BOUNDARY LAYERS IN THE PRESENCE OF CHARACTERISTIC POINTS

Roger Temam

Indiana University, Bloomington

http://mypage.iu.edu/˜temam/

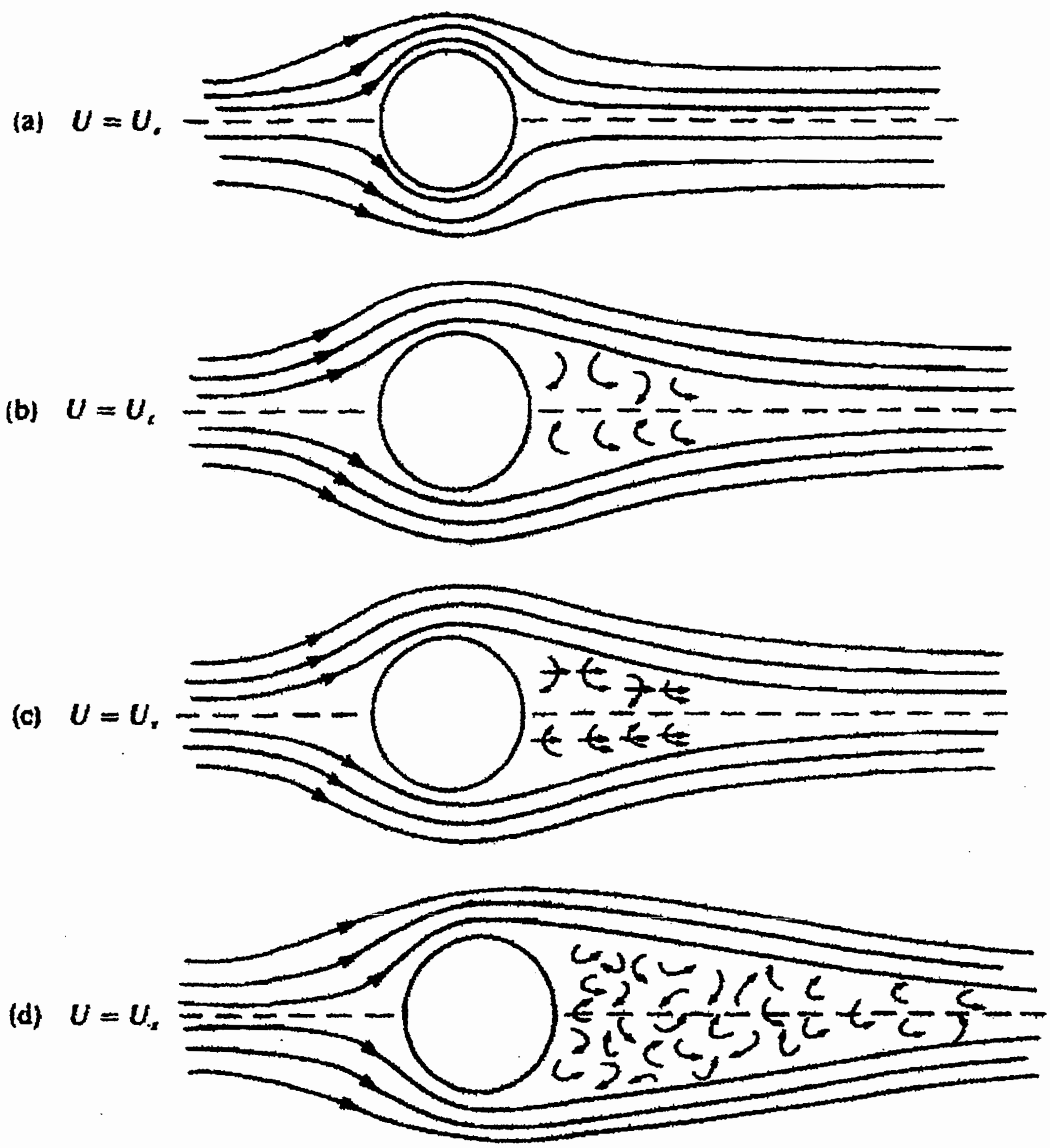
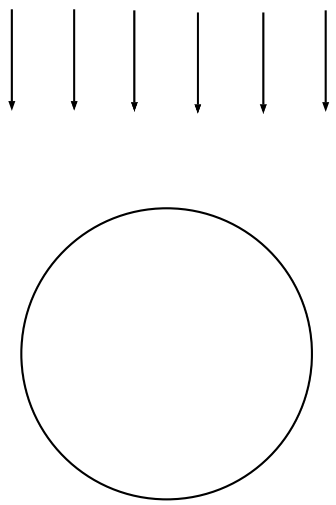


