boldsymbol{u}\cdot\nabla\boldsymbol{u}_\perp} + \frac{1}{Ro}\widehat{\boldsymbol{z}}\times\boldsymbol{u}_\perp = -Eu\nabla p + \frac{1}{Fr_L}b\widehat{\boldsymbol{z}} + \frac{1}{Re}\nabla^2\boldsymbol{u}_\perp + F_\perp$$

$$\nabla\cdot\boldsymbol{u} = 0$$

Primitive Equations

typical for GCM's

$$\partial_t b + \boldsymbol{u}\cdot\nabla b + \frac{1}{Fr_L}N^2(z)w = \frac{1}{Pe}\nabla^2 b$$

Rossby, Froude numbers;

$$Ro = \frac{U}{2\Omega L}, \quad Fr_L = \frac{U}{NL}, \quad Fr_v = \frac{U}{NH}$$

GFD setting

$$\left(\frac{N}{2\Omega}\right)_{Earth} \sim 100,\ 2\Omega \sim 10^{-4}\mathrm{rad/s}$$

$$N > 2\Omega \;\Rightarrow\; Fr_L < Ro \;\Rightarrow\; l_b < l_\Omega$$

RST $\Rightarrow$ ST $\Rightarrow$ HT

$$(i)\ \text{Hydrostatic balance:}\quad Fr_L \ll 1 \;\Rightarrow\; Eu\partial_z \sim \frac{1}{Fr_L},\ \ w \sim Fr_L$$

$$(ii)\ \text{PGF:}\quad Eu \sim 1 \;\Rightarrow\; \partial_z \sim \frac{1}{Fr_L}$$

$$(iii)\ \text{Rotation:}\quad Ro \sim 1 \;\Rightarrow\; L_d = \frac{N}{2\Omega}H$$

Eddy - Wave dynamics for RST

$$\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \frac{1}{Ro} \widehat{\boldsymbol{z}} \times \boldsymbol{u} = -Eu \nabla \pi + \frac{1}{Fr_L} b \widehat{\boldsymbol{z}} + \frac{1}{Re} \nabla^2 \boldsymbol{u} + \boldsymbol{F}$$

$$\nabla \cdot \boldsymbol{u} = 0$$

$$\partial_t b + \boldsymbol{u} \cdot \nabla b + \frac{1}{Fr_L} w = \frac{1}{Pe} \nabla^2 b$$

Rossby, Froude numbers;

$$Ro = \frac{U}{2\Omega L}, \quad Fr_L = \frac{U}{NL}, \quad Fr_v = \frac{U}{NH}$$

Eddy - Wave dynamics for RST

Inertial-gravity waves have bounded spectrum

$$\partial_{tt}\nabla^2 w + \frac{1}{Ro^2}\partial_{zz} w + \frac{1}{Fr_L^2}\nabla_\perp^2 w \approx 0$$

$$\omega_f^2 = \frac{1}{Ro^2}\sin^2\phi + \frac{1}{Fr^2}\cos^2\phi$$

Primitive Equations

Filter isotropic IGW

$$\omega_f \in [1/Ro_L, 1/Fr_L] \to [2\Omega, N]$$

Slow dynamics in geostrophic and hydrostatic balance (Thermal wind balance)

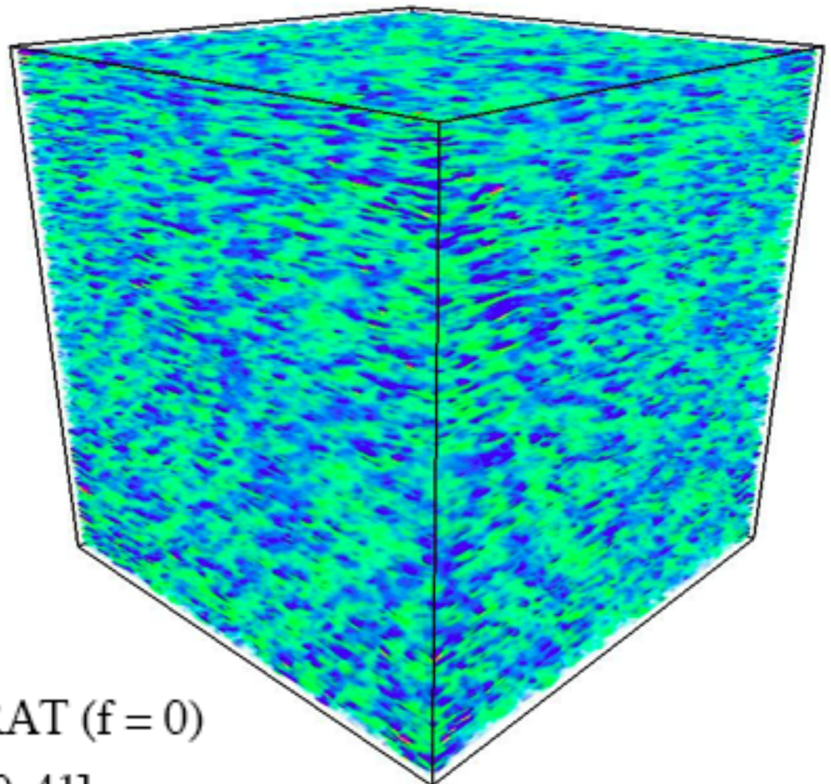
reduction: all field can be related to the pressure

$$\frac{1}{Ro}\hat{\boldsymbol{z}} \times \boldsymbol{u} \approx -Eu\nabla_\perp p$$

$$0 \approx -Eu\partial_z p + \frac{1}{Fr_L} b$$

$$\nabla_\perp \cdot \boldsymbol{u}_\perp = 0$$

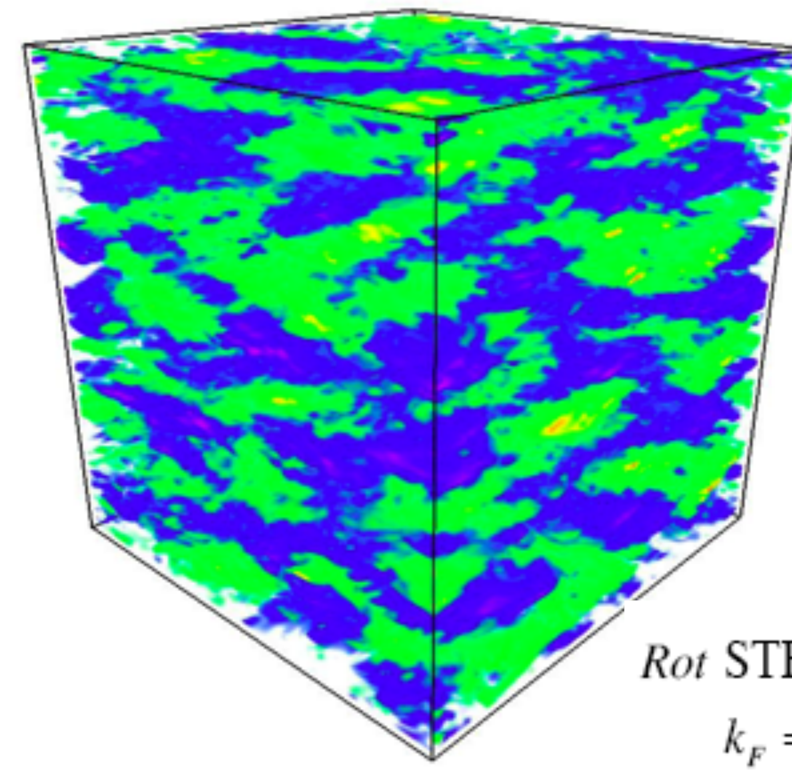
Rotating Stratified Turbulence (Marino et al EPL 2013)



Purely STRAT (f = 0)

