On the regularity of spatial convolution kernels for linear feedback control & estimation of perturbations to nearly-parallel flows

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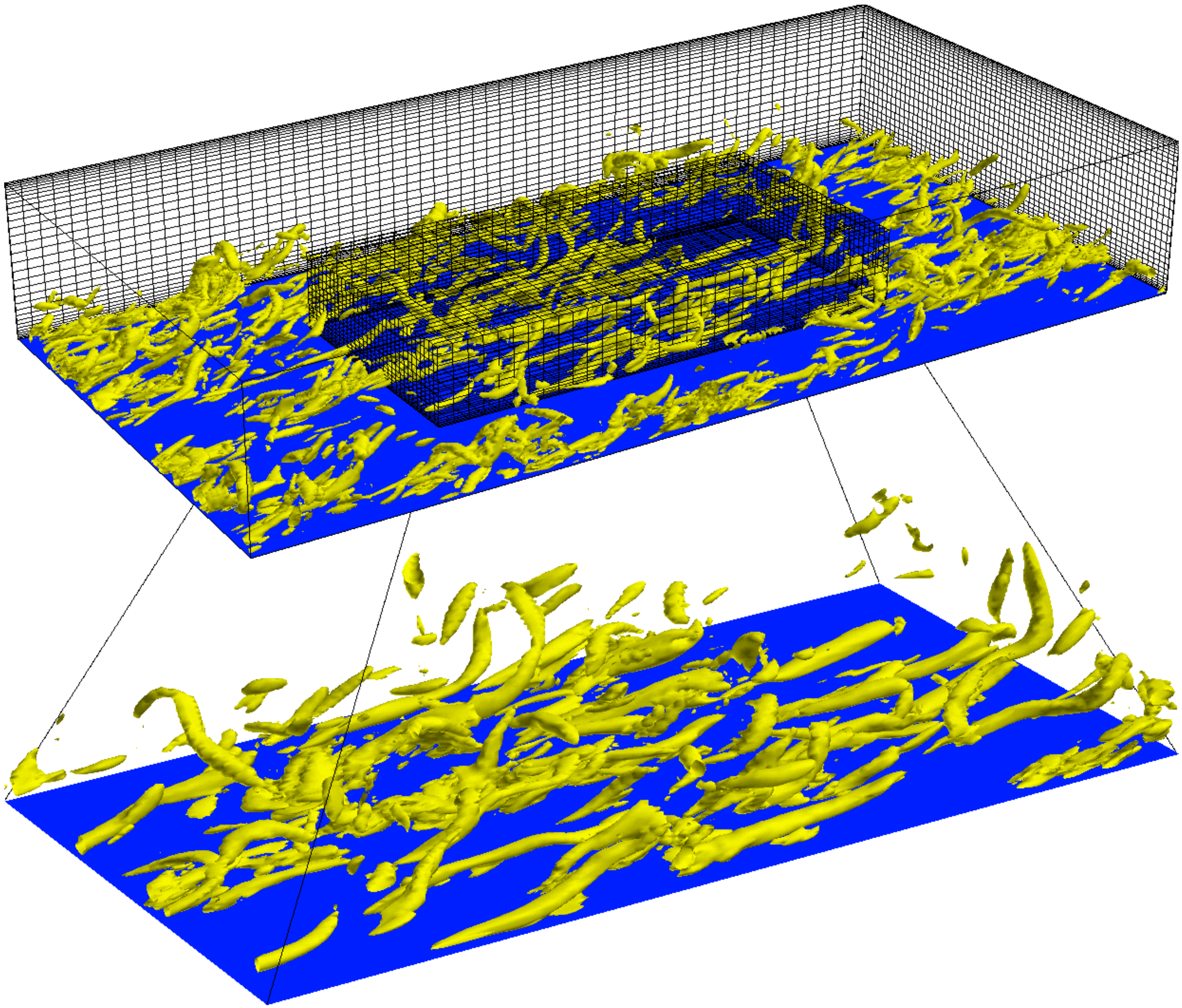
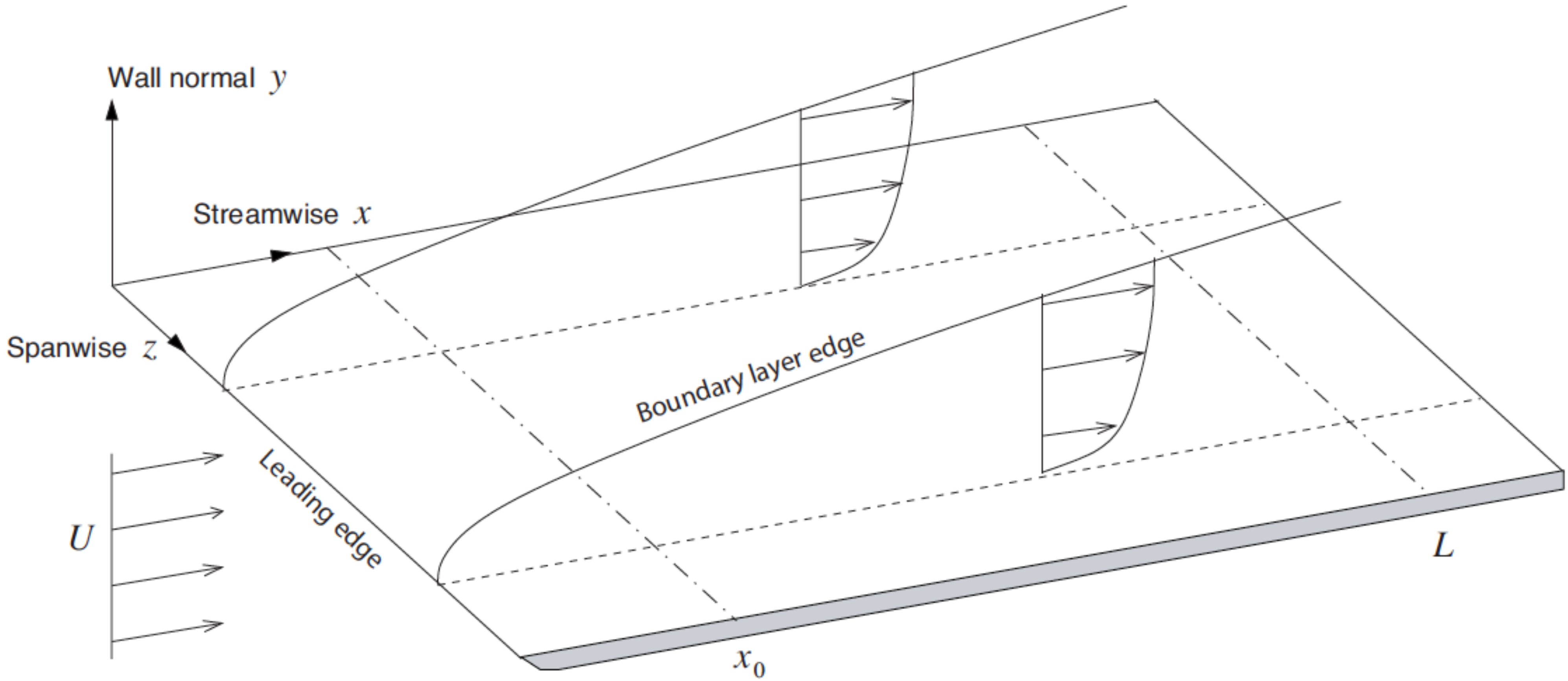


Methods developed extend immediately to boundary-layer flows



Benchmark PDE system: channel flow

State equation (Navier-Stokes): $\boxed{E\dot{\mathbf{q}} = N(\mathbf{q},\mathbf{f})}$ $\quad \mathbf{q} = \begin{pmatrix} \mathbf{u}(x,y,z,t) \\ p(x,y,z,t) \end{pmatrix}, \quad \mathbf{f} = P_x$

$$E = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad N(\mathbf{q},\mathbf{f}) = \begin{pmatrix} -(\mathbf{u}\cdot\nabla)\mathbf{u} - \nabla p + \nu\Delta\mathbf{u} + \mathbf{i}P_x \\ \nabla\cdot\mathbf{u} \end{pmatrix}$$

$\Rightarrow$ 3D system requires discretization on $O(10^6)$ to $O(10^7)$ gridpoints.

Linearization (Orr-Sommerfeld/Squire): $\boxed{\hat{E}\dot{\hat{\mathbf{q}}}' = \hat{A}\hat{\mathbf{q}}'}$, $\quad \hat{\mathbf{q}}' = \begin{pmatrix} \hat{v}'_{\{k_x,k_z\}}(y,t) \\ \hat{\omega}'_{\{k_x,k_z\}}(y,t) \end{pmatrix}$