Figure 2.2. Flow past a sphere. (a) Laminar flow (small Reynolds number). (b) Appearance of the von Kármán vortices in the wake behind the sphere (stationary flow). (c) Time-periodic flow: the vortices behind the sphere are moving to the right in an (apparently) time-periodic manner. (d) Fully turbulent flow in the wake behind the sphere at large Reynolds numbers.



Joint work with C.Y. Jung

[JT11] C.Y. Jung and R. Temam, Convection-diffusion equations in a circle: The compatible case, *J. Math. Pures Appl.*, **96**, 2011, 88-107.

[JT12] C.Y. Jung and R. Temam, Convection-diffusion equations in a circle: The generic non compatible case, *SIAM Journal on Mathematical Analysis*, **44**, No. 6, 2012, 4274-4296, DOI: 10.1137/110839515.

[JT14] C.-Y. Jung and R. Temam, Boundary layer theory for convection-diffusion equations in a circle, special volume in memory of Mark Vishik, *Russian Mathematical Survey*, **69**:3, 2014, 435-480. *Uspekhi Mat. Nauk*, **69**:3, 2014, 43-86. DOI:10.4213/rm9584.

Also

[JT06] C.Y. Jung and R. Temam, On parabolic boundary layers for convection diffusion equations in a channel: Analysis and numerical applications, *J. of Scientific Computing*, **28**, No. 1, 2006, 361-410.

1. Introduction. The Compatibility Conditions

The model problem

(1)
$$\begin{cases} -\varepsilon\Delta\, u^\varepsilon - u^\varepsilon_y = f \quad \text{in} \quad \Omega \\ \cdot \qquad u^\varepsilon = 0 \quad \text{on} \quad \partial\Omega. \end{cases}$$

(2)
$$\begin{cases} - \;\; u^0_y = f \quad \text{in} \quad \Omega \\ \quad u^0 = 0 \quad \text{on} \quad \Gamma_u = \partial\Omega \cap \{y > 0\} \end{cases}$$

Ω is the circle $x^2 + y^2 < 1$.

Motivation: Linearization of the stationary Navier-Stokes equations at small viscosity ε, around the flow $(0, -1)$. Hence the problem serves as a model of the difficulties with the Navier-Stokes equations at vanishing viscosity.

References

Bardos, 1970, $u^\varepsilon \to u^0$ as $\varepsilon \to 0$ (semigroup theory evolution problem)

Levinson, 1950, $\varepsilon \Delta u + a(x,y)u_x + b(x,y)u_y + c(x,y) = d(x,y)$

(maximum principle)

Eckhaus and Jager, 1966, very partial results

J. Grassman, 1971.

Temme, 2007, using explicit solution with Bessel functions.

Note: *None of these articles describes the behavior of* u^ε *near the characteristic points* $(\pm 1, 0)$.

Related (“easier") cases

1) $\Omega =$ channel, $x \in \mathbb{R}, 0 < y < 1$

u^ε, u^0 are periodic in x, $u^0 = 0$ at $y = 1$,

$u^\varepsilon \to u^0$ in $L^2(\Omega)$, $u^\varepsilon \nrightarrow u^0$ in $H^1_0(\Omega)$

$u^\varepsilon - u^0 - \theta^\varepsilon \to 0$ in $H^1_0(\Omega)$

where θ^ε is a *classical* corrector (corrector of the boundary condition of u^0):

$$\theta^\varepsilon = \bar{\theta}^\varepsilon + e.s.t., \;\; \bar{\theta}^\varepsilon = -u^0(0,y)e^{-y/\varepsilon} - u(1,y)e^{-(1-y)/\varepsilon}$$

with a boundary layer of thickness ε.