$k_F = [40, 41]$

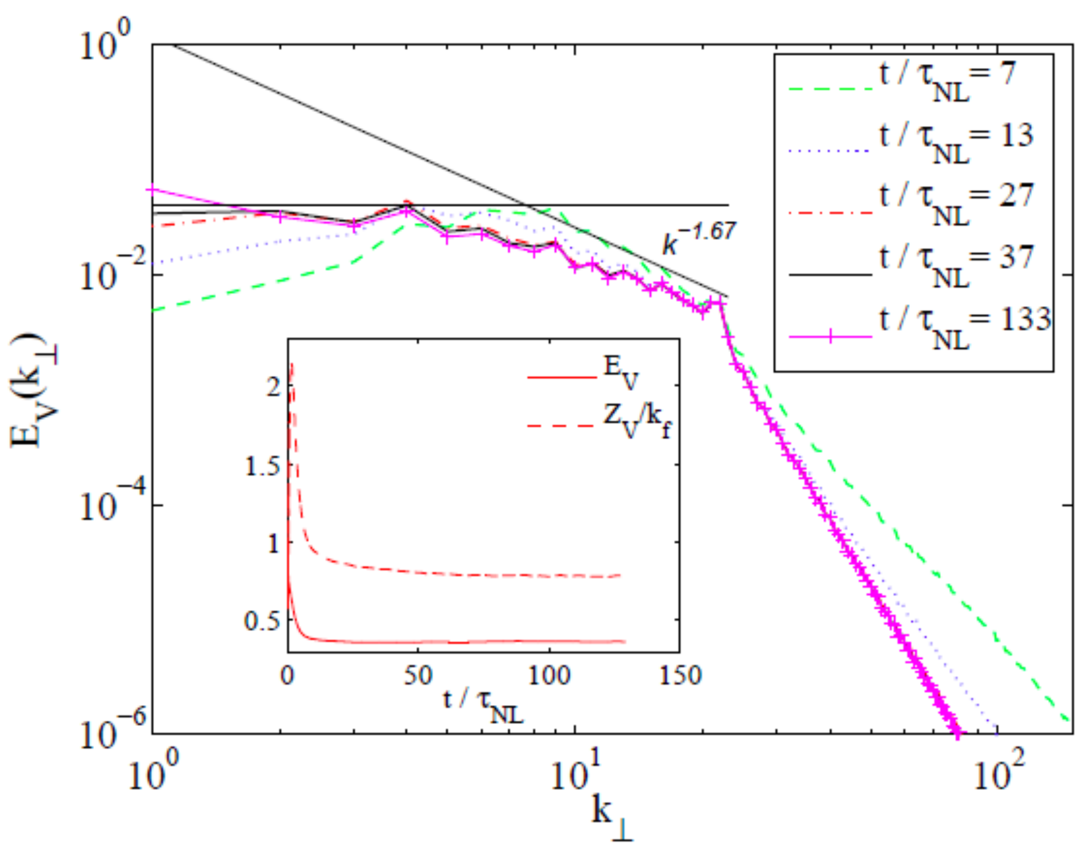
$Fr = 0.04$

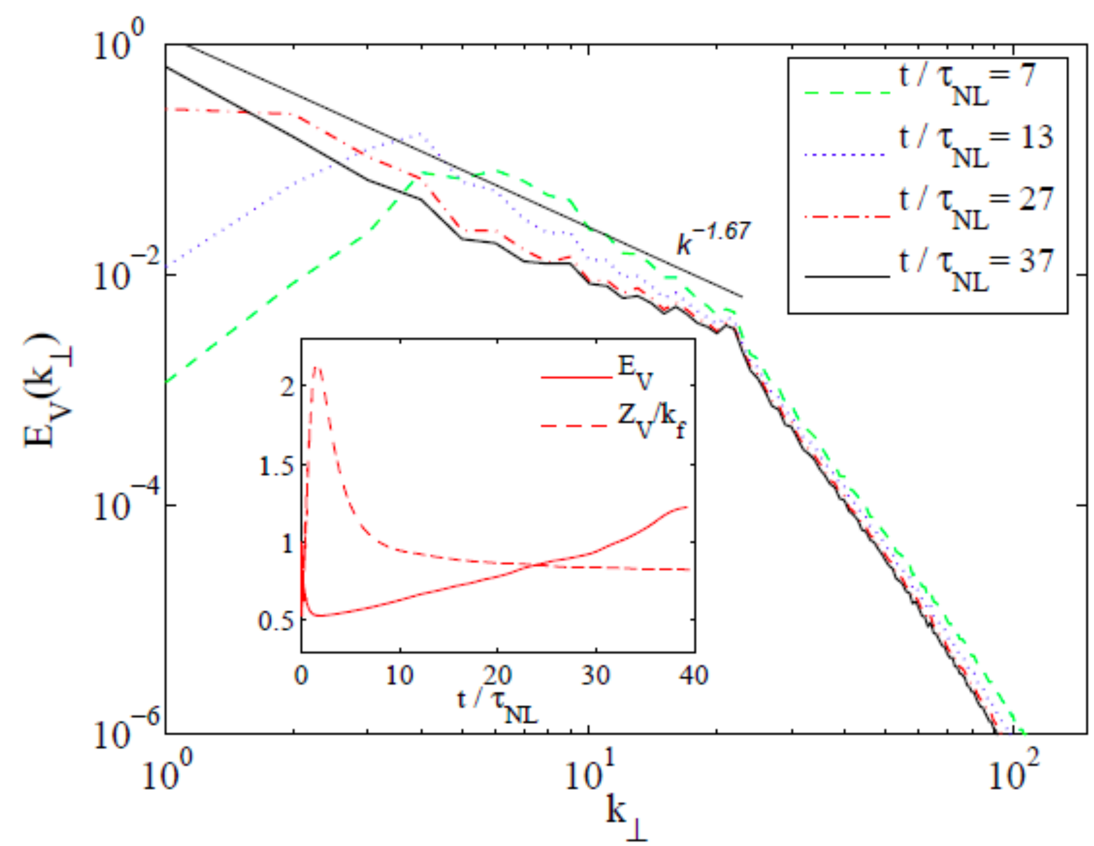


Rot STRAT (N/f = 2)

$k_F = [40, 41]$

$Fr = 0.04$

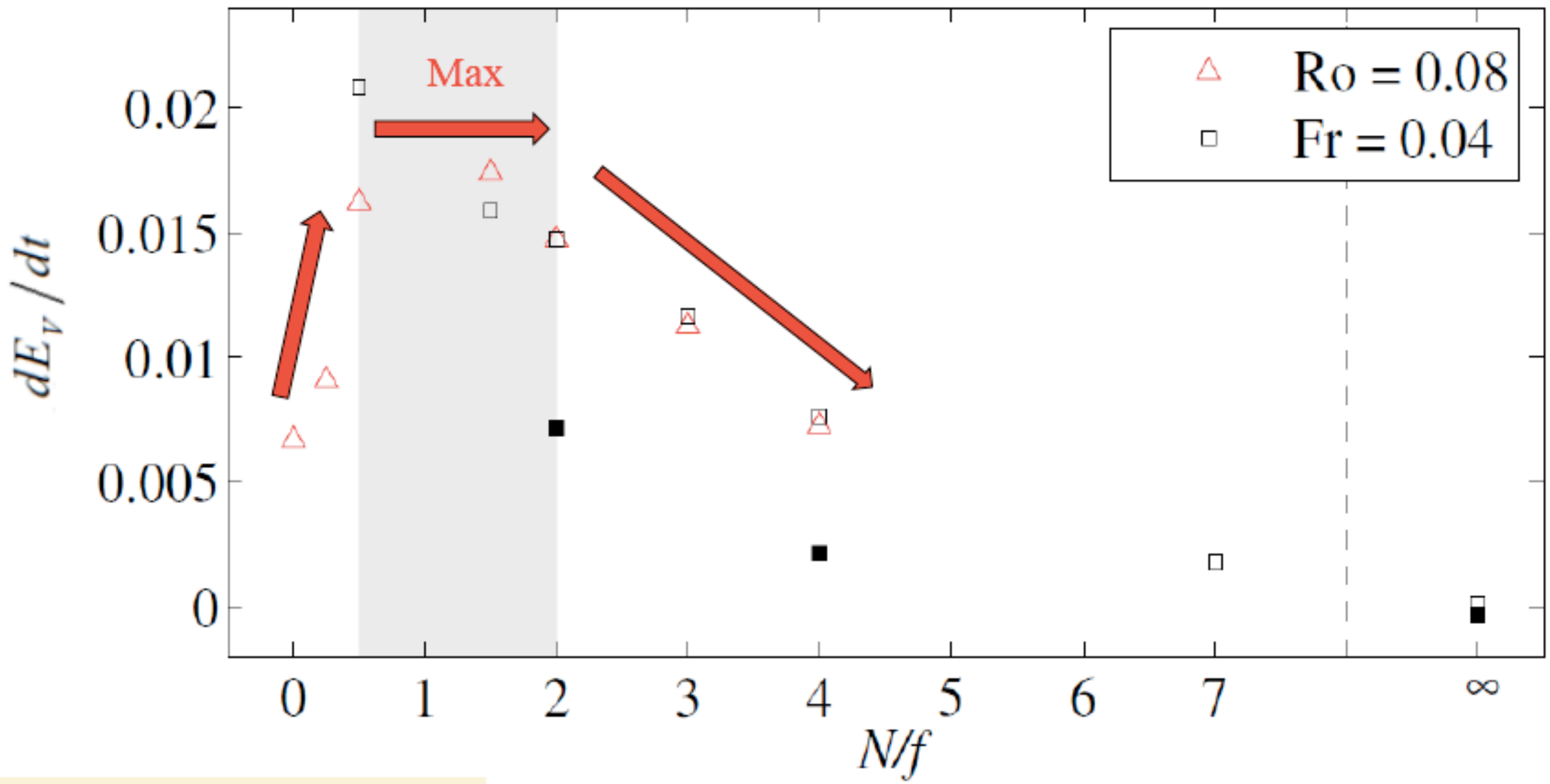




Inverse cascade for nonzero rotation rates

dE_V/dt computed for $10 \le t/\tau_{NL} \le 30$

$\frac{1}{2} \le N/f \le 2 \rightarrow$ No wave resonces (Smith & Waleffe, 2002)



Inertial-gravity waves

$$\sigma(k) = k^{-1}\sqrt{N^2k_\perp^2 + f^2k_\parallel^2}$$

for $N/f = 1 \rightarrow \omega_k = \pm N \rightarrow \cancel{\omega_k + \omega_p + \omega_q = 0}$

Marino et al., *EPL 2013*

Inverse cascade for nonzero rotation rates

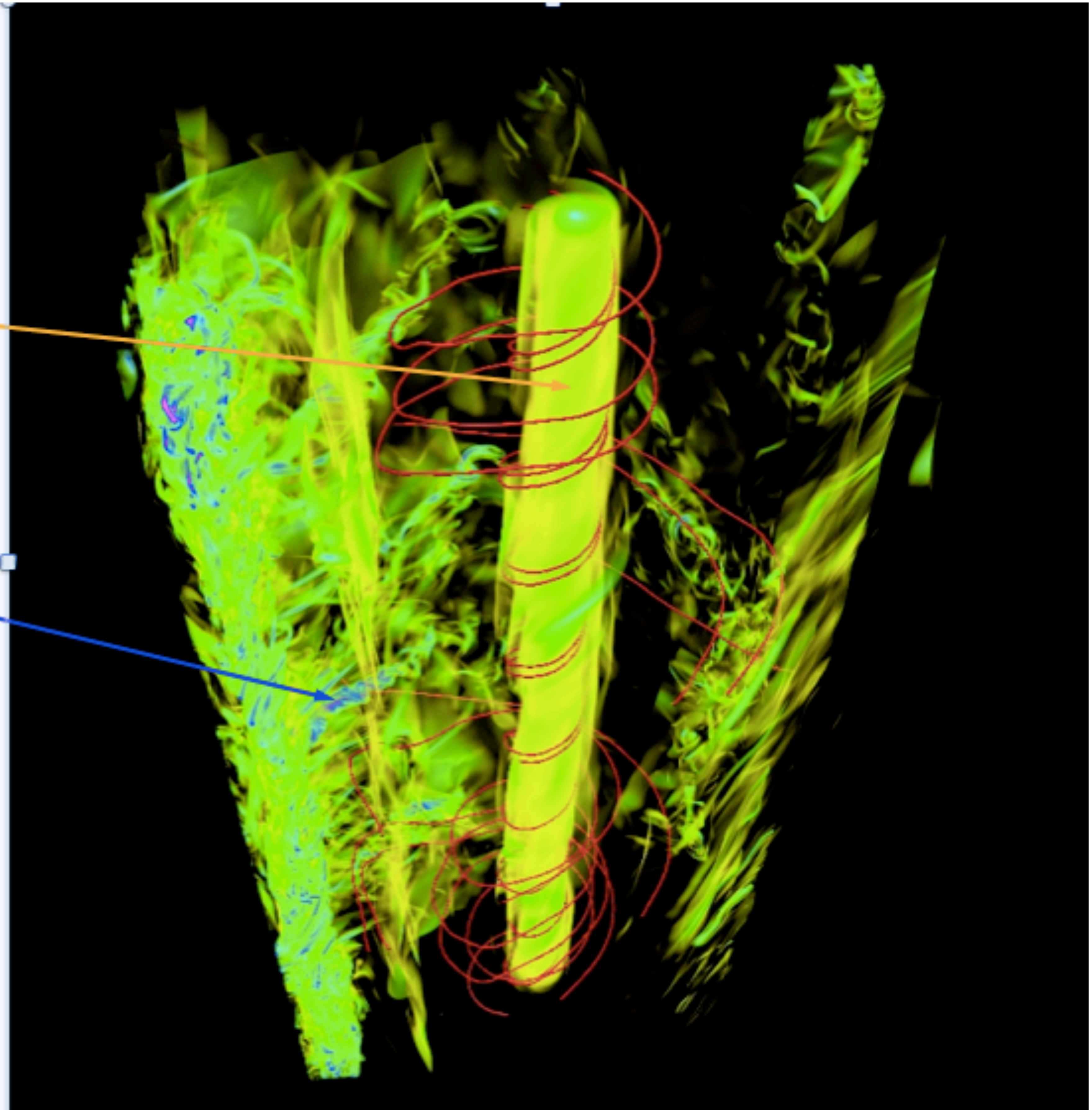
Mininni & AP,
Phys. Fluids 22 (2010)

Zoom on a Beltrami core vortex

amidst a tangle of smaller-scale vortex filaments

Together with particle trajectories

1536^3 grid, $k_F=7$,
Re=5100,
Ro=0.06, 1536^3 grid



Geostrophic - Nonhydrostatic Dynamics?

$$\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \frac{1}{Ro} \widehat{\boldsymbol{z}} \times \boldsymbol{u} = -Eu \nabla \pi + \frac{1}{Fr_L} b \widehat{\boldsymbol{z}} + \frac{1}{Re} \nabla^2 \boldsymbol{u} + \boldsymbol{F}$$

$$\nabla \cdot \boldsymbol{u} = 0$$

$$\partial_t b + \boldsymbol{u} \cdot \nabla b + \frac{1}{Fr_L} w = \frac{1}{Pe} \nabla^2 b$$

For rotating, stably stratified flows, primary modeling tools are - Primitive equations and QuasiGesotrophic equation

What about unstably stratified flows?