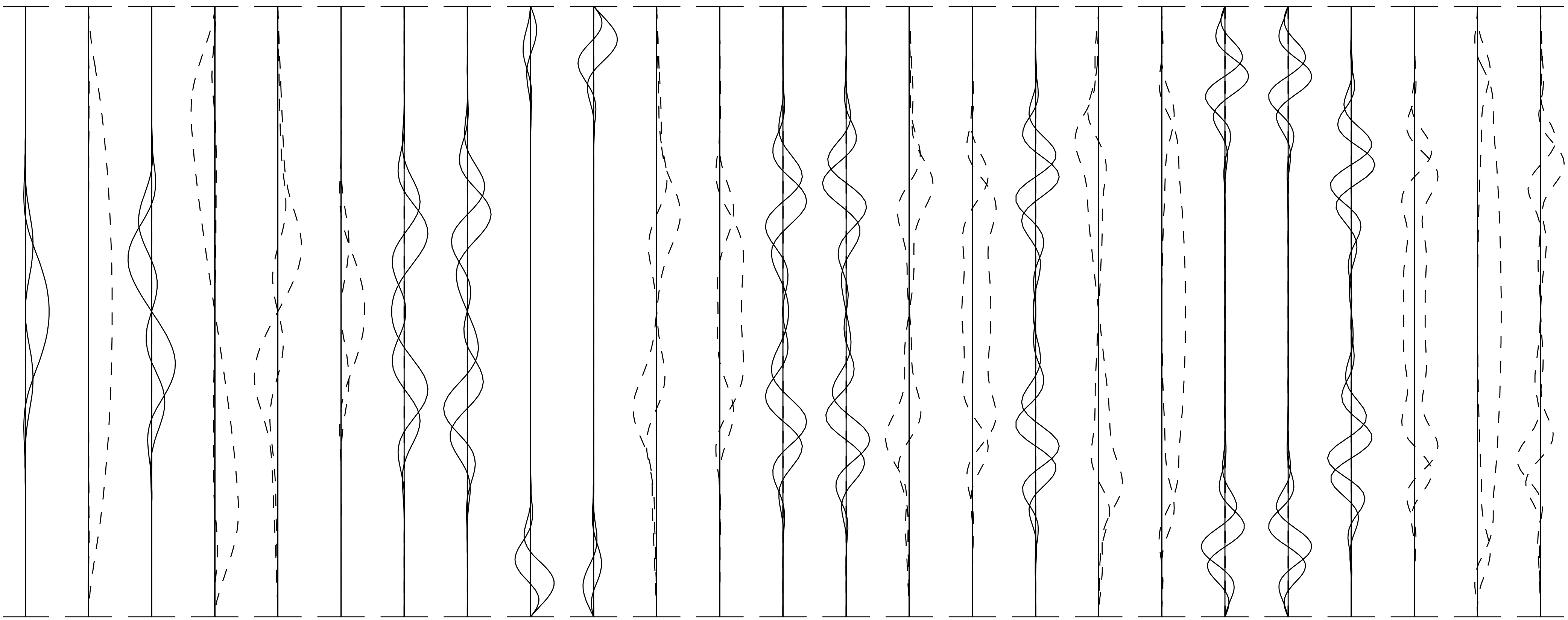
$$\hat{E} = \begin{pmatrix} \hat{\Delta} & 0 \\ 0 & I \end{pmatrix}, \quad \hat{A} = \begin{pmatrix} \hat{\Delta}(\hat{\Delta}/Re) - ik_xU\hat{\Delta} + ik_xU'' & 0 \\ -ik_zU' & \hat{\Delta}/Re - ik_xU \end{pmatrix}, \quad \hat{E}^{-1}\hat{A} = \begin{pmatrix} L & 0 \\ C & S \end{pmatrix}$$

$\Rightarrow$ Linearization about mean flow U decouples each $\{k_x,k_z\}$ mode.

- **Boundary control**: $\phi(x,z,t)$ (blowing/suction $\Rightarrow$ $\mathbf{u} = -\phi\mathbf{n}$ on walls).
- **Distributed disturbance forcing**: $\psi(x,y,z,t)$ added to RHS of PDE.
- **Measurements**: $\mathbf{y}(x,z,t)$ (skin friction and pressure on walls).

First 25 evecs of Orr-Sommerfeld/Squire at {kx,kz} = {1,0}, Re_B = 1429

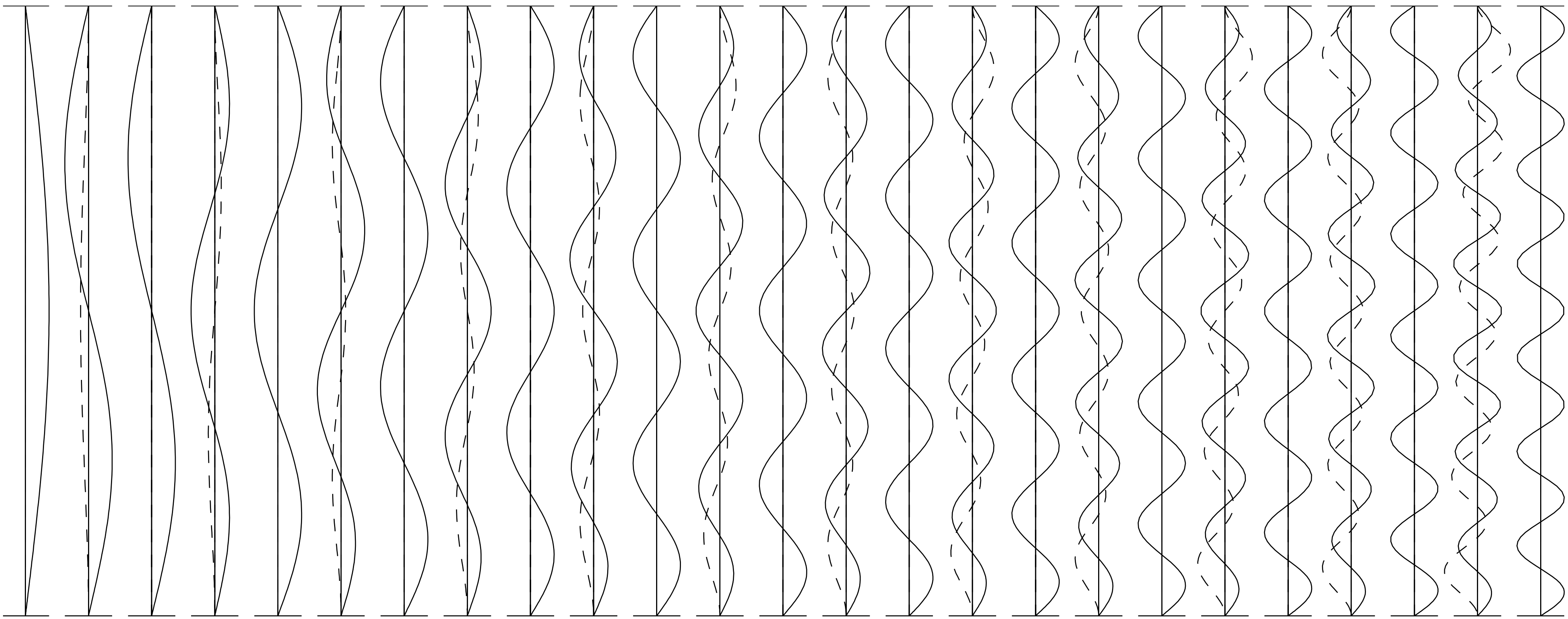
[B, Progress in Aerospace Sciences, 2001]



Real and imaginary parts of the ω component of the least-stable eigenvectors (solid), and real and imaginary parts of the corresponding v components (dashed)

First 25 evecs of Orr-Sommerfeld/Squire at {kx,kz} = {0,2}, Re_B = 1429

[B, Progress in Aerospace Sciences, 2001]



Real part of the ω component of the least-stable eigenvectors (solid), and 200 times the imaginary part of the corresponding v components (dashed).

Definition of 2-norm of transfer function

$$\begin{aligned} \mathbf{x}'(t) &= A\mathbf{x}(t) + B\mathbf{w}(t) \\ \mathbf{z}(t) &= C\mathbf{x}(t) + D\mathbf{w}(t) \end{aligned} \quad \Rightarrow \quad \begin{aligned} \mathbf{Z}(s) &= T(s)\mathbf{W}(s), \\ T(s) &= C(sI - A)^{-1}B + D. \end{aligned}$$

$$\begin{aligned} \|T(s)\|_2^2 &\triangleq \frac{1}{2\pi}\int_{-\infty}^{\infty} \|T(\mathrm{i}\omega)\|_F^2 \, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{trace}\left[T^H(\mathrm{i}\omega)T(\mathrm{i}\omega)\right] d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \sum_i \sigma_i^2\left[T(\mathrm{i}\omega)\right] d\omega, \end{aligned}$$

• The square of the transfer function 2-norm is the **total energy** of the output $\mathbf{z}(t)$ of the system when the input $\mathbf{w}(t)$ contains a sequence of unit impulses in each component.

• The square of the transfer function 2-norm is also the **expected mean energy** of the output, $\mathcal{E}\{\mathbf{z}^H(t)\mathbf{z}(t)\}$, when the system is excited with a zero mean white random process $\mathbf{w}(t)$ with unit spectral density.

Definition of infinity-norm of transfer function

$$\begin{aligned} \mathbf{x}'(t) &= A\mathbf{x}(t) + B\mathbf{w}(t) \\ \mathbf{z}(t) &= C\mathbf{x}(t) + D\mathbf{w}(t) \end{aligned} \quad \Rightarrow \quad \begin{aligned} \mathbf{Z}(s) &= T(s)\mathbf{W}(s), \\ T(s) &= C(sI - A)^{-1}B + D. \end{aligned}$$

$$\|T(s)\|_\infty \triangleq \sup_{0 \le \omega < \infty} \|T(\mathrm{i}\omega)\|_{i2} = \sup_{0 \le \omega < \infty} \sigma_{\max}\left[T(\mathrm{i}\omega)\right],$$

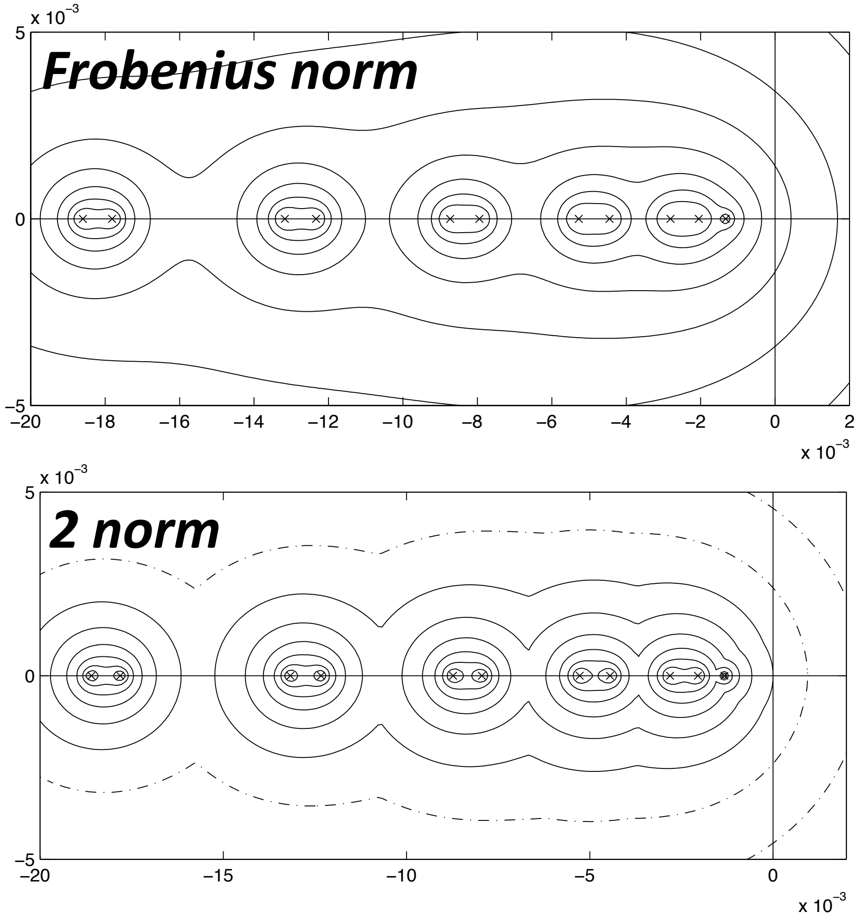
• In the *frequency* domain, the transfer function ∞-norm is the maximum over all frequencies of the gain of the corresponding Bode plot.