Easier cases (continued)

2) $\Omega =$ channel $0 < x < 1, y \in \mathbb{R}$,

u^ε, u^0 are periodic in y, $\int_0^1 u^0(x,y)dy = 0$,

$u^\varepsilon \to u^0$ in $L^2(\Omega), u^\varepsilon \nrightarrow u^0$ in $H_0^1(\Omega)$

$u^\varepsilon - u^0 - \varphi^\varepsilon \to 0$ in $H_0^1(\Omega)$

where φ^ε is a *parabolic* corrector $\varphi^\varepsilon = \bar{\varphi}^\varepsilon +$ e.s.t. $, \bar{\varphi}^\varepsilon = \bar{\varphi}_\ell^\varepsilon + \bar{\varphi}_u^\varepsilon$ where e.g. $\bar{\varphi}_\ell^\varepsilon = \varphi(\frac{x}{\sqrt{\varepsilon}}, y)$ solves the parabolic equation

$$\begin{cases} -\varepsilon\varphi_{xx} - \varphi_y = 0 \\ \quad \varphi = -u^0(0,y) \text{ at } x = 0, \ \ \varphi \to 0 \text{ as } x \to +\infty \\ \quad \varphi \text{ is periodic in } y \end{cases}$$

The Circle

First difficulty:

u^0 is generally singular at the characteristic points $(\pm 1, 0)$:

$u^0_y = 0$ at these points but not f in general

These singularities are not present if

$$f(-1,0) = f(1,0) = 0$$

The Compatibility Conditions

More generally

Theorem 1

If $f \in \mathcal{C}^\infty(\bar{\Omega})$ is infinitely flat at $(\pm 1, 0)$, then $u^0 \in \mathcal{C}^\infty(\bar{\Omega})$ and u^0 is infinitely flat at $(\pm 1, 0)$.
Proof.
We need to estimate $\partial^{i+m} u^0 / \partial x^i \partial y^m$ near the characteristic points. But, for $m \geq 1$

$$\frac{\partial^{i+m} u^0}{\partial x^i \partial y^m} = -\frac{\partial^{i+m-1} f}{\partial x^i \partial y^{m-1}},$$

is smooth and so we only need to consider the case where $m = 0$.

Then, we write

$$u^0 = \int_y^{C_u(x)} f(x,s)ds = C_u(x)^{2M} \int_y^{C_u(x)} \tilde{f}(x,s)ds, \tag{3}$$
$$\tilde{f}(x,y) = C_u(x)^{-2M} f(x,y), C_u(x) = \sqrt{1-x^2}, M \text{ integer},$$

$\tilde{f}$ is also in $\mathcal{C}^\infty(\bar{\Omega})$, and infinitely flat at $(\pm 1, 0)$.
Then we differentiate the RHS of (1) to compute $\partial^i u^0/\partial x^i$, observing that, for $-1 < x < 1$

$$\frac{d^n C^m(x)}{dx^n} \leq \kappa_n \begin{cases} 1, \text{ if } m \text{ is even}, n \leq m \ (0 \text{ if } n > m) \\ C_u^{m-2n}(x) \text{ if } m \text{ is odd} \end{cases} \tag{4}$$

The full outer expansion

$$u^\varepsilon \simeq \sum_{j=0}^{\infty} \varepsilon^j u^j \tag{5}$$

with $u^{-1} = f$ and for $j \geq 1$ $\begin{cases} -u^j_y & = \Delta u^{j-1} \quad \text{in } \Omega \\ u^j & = 0 \quad \text{on} \quad \Gamma_u \end{cases}$

$$u^j(x,y) = \int_y^{c_u(x)} \Delta u^{j-1}(x,s)ds$$

Similarly

(6) $\left\{ \begin{array}{l} \text{If } f \in \mathcal{C}^\infty(\bar{\Omega}) \text{ is infinitely flat at } (\pm 1, 0), \text{ then} \\ \text{all the } u^j \text{ belong to } \mathcal{C}^\infty(\bar{\Omega}) \text{ and are infinitely} \\ \text{flat at } (\pm 1, 0). \end{array} \right.$

