Local Approximations of the Pressure Hessian near Vorticity Concentrations in Incompressible Flows

C. David Levermore
Department of Mathematics *and*
Institute for Physical Science and Technology
University of Maryland, College Park, MD
lvrmr@math.umd.edu

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Introduction

Incompressible (low Mach number) fluid flows are commonly modeled by initial-value problems for a Navier-Stokes system that take the form

$$\nabla_x \cdot u = 0\,, \tag{1a}$$

$$\partial_t u + u \cdot \nabla_x u + \nabla_x p = \nu \Delta_x u\,, \tag{1b}$$

$$u(x, 0) = u_o(x)\,. \tag{1c}$$

Of course, the spatial domain and appropriate boundary conditions must also be specified. This system was introduced by Navier (1823). It built upon the system introduced by Euler (1757), which is obtained by setting $\nu = 0$ in (1b). For many such problems a unique classical solution is known to exist for a finite time whenever the initial data u_o is sufficiently regular. For $D > 2$ the solutions of these systems can develop velocity gradient concentrations. It is unkown if these concentrations can blow-up. In this talk we explore models of this potential blow-up.

We consider the dynamics of the gradient tensor $A(x,t) = \nabla_x u(x,t)$ for solutions $u(x,t)$ of the INS system (1). Taking the gradient of (1b) yields

$$\partial_t A + u \cdot \nabla_x A + A^2 + \nabla_x^2 p = \nu \Delta_x A \,, \qquad A(x,0) = \nabla_x u_o(x) \,. \tag{2}$$

Here $\nabla_x^2 p$ is the Hessian matrix of p, not its Laplacian! Equation (1a) implies that $\mathrm{tr}(A(x,t)) = 0$, whereby taking the trace of (2) yields the so-called *pressure equation,*

$$-\Delta_x p = -\,\mathrm{tr}\Big(\nabla_x^2 p\Big) = \mathrm{tr}\Big(A^2\Big) \,. \tag{3}$$

By combining this with the dynamical equation in (2) we obtain the *velocity gradient dynamics* equation

$$\begin{aligned} \partial_t A + u \cdot \nabla_x A + A^2 - \tfrac{1}{D}\,\mathrm{tr}\Big(A^2\Big) I \\ + \nabla_x^2 p - \tfrac{1}{D} \Delta_x p \, I = \nu \Delta_x A \,. \end{aligned} \tag{4}$$

Patrick Vieillefosse (1982, 1984) proposed to model singularity formation in the three-dimensional incompressible Euler system — and therefore to model singularity or near singularity formation in the three-dimensional INS system by making the approximations

$$\nabla_x^2 p - \tfrac{1}{D}\Delta_x p\, I = 0\,, \qquad \nu\Delta_x u = 0\,. \tag{5}$$

The first states that the Hessian of p is nearly isotropic compared to the anisotropies that arise from A. The second states that the viscosity will have little effect on singularity formation.

The result is the so-called *Restricted Euler* (RE) equation

$$\partial_t A + u\cdot\nabla_x A + A^2 - \tfrac{1}{D}\,\mathrm{tr}\big(A^2\big) I = 0\,. \tag{6}$$

Because this equation does not preserve the relationship that the three-tensor $\nabla_x A$ is symmetric in its last two indices, which is required for A to be given as $A = \nabla_x u$ for some u, the approximation that $\nabla_x^2 p$ is isotropic will generally not be valid globally. See Meneveau (2011) for a review article.

The Vieillefosse model can be extended to include an anisotropic pressure Hessian. For example, we can consider the family of models in the form

$$\begin{aligned}\partial_t A + u\cdot\nabla_x A + A^2 - \tfrac{1}{D}\,\mathrm{tr}\Big(A^2\Big)I \qquad\qquad& \\ + \eta_1\bigg(A^TA + AA^T - \tfrac{2}{D}\,\mathrm{tr}\Big(A^TA\Big)I\bigg)& \\ + \eta_2\bigg(A^2 + A^{T2} - \tfrac{2}{D}\,\mathrm{tr}\Big(A^2\Big)I\bigg)& \\ + \eta_3\Big(A^TA - AA^T\Big) = 0\,.&\end{aligned} \tag{7}$$

where η_1, η_2, and η_3 are unitless scalar coefficients that can depend upon unitless combinations of A. Every member of this family preserves all of the dilational, rotational, and Galilean symmetries of the velocity gradient dynamics equation being modeled (4).

Here we select one member of this family by considering the form of $\nabla_x^2 p$ that follows from the pressure equation (3) for finite energy solutions u of the Euler or Navier-Stokes system (1).

More specifically, we propose replacing the approximation that $\nabla_x^2 p$ is isotropic with an approximation based upon a decomposition of $\nabla_x^2 p$ into

1. a near-field approximation that depends locally on x,
2. a near-field correction that depends nonlocally on x,
3. a far-field correction that depends nonlocally on x.

The near-field approximation will yield an anisotropic model in the form (7) with constant coefficients.