Reduced PDE Models
Balanced Dynamics

Unified QuasiGeostrophic (QG) Theory

Anisotropic rescaling of BE: $A = H/L$

$$D_t^{\perp}\boldsymbol{u}_{\perp} + \frac{1}{A}w\partial_z\boldsymbol{u}_{\perp} + \frac{1}{Ro}\widehat{\boldsymbol{z}} \times \boldsymbol{u}_{\perp} = -Eu\nabla_{\perp}p + \frac{1}{Re}\left(\nabla_{\perp}^2 + \frac{1}{A^2}\partial_{zz}\right)\boldsymbol{u}_{\perp}$$

$$D_t^{\perp}w + \frac{1}{A}w\partial_z w = -\frac{Eu}{A}\partial_z p + \frac{1}{Fr_L}b + \frac{1}{Re}\left(\nabla_{\perp}^2 + \frac{1}{A^2}\partial_{zz}\right)w$$

$$D_t^{\perp}b + \frac{1}{A}w\partial_z b + \frac{1}{Fr_L}wN^2(z) = \frac{1}{Pe}\left(\nabla_{\perp}^2 + \frac{1}{A^2}\partial_{zz}\right)b$$

$$\nabla_{\perp} \cdot \boldsymbol{u}_{\perp} + \frac{1}{A}\partial_z w = 0$$

Separate NSE into horizontal and vertical components

Unified QG Theory

Geostrophy (diagnostic balance): $\boxed{Eu \sim \frac{1}{Ro}}$

$$\boldsymbol{u}_\perp = \boldsymbol{u}_\perp^G + \underbrace{\boldsymbol{u}_\perp^{AG}}_{\text{small}}$$

$$D_t^\perp \boldsymbol{u}_\perp + \frac{1}{A} w \partial_z \boldsymbol{u}_\perp + \boxed{\frac{1}{Ro} \widehat{\boldsymbol{z}} \times \boldsymbol{u}_\perp = -Eu \nabla_\perp p} + \frac{1}{Re} \left(\nabla_\perp^2 + \frac{1}{A^2} \partial_{zz} \right) \boldsymbol{u}_\perp$$

$$D_t^\perp w + \frac{1}{A} w \partial_z w = -\frac{Eu}{A} \partial_z p + \frac{1}{Fr_L} b + \frac{1}{Re} \left(\nabla_\perp^2 + \frac{1}{A^2} \partial_{zz} \right) w$$

$$D_t^\perp b + \frac{1}{A} w \partial_z b + \frac{1}{Fr_L} w N^2(z) = \frac{1}{Pe} \left(\nabla_\perp^2 + \frac{1}{A^2} \partial_{zz} \right) b$$

$$\boxed{\nabla_\perp \cdot \boldsymbol{u}_\perp} + \frac{1}{A} \partial_z w = 0$$

dynamics require ageostrophic motions

Unified QG Theory

Geostrophy (diagnostic balance): $\boldsymbol{u}_\perp^G = \nabla^\perp p^G$

$$D_t^\perp \boldsymbol{u}_\perp^G + \frac{1}{A} w \partial_z \boldsymbol{u}_\perp^G + \frac{1}{Ro} \widehat{\boldsymbol{z}} \times \boldsymbol{u}_\perp^{AG} = -\frac{1}{Ro} \nabla_\perp p^{AG} + \frac{1}{Re} \left(\nabla_\perp^2 + \frac{1}{A^2} \partial_{zz} \right) \boldsymbol{u}_\perp^G$$

$$D_t^\perp w + \frac{1}{A} w \partial_z w = -\frac{1}{ARo} \partial_z p^G + \frac{1}{Fr_L} b + \frac{1}{Re} \left(\nabla_\perp^2 + \frac{1}{A^2} \partial_{zz} \right) w$$

$$D_t^\perp b + \frac{1}{A} w \partial_z b + \frac{1}{Fr_L} w N^2(z) = \frac{1}{Pe} \left(\nabla_\perp^2 + \frac{1}{A^2} \partial_{zz} \right) b$$

$$\nabla_\perp \cdot \boldsymbol{u}_\perp^{AG} + \frac{1}{A} \partial_z w = 0$$

dynamics require ageostrophic motions

$$\boldsymbol{u}_\perp^{AG}, p^{AG} \sim Ro \quad \Rightarrow \quad w \sim ARo$$

$$A = H/L$$

Unified QG Theory

Geostrophy (diagnostic balance): $\boldsymbol{u}_\perp^G = \nabla^\perp p^G$

$$D_t^\perp \boldsymbol{u}_\perp^G + \frac{1}{A} w \partial_z \boldsymbol{u}_\perp^G + \frac{1}{Ro} \widehat{\boldsymbol{z}} \times \boldsymbol{u}_\perp^{AG} = -\frac{1}{Ro} \nabla_\perp p^{AG} + \frac{1}{Re} \left(\nabla_\perp^2 + \frac{1}{A^2} \partial_{zz} \right) \boldsymbol{u}_\perp^G$$

$$D_t^\perp w + \frac{1}{A} w \partial_z w = -\frac{1}{ARo} \partial_z p^G + \frac{1}{Fr_L} b + \frac{1}{Re} \left(\nabla_\perp^2 + \frac{1}{A^2} \partial_{zz} \right) w$$

$$D_t^\perp b + \frac{1}{A} w \partial_z b + \frac{1}{Fr_L} w N^2(z) = \frac{1}{Pe} \left(\nabla_\perp^2 + \frac{1}{A^2} \partial_{zz} \right) b$$

$$\nabla_\perp \cdot \boldsymbol{u}_\perp^{AG} + \frac{1}{A} \partial_z w = 0$$

dynamics require ageostrophic motions

$$\boldsymbol{u}_\perp^{AG}, p^{AG} \sim Ro \quad \Rightarrow \quad w \sim ARo$$

universal scaling in QG theory

$A = H/L$

Unified QG Theory

Geostrophy (diagnostic balance): $\boldsymbol{u}_\perp^G = \nabla^\perp p^G$

$$D_t^\perp \boldsymbol{u}_\perp^G + Row\partial_z \boldsymbol{u}_\perp^G + \widehat{\boldsymbol{z}} \times \boldsymbol{u}_\perp^{AG} = -\nabla_\perp p^{AG} + \frac{1}{Re}\left(\nabla_\perp^2 + \frac{1}{A^2}\partial_{zz}\right)\boldsymbol{u}_\perp^G$$

$$(ARo)^2\left(D_t^\perp w + Row\partial_z w\right) = -\,\partial_z p^G + \left[\frac{ARo}{Fr_L}\right] b + \frac{(ARo)^2}{Re}\left(\nabla_\perp^2 + \frac{1}{A^2}\partial_{zz}\right) w$$

$$D_t^\perp b + Row\partial_z b + \left[\frac{ARo}{Fr_L}\right] wN^2(z) = \frac{1}{Pe}\left(\nabla_\perp^2 + \frac{1}{A^2}\partial_{zz}\right) b$$

$$\nabla_\perp \cdot \boldsymbol{u}_\perp^{AG} + \partial_z w = 0$$

Rescale vertical motions

Unified QG Theory

Geostrophy (diagnostic balance): $$\boldsymbol{u}_\perp^G = \nabla^\perp p^G$$

$$D_t^\perp \boldsymbol{u}_\perp^G + \cancel{Row\partial_z \boldsymbol{u}_\perp^G} + \widehat{\boldsymbol{z}} \times \boldsymbol{u}_\perp^{AG} = -\nabla_\perp p^{AG} + \frac{1}{Re}\left(\nabla_\perp^2 + \frac{1}{A^2}\partial_{zz}\right)\boldsymbol{u}_\perp^G$$

$$(ARo)^2\left(D_t^\perp w + \cancel{Row\partial_z w}\right) = -\,\partial_z p^G + \left[\frac{ARo}{Fr_L}\right] b + \frac{(ARo)^2}{Re}\left(\nabla_\perp^2 + \frac{1}{A^2}\partial_{zz}\right) w$$

$$D_t^\perp b + \cancel{Row\partial_z b} + \left[\frac{ARo}{Fr_L}\right] wN^2(z) = \frac{1}{Pe}\left(\nabla_\perp^2 + \frac{1}{A^2}\partial_{zz}\right) b$$

$$\nabla_\perp \cdot \boldsymbol{u}_\perp^{AG} + \partial_z w = 0$$

Vertical advection is subdominant

Inclusion of buoyancy driving requires $$Fr_L = ARo$$

-- a measure of the stratification strength

Unified QG Theory

Geostrophy (diagnostic balance): $$\boldsymbol{u}_\perp^G = \nabla^\perp p^G$$

Closed Balanced Model

$$D_t^\perp \boldsymbol{u}_\perp^G + \widehat{\boldsymbol{z}} \times \boldsymbol{u}_\perp^{AG} = -\nabla_\perp p^{AG} + \frac{1}{Re}\left(\nabla_\perp^2 + \frac{1}{A^2}\partial_{zz}\right)\boldsymbol{u}_\perp^G$$

$$(ARo)^2 D_t^\perp w = -\ \partial_z p^G + b + \frac{(ARo)^2}{Re}\left(\nabla_\perp^2 + \frac{1}{A^2}\partial_{zz}\right) w$$

$$D_t^\perp b + wN^2(z) = \frac{1}{Pe}\left(\nabla_\perp^2 + \frac{1}{A^2}\partial_{zz}\right) b$$

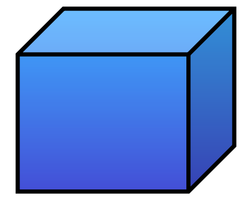
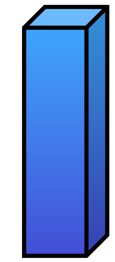
$$\nabla_\perp \cdot \boldsymbol{u}_\perp^{AG} + \partial_z w = 0$$

Measure of stratification $Fr_L = ARo$

Fr governs whether the dynamics are hydrostatic or nonhydrostatic

Unified QG model capturing balanced dynamics irrespective of spatial anisotropy

$Ro \ll 1$

Stratification	$Fr = ARo \ll 1$	$Fr \sim Ro$	$Fr \sim 1$
	QG	**Intermediate**	**Convection**
$w \sim (ARo)u$			
$A =$	$H/L << 1$	$H/L = O(1)$	$H/L >> 1$
	Charney (1948)	Embid & Majda (1998)	Julien et al. (1998)

QG Summary

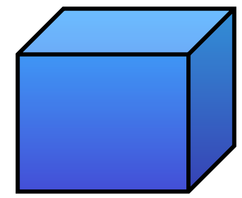
$$D_t^{\perp}\zeta - \partial_z w = \frac{1}{Re}\left(\nabla_{\perp}^2 + \frac{1}{A^2}\partial_{zz}\right)\zeta$$

$$(ARo)^2\, D_t^{\perp} w = -\,\partial_z \psi + b + \frac{(ARo)^2}{Re}\left(\nabla_{\perp}^2 + \frac{1}{A^2}\partial_{zz}\right) w$$

$$D_t^{\perp} b + wN^2(z) = \frac{1}{Pe}\left(\nabla_{\perp}^2 + \frac{1}{A^2}\partial_{zz}\right) b$$

Julien et al JFM 2006

Unified QG model capturing balanced dynamics irrespective of spatial anisotropy

Stratification	$Fr = ARo \ll 1$	$Fr \sim Ro$	$Fr \sim 1$	$Ro \ll 1$
	QG	**Intermediate**		
$w \sim (ARo)u$				
$A =$	$H/L << 1$ Charney (1948)	$H/L = O(1)$ Embid & Majda (1998)		

QG Summary

$$D_t^{\perp}\zeta - \partial_z w = \frac{1}{Re}\left(\nabla_\perp^2 + \frac{1}{A^2}\partial_{zz}\right)\zeta$$

$$\cancel{(ARo)^2 D_t^{\perp} w} = -\,\partial_z\psi + b + \cancel{\frac{(ARo)^2}{Re}\left(\nabla_\perp^2 + \frac{1}{A^2}\partial_{zz}\right) w}$$