• In the *time* domain, the infinity norm quantifies the response of the system to the "most disturbing" input $\mathbf{w}$, that is,

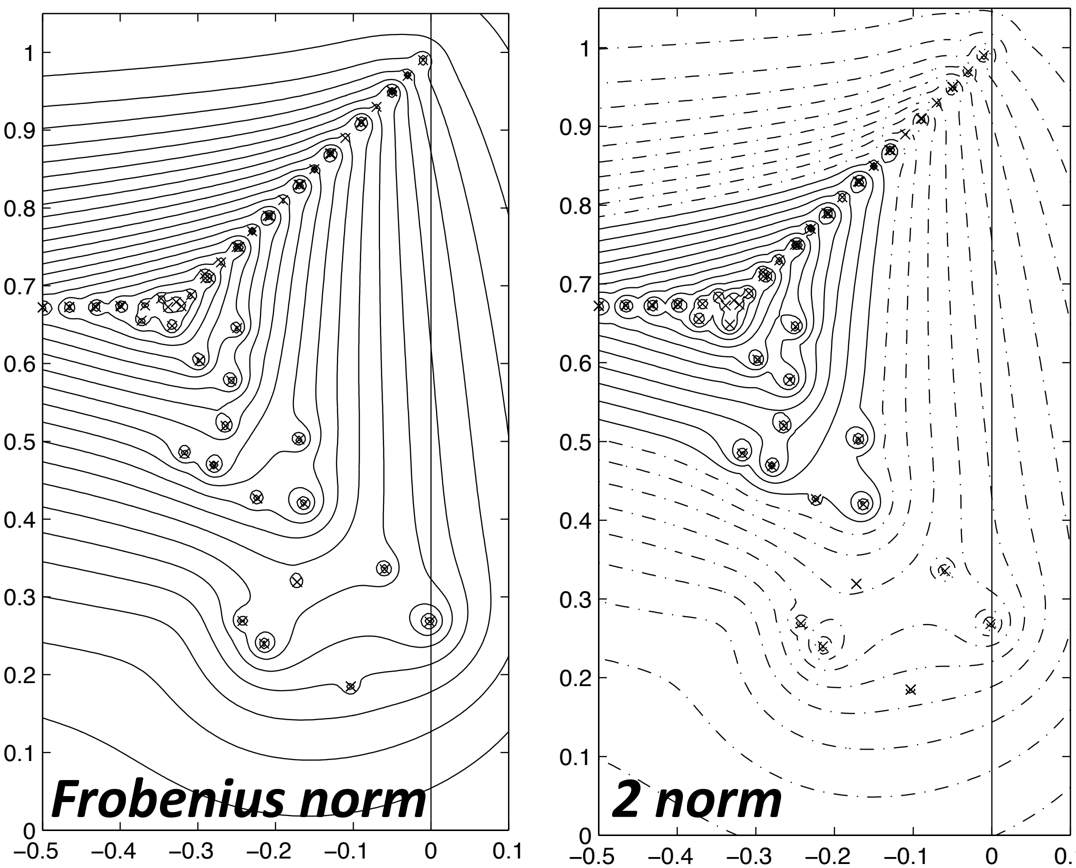
$$\|T(s)\|_\infty = \max_{\mathbf{w}(t) \neq 0} \frac{\|\mathbf{z}(t)\|_2}{\|\mathbf{w}(t)\|_2} = \max_{\|\mathbf{w}(t)\|_2 = 1} \|\mathbf{z}(t)\|_2 .$$

Isosurfaces of transfer fn norms of Re_B = 1429 Orr-Sommerfeld/Squire

[B, Progress in Aerospace Sciences, 2001]



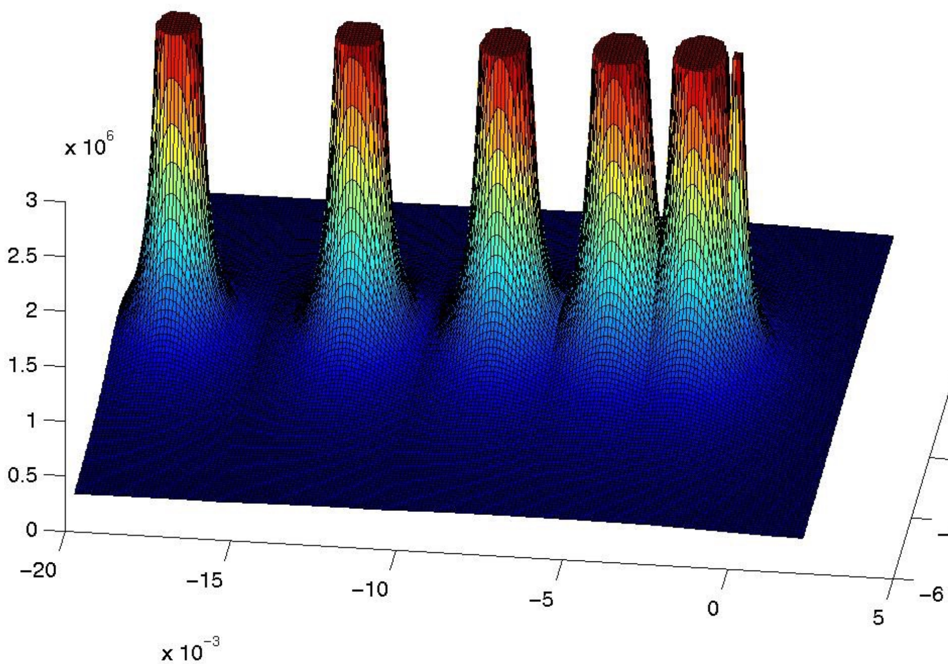
{kx,kz} = {0,2}



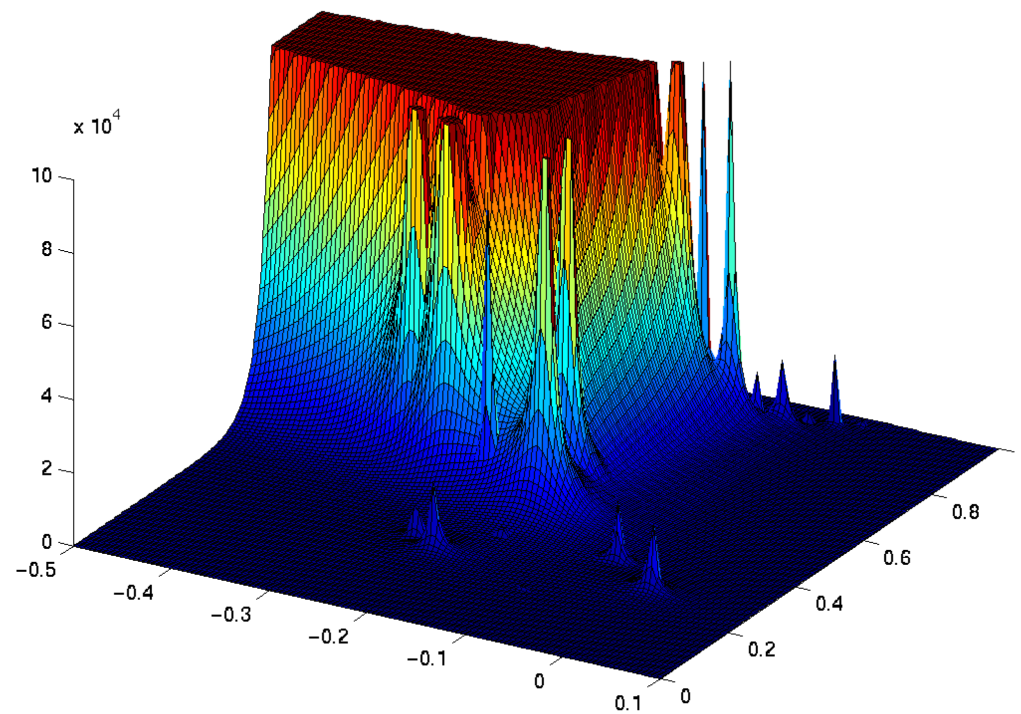
{kx,kz} = {1,0}

Isosurfaces of transfer fn 2 norms

[B, Progress in Aerospace Sciences, 2001]



{kx,kz} = {0,2}



{kx,kz} = {1,0}

A brief introduction to control theory

The transfer fn 2-norm and infinity norm measure physically relevant quantities.
H_2 control minimizes the transfer function 2-norm.
H_inf control minimizes the transfer function infinity norm.

Model predictive control (MPC) minimizes a relevant cost function via iterative state and adjoint analysis and gradient-based optimization.

In the following 4 pages, we briefly introduce MPC, H_2, and H_inf state-feedback control theory.

An introduction to estimation theory was given in my previous talk at IPAM (recording available at IPAM website).

Adjoint analysis for gradient-based optimization

State equation:

$$\boxed{\begin{array}{ll} E\dot{\mathbf{q}} = N(\mathbf{q},\mathbf{f},\phi,\psi) & \text{on} \quad 0<t<T \\ \mathbf{q} = \mathbf{q}_0 & \text{at} \quad t=0 \end{array}}$$

with: $\mathbf{q} =$ state, $\mathbf{f} =$ external force, $\phi =$ control, $\psi =$ disturbance.

Perturbation equation:

$$\left\{ \begin{array}{ll} \mathcal{L}\mathbf{q}' = B_\phi \phi' + B_\psi \psi' & \text{on} \quad 0<t<T \\ \mathbf{q}' = 0 & \text{at} \quad t=0 \end{array} \right\} \quad \Rightarrow$$

Small perturbations ϕ' to control ϕ & small perturbations ψ' to disturbance ψ cause small perturbation $\mathbf{q}'$ to state $\mathbf{q}$.