Intermediate levels of regularity
They require compatibility (flatness) conditions of the type

$$(7) \qquad \frac{\partial^{p+q} f}{\partial x^p \partial y^q} = 0 \text{ at } (\pm 1, 0), 0 \leq 2p + q \leq 1 + 3j, p, q \geq 0.$$

And we have e.g.

$$(8) \qquad \begin{cases} \text{If (7) holds for } 0 \leq 2p + q \leq 2(r-1) + 3j \\ \text{ (no condition if RHS } \leq 0) \text{ then} \\ u^j \in \mathcal{C}^r(\bar{\Omega}). \end{cases}$$

We call this the *compatible case studied in [JT11]*.

2. The Compatible Case [JT11]

First corrector for the compatible case
We look for θ^0 such that

$$u^\varepsilon - u^0 - \theta^0 \to 0 \quad \text{in} \quad H_0^1(\Omega) \text{ as } \varepsilon \to 0$$

We introduce the boundary-fitted (polar) coordinates

$$\begin{aligned} &x = (1-\xi)\cos\eta, \quad y = (1-\xi)\sin\eta, \\ &0 < \eta < 2\pi, \quad 0 < \xi = 1 - r < 1 \end{aligned}$$

Writing the equation (1) in these coordinates we find that the boundary layer is $\mathcal{O}(\varepsilon^{1/2})$ and that the first corrector is solution of

$$\text{(9)} \qquad \begin{cases} -\varepsilon \frac{\partial^2\theta^0}{\partial\xi^2} + \sin\eta \frac{\partial\theta^0}{\partial\xi} = 0, \quad \text{for } \xi > 0, \pi < \eta < 2\pi, \\ \theta^0 = -u^0(\cos\eta, \sin\eta) \quad \text{at } \xi = 0, \\ \theta^0 \to 0 \text{ as } \xi \to +\infty. \end{cases}$$

Hence $\theta^0 = \bar{\theta}^0 +$ e.s.t., with

$$\bar{\theta}^0 = -u^0(\cos\eta, \sin\eta)\exp(\frac{\sin\eta}{\varepsilon}\xi)\delta(\xi)\chi_{[\pi,2\pi]}(y),$$

$\delta = 1$ for $\xi \in (0, 1/4)$, $= 0$ for $\xi \in (1/2, 1)$.

Convergence Theorem (at order 0)

Theorem 2

Assume that f is sufficiently regular and

(10)
$$f = 0 \quad \text{at} \quad (\pm 1, 0).$$

Then, as $\varepsilon \to 0$:

(11)
$$\begin{cases} |u^\varepsilon - u^0 - \bar{\theta}^0|_{L^2(\Omega)} \leq \kappa \varepsilon^{1/2}, \\ |u^\varepsilon - u^0 - \bar{\theta}^0|_{H^1(\Omega)} \leq \kappa, \end{cases}$$

and thus

(12)
$$|u^\varepsilon - u^0|_{L^2(\Omega)} \leq \kappa \varepsilon^{1/2}.$$

3. The generic non compatible case [JT12]

The non compatible case

This occurs when $f \neq 0$ at $(\pm 1, 0)$, and more generally (at higher orders), when f is not sufficiently flat at $(\pm 1, 0)$.
Using the Taylor expansion $\hat{f}$ of f near the characteristic points $(\pm 1, 0)$ (up to a certain order), we write

$$f = \hat{f} + (f - \hat{f}),$$

where $f - \hat{f}$ is compatible (up to a certain order).
The previous results on the compatible case apply to $f = (f - \hat{f})$, and, by linearity, we are led to consider the case where $f = \hat{f}$ is a polynomial in x, y.

Further reduction

We are led to consider $f = \hat{f} = (1 \pm x)C_u(x)^{2p}y^q, p, q \geq 0$ integers. With an emphasis on the point $(\pm 1, 0)$, we consider:

$$\hat{f} = (1 + x)C_u(x)^{2p}y^q = (1 + x)(1 - x^2)^p y^q, \tag{13}$$

for which

$$u^0(x, y) = \frac{1}{q+1}(1 + x)C_u(x)^{2p}(C_u(x)^{q+1} - y^{q+1}). \tag{14}$$

- When q is odd, u^0 is smooth and vanish on all of $\partial\Omega \ (y = \pm C_u(x))$.