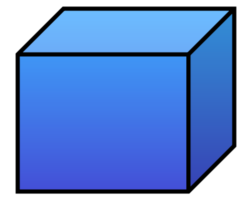
$$D_t^{\perp} b + wN^2(z) = \frac{1}{Pe}\left(\nabla_\perp^2 + \frac{1}{A^2}\partial_{zz}\right) b$$

Julien et al JFM 2006

Unified QG model capturing balanced dynamics irrespective of spatial anisotropy

Stratification $Fr = ARo \ll 1$ $\quad Fr \sim Ro$ $\quad Fr \sim 1$ $\quad Ro \ll 1$

$w \sim (ARo)u$

QG	Intermediate
	
$H/L << 1$	$H/L = O(1)$
Charney (1948)	Embid & Majda (1998)

$A =$

QG-H

$$D_t^{\perp} q = \left(\nabla_{\perp}^2 + \frac{1}{A^2}\partial_{zz} \right) \left[\frac{1}{Re}\zeta + \frac{1}{Pe}\partial_z \left(\frac{\partial_z \psi}{N^2(z)} \right) \right]$$

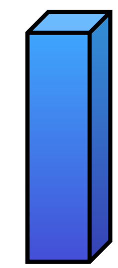
$$q = \left[\zeta + \partial_z \left(\frac{\partial_z \psi}{N^2(z)} \right) \right]$$

Conservation: Energy, Enstrophy $\quad$ dual cascade

Unified QG model capturing balanced dynamics irrespective of spatial anisotropy

Stratification $Fr = ARo \ll 1$ $Fr \sim Ro$ $Fr \sim 1$ $Ro \ll 1$

Convection



$H/L >> 1$

Julien et al. (1998)

$w \sim (ARo)u$

$A =$

QG-NH

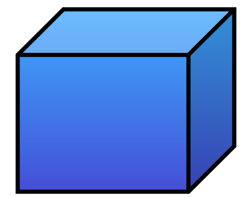
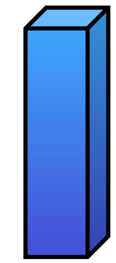
$$D_t^{\perp}\zeta - \partial_z w = \frac{1}{Re}\left(\nabla_{\perp}^2 + \frac{1}{A^2}\partial_{zz}\right)\zeta$$

$$(ARo)^2\, D_t^{\perp} w = -\,\partial_z \psi + b + \frac{(ARo)^2}{Re}\left(\nabla_{\perp}^2 + \frac{1}{A^2}\partial_{zz}\right) w$$

$$D_t^{\perp} b + wN^2(z) = \frac{1}{Pe}\left(\nabla_{\perp}^2 + \frac{1}{A^2}\partial_{zz}\right) b$$

Julien et al JFM 2006

Conservation: Energy only, dual cascade still present.

Stratification	$Fr = ARo \ll 1$	$Fr \sim Ro$	$Fr \sim 1$	$Ro \ll 1$
	QG	**Intermediate**	**Convection**	
$w \sim (ARo)u$				
$A =$	$H/L << 1$	$H/L = O(1)$	$H/L >> 1$	
	Charney (1948)	Embid & Majda (1998)	Julien et al. (1998)	

QG Summary

$$D_t^{\perp}\zeta - \partial_z w = \frac{1}{Re}\left(\nabla_{\perp}^2 + \frac{1}{A^2}\partial_{zz}\right)\zeta$$

$$(ARo)^2 D_t^{\perp} w = -\partial_z \psi + b + \frac{(ARo)^2}{Re}\left(\nabla_{\perp}^2 + \frac{1}{A^2}\partial_{zz}\right) w$$

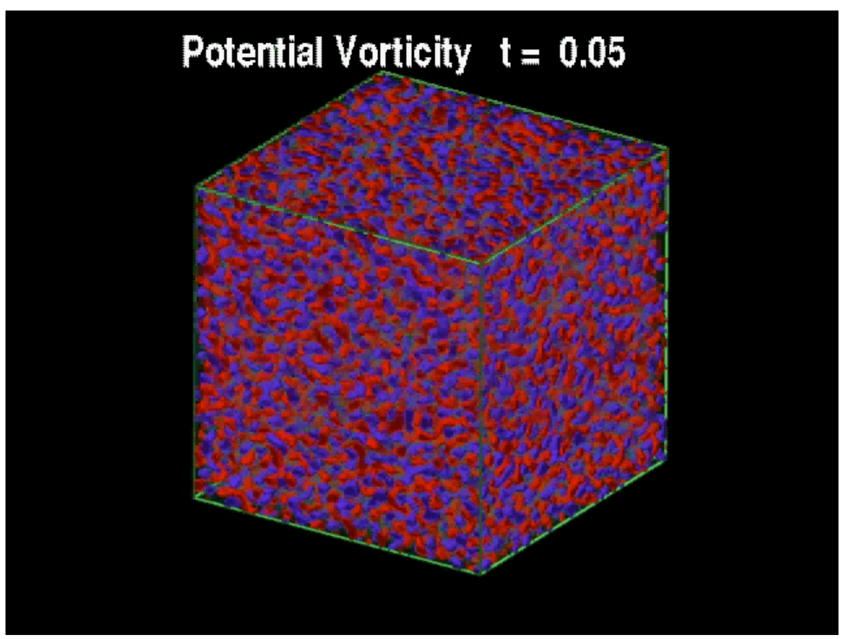
$$D_t^{\perp} b + w N^2(z) = \frac{1}{Pe}\left(\nabla_{\perp}^2 + \frac{1}{A^2}\partial_{zz}\right) b$$