Outline

- Green Function for the Pressure
- Local Approximation of the Pressure Hessian
- New Model of Velocity Gradient Dynamics
- Nonlocal Near-Field and Far-Field Corrections
- Concluding Remarks

Green Function for the Pressure

The pressure equation (3) can be written as

$$-\Delta_x p = \nabla_x^2 : u^{\vee 2} \,. \tag{8}$$

Here $\vee$ is symmetric tensor product. For $u \in C_c^\infty(\mathbb{R}^D; \mathbb{R}^D)$ a solution is

$$p = g * \left(\nabla_x^2 : u^{\vee 2}\right) \,, \tag{9}$$

where $*$ denotes convolution and g is the Green function given by

$$g(x) = \begin{cases} \dfrac{1}{D-2} \, \dfrac{1}{|\mathbb{S}^{D-1}|} \, \dfrac{1}{|x|^{D-2}} & \text{for } D > 2 \,, \\ -\dfrac{1}{2\pi} \log(|x|) & \text{for } D = 2 \,. \end{cases} \tag{10}$$

Every other solution of (8) is obtained from the one given in (9) by adding an harmonic function. Boundary conditions need to be specified to make this choice unique. For $D > 2$ solution (9) is the unique solution that decays as $|x| \to \infty$.

For every $D \geq 2$ the first four gradient tensors of g are

$$\nabla_x g(x) = -\frac{1}{|\mathbb{S}^{D-1}|} \frac{x}{|x|^D}, \tag{11a}$$

$$\nabla_x^2 g(x) = \frac{1}{|\mathbb{S}^{D-1}|} \left(D \frac{x^{\vee 2}}{|x|^{D+2}} - \frac{\delta}{|x|^D} \right) \tag{11b}$$

$$\nabla_x^3 g(x) = \frac{1}{|\mathbb{S}^{D-1}|} \left(\frac{3\delta \vee x}{|x|^{D+2}} - (D+2) \frac{x^{\vee 3}}{|x|^{D+4}} \right), \tag{11c}$$

$$\begin{aligned} \nabla_x^4 g(x) = \frac{1}{|\mathbb{S}^{D-1}|} \bigg(& \frac{3\delta^{\vee 2}}{|x|^{D+2}} - (D+2) \frac{6\delta \vee x^{\vee 2}}{|x|^{D+4}} \\ & + (D+2)(D+4) \frac{x^{\vee 4}}{|x|^{D+6}} \bigg), \end{aligned} \tag{11d}$$

where δ is the Kronecker two-tensor with entries

$$\delta_{ij} = \begin{cases} 1 & \text{when } i = j, \\ 0 & \text{when } i \neq j. \end{cases}$$

Because these gradient tensors are traceless and symmetric, they can be expressed in terms of spherical harmonic tensors. We define the spherical harmonic tensor $Y^m(\omega)$ over $\omega \in \mathbb{S}^{D-1}$ to be the traceless part of the symmetric tensor $\omega^{\vee m}$. Then the gradient tensors are

$$\nabla_x g(x) = -\frac{1}{|\mathbb{S}^{D-1}|} \frac{1}{|x|^{D-1}} Y^1\left(\frac{x}{|x|}\right), \tag{12a}$$

$$\nabla_x^2 g(x) = \frac{D}{|\mathbb{S}^{D-1}|} \frac{1}{|x|^{D}} Y^2\left(\frac{x}{|x|}\right), \tag{12b}$$

$$\nabla_x^3 g(x) = -\frac{D+2}{|\mathbb{S}^{D-1}|} \frac{1}{|x|^{D+1}} Y^3\left(\frac{x}{|x|}\right), \tag{12c}$$

$$\nabla_x^4 g(x) = \frac{(D+2)(D+4)}{|\mathbb{S}^{D-1}|} \frac{1}{|x|^{D+2}} Y^4\left(\frac{x}{|x|}\right). \tag{12d}$$

Notice that only $\nabla_x g$ is locally integrable at the origin.

We will use the spherical harmonic tensors through order 4, which are

$$\begin{aligned}
Y^0(\omega) &= 1\,, \\
Y^1(\omega) &= \omega\,, \\
Y^2(\omega) &= \omega^{\vee 2} - \tfrac{1}{D}\,\delta\,, \\
Y^3(\omega) &= \omega^{\vee 3} - \tfrac{3}{D+2}\,\delta \vee \omega\,, \\
Y^4(\omega) &= \omega^{\vee 4} - \tfrac{6}{D+4}\,\delta \vee \omega^{\vee 2} + \tfrac{3}{(D+2)(D+4)}\,\delta^{\vee 2}\,.
\end{aligned}$$

Spherical harmonic tensors of different orders are orthogonal in $L^2(\mathrm{d}\omega)$, where $\mathrm{d}\omega$ is the Lebesgue measure on $\mathbb{S}^{D-1}$. More precisely, we have

$$\left\langle Y^m_{i_1\cdots i_m} Y^n_{j_1\cdots j_n} \right\rangle = 0 \qquad \text{when } m \neq n\,,$$

where the subscripts are the indices of the associated tensor entries and

$$\langle \Phi \rangle = \frac{1}{|\mathbb{S}^{D-1}|} \int_{\mathbb{S}^{D-1}} \Phi(\omega)\, \mathrm{d}\omega\,.$$