$\mathcal{L}\mathbf{q}' \triangleq \left(E\frac{d}{dt} - A\right)\mathbf{q}'$ is the linearization of the state eqn about the trajectory $\mathbf{q}(\phi,\psi)$.

Cost function (minimize w.r.t. ϕ and maximize w.r.t. ψ):

$$\boxed{\mathcal{J} = \frac{1}{2}\int_0^T (\mathbf{q}^* Q \mathbf{q} + \ell^2 \phi^* \phi - \gamma^2 \psi^* \psi)\, dt} \quad \Rightarrow \quad \mathcal{J}' = \int_0^T (\mathbf{q}^* Q \mathbf{q}' + \ell^2 \phi^* \phi' - \gamma^2 \psi^* \psi')\, dt.$$

Statement of adjoint identity. Define inner product $\langle \mathbf{r}, \mathbf{q}' \rangle = \int_0^T \mathbf{r}^* \mathbf{q}' \, dt$. Then:

$$\boxed{\langle \mathbf{r}, \mathcal{L}\mathbf{q}' \rangle = \langle \mathcal{L}^*\mathbf{r}, \mathbf{q}' \rangle + \mathbf{b}}$$

with: $\mathbf{r} =$ adjoint, $\mathcal{L}^*\mathbf{r} = \left(-E^*\frac{d}{dt} - A^*\right)\mathbf{r}$, $\mathbf{b} = \mathbf{r}^* E \mathbf{q}'\Big|_{t=T} - \mathbf{r}^* E \mathbf{q}'\Big|_{t=0}$.

Definition of adjoint equation. Adjoint field easy to compute, though $A = A(\mathbf{q})$.

$$\left\{ \begin{array}{ll} \mathcal{L}^*\mathbf{r} = Q\mathbf{q} & \text{on} \quad 0 < t < T \\ \mathbf{r} = 0 & \text{at} \quad t = T \end{array} \right\} \Leftrightarrow \boxed{\begin{array}{ll} -E^*\dot{\mathbf{r}} = A^*\mathbf{r} + Q\mathbf{q} & \text{on} \quad 0 < t < T \\ \mathbf{r} = 0 & \text{at} \quad t = T \end{array}}$$

Extraction of gradients. Combining equations, we have:

$$\langle \mathbf{r}, B_\phi \phi' + B_\psi \psi' \rangle = \langle Q\mathbf{q}, \mathbf{q}' \rangle \quad \Rightarrow \quad \int_0^T \mathbf{q}^* Q \mathbf{q}' \, dt = \int_0^T \mathbf{r}^* (B_\phi \phi' + B_\psi \psi') \, dt.$$

$$\mathcal{J}' = \int_0^T \left[\left(B_\phi^* \mathbf{r} + \ell^2 \phi \right)^* \phi' + \left(B_\psi^* \mathbf{r} - \gamma^2 \psi \right)^* \psi' \right] dt \triangleq \int_0^T \left[\left(\frac{\mathscr{D}\mathcal{J}}{\mathscr{D}\phi} \right)^* \phi' + \left(\frac{\mathscr{D}\mathcal{J}}{\mathscr{D}\psi} \right)^* \psi' \right] dt$$

As ϕ' and ψ' are arbitrary, **the gradient is:** $\dfrac{\mathscr{D}\mathcal{J}}{\mathscr{D}\phi} = B_\phi^* \mathbf{r} + \ell^2 \phi$, $\dfrac{\mathscr{D}\mathcal{J}}{\mathscr{D}\psi} = B_\psi^* \mathbf{r} - \gamma^2 \psi$.

Riccati analysis for coordinated feedback control

Characterization of saddle point. The control ϕ which minimizes $\mathscr{J}$ and the disturbance ψ which maximizes $\mathscr{J}$ are given by

$$\frac{\mathscr{D}\mathscr{J}}{\mathscr{D}\phi} = 0, \quad \frac{\mathscr{D}\mathscr{J}}{\mathscr{D}\psi} = 0 \quad \Rightarrow \quad \phi = -\frac{1}{\ell^2}B_\phi^*\mathbf{r}, \quad \psi = \frac{1}{\gamma^2}B_\psi^*\mathbf{r}.$$

Combined matrix form. Combining the perturbation and adjoint eqns at the saddle point determined above, assuming $E = I$, gives:

$$\begin{matrix}\text{Perturbation equation} \rightarrow \\ \text{Adjoint equation} \rightarrow\end{matrix} \quad \begin{bmatrix} \dot{\mathbf{q}}' \\ \mathbf{r} \end{bmatrix} = \begin{bmatrix} A & \overbrace{-\frac{1}{\ell^2}B_\phi B_\phi^* + \frac{1}{\gamma^2}B_\psi B_\psi^*}^{\text{control and disturbance at saddle point}} \\ -Q & -A^* \end{bmatrix} \begin{bmatrix} \mathbf{q}' \\ \mathbf{r} \end{bmatrix}$$

Solution Ansatz. Relate perturbation $\mathbf{q}' = \mathbf{q}'(t)$ and adjoint $\mathbf{r} = \mathbf{r}(t)$:

$$\boxed{\mathbf{r} = X\mathbf{q}'}, \quad \text{where } X = X(t).$$

Riccati equation. Inserting solution ansatz into the combined matrix form to eliminate $\mathbf{r}$ and combining rows to eliminate $\dot{\mathbf{q}}'$ gives:

$$\left[-\dot{X} = A^*X + XA + X\left(\tfrac{1}{\gamma^2}B_\psi B_\psi^* - \tfrac{1}{\ell^2}B_\phi B_\phi^*\right)X + Q\right]\mathbf{q}'.$$

As this equation is valid for all $\mathbf{q}'$, it follows that:

$$\boxed{-\dot{X} = A^*X + XA + X\left(\tfrac{1}{\gamma^2}B_\psi B_\psi^* - \tfrac{1}{\ell^2}B_\phi B_\phi^*\right)X + Q}.$$

Due to the terminal conditions on $\mathbf{r}$, we must have $\boxed{X = 0 \quad \text{at} \quad t = T}$.

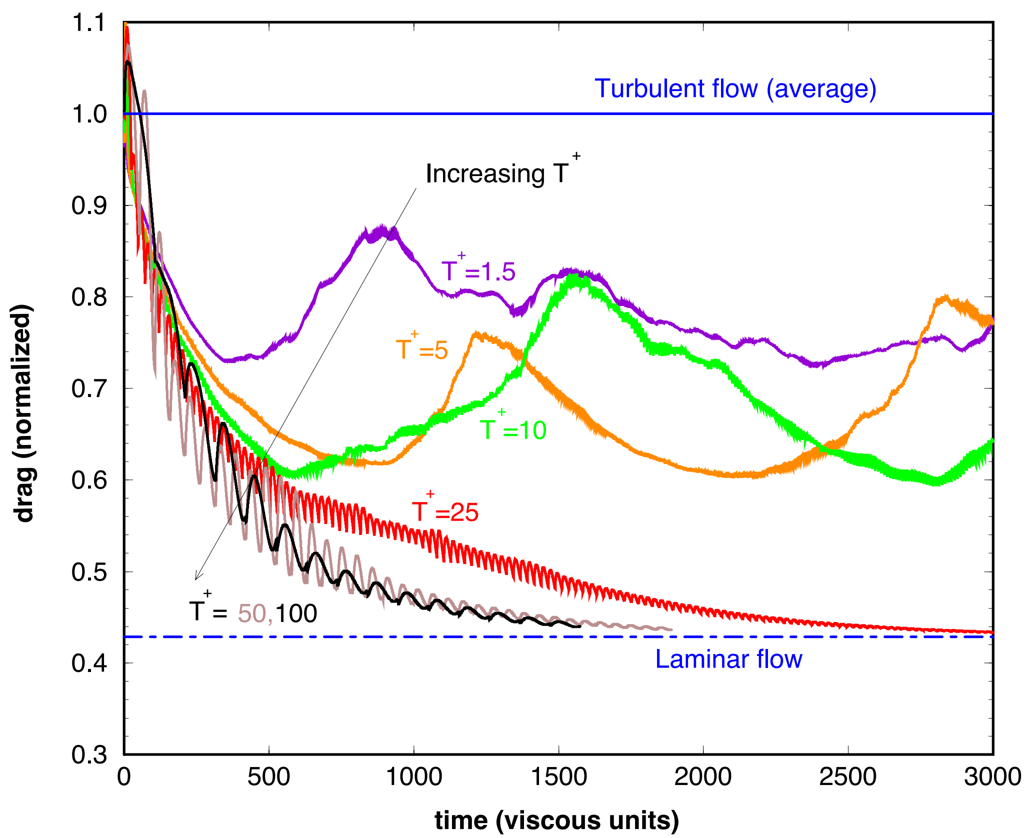
Note solutions of this matrix equation satisfy $X^* = X$.