- When q is even, the derivatives of u^0 are singular at $(\pm 1, 0)$.

Indeed using $C_u(x)' = -xC_u(x)^{-1}$ near $x = 1$, we find that for $p = q = 0, u^0_x = \mathcal{O}((1-x)^{-1/2}), u^0_{xx} = \mathcal{O}((1-x)^{-3/2})$.

Performing a further reduction, we consider a cut off function ρ with

$$0 \leq \rho \leq 1 \quad \text{for} \quad 1 - \tilde{\sigma} \leq x \leq 1, \quad \text{and}$$
$$\rho = 1 \quad \text{for} \quad -1 \leq x \leq 1 - \tilde{\sigma}, \tilde{\sigma} = 1 - \cos\sigma.$$

Writing (with $\hat{f}$ as in (13))

$$\hat{f} = f^* + f^{**}, \; f^*(x,y) = \rho(x)\check{\rho}(x)\hat{f}(x,y),$$

we see that f^* vanishes identically near $(\pm 1, 0)$ and is thus a “compatible" data.

Direction
We will now study separately the solutions $u^{*\varepsilon}$ and $u^{**\varepsilon}$ corresponding to f^* and to f^{**}.

Study of $u^{*\varepsilon}, f^*$

We thus have

$$(15) \qquad \begin{cases} -u_y^{*\,0} = \; f^* = \; \rho(x)\check{\rho}(x)C_u(x)^{2p} \; y^q \quad \text{in } \Omega \\ u^{*\,0} = 0 \quad \text{on} \quad \Gamma_u \end{cases}$$

and for $j \geq 1$

$$(16) \qquad \begin{cases} -u_y^{*\,j} = \Delta u^{*(j-1)} \quad \text{in} \quad \Omega \\ u^{*\,j} = 0 \quad \text{on} \quad \Gamma_u \end{cases}$$

Suitable estimates on the $u^{*\,j}$ are derived in [JT12].

Correctors

To compare $u^{*\varepsilon}$ to u^{*0}, we need to introduce suitable correctors

Zeroth order corrector
As before the zeroth-order corrector θ^{*0} reads

$$\theta^{*0} = -u^{*0}(\cos\eta, \sin\eta) exp \ (\frac{sin \ \eta}{\varepsilon}\xi)\chi_{[\pi,2\pi]}(\eta),$$

which is solution of

$$-\varepsilon\frac{\partial^2\theta^{*0}}{\partial\xi^2} + \sin\eta\frac{\partial\theta^{*0}}{\partial\xi} = 0,$$

and the simplified corrector (up to an e.s.t.) reads:

$$\bar{\theta}^{*0} = \theta^{*0}\delta(\xi)$$

$\delta = 0$ if $\xi \geq 1/2, = 1$ if $\xi \leq 1/4$.

High order correctors

More generally, we derived in [JT12] the correctors $\theta^{*j}, \bar{\theta}^{*j}$ associated with $u^{*\varepsilon}$ and f^*:
We find

$$\theta^{*j} = P^{*j}(\eta, \bar{\xi}) \exp\left((\sin\eta)\bar{\xi}\right) \chi_{[\pi,2\pi]}(\eta),\ j \geq 0, \tag{17}$$

where

$$P^{*j}(\eta, \bar{\xi}) = \sum_{i=0}^{j} \sum_{k=0}^{2j-2i} a^*_{i,3j-3i-k}(\eta)\bar{\xi}^k, \tag{18}$$

$$a^*_{i,q}(\eta) = \sum_{\substack{m+r\leq q,\\ m,r\geq 0}} \frac{c_{m,r}}{\sin^m \eta} \frac{d^r v^{*i}(\eta)}{d\eta^r}, \tag{19}$$

and $v^{*j}(\eta) = -u^{*j}(\cos\eta, \sin\eta)$. Here the coefficients $c_{m,r} = c_{m,r}(\eta) \in C^\infty([0, 2\pi))$.