We will use the orthogonality relations for orders 0, 1, 2, and 3 given by

$$\left\langle Y^0 Y^0 \right\rangle = 1\,,$$

$$\left\langle Y^1_{i_1} Y^1_{j_1} \right\rangle = \tfrac{1}{D}\,\delta_{i_1 j_1}\,,$$

$$\left\langle Y^2_{i_1 i_2} Y^2_{j_1 j_2} \right\rangle = \tfrac{1}{D(D+2)}\Big[\delta_{i_1 j_1}\delta_{i_2 j_2} + \delta_{i_1 j_2}\delta_{i_2 j_1} - \tfrac{2}{D}\,\delta_{i_1 i_2}\delta_{j_1 j_2}\Big]\,,$$

$$\left\langle Y^3_{i_1 i_2 i_3} Y^3_{j_1 j_2 j_3} \right\rangle = \tfrac{1}{D(D+2)(D+4)}$$

$$\Big[\delta_{i_1 j_1}\delta_{i_2 j_2}\delta_{i_3 j_3} + \delta_{i_1 j_2}\delta_{i_2 j_3}\delta_{i_3 j_1} + \delta_{i_1 j_3}\delta_{i_2 j_1}\delta_{i_3 j_2}$$

$$+ \delta_{i_1 j_2}\delta_{i_2 j_1}\delta_{i_3 j_3} + \delta_{i_1 j_3}\delta_{i_2 j_2}\delta_{i_3 j_1} + \delta_{i_1 j_1}\delta_{i_2 j_3}\delta_{i_3 j_2}$$

$$- \tfrac{2}{D+2}\Big[\delta_{i_1 i_2}\delta_{i_3 j_3}\delta_{j_1 j_2} + \delta_{i_1 i_2}\delta_{i_3 j_1}\delta_{j_2 j_3} + \delta_{i_1 i_2}\delta_{i_3 j_2}\delta_{j_3 j_1}$$

$$+ \delta_{i_2 i_3}\delta_{i_1 j_3}\delta_{j_1 j_2} + \delta_{i_2 i_3}\delta_{i_1 j_1}\delta_{j_2 j_3} + \delta_{i_2 i_3}\delta_{i_1 j_2}\delta_{j_3 j_1}$$

$$+ \delta_{i_3 i_1}\delta_{i_2 j_3}\delta_{j_1 j_2} + \delta_{i_3 i_1}\delta_{i_2 j_1}\delta_{j_2 j_3} + \delta_{i_3 i_1}\delta_{i_2 j_2}\delta_{j_3 j_1}\Big]\Big]\,.$$

Local Approximations of the Pressure Hessian

The pressure Hessian is given by

$$\begin{aligned}
\nabla_x^2 p &= g * \left(\nabla_x^4 : u^{\vee 2}\right) = \nabla_x g * \vee \left(\nabla_x^3 : u^{\vee 2}\right) \\
&= \int \nabla_x g(x-y) \vee \left(\nabla_y^3 : u^{\vee 2}(y)\right) \, \mathrm{d}y \\
&= \int_{|y-x|\leq R} \nabla_x g(x-y) \vee \left(\nabla_y^3 : \left(u^{\vee 2}(y) - \mathcal{T}_x^2 u^{\vee 2}(y)\right)\right) \, \mathrm{d}y \\
&\quad + \int_{|y-x|>R} \nabla_x g(x-y) \vee \left(\nabla_y^3 : u^{\vee 2}(y)\right) \, \mathrm{d}y \,,
\end{aligned} \tag{13}$$

where $R > 0$ is arbitrary and $\mathcal{T}_x^2 u^{\vee 2}(y)$ denotes the second-order Taylor approximation of $u^{\vee 2}(y)$ centered at x, which is given by

$$\begin{aligned}
\mathcal{T}_x^2 u^{\vee 2}(y) = u^{\vee 2}(x) &+ (y-x) \cdot \nabla_x u^{\vee 2}(x) \\
&+ \tfrac{1}{2}(y-x)^{\vee 2} : \nabla_x^2 u^{\vee 2}(x) \,.
\end{aligned} \tag{14}$$