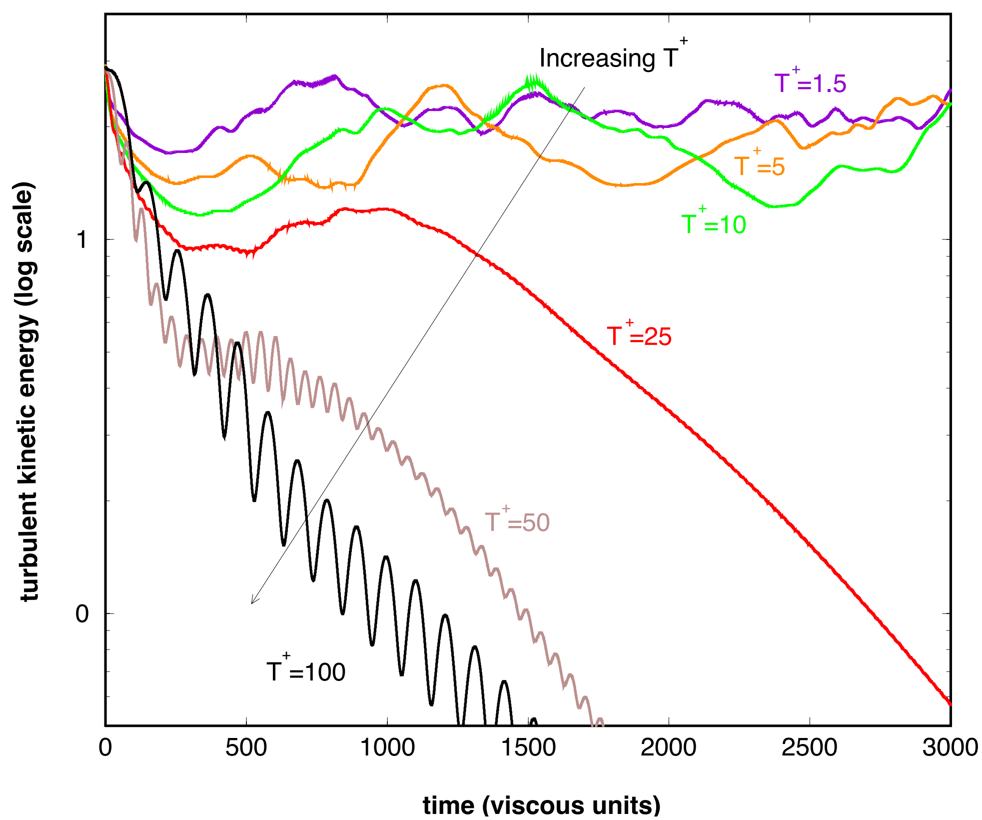
Note also that, by the characterization of the saddle point, we have

$$\psi = \frac{1}{\gamma^2}B_\psi^*\mathbf{r} \quad \text{and} \quad \boxed{\phi = K\mathbf{q}' \quad \text{where} \quad K = -\frac{1}{\ell^2}B_\phi^* X}.$$

This is **the finite-horizon $\mathcal{H}_\infty$ control solution**, and may be solved for linear time-varying (LTV) systems or marched to steady state.

Relaminarization of fully-developed Re_B = 1429 channel-flow turbulence via adjoint-based MPC [B, Moin, & Temam, JFM 2001]





Simplification of feedback control problem via Fourier transform

3D O.S./Squire control/estimation problems solved in Fourier space, where decoupling simplifies to several 1D problems. [B & Liu, JFM, 1998]

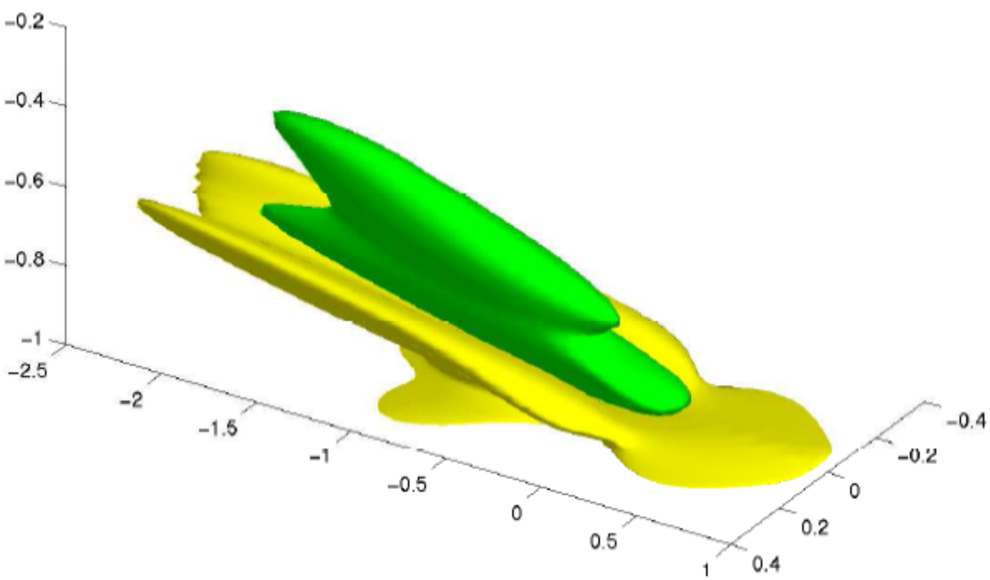
Result inverse transformed to physical space, yields localized convolution kernels. [related work: Bamieh, Paganini, & Dahleh, TAC, 2002]

Procedure highly sensitive to nuances of numerical discretization. Spurious eigenvalues must be addressed. [Huang & Sloan 1993, JCP 111]

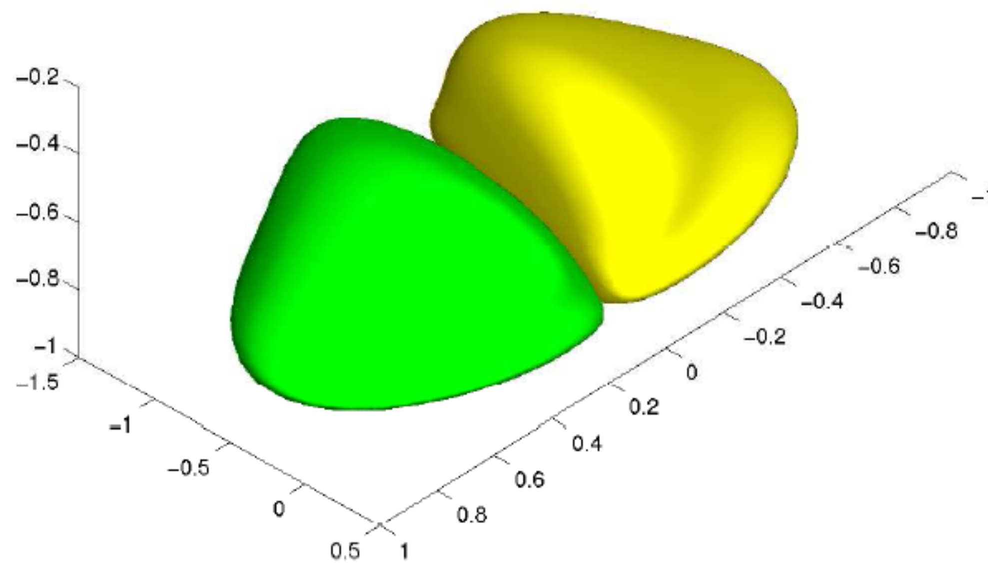
Kernels relating v & ω fluctuations to blowing/suction control ϕ

[Hogberg, B, Henningson, JFM 2003a]

v



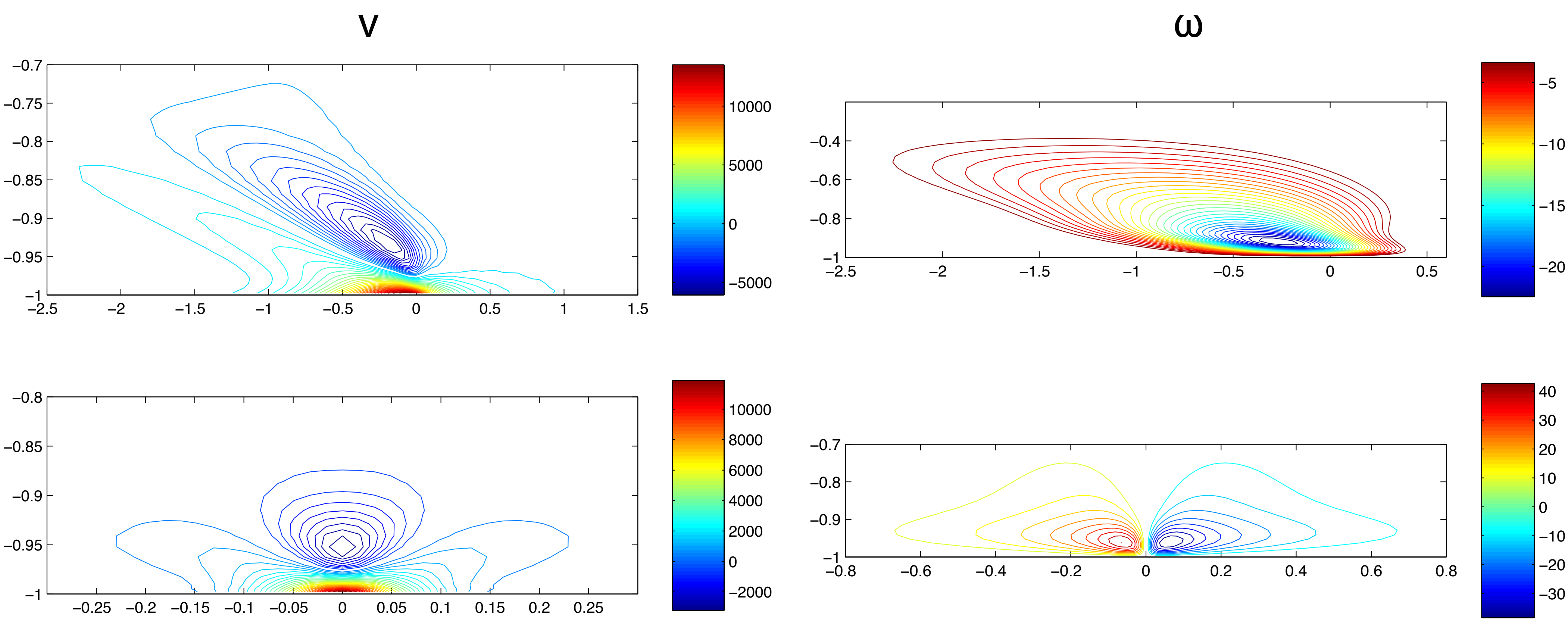
ω



Visualized are the positive (green) and negative (yellow) iso-surfaces with iso-values of ± 5% of the maximum amplitude for each kernel illustrated.