Julien et al JFM 2006

Unified QG model capturing balanced dynamics irrespective of spatial anisotropy

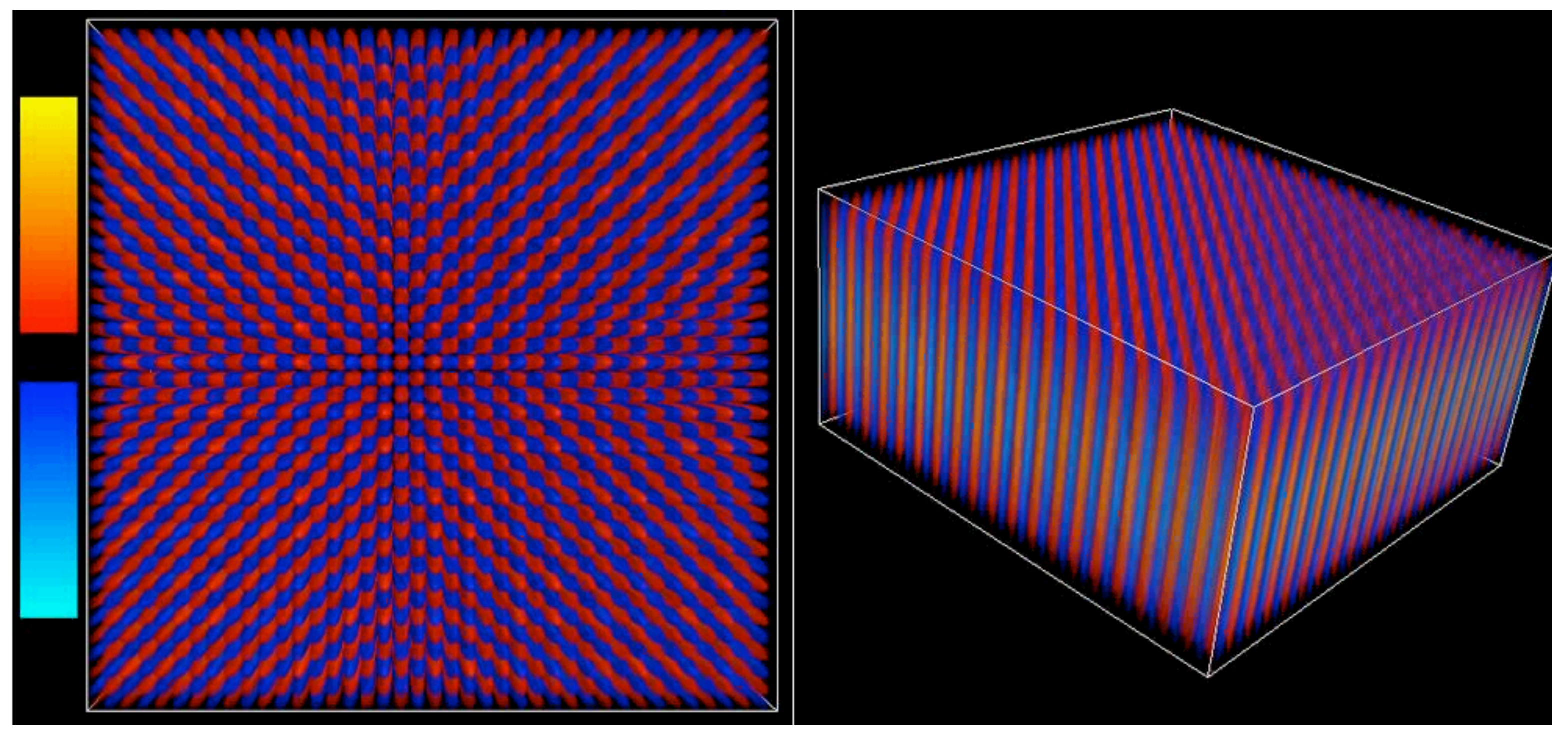
QGE modeling capabilities

H: unforced decay



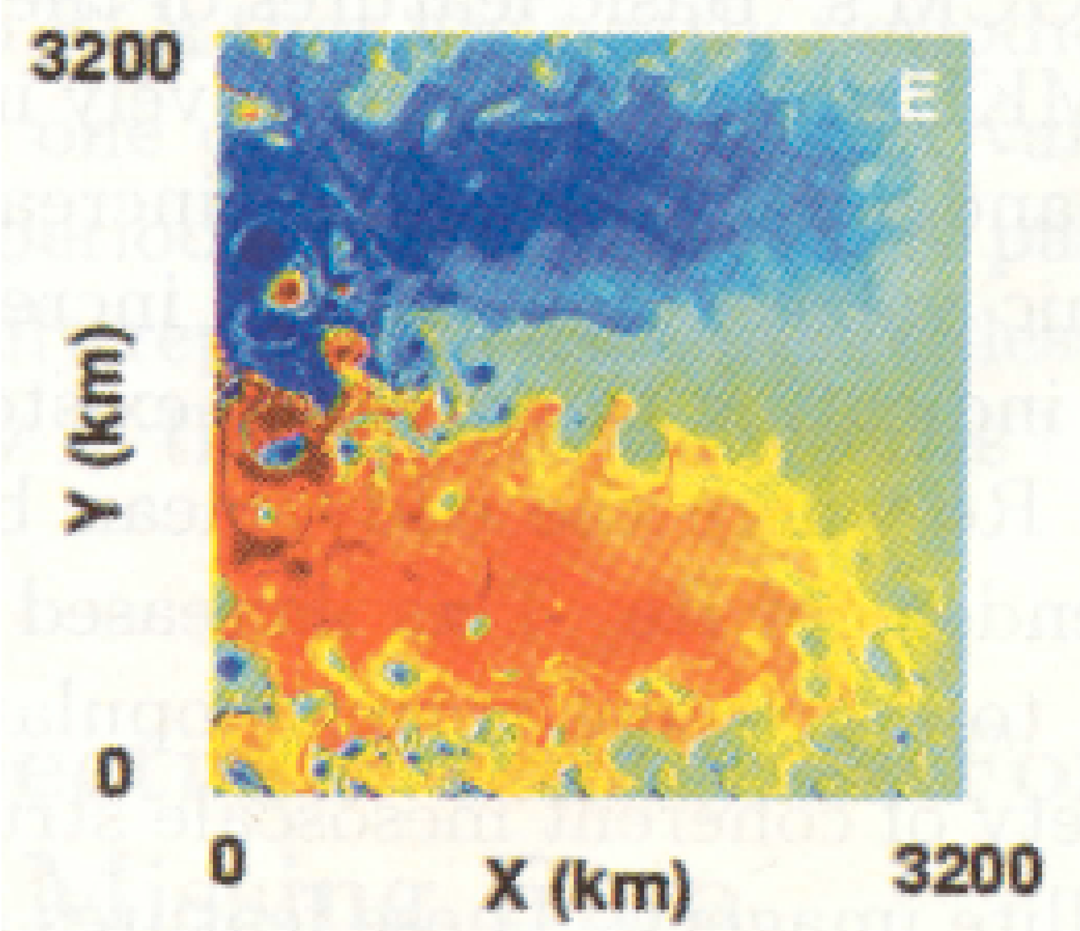
NH: Rotating Convection

$RaE^{4/3} = 40, \sigma = 7$ Sprague et al JFM '06 Groom et al PRL '10



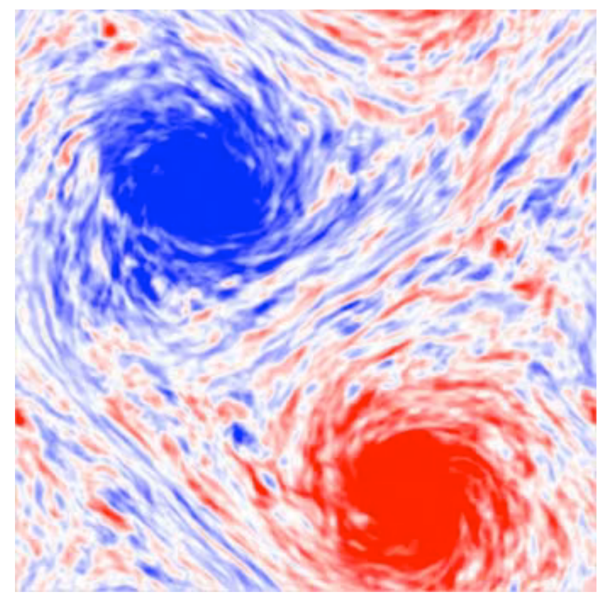
QGE modeling capabilities

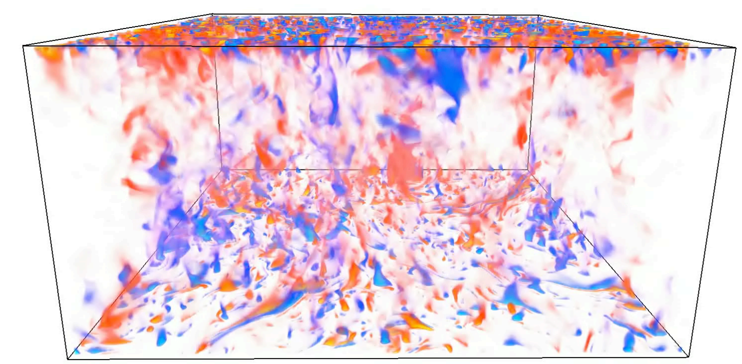
Siegel et al GRL 2001



NH: Rotating Convection

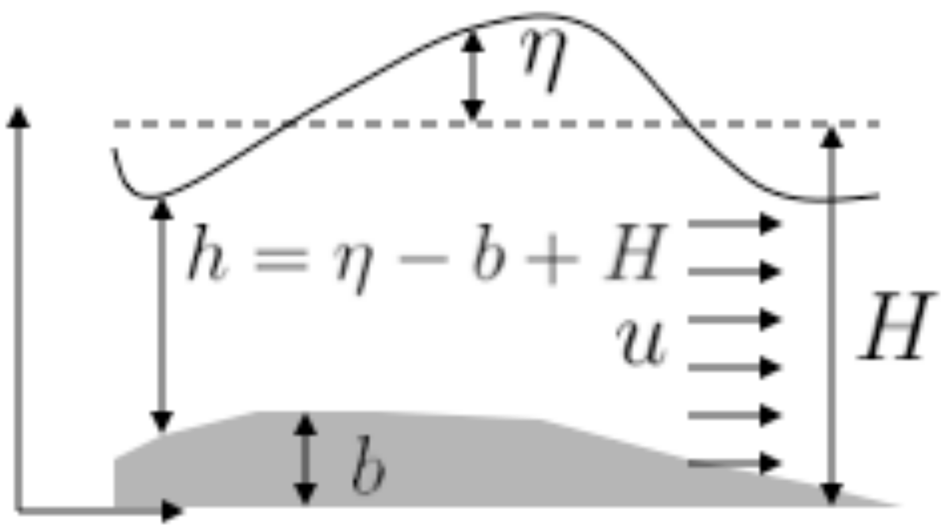
Julien et al GAFD '12 Julien et al PRL '14 $RaE^{4/3} = 100, \sigma = 1$





Shallow Water Equations

Simplest setting to explore rotating dynamics



hydrostatic balance - barotropic dynamics

$$-\partial_z p - \rho_0 g = 0$$

$$p = P_{atm} + \rho_0 g(\eta + H - z)$$

$$h = \eta - b + H$$

$$D_t^{\perp} \boldsymbol{u}_{\perp} + f(y)\widehat{\boldsymbol{z}} \times \boldsymbol{u}_{\perp} = -g\nabla_{\perp}\eta$$

$$\partial_t \eta + \nabla_{\perp} \cdot (h\boldsymbol{u}_{\perp}) = 0$$

$$f(y) = f_0 + \beta y$$
$$= 2\Omega\left(\sin\vartheta_0 + \frac{\cos\vartheta_0}{R_e} y\right)$$

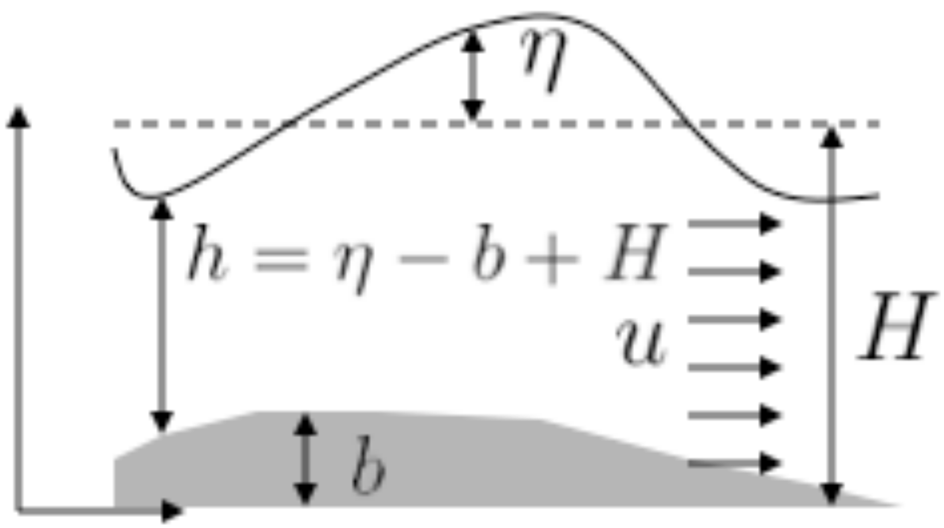
Vertically integrated continuity equation

Potential vorticity q is conserved

$$D_t q = 0, \qquad q = \left(\frac{\zeta + f}{h}\right), \quad \zeta = \widehat{\boldsymbol{z}} \cdot \nabla \times \boldsymbol{u}_{\perp}$$

Shallow Water Equations

Simplest setting to explore rotating dynamics



hydrostatic balance - barotropic dynamics

$$-\partial_z p - \rho_0 g = 0$$

$$p = P_{atm} + \rho_0 g(\eta + H - z)$$

$$h = \eta - b + H$$

$$D_t^{\perp} \boldsymbol{u}_{\perp} + \frac{1}{Ro}\hat{f}(y)\hat{\boldsymbol{z}} \times \boldsymbol{u}_{\perp} = -\frac{1}{Fr^2}\nabla_{\perp}\eta$$

$$\partial_t \eta + \nabla_{\perp} \cdot (h\boldsymbol{u}_{\perp}) = 0$$

$$\hat{f}(y) = \hat{f}_0 + Ro\beta y$$

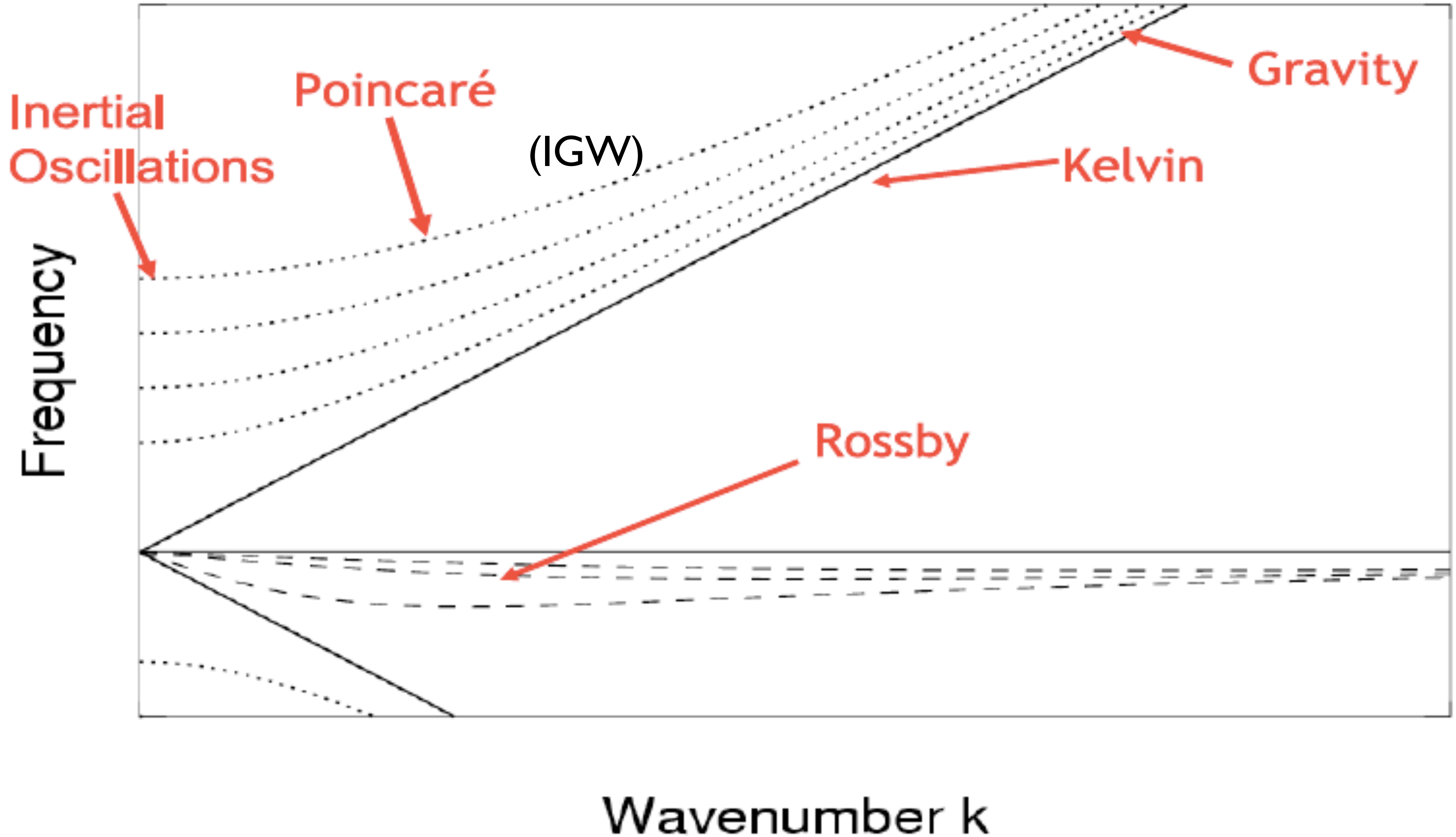
$$Ro = \frac{U}{f_0 L} \quad Fr = \frac{U}{\sqrt{gH}}$$