Now we integrate the last two integrals in (13) by parts using the identity

$$
\begin{aligned}
\nabla_x g(x-y) \vee \left(\nabla_y^3 : W(y)\right) = \nabla_y \vee \left(\nabla_x g(x-y) \nabla_y^2 : W(y)\right) \\
+ \nabla_y \cdot \left(\nabla_x^2 g(x-y) \nabla_y \cdot W(y)\right) \\
+ \nabla_y \cdot \left(\nabla_x^3 g(x-y) \cdot W(y)\right) \\
+ \nabla_x^4 g(x-y) : W(y)\,,
\end{aligned}
\tag{15}
$$

with $W(y) = u^{\vee 2}(y) - \mathcal{T}_x^2 u^{\vee 2}(y)$ and $W(y) = u^{\vee 2}(y)$ respectively. The contributions of $u^{\vee 2}(y)$ to the resulting boundary integrals will cancel, leaving only the contribution of $\mathcal{T}_x^2 u^{\vee 2}(y)$ from the first boundary integral. To evaluate this remaining boundary integral contribution we use

$$
\begin{aligned}
\nabla_y \cdot \left(\mathcal{T}_x^2 u^{\vee 2}(y)\right) = \nabla_x \cdot u^{\vee 2}(x) + (y-x) \cdot \nabla_x^2 \cdot u^{\vee 2}(x)\,, \\
\nabla_y^2 : \left(\mathcal{T}_x^2 u^{\vee 2}(y)\right) = \nabla_x^2 : u^{\vee 2}(x)\,.
\end{aligned}
\tag{16}
$$

We obtain

$$\begin{aligned}\nabla_x^2 p = &\int_{S_R(x)} \nabla_x g(x-y) \vee \frac{x-y}{|x-y|} \, \mathsf{dS}(y) \left(\nabla_x^2 : u^{\vee 2}(x)\right) \\ &+ \int_{S_R(x)} \nabla_x^2 g(x-y) \left(\frac{(x-y)^{\vee 2}}{|x-y|} : \left(\nabla_x^2 \cdot u^{\vee 2}(x)\right) \right) \mathsf{dS}(y) \\ &+ \int_{S_R(x)} \nabla_x^3 g(x-y) \cdot \left(\frac{(x-y)^{\vee 3}}{|x-y|} \,\vdots\, \left(\nabla_x^2 \otimes u^{\vee 2}(x)\right) \right) \mathsf{dS}(y) \\ &+ \int_{|y-x| \le R} \nabla_x^4 g(x-y) : \left(u^{\vee 2}(y) - \mathcal{T}_x^2 u^{\vee 2}(y) \right) \mathsf{d}y \\ &+ \int_{|y-x| > R} \nabla_x^4 g(x-y) : u^{\vee 2}(y) \, \mathsf{d}y \,, \end{aligned} \tag{17}$$

where $\otimes$ denotes tensor product, $S_R(x)$ denotes the sphere of radius R centered at x, and $\mathsf{dS}(y)$ denotes the Lebesgue measure on $S_R(x)$. When $y \in S_R(x)$ is expressed as $y = x + R\omega$ for some $\omega \in \mathbb{S}^{D-1}$ then $\mathsf{dS}(y) = R^{D-1} \mathsf{d}\omega$, where $\mathsf{d}\omega$ is the Lebesgue measure on $\mathbb{S}^{D-1}$.

The associated integrands are then expressed as

$$\nabla_x g(x-y) \vee \frac{x-y}{|x-y|} = -\frac{1}{|\mathbb{S}^{D-1}|}\frac{1}{R^{D-1}}\, Y^1(\omega) \vee \omega\,,$$

$$\nabla_x^2 g(x-y) \otimes \frac{(x-y)^2}{|x-y|} = \frac{D}{|\mathbb{S}^{D-1}|}\frac{1}{R^{D-1}}\, Y^2(\omega) \otimes \omega^{\vee 2}\,,$$

$$\nabla_x^3 g(x-y) \otimes \frac{(x-y)^3}{|x-y|} = -\frac{D+2}{|\mathbb{S}^{D-1}|}\frac{1}{R^{D-1}}\, Y^3(\omega) \otimes \omega^{\vee 3}\,.$$

The orthogonality properties of the spherical harmonic tensors imply

$$\int_{S_R(x)} \nabla_x g(x-y) \vee \frac{x-y}{|x-y|}\, \mathsf{dS}(y) = -\left\langle Y^1 \vee Y^1 \right\rangle,$$

$$\int_{S_R(x)} \nabla_x^2 g(x-y) \otimes \frac{(x-y)^2}{|x-y|}\, \mathsf{dS}(y) = D\left\langle Y^2 \otimes Y^2 \right\rangle,$$

$$\int_{S_R(x)} \nabla_x^3 g(x-y) \otimes \frac{(x-y)^3}{|x-y|}\, \mathsf{dS}(y) = -(D+2)\left\langle Y^3 \otimes Y^3 \right\rangle.$$

Therefore

$$\begin{aligned}\nabla_x^2 p = &-\Big\langle Y^1 \vee Y^1\Big\rangle \left(\nabla_x^2 : u^{\vee 2}\right) + D\Big\langle Y^2 \otimes Y^2\Big\rangle : \left(\nabla_x^2 \cdot u^{\vee 2}\right) \\ &- (D+2)\Big\langle Y^3 \otimes Y^3\Big\rangle \cdot \vdots \left(\nabla_x^2 \otimes u^{\vee 2}\right) \\ &+ \text{F.P.} \int \nabla_x^4 g(x-y) : u^{\vee 2}(y)\,\mathrm{d}y\,,\end{aligned}$$