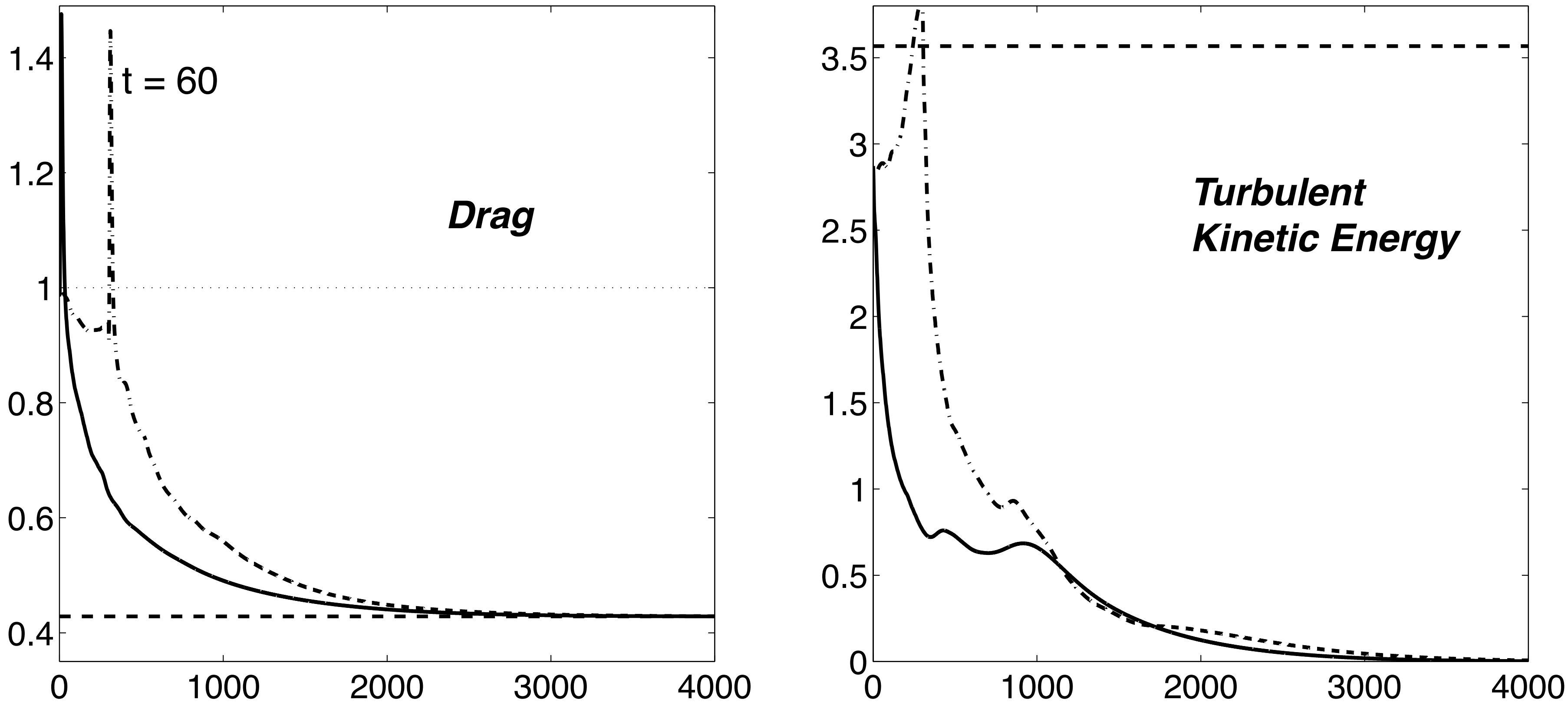
Kernels relating v & ω fluctuations to blowing/suction control ϕ

[Hogberg, B, Henningson, JFM 2003a]



Relaminarization of fully-developed turbulence via linear feedback

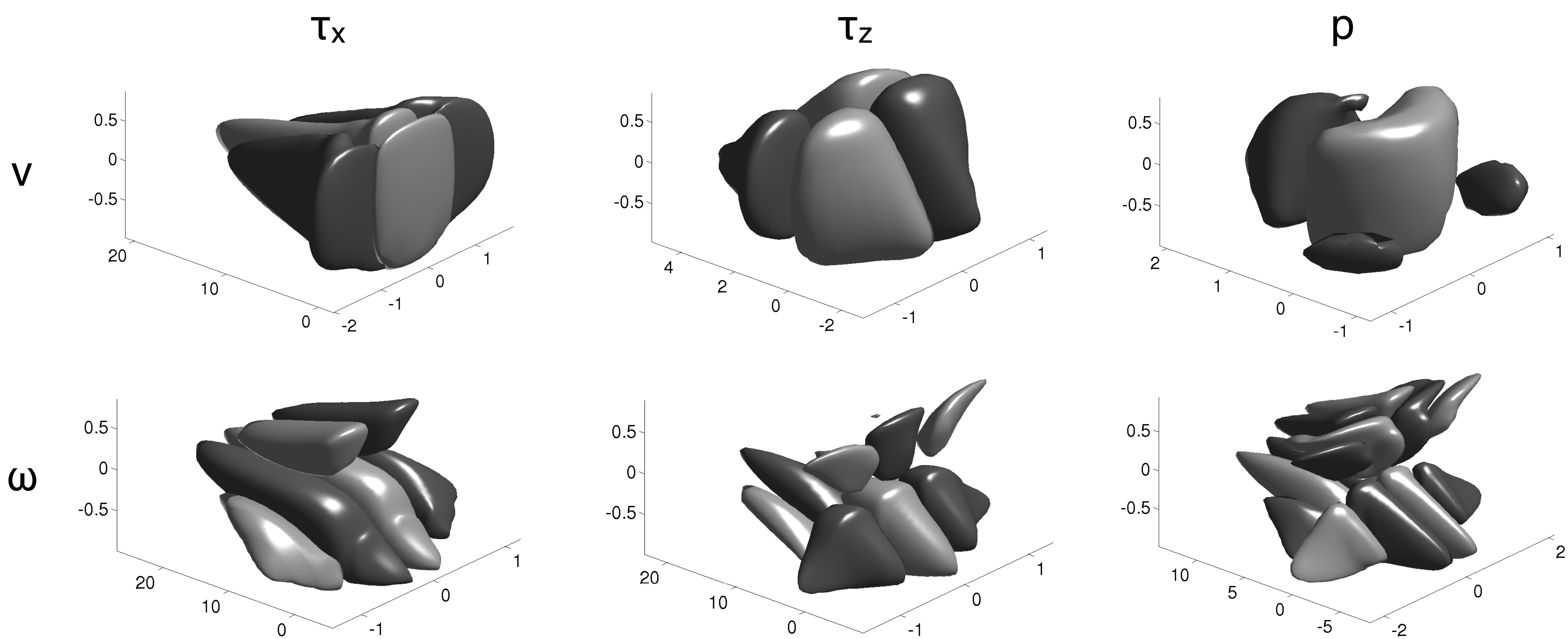
[Hogberg, B, Henningson, JFM 2003b]



Relaminarization of fully-developed channel-flow turbulence at Re_B = 1429.

Kernels relating τ_x, τ_z, & p measurements to v, ω forcing of estimator

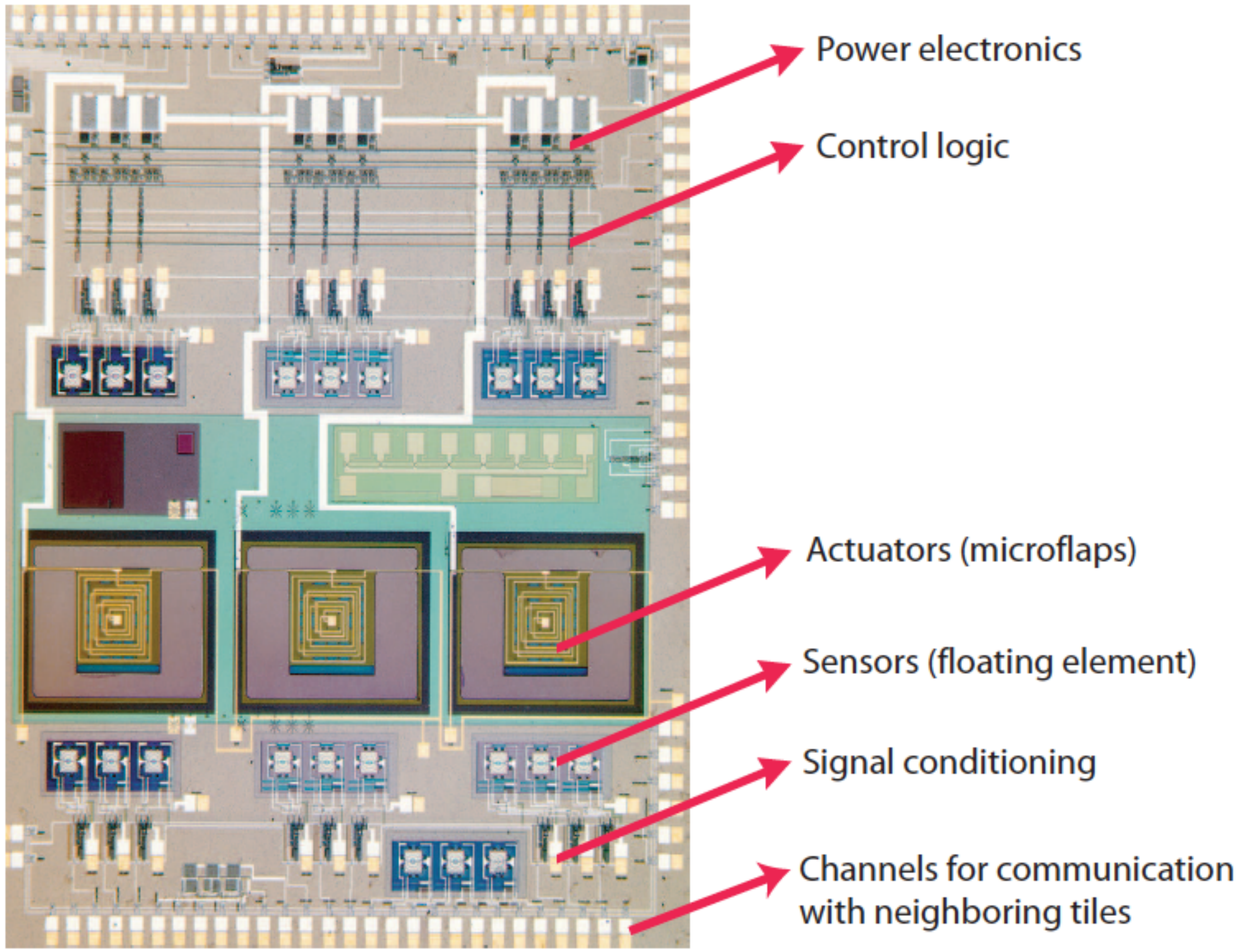
[Hoepffner, Chevalier, B, Henningson, JFM 2005]



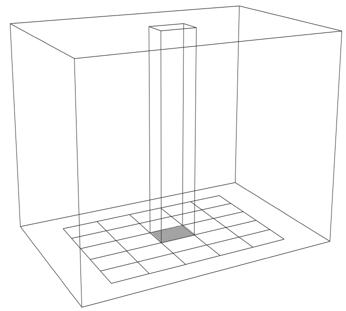
Visualized are the positive (dark) and negative (light) iso-surfaces with iso-values of ± 5% of the maximum amplitude for each kernel illustrated.