The analysis concerning $f^* = \rho(x)\check{\rho}(x)(1+x)C_u(x)^{2p}y^q$ was fully developed at any order in [JT12] and the following theorem was proven therein:

Theorem 3

For $p, q \geq 0$, let $u^{*\varepsilon}$ be the solution of Eq. (1), and let $u^*_{\varepsilon n} = \sum_{j=0}^{n} \varepsilon^j u^{*j}$, and $\bar{\theta}^*_{\varepsilon n} = \sum_{j=0}^{n} \varepsilon^j \bar{\theta}^{*j}$ where the u^{*j} are as in (5) and $\bar{\theta}^{*j} = \theta^{*j}\delta(\xi)$ is the approximate form of θ^{*j} as in (17), with f replaced by f^* everywhere.

Then we have:

(20)

$$\|u^\varepsilon - u^*_{\varepsilon n} - \bar{\theta}^*_{\varepsilon n}\|_\varepsilon \leq$$

$$\kappa\varepsilon^{n+1} \begin{cases} \sigma^{-3n+2p+q}(\sigma^{-1} + \min\{\varepsilon^{-\frac{1}{2}}, \sigma^{-2}\}) & \text{if } 2p+q \leq 3n-1, \\ \sigma^{-1} + \min\{\varepsilon^{-\frac{1}{2}}(-\ln\sigma)^{\frac{1}{2}}, \sigma^{-2}\} & \text{if } 2p+q = 3n, \\ (-\ln\sigma)^{\frac{1}{2}} + \min\{\varepsilon^{-\frac{1}{2}}, \sigma^{-1}\} & \text{if } 2p+q = 3n+1, \\ \min\{\varepsilon^{-\frac{1}{2}}, (-\ln\sigma)^{\frac{1}{2}}\} & \text{if } 2p+q = 3n+2, \\ 1 & \text{if } 2p+q \geq 3n+3. \end{cases}$$

with $||u||_\varepsilon = \sqrt{\varepsilon}|\nabla u|_{L^2} + |u|_{L^2}$.

Study of $u^{**\varepsilon}$ and f^{**}, with $f^{**} = \hat{f} - f^*$

Reminder: f^{**} is supported near $(1, 0)$ $(1 - \sigma < x < 1)$ and $\hat{f}$ is a typical monomial term of the Taylor expansion $\hat{f}$ of f (at a certain order).

Direction: It is suggested by Grasman, to introduce the following stretched variables near $(+1, 0)$:

$$\hat{\xi} = \frac{\xi}{\varepsilon^{2/3}} \ , \ \ \hat{\eta} = \frac{\eta}{\varepsilon^{1/3}},$$

and

$$\bar{\xi} = \frac{\xi}{\varepsilon} \ , \ \ \bar{\eta} = \frac{\eta}{\varepsilon}.$$

From this we derive a *parabolic boundary layer*.

Parabolic boundary layer at order 0

The equations for the correctors $\varphi^0, \tilde{\varphi}^0$ read

$$-\varepsilon\frac{\partial^2\varphi^0}{\partial\xi^2}+\eta\frac{\partial\varphi^0}{\partial\xi}-\frac{\partial\varphi^0}{\partial\eta}=0, \tag{21}$$

$$-\varepsilon\frac{\partial^2\tilde{\varphi}^0}{\partial\eta^2}-\varepsilon\frac{\partial^2\tilde{\varphi}^0}{\partial\xi^2}-\frac{\partial\tilde{\varphi}^0}{\partial\eta}=0. \tag{22}$$

However our analysis is different than that of Grasman, and we only need $\varphi^0(=\theta^{**0})$.
Similarly we construct and estimate in [JT12] the correctors φ^j $(=\theta^{**j})$ at all orders.