where the solo $\cdot$ in the third term is associated with the left-most Y^3 and the right-most u. Here the F.P. indicates a *Hadamard finite part* integral. Because $\nabla_x^4 g(x)$ has a $|x|^{D+2}$ singularity, this integral takes the form

$$\begin{aligned}&\text{F.P.} \int \nabla_x^4 g(x-y) : u^{\vee 2}(y)\,\mathrm{d}y \\ &\quad = \int_{|y-x|\leq R} \nabla_x^4 g(x-y) : \left(u^{\vee 2}(y) - \mathcal{T}_x^2 u^{\vee 2}(y)\right) \mathrm{d}y \\ &\qquad + \int_{|y-x|>R} \nabla_x^4 g(x-y) : u^{\vee 2}(y)\,\mathrm{d}y\,.\end{aligned} \tag{18}$$

By using the orthogonality relations for spherical harmonic tensors with the same order, we find that

$$\begin{aligned}
\left\langle Y^1 \vee Y^1 \right\rangle \left(\nabla_x^2 : u^{\vee 2}\right) &= \tfrac{1}{D}\, \delta\, \nabla_x^2 : u^{\vee 2}\,, \\
\left\langle Y^2 \otimes Y^2 \right\rangle : \left(\nabla_x^2 \cdot u^{\vee 2}\right) &= \tfrac{2}{D(D+2)} \Big[\nabla_x \vee \left(\nabla_x \cdot u^{\vee 2}\right) - \tfrac{1}{D}\, \delta\, \nabla_x^2 : u^{\vee 2} \Big]\,, \\
\left\langle Y^3 \otimes Y^3 \right\rangle \cdot\vdots \left(\nabla_x^2 \otimes u^{\vee 2}\right) &= \tfrac{2}{D(D+2)(D+4)} \Big[\nabla_x^2 |u|^2 - \tfrac{1}{D}\, \delta\, \Delta_x |u|^2 \Big] \\
&\quad + \tfrac{4}{D(D+2)(D+4)} \Big[\nabla_x \vee \left(\nabla_x \cdot u^{\vee 2}\right) - \tfrac{1}{D}\, \delta\, \nabla_x^2 : u^{\vee 2} \Big)\Big]\,.
\end{aligned}$$

Therefore

$$\begin{aligned}
\nabla_x^2 p = &-\tfrac{1}{D}\, \delta \left(\nabla_x^2 : u^{\vee 2}\right) - \tfrac{2}{D(D+4)} \Big[\nabla_x^2 |u|^2 - \tfrac{1}{D}\, \delta\, \Delta_x |u|^2 \Big] \\
&+ \tfrac{2D^2+4D-8}{D(D+2)(D+4)} \Big[\nabla_x \vee \left(\nabla_x \cdot u^{\vee 2}\right) - \tfrac{1}{D}\, \delta\, \nabla_x^2 : u^{\vee 2} \Big)\Big] \\
&+ \mathrm{F.P.} \int \nabla_x^4 g(x-y) : u^{\vee 2}(y)\, \mathrm{d}y\,.
\end{aligned}$$

In summary, the pressure Hessian decomposes as

$$\nabla_x^2 p(x) = \mathcal{A}(x) + \mathcal{C}(x)\,, \tag{19a}$$

where $\mathcal{A}(x)$ is the local *approximation* given by

$$\begin{aligned}\mathcal{A}(x) = &-\tfrac{1}{D}\,\delta\left(\nabla_x^2 : u^{\vee 2}\right) - \tfrac{2}{D(D+4)}\left[\nabla_x^2|u|^2 - \tfrac{1}{D}\,\delta\,\Delta_x|u|^2\right] \\ &+ \tfrac{2D^2+4D-8}{D(D+2)(D+4)}\left[\nabla_x \vee \left(\nabla_x \cdot u^{\vee 2}\right) - \tfrac{1}{D}\,\delta\,\nabla_x^2 : u^{\vee 2}\right)\right]\,,\end{aligned} \tag{19b}$$

and $\mathcal{C}(x)$ is the nonlocal *correction* given by

$$\mathcal{C}(x) = \text{F.P.}\int \nabla_x^4 g(x-y) : u^{\vee 2}(y)\,\mathrm{d}y\,. \tag{19c}$$

In particular, for $D = 3$ the local approximation is

$$\begin{aligned}\mathcal{A}(x) = &-\tfrac{1}{3}\,\delta\left(\nabla_x^2 : u^{\vee 2}\right) - \tfrac{2}{21}\left[\nabla_x^2|u|^2 - \tfrac{1}{3}\,\delta\,\Delta_x|u|^2\right] \\ &+ \tfrac{22}{105}\left[\nabla_x \vee \left(\nabla_x \cdot u^{\vee 2}\right) - \tfrac{1}{3}\,\delta\,\nabla_x^2 : u^{\vee 2}\right)\right]\,.\end{aligned} \tag{20}$$

New Model of Velocity Gradient Dynamics

We can build a model for velocity gradient dynamics based upon the local approximation to the pressure Hessian (19b). If the velocity field $u(x,t)$ is nearly linear in a neighborhood of a parcel trajectory $X(t)$ then for x sufficiently near $X(t)$ we have

$$u(x,t) = u(X(t),t) + (x - X(t)) \cdot \nabla_x u(X(t),t) + o\Big(|x - X(t)|^2\Big) .$$

Set $A(t) = \nabla_x u(X(t),t)$. Then for x sufficiently near $X(t)$ we have

$$\begin{aligned} \nabla_x u(x,t) &= A(t) + o\Big(|x - X(t)|\Big) , \\ \nabla_x^2 u(x,t) &= o\Big(|x - X(t)|^0\Big) . \end{aligned}$$

This approximation can only hold along very special trajectories. Perhaps it is too strong.