Photograph of MEMS tile suitable for decentralized control

(Chih-Ming Ho *et al.* (UCLA) and Yu-Chong Tai *et al.* (Caltech))



Implementation
[B, PAS, 2001]



Experimental apparatus

actuator inputs on tile $\{i, j\}$ only

sensor measurements on tile $\{i, j\}$ only

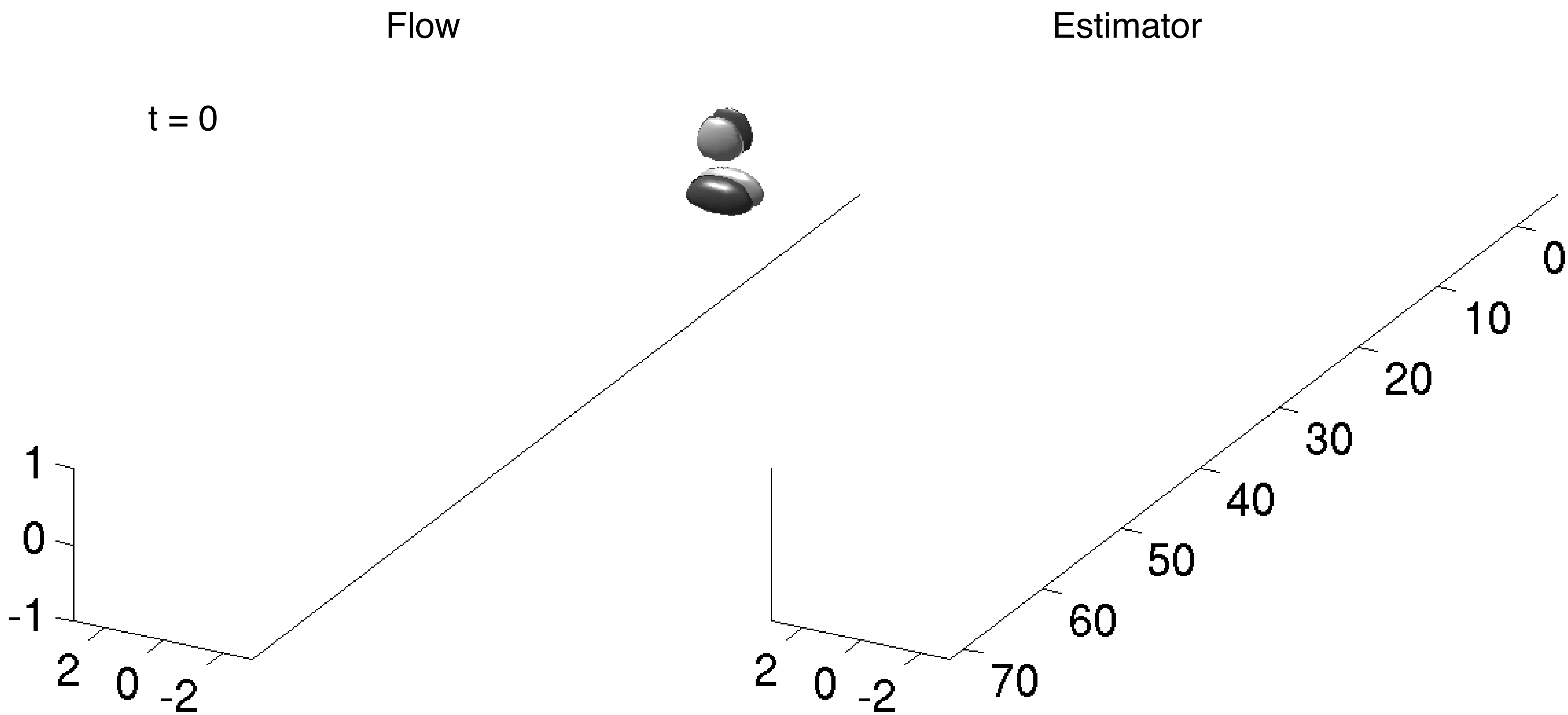
Localized physical-space compensation:
estimation of state above tile $\{i, j\}$,
computation of control on tile $\{i, j\}$

Decentralized logic circuit replicated on each tile

Communication with neighboring tiles about nearby sensor measurements and state estimates

Evolution of small disturbance to state (left) and estimate (right)

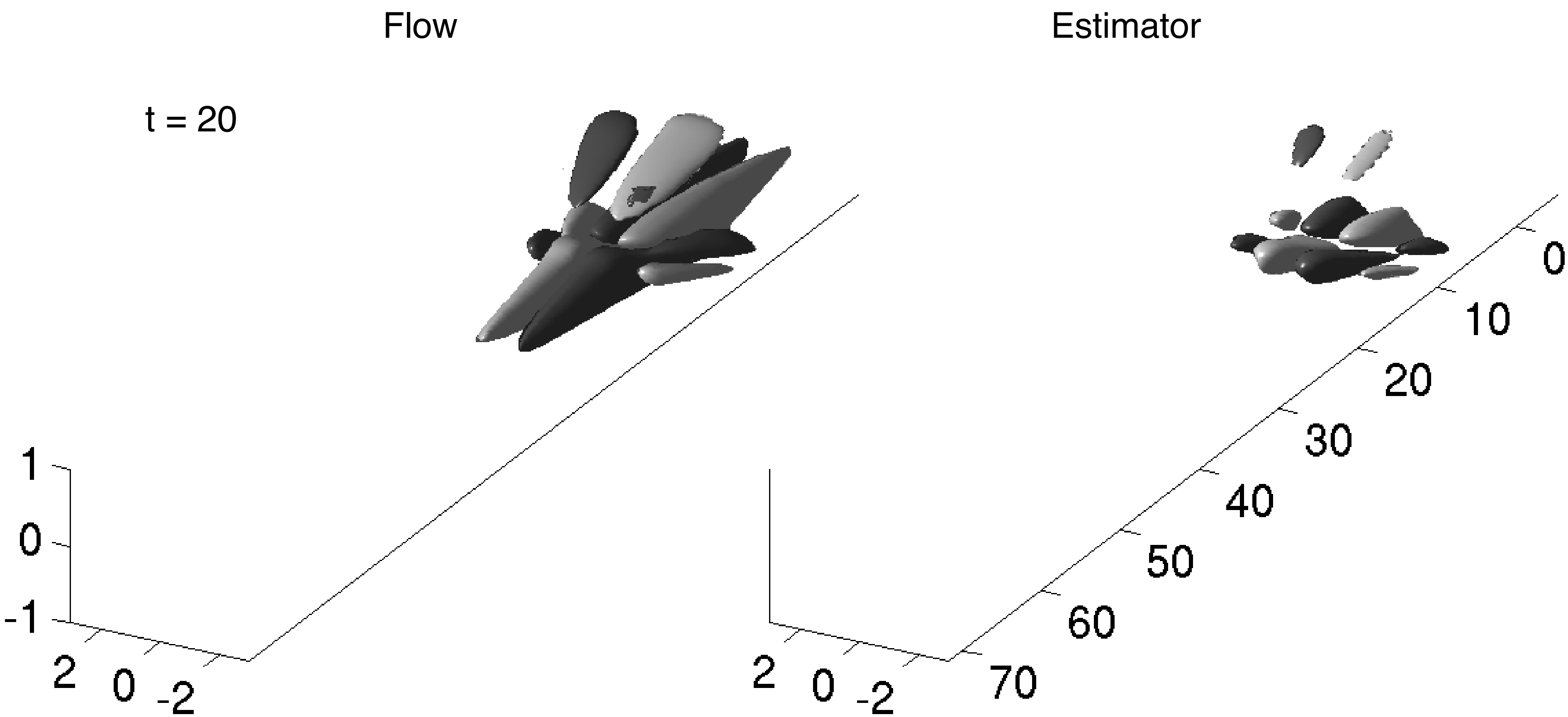
[Hoepffner, Chevalier, B, Henningson, JFM 2005]



Positive (light) and negative (dark) iso-surfaces of the streamwise component of velocity. Iso-values at ± 10% of the maximum streamwise velocity of the flow during interval shown.

Evolution of small disturbance to state (left) and estimate (right)

[Hoepffner, Chevalier, B, Henningson, JFM 2005]



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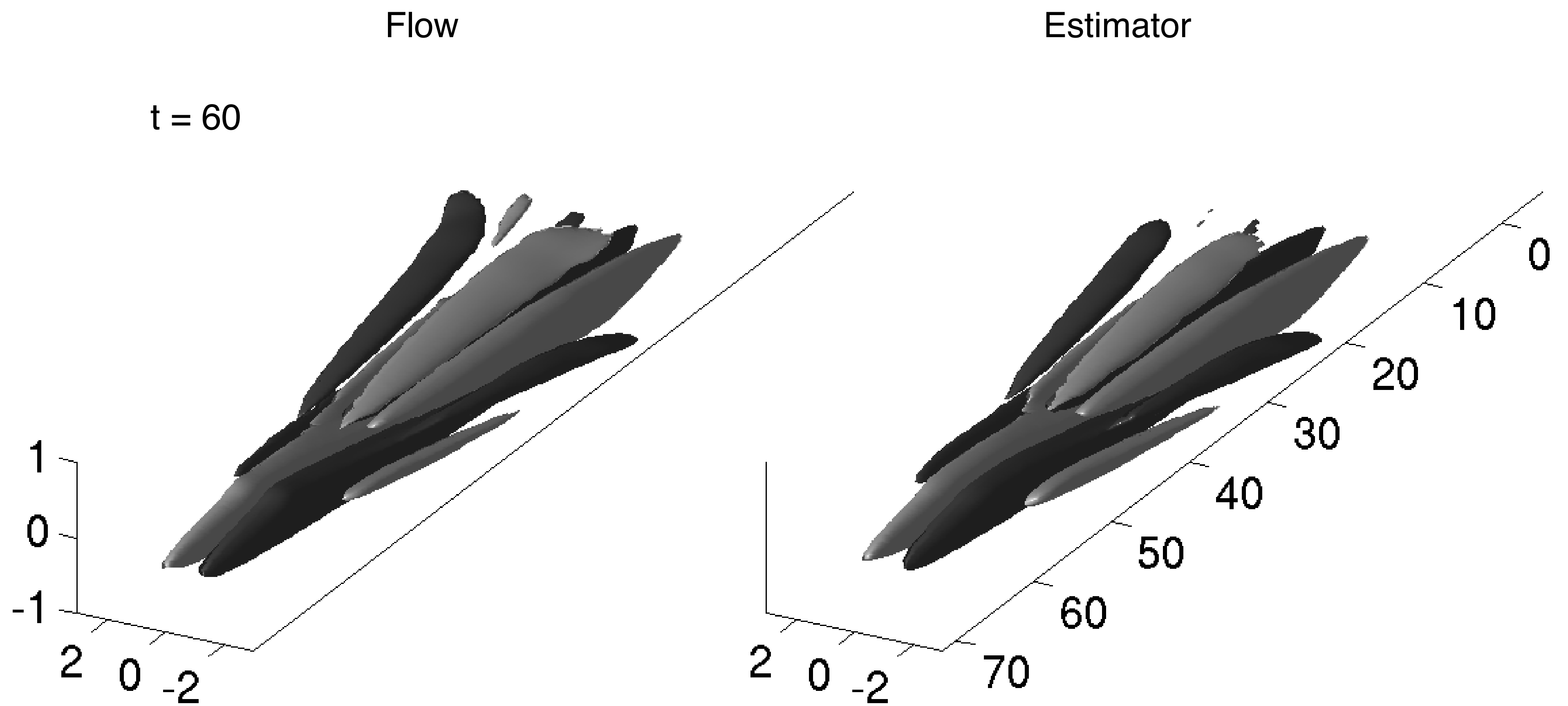