Vertically integrated continuity equation

Potential vorticity q is conserved

$$D_t q = 0, \qquad q = \left(\frac{\zeta + f}{h}\right), \quad \zeta = \hat{\boldsymbol{z}} \cdot \nabla \times \boldsymbol{u}_{\perp}$$

Shallow water dispersion relations



Plane wave solutions: $\eta \propto e^{i(\boldsymbol{k}_\perp \cdot \boldsymbol{x}_\perp - \omega t)}$

$\eta \propto e^{-\frac{y}{L_d}} e^{i(k_x x - \omega t)}, \ L_d = \sqrt{gH}/f_0$

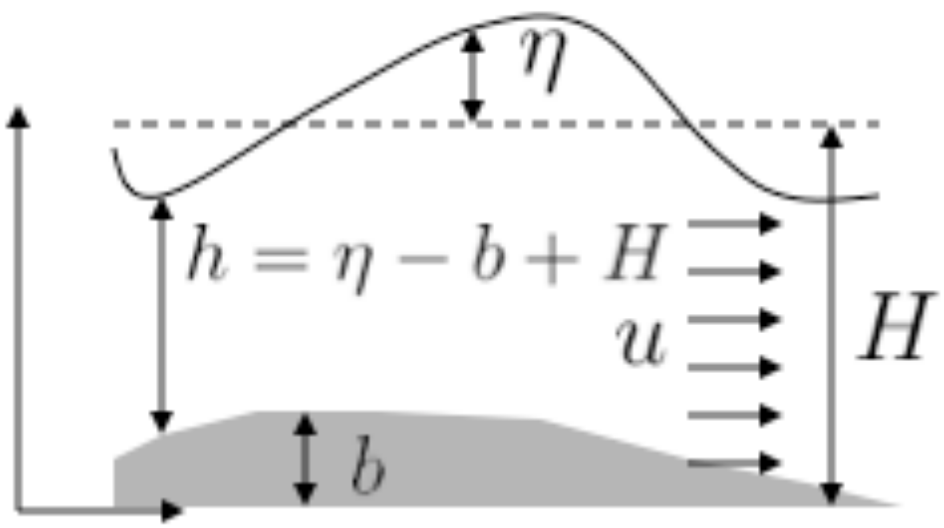
$$\omega_{IGW}^2 = f_0^2 + (gH)k_\perp^2$$

$$\omega_{RW} = -\beta k_x / k_\perp^2$$

$$\omega_{KW} = (gH)k_x$$

boundary wave

Equatorial Beta plane $\widehat{f}_0 = 0$



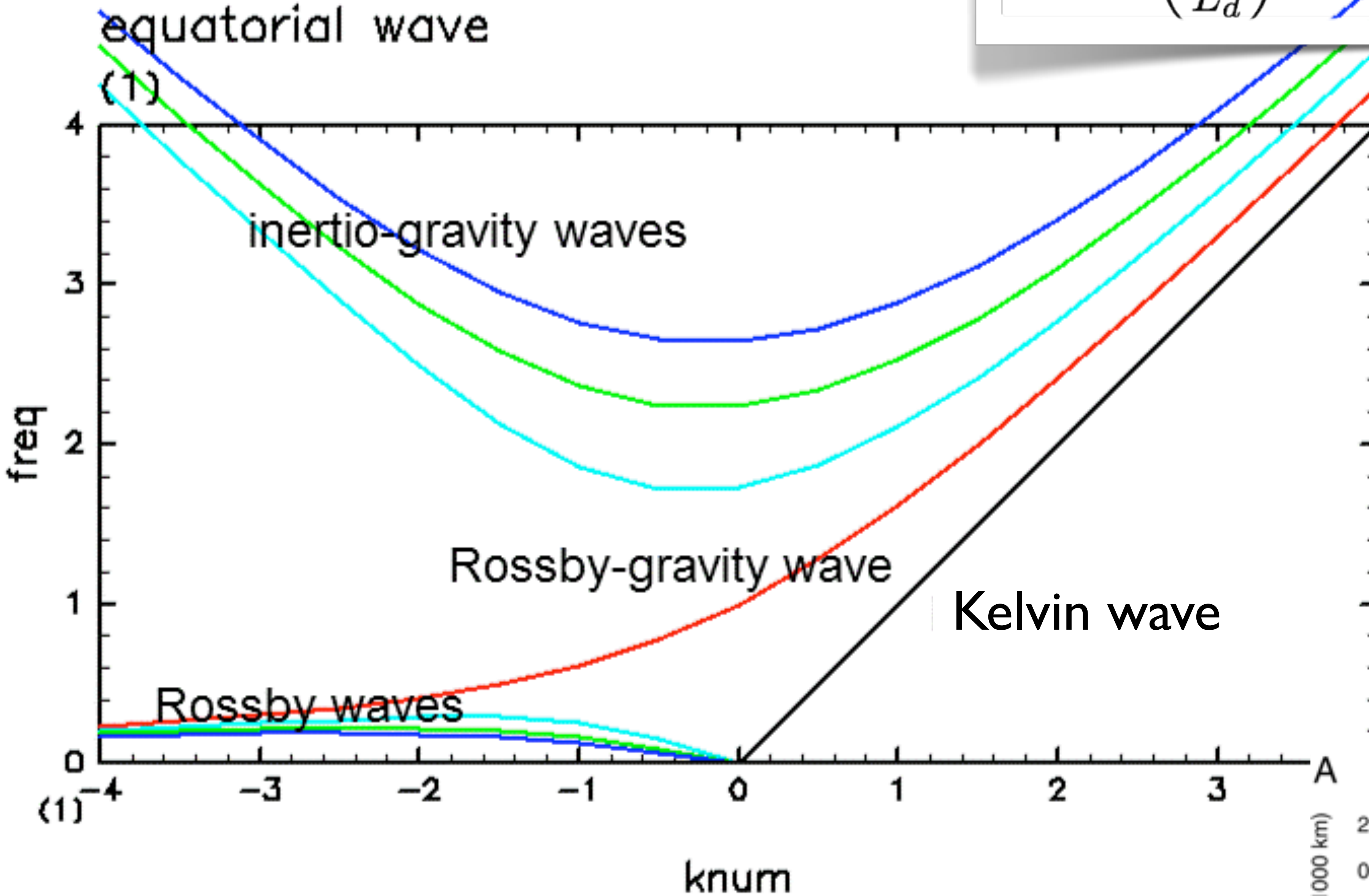
$$D_t^{\perp} \boldsymbol{u}_{\perp} + \beta y \widehat{\boldsymbol{z}} \times \boldsymbol{u}_{\perp} = -\frac{1}{Fr^2} \nabla_{\perp} \eta$$

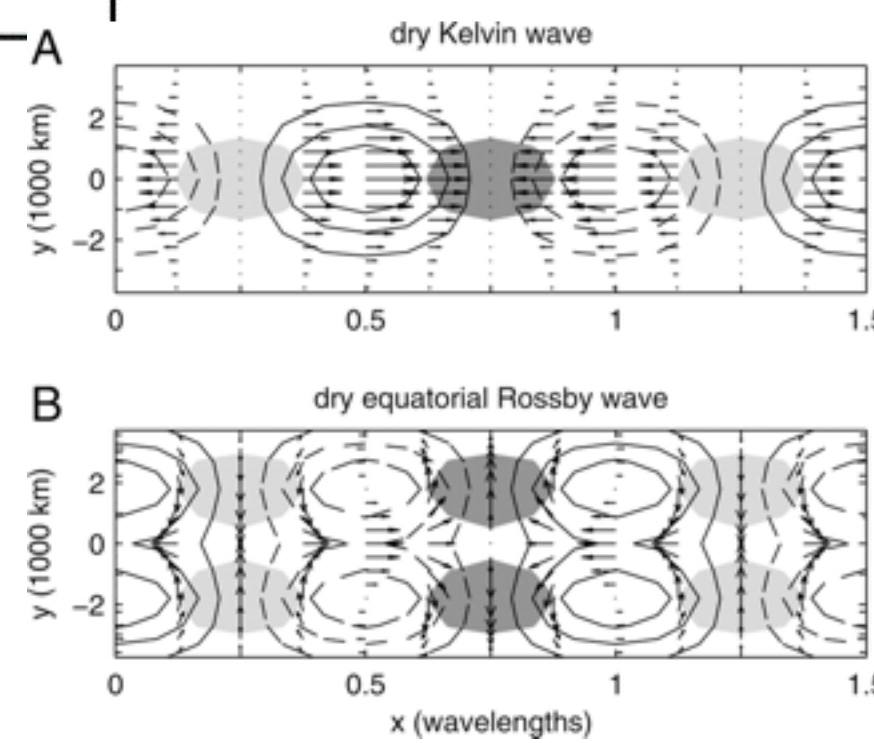
$$\partial_t \eta + \nabla_{\perp} \cdot (h \boldsymbol{u}_{\perp}) = 0$$

Equatorial wave solutions:

$$\eta \propto H_n \left(\frac{y}{L_d} \right) e^{-\frac{y^2}{2L_d}} e^{i(k_x x - \omega t)}, \quad L_d = \left(\frac{gH}{\beta^2} \right)^{1/4}$$