Convergence analysis for $u^{*\varepsilon}, f^*$

Let $u^{**\varepsilon}$ be the solution of

(23)

$$\begin{cases} L_\varepsilon u^{**\varepsilon} = \hat{f} - f^* = f^{**} = (1-\rho(x)\check{\rho}(x))(1+x)C_u(x)^{2p}y^q \text{ in } D, \\ u^{**\varepsilon} = 0 \text{ on } \partial D. \end{cases}$$

Let

(24)
$$\tilde{\varphi}_{\varepsilon n} = \sum_{i=0}^{n} \sigma^{2i}\varphi^{2p+q+2i}\delta(\xi)\rho(\eta).$$

Our results are as follows:

Theorem 4

Let $u^{**\varepsilon}$ be the solution of Eq. (23), let $\bar{\varphi}_{\varepsilon n}$ be as in (24), and assume that

$$\sigma^6 \leq \varepsilon \leq \sigma^3 \iff \varepsilon^{1/3} \leq \sigma \leq \varepsilon^{1/6}. \tag{25}$$

Then there exists a constant $\kappa > 0$ independent of σ, ε such that

$$\|u^\varepsilon - \bar{\varphi}_{\varepsilon n}\|_\varepsilon \leq \kappa(\varepsilon^{-1}\sigma^6)^{\frac{n}{2}}\varepsilon^{-\frac{1}{2}}\sigma^{2p+q+3}. \tag{26}$$

Then we deduce the following theorem.

Theorem 5

Let $\sigma = \varepsilon^{\frac{1}{4}}$ which is consistent with (25), and let $u^{**\varepsilon}$ be the solution of Eq. (23). Then there exists a constant $\kappa > 0$ independent of ε such that

$$\|u^{*\varepsilon} - u^*_{\varepsilon n} - \bar{\theta}^*_{\varepsilon n} - \bar{\varphi}_{\varepsilon n}\|_\varepsilon \leq \kappa \varepsilon^{\frac{n+2p+q+1}{4}}, \tag{27}$$

where for $2p + q < 3n$, $\bar{\varphi}_{\varepsilon n} = \sum_{i=0}^{n} \sigma^{2i} \bar{\varphi}^{,2i} = \sum_{i=0}^{n} \varepsilon^{\frac{i}{2}} \bar{\varphi}^{,2i}$, and for $2p + q \geq 3n$, $\bar{\varphi}_{\varepsilon n} = 0$.

3. The general non compatible case

We wrote $f = f - \hat{f} + \hat{f}$, with $\hat{f}$ polynomial (Taylor expansion), $\hat{f} = f^* + f^{**}$. We now want to consider the case where f is replaced by $f - \hat{f}$ and then collect all results. This is done in [JT14].

Remark We could think at writing directly

$$f = \rho\hat{\rho}f + (1 - \rho\hat{\rho})f$$

where $\rho\hat{\rho}f$ is flat at $(\pm 1, 0)$. However in this case we cannot estimate the correctors which are not explicit, unlike the case when $f = \hat{f}$ is polynomial.

4. Summary of results at first order

• For f sufficiently flat (satisfying $f = 0$ at $(\pm 1, 0)$) :

$$||u^\varepsilon - u^0 - \bar{\theta}^0||_\varepsilon \leq \kappa \varepsilon^{1/2},$$

implying

$$|u^\varepsilon - u^0|_{L^2} \leq \kappa \varepsilon^{1/2},$$

where

$$||g||_\varepsilon = \sqrt{\varepsilon}|\nabla g|_{L^2} + |g|_{L^2}$$

• For f smooth but not flat

$$||u^\varepsilon - u^0 - \bar{\theta}^0 - \bar{\vartheta}^0||_\varepsilon \leq \kappa_\varepsilon^{4/7}$$

with a suitable $\bar{\vartheta}^0$ constructed using φ^0. This implies convergence in H^1 :

$$||u^\varepsilon - u^0 - \bar{\theta}^0 - \bar{\vartheta}^0||_{H^1} \leq \kappa\varepsilon^{1/14}.$$

Conclusion

What to retain?

- The difficulties related to the (non) flatness of f
- The semi-classical boundary layer(s) on Γ_ℓ.
- The parabolic boundary layer(s) at the characteristic points

And finally: this is a very “simplified"(linearized) version of the stationary Navier Stokes equations in a circle, in 2D.

IPAMCIRCLE2014.tex