Positive (light) and negative (dark) iso-surfaces of the streamwise component of velocity. Iso-values at ± 10% of the maximum streamwise velocity of the flow during interval shown.

Properties of feedback convolution kernels

Kernels are independent of the box size in which they were computed, so long as the computational box is sufficiently large.

-> Nonphysical assumption of spatial periodicity is relaxed.

Kernels are well-resolved with grid resolutions appropriate for the simulation of the physical system of interest. -> Grid independent.

Kernels eventually decay exponentially, and may be truncated to any desired degree of precision.

-> Truncated kernels are spatially compact with finite support. Implementable!

Kernel structure is physically tenable, but not imposed a priori:

-> Control convolution kernels angle away from the actuator upstream.

-> Estimation convolution kernels extend well downstream of sensor.

Open questions

Appropriate regularization in cost function for control problem, and disturbance modeling in estimation problem, are essential to obtain meaningful results (i.e., smooth enough to obtain “convergence upon grid refinement”)!

Q: How much “smoothing” is needed? What is its precise effect?

Achieving convergence in controller

Taking J as linear combination of TKE and 2-norm of control **failed** to achieve convergence upon grid refinement (nonsmooth kernel, strong high-frequency components - not even in L_2?)

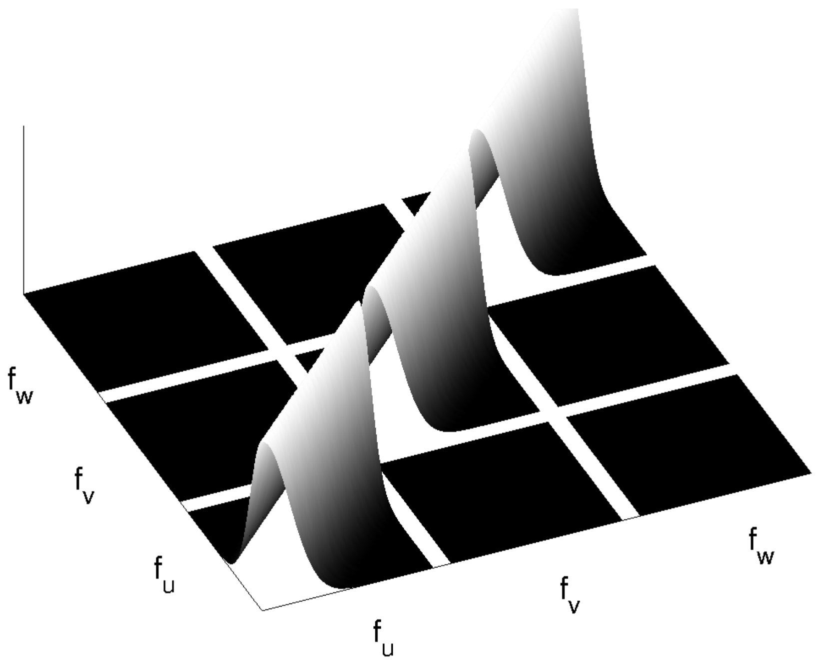
Taking J as linear combination of TKE and 2-norm of **time-derivative** of control succeeded in achieving convergence upon grid refinement (smooth kernel, nicely decaying high-frequency components).

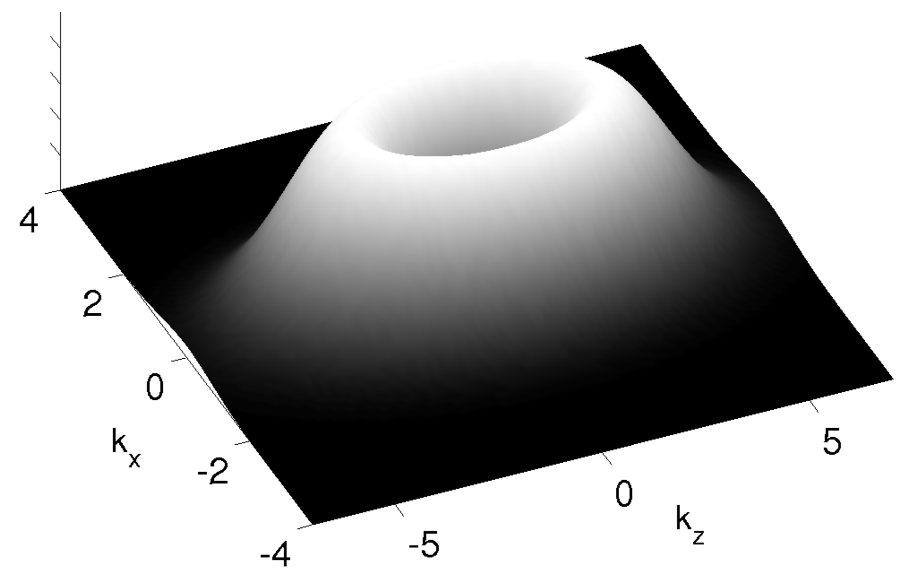
Why?? Trace theorem is one hint.

By the NSE, one time derivative = two space derivatives. So, is taking J as linear combination of TKE and 2-norm of **gradient** of velocity field sufficent??

Achieving convergence in estimator (transitional flow)

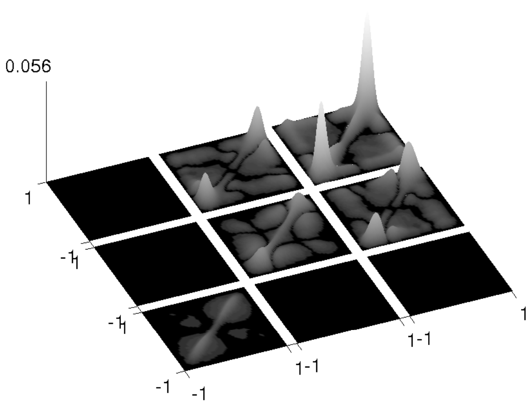
Taking Q=I, model of covariance doesn't converge upon grid refinement to some smooth function. Hoepffner, Chevalier, B, & Henningson thus proposed taking Q as a discretization of some smooth yet ad hoc diagonally-dominant shape functions, with ad hoc weighting between various $\{k_x, k_z\}$.



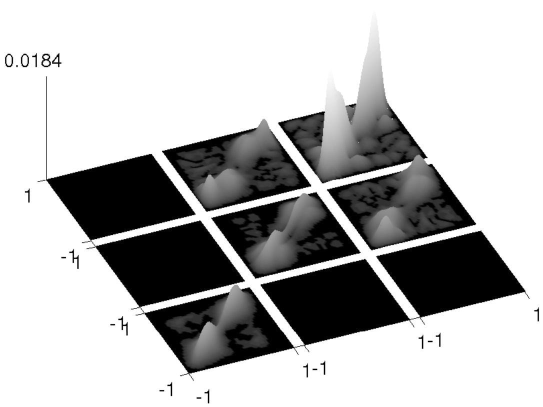


Achieving convergence in estimator (turbulent flow)

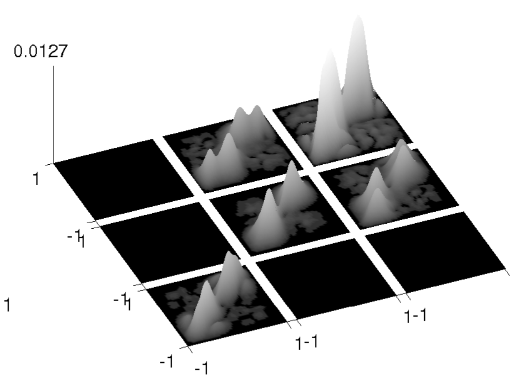
Taking full NSE as LNSE+f, compute the statistics of f from a turbulent database. Use those covariance statistics Q to compute feedback kernels for the estimator.



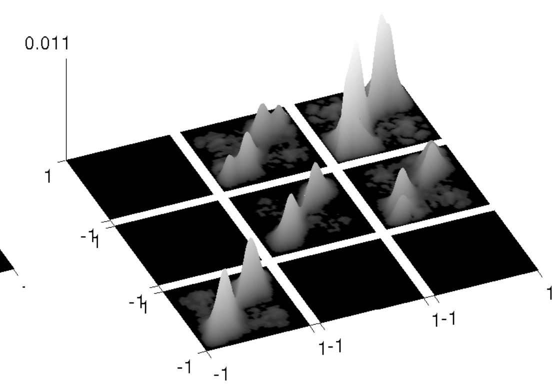
{kx,kz} = {1,3}



{kx,kz} = {3,1.5}



{kx,kz} = {0,1.5}



{kx,kz} = {4,4.5}