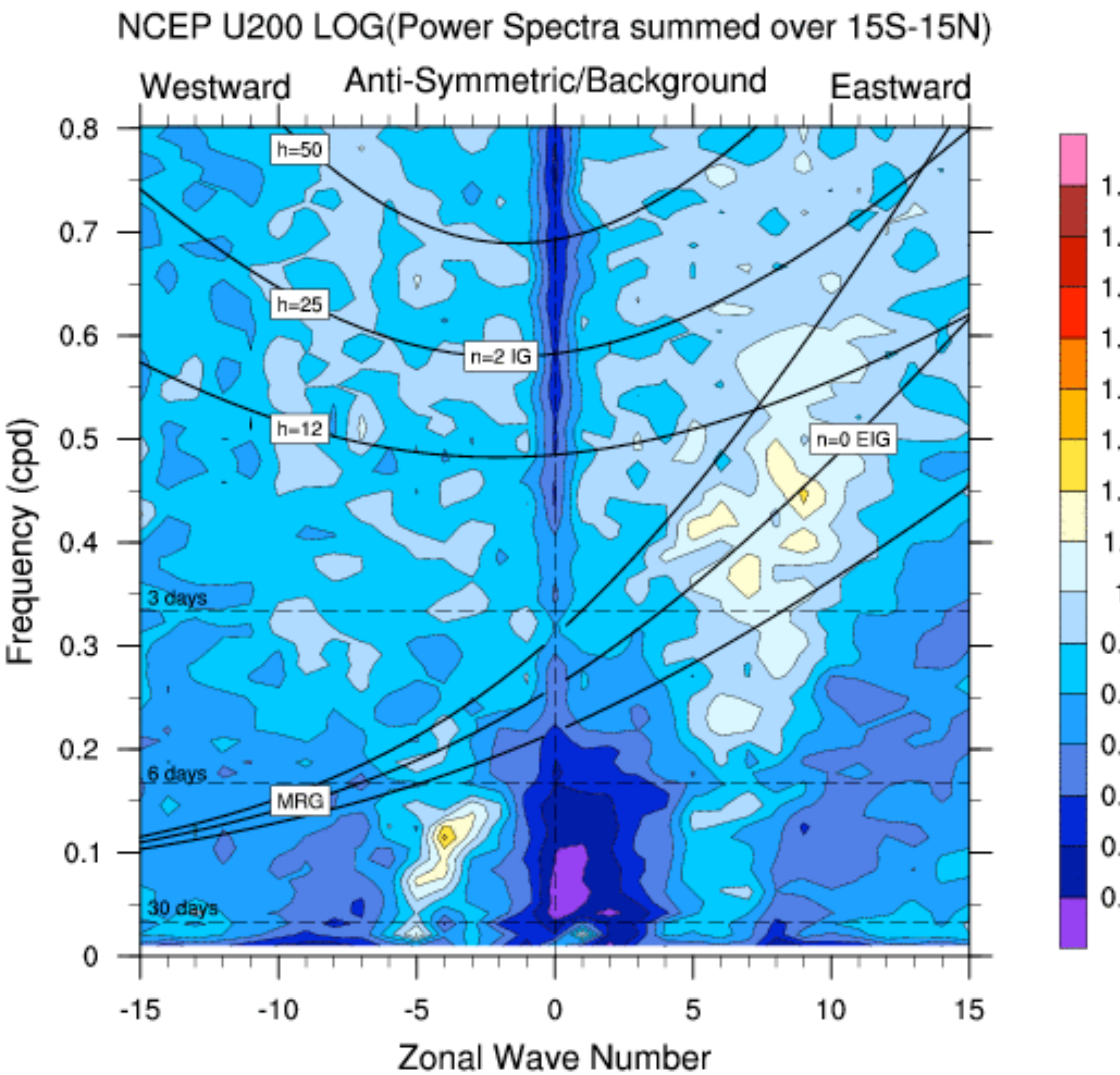
$$\eta \propto H_n\left(\frac{y}{L_d}\right) e^{-\frac{y^2}{2L_d}} e^{i(k_x x - \omega t)}, \quad L_d = \left(\frac{gH}{\beta^2}\right)^{1/4}$$

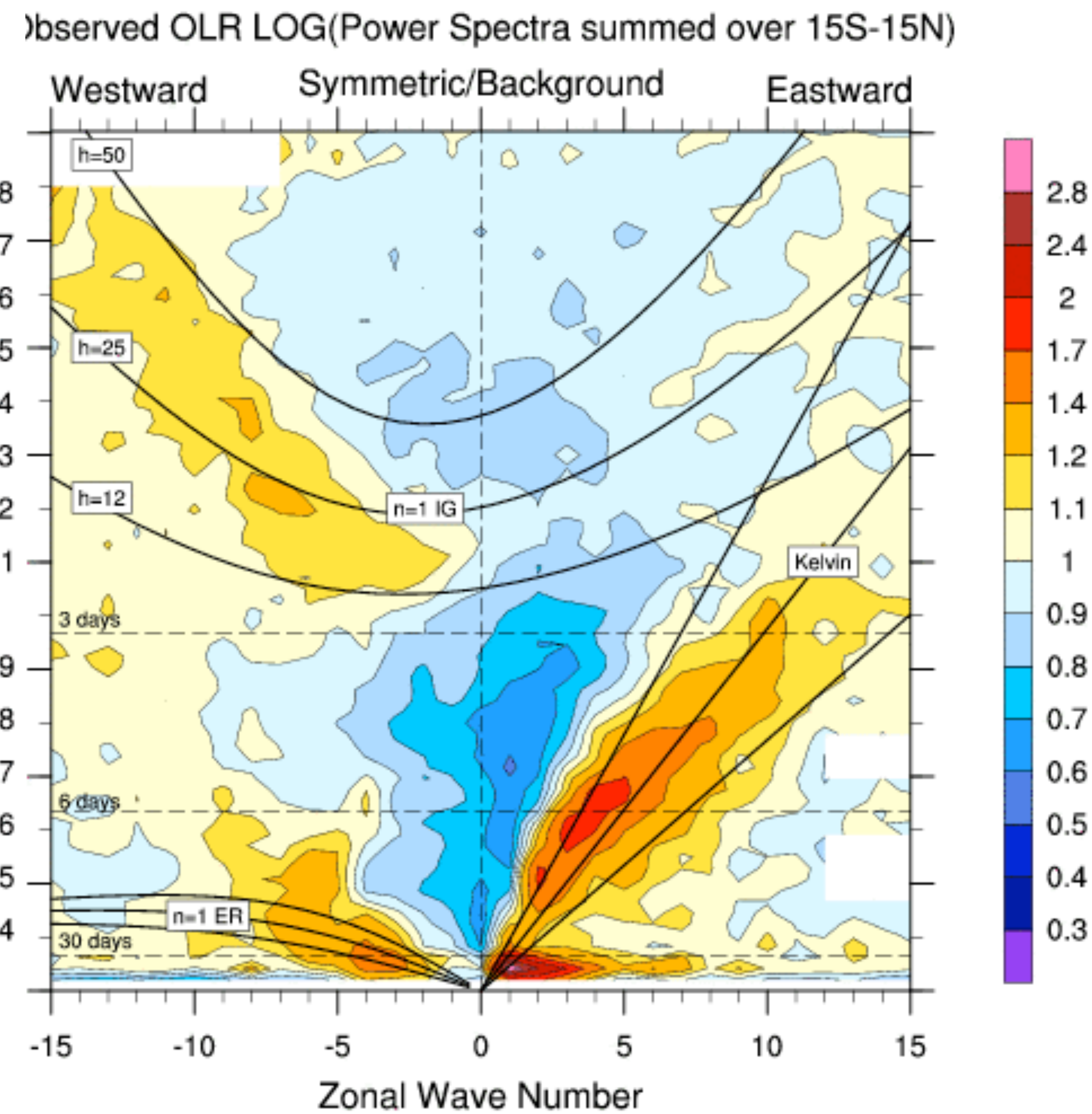




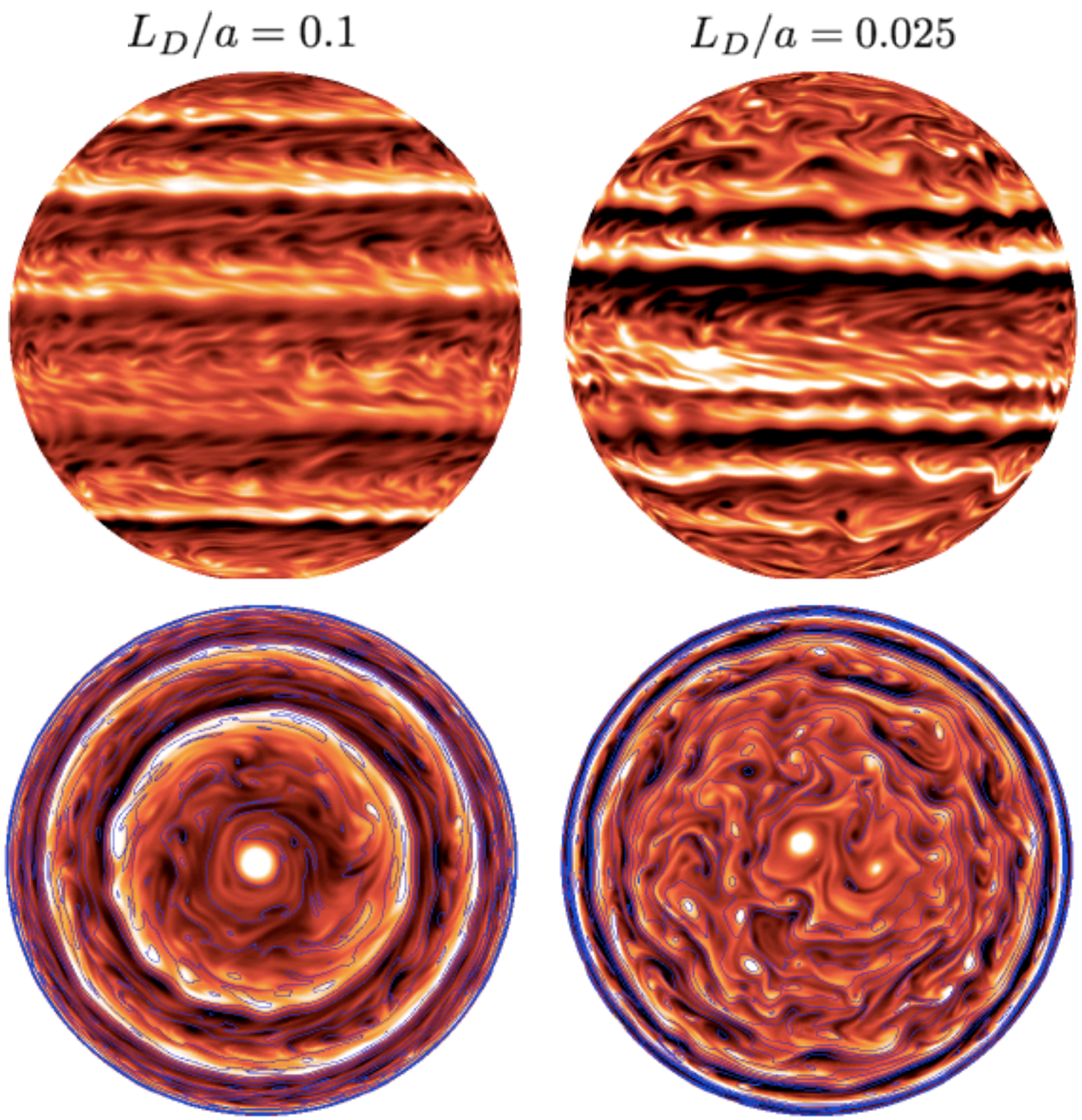
$$\omega_{KW} = \sqrt{gH}k_x$$

$$\omega_{IGW,RW}: \quad \frac{\omega^2}{gH} - k_x^2 - \frac{\beta k_x}{\omega} = \frac{2n+1}{L_d^2}$$


NCEP U200 LOG(Power Spectra summed over 15S-15N)
Westward
Anti-Symmetric/Background
Eastward
Frequency (cpd)
h=50
h=25
n=2 IG
h=12
n=0 EIG
3 days
6 days
MRG
30 days
Zonal Wave Number


bserved OLR LOG(Power Spectra summed over 15S-15N)
Westward
Symmetric/Background
Eastward
h=50
h=25
h=12
n=1 IG
Kelvin
3 days
6 days
n=1 ER
30 days
Zonal Wave Number