If we neglect the far-field correction and use the approximation $\nabla_x u(x,t) \approx A(t)$ in all other terms of our local approximation to the pressure Hessian then we obtain

$$\begin{aligned}\nabla_x^2 p(X(t),t) \approx -\tfrac{1}{D} I\,\mathrm{tr}(A^2) - \tfrac{4}{D(D+4)}\Big[A^T A - \tfrac{1}{D} I\,\mathrm{tr}(A^T A)\Big] \\ + \tfrac{D^2+2D-4}{D(D+2)(D+4)}\Big[A^2 + A^{T2} - \tfrac{2}{D} I\,\mathrm{tr}(A^2)\Big] .\end{aligned}$$

When this approximation is placed into the velocity gradient equation (4) then we obtain the model

$$\begin{aligned}\partial_t A + u\cdot\nabla_x A + A^2 - \tfrac{1}{D} I\,\mathrm{tr}(A^2) \\ + \beta\Big[A^T A - \tfrac{1}{D} I\,\mathrm{tr}(A^T A)\Big] \\ + \gamma\Big[A^2 + A^{T2} - \tfrac{2}{D} I\,\mathrm{tr}(A^2)\Big] = 0 ,\end{aligned} \tag{21a}$$

where

$$\beta = -\tfrac{4}{D(D+4)} , \qquad \gamma = \tfrac{D^2+2D-4}{D(D+2)(D+4)} . \tag{21b}$$

Model (21) is the member of the family of models (7) with

$$\eta_1 = \eta_3 = \tfrac{1}{2}\beta = -\tfrac{2}{D(D+4)}\,, \qquad \eta_2 = \gamma = \tfrac{D^2+2D-4}{D(D+2)(D+4)}\,.$$

The Vieillefosse model, which corresponds to (7) with $\eta_1 = \eta_2 = \eta_3 = 0$, can be integrated for every D — see Levermore (2014). In contrast, model (21) is known to be integrable only for $D = 2$, where it has linear dynamics.

For $D = 2$ model (21) corresponds to (7) with

$$\eta_1 = \eta_3 = -\tfrac{1}{6}\,, \qquad \eta_2 = \tfrac{1}{12}\,.$$

This leaves the vorticity field constant and gives an oscillatory dynamics to the strain rate field. This contrasts sharply with the Vieillefosse model, which gives no dynamics for $D = 2$.

For $D = 3$ model (21) corresponds to (7) with

$$\eta_1 = \eta_3 = -\tfrac{2}{21}\,, \qquad \eta_2 = \tfrac{11}{105}\,.$$

We see that this model gives a 10% correction to the Vieillefosse model.

- We expect our model to be valid only near localized velocity gradient concentrations where the flow is locally linear.

- We do not expect the model to be valid when $D = 2$ because then concentrations do not happen. Indeed, in that case the dominant structures are vortex pairs, which are not captured by the model.

- For $D > 2$ we suspect (but have not proved) that the velocity gradient governed by this model will blow-up for most initial data, just as for the Vieillefosse model. It is less clear that these blow-ups give good local quantitative descriptions of velocity gradient concentrations for solutions of the Euler or Navier-Stokes systems.

Nonlocal Near-Field and Far-Field Corrections

Because the correction $\mathcal{C}(x)$ is given by a *Hadamard finite part* integral, each choice of $R > 0$ gives a natrual decomposition of $\mathcal{C}(x)$ into a near-field part and a far-field part. Specifically,

$$\mathcal{C}(x) = \mathcal{C}_R^{\mathsf{nf}}(x) + \mathcal{C}_R^{\mathsf{ff}}(x)\,, \tag{22a}$$

where the near-field correction is given by

$$\mathcal{C}_R^{\mathsf{nf}}(x) = \int_{|y-x|\leq R} \nabla_x^4 g(x-y) : \Big(u^{\vee 2}(y) - \mathcal{T}_x^2 u^{\vee 2}(y)\Big)\,\mathsf{d}y\,, \tag{22b}$$

and the far-field correction is given by

$$\mathcal{C}_R^{\mathsf{ff}}(x) = \int_{|y-x|>R} \nabla_x^4 g(x-y) : u^{\vee 2}(y)\,\mathsf{d}y\,. \tag{22c}$$

We would like to bound the size of these corrections.

The far-field correction given by (22c) is easier to bound, so we treat it first. It is

$$\mathcal{C}_R^{\mathsf{ff}}(x) = \int_{|y-x|>R} \nabla_x^4 g(x-y) : u^{\vee 2}(y)\, \mathsf{d}y\,.$$

It satisfies the uniform bound

$$\left|\mathcal{C}_R^{\mathsf{ff}}(x)\right| \leq \frac{1}{R^{D+2}}\, C_D^{\mathsf{ff}}\, \|u\|^2_{L^2(\mathsf{d}x)}\,, \tag{23a}$$

where the constant C_D^{ff} is given by

$$C_D^{\mathsf{ff}} = \frac{(D+2)(D+4)}{|\mathbb{S}^{D-1}|} \left\|Y^4\right\|_{L^\infty(\mathsf{d}\omega)}\,. \tag{23b}$$

Bound (23b) is natural because $\|u\|^2_{L^2(\mathsf{d}x)}$ is bounded by its initial value for every classical solution u of either the Euler or Navier-Stokes system. It is a strictly decreasing function of R that vanishes as $R \to \infty$.

The near field correction given by (22b) is

$$\mathcal{C}_R^{\mathsf{nf}}(x) = \int_{B_R(x)} \nabla_x^4 g(x-y) : \Big(u^{\vee 2}(y) - \mathcal{T}_x^2 u^{\vee 2}(y)\Big)\, \mathrm{d}y\,, \tag{24a}$$

where $B_R(x)$ denotes the ball of radius R centered at x. This correction can also be expressed as

$$\mathcal{C}_R^{\mathsf{nf}}(x) = \int_{B_R(x)} \nabla_x^4 g(x-y) : \Big(u^{\vee 2}(y) - \mathcal{T}_x^3 u^{\vee 2}(y)\Big)\, \mathrm{d}y\,, \tag{24b}$$

where $\mathcal{T}_x^3 u^{\vee 2}(y)$ denotes the third-order Taylor approximation of $u^{\vee 2}(y)$ centered at x, which is given by

$$\begin{aligned} \mathcal{T}_x^3 u^{\vee 2}(y) = u^{\vee 2}(x) &+ (y-x)\cdot \nabla_x u^{\vee 2}(x) \\ &+ \tfrac{1}{2}(y-x)^{\vee 2} : \nabla_x^2 u^{\vee 2}(x) \\ &+ \tfrac{1}{6}(y-x)^{\vee 3} \vdots\, \nabla_x^3 u^{\vee 2}(x)\,. \end{aligned} \tag{25}$$

The Cauchy form for the Taylor remainder applied to $u^{\vee 2}$ gives

$$u^{\vee 2}(y) = \mathcal{T}_x^m u^{\vee 2}(y) + (y-x)^{\vee(m+1)} \odot \mathcal{R}_x^m u^{\vee 2}(y)\,, \tag{26a}$$

where $\odot$ denotes a total contraction of symmetric tensors,

$$\mathcal{T}_x^m u^{\vee 2}(y) = \sum_{k=0}^{m} \frac{1}{k!}\,(y-x)^{\vee k} \odot \nabla_x^k u^{\vee 2}(x)\,, \tag{26b}$$

and

$$\mathcal{R}_x^m u^{\vee 2}(y) = \frac{1}{m!}\int_0^1 \nabla_x^{m+1} u^{\vee 2}(x+s(y-x))(1-s)^m\,\mathrm{d}s\,. \tag{26c}$$

For every $q \in (D,\infty]$ we have the bound

$$\left\|\mathcal{R}_x^m u^{\vee 2}\right\|_{L^q(B_R(x))} \leq \frac{1}{m!}\int_0^1 s^{-\frac{D}{q}}(1-s)^m\,\mathrm{d}s\,\left\|\nabla_x^{m+1} u^{\vee 2}\right\|_{L^q(B_R(x))}\,. \tag{27}$$

We will apply (26) and (27) with $m=2$ and with $m=3$ in order to bound $\mathcal{C}_R^{\mathsf{nf}}(x)$ as given by (24a) and as given by (24b) respectively.

To bound $\mathcal{C}_R^{\mathsf{nf}}(x)$ we use (26) to express (24a) and (24b) as

$$\mathcal{C}_R^{\mathsf{nf}}(x) = \int_{B_R(x)} \nabla_x^4 g(x-y) : \Big((x-y)^{\vee(m+1)} \odot \mathcal{R}_x^m u^{\vee 2}(y) \Big) \, \mathrm{d}y \,,$$

with $m = 2$ and $m = 3$ respectively. Then because $q \in (D, \infty]$ and $\frac{1}{q} + \frac{1}{q^*} = 1$ implies that $q^* \in [1, \frac{D}{D-1})$, and because

$$\nabla_x^4 g(x) \otimes x^{\vee(m+1)} = O\bigg(\frac{1}{|x|^{D+1-m}} \bigg) \quad \text{as } x \to 0 \,,$$

we see $\nabla_x^4 g(x) \otimes x^{\vee(m+1)}$ is in $L^{q^*}(B_R(0))$ for $m = 2$ and $m = 3$. Therefore by the Hölder inequality we have the bound

$$\Big| \mathcal{C}_R^{\mathsf{nf}}(x) \Big| \leq \Big\| \nabla_x^4 g(x) \otimes x^{\vee(m+1)} \Big\|_{L^{q^*}(B_R(0))} \Big\| \mathcal{R}_x^m u^{\vee 2} \Big\|_{L^q(B_R(x))} . \tag{28}$$

The last factor was bounded by (27). We evaluate the first factor next.