Forced shallow water on sphere (Scott, Polvani JAS 2007)



vorticity

$$L_D \sim \frac{\sqrt{gH}}{f}$$

$$L_{Rh} = \sqrt{U/\beta(1-\alpha)},\ \alpha = U/\beta L_D^2$$

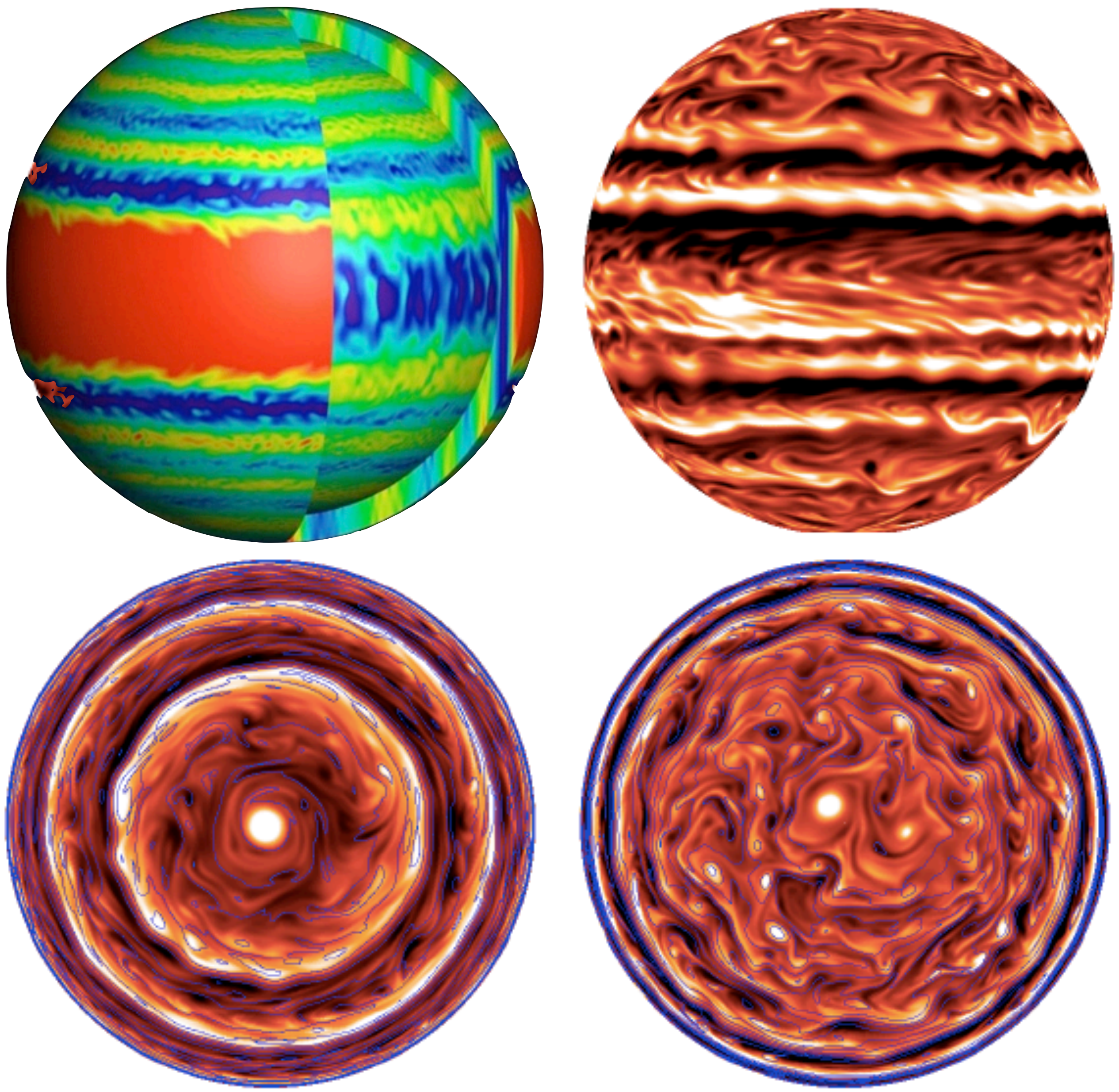
- Rhines mechanism operating on sphere
  - ~ complex int’n waves & jets
  - ~ dependent of deformation radius which increases towards equator
  - ~ strong mixing btw prograde jets

- High latitude jets - increasingly undular
  - ~ no jets for α > 1, eddies
  - ~ Jupiter, Saturn $L_D/a \sim 0.0.25$ SWE $\Rightarrow$ deeper atmosphere

- Equatorial superrotation depends on $|L_D/a|$
  - ~ deep convection vs SW theory?

Forced shallow water on sphere (Scott, Polvani JAS 2007)

$L_D/a = 0.1$ $\qquad$ $L_D/a = 0.025$



vorticity

$$L_D \sim \frac{\sqrt{gH}}{f}$$

$$L_{Rh} = \sqrt{U/\beta(1-\alpha)},\ \alpha = U/\beta L_D^2$$

- Rhines mechanism operating on sphere
  - ~ complex int'n waves & jets
  - ~ dependent of deformation radius which increases towards equator
  - ~ strong mixing btw prograde jets

- High latitude jets - increasingly undular
  - ~ no jets for $\alpha > 1$, eddies
  - ~ Jupiter, Saturn $L_D/a \sim 0.0.25$ SWE $\Rightarrow$ deeper atmosphere

- Equatorial superrotation depends on $|L_D/a|$
  - ~ deep convection vs SW theory?

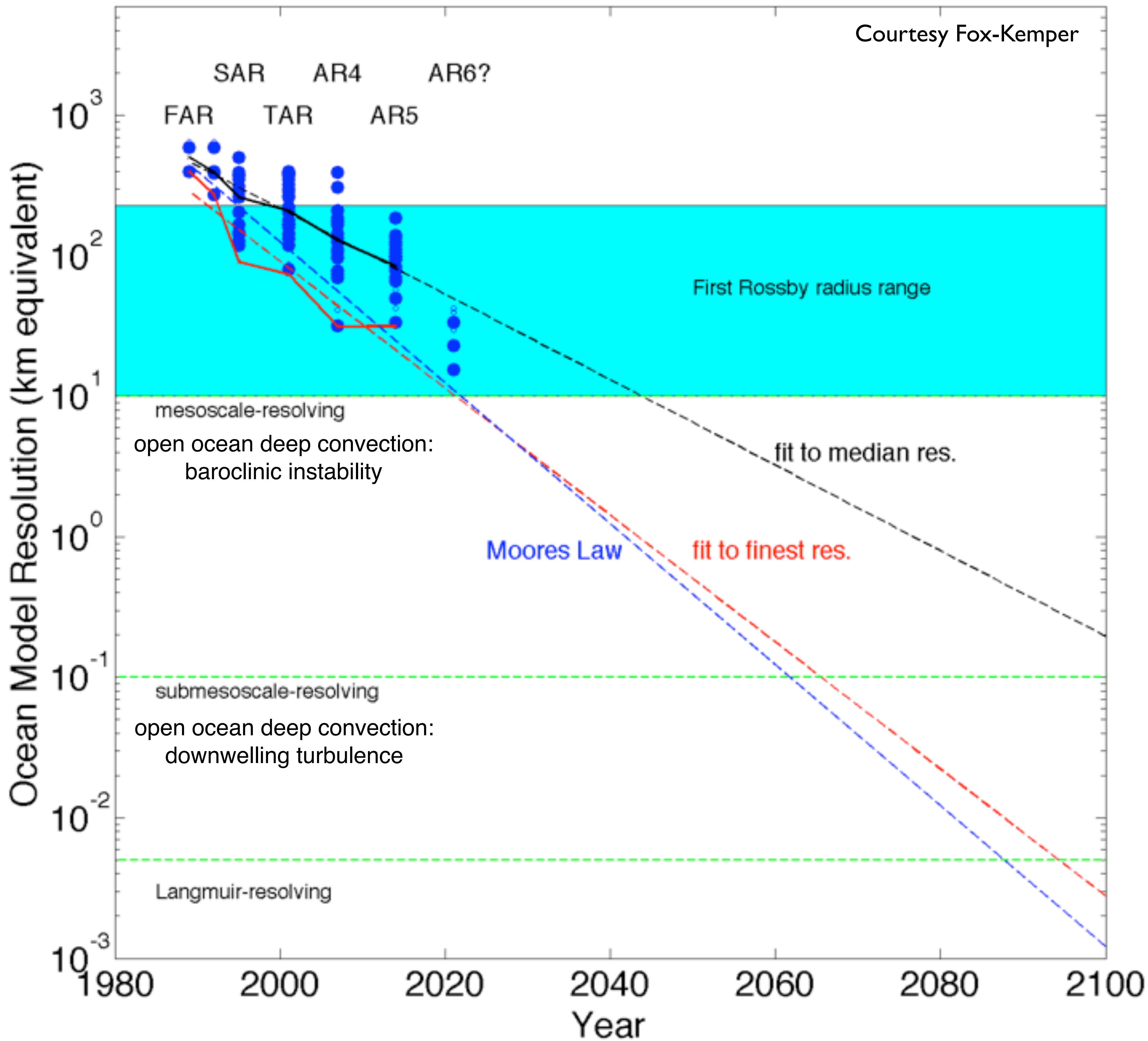
GAFD Multiscale Challenge

- Dynamics captured by NSE for GAFD systems
  - ~ Theoretically too rich without idealized investigations
  - ~ Computationally - always unresolved physics

- Reductions of NSE
  - ~ reduced/Balanced models isolate appropriate physics and have computational advantages + theoretical insights
  - ~ large scale or small scale phenomena not captured, thus absence of turbulent couplings potentially reduce the models fidelity.

- Open question: strategies of how to proceed!

Resolution of Ocean Component of Coupled IPCC models



Direct Numerical Simulations not possible for several centuries!

GAFD Multiscale Challenge

Parameterization of unresolved scales: $S_j = \overline{u}_j\overline{\phi}_j - \overline{u_j\phi} \equiv \overline{u'_j\phi'}, \quad \phi = \overline{\phi} + \phi' \ldots$

- Phenomenological approach guided by
  - ~ physical intuition
  - ~ observations
  - ~ idealized models and simulation studies

- Some examples
  - ~ Atmosphere - Cloud resolving models (Grabowski, Wu JAS 1998)
  - ~ Ocean mixing - GM (Gent-McWilliams) parameterization favoring isopynical mixing (GM JPO 1990, Bachman, Fox-Kemper Ocean Modeling 2013)

- Astrophysical and Planetary Sciences
  - ~ issue of unresolved scales typically not addressed
  - ~ limitations placed on accessing extreme parameter regimes

GAFD Multiscale Challenge

Multiscale asymptotic approach: $\partial_t = \partial_\tau + \epsilon \partial_T, \quad \nabla = \nabla_f + \epsilon \nabla_s$

- Reduction from master equations
  - ~ underlying physics carried along thru asymptotics
  - ~ coupling between scales occurs naturally
  - ~ computational challenge still significant and naturally leads about parameterizations of filtered (fine grid) scales
  - ~ modal vs stochastic approaches (Grooms & Majda, PNAS 2013)

- Some examples
  - ~ Tropics - convectively coupled waves (many classes of coupled balanced models) - (review Khouider, Majda & Stechmann Nonlinearity 2013)
  - ~ Nonhydrostatic ocean mixing - (Malecha, Chini, J JCP 2013)

GAFD Multiscale Challenge

Multiscale computational approaches - development of efficient massively llel algorithms

- Non-uniform grid, AMR
- Multiscale numerical strategies - llel in time methods (Haut & Wingate SIAM 2014)

Mathematics of Turbulence

Workshop III: Geophysical and Astrophysical Turbulence

October 27 - 31, 2014

Organizing Committee | Scientific Overview | Speaker List

Application/Registration | Contact Us

Organizing Committee

Jon Aurnou (University of California, Los Angeles (UCLA))
Oliver Bühler (New York University, Courant Institute of Mathematical Science)
Baylor Fox-Kemper (Brown University)
Pascale Garaud (University of California, Santa Cruz (UC Santa Cruz))
Keith Julien (University of Colorado, Boulder)

Back to Top

Scientific Overview

This workshop will cover a number of selected topics that are common to Oceanography, Planetary Geophysics, Atmospheric Dynamics and Astrophysics including turbulent convection, turbulence induced by baroclinic instabilities, shear turbulence (both stratified and unstratified), double-diffusive convection, and wave-induced turbulence. In all these cases geophysicists and astrophysicists strive to model the effects of the specific type of turbulence considered on heat, compositional and momentum transport. In many instances progress on common problems has been made in parallel without much interaction between the scientific communities.

This workshop will bring together scientists from different backgrounds to share their most recent results, attempting to bridge the subject-gap, and foster fruitful collaborations.

This workshop will include a poster session; a request for posters will be sent to registered participants in