Because

$$\nabla_x^4 g(x) \otimes x^{\vee(m+1)} = \frac{(D+2)(D+4)}{|\mathbb{S}^{D-1}|} \frac{Y^4(\omega) \otimes \omega^{\vee(M+1)}}{r^{D+1-m}},$$

with $r = |x|$ and $\omega = x/|x|$, we obtain the evaluation

$$\left\| \nabla_x^4 g(x) \otimes x^{\vee(m+1)} \right\|_{L^{q^*}(B_R(0))} = C_g \, R^{m-1-\frac{D}{q}}, \tag{29a}$$

where $m = 2$ or $m = 3$ and

$$C_g = \frac{(D+2)(D+4)}{|\mathbb{S}^{D-1}|} \left\| Y^4(\omega) \otimes \omega^{\vee(m+1)} \right\|_{L^{q^*}(\mathrm{d}\omega)} \left(\int_0^1 \frac{s^{D-1}}{s^{(D+1-m)q^*}} \, \mathrm{d}s \right)^{\frac{1}{q^*}}. \tag{29b}$$

The last integral will be evaluated on the next slide.

When bound (27) and evaluation (29) are placed into bound (28), we obtain the bound on the near-field correction given by

$$\left|\mathcal{C}_R^{\mathsf{nf}}(x)\right| \leq R^{m-1-\frac{D}{q}} C_{D,q,m}^{\mathsf{nf}} \left\|\nabla_x^{m+1} u^{\vee 2}\right\|_{L^q(B_R(x))}, \tag{30a}$$

where $m = 2$ or $m = 3$ and

$$\begin{aligned} C_{D,q,m}^{\mathsf{nf}} = \frac{(D+2)(D+4)}{|\mathbb{S}^{D-1}|} \left\|Y^4(\omega) \otimes \omega^{\vee(m+1)}\right\|_{L^{q^*}(\mathrm{d}\omega)} \\ \left(\frac{1}{(m-1-\frac{D}{q})q^*}\right)^{\frac{1}{q^*}} \frac{1}{m!} \int_0^1 s^{-\frac{D}{q}} (1-s)^m \, \mathrm{d}s \,. \end{aligned} \tag{30b}$$

Bound (30a) is a strictly increasing function of R that vanishes as $R \to 0^+$. It is less natural than the bound (23) we obtained for the far-field correction because the L^{q^*} norm of $\nabla_x^{m+1} u^{\vee 2}$ is not controlled for classical solutions u of either the Euler or Navier-Stokes system.

The bounds on the near-field and far-field corrections given by (30) and (23) imply that the total correction given by (22a) is bounded by

$$|\mathcal{C}(x)| \le R^{m-1-\frac{D}{q}} C^{\mathsf{nf}}_{D,q,m} \left\| \nabla_x^{m+1} u^{\vee 2} \right\|_{L^q(\mathrm{d}x)} + \frac{1}{R^{D+2}} C^{\mathsf{ff}}_{D} \left\| u \right\|^2_{L^2(\mathrm{d}x)} ,$$

where $m = 2$ or $m = 3$. Here we have replaced the $L^q(B_R(x))$-norm of $\nabla_x^{m+1} u^{\vee 2}$ by its $L^q(\mathrm{d}x)$-norm. This replacement does not change much when the major contribution to this norm is concentrated within $B_R(x)$.

Finally, by optimizing this bound over R we obtain the bound

$$|\mathcal{C}(x)| \le \left(r\, C^{\mathsf{nf}}_{D,q,m} \right)^{\frac{1}{r}} \left(r^* C^{\mathsf{ff}}_{D} \right)^{\frac{1}{r^*}} \left\| \nabla_x^{m+1} u^{\vee 2} \right\|^{\frac{1}{r}}_{L^q(\mathrm{d}x)} \left\| u \right\|^{2\frac{1}{r^*}}_{L^2(\mathrm{d}x)} , \tag{31a}$$

where $m = 2$ or $m = 3$ and

$$\frac{1}{r} = \frac{2 + D}{m + 1 + \frac{D}{q^*}} , \qquad \frac{1}{r^*} = \frac{m - 1 - \frac{D}{q}}{m + 1 + \frac{D}{q^*}} . \tag{31b}$$

Concluding Remarks

- At this time there is no evidence that the local approximation to the pressure Hessian is a good one.

- The foregoing bounds on the correction are very crude. Better bounds can be sought if numerics suggest that the approximation is good.

- At this time there is no evidence that the velocity gradient dynamics model is a good one.

Thank You!

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