Wireless Network Information Flow

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ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

Network communication challenges

Information flow over shared networks

- Unicast, multicast, multiple unicast (multicommodity flow).
- Significant progress for graphs (routing, network coding etc).
- Less understood for flows over wireless networks.

Network data compression: Motivation $\longrightarrow$ sensor networks.

- Some successes in side-information coding: Slepian-Wolf, Wyner Ziv etc.
- Many unresolved questions: Distributed source coding, multiple description coding.

Question: How can we make progress to fundamentally characterize flow of information over networks?

Key distinctions between wired and wireless channels

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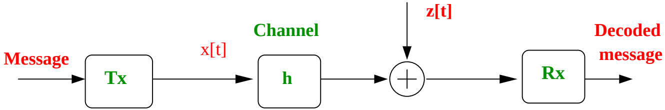
Question: Can we develop cooperative mechanisms to utilize signal interaction?

Gaussian point-to-point channel



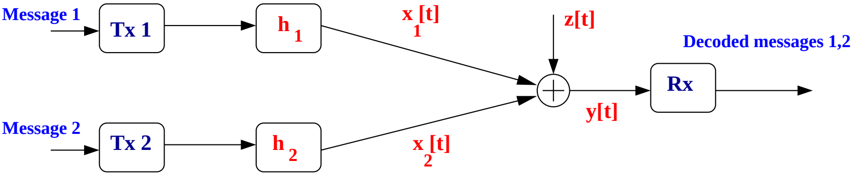
NOISY POINT−TO−POINT CHANNEL (SHANNON, 1948)

$$y[t] = hx[t] + z[t]$$

- Linear channel model and additive noise model.
- Transmit power constraint: $\mathbb{E}[|x|^2] \leq P$
- Gaussian channel due to noise $z[t] \sim \mathcal{N}(0, 1)$.

Capacity: $C = \frac{1}{2}\log(1 + |h|^2 P)$.

Gaussian multiple access channel



MULTIPLE ACCESS CHANNEL: Ahlswede−Liao (1971)

$$y[t] = h_1 x_1[t] + h_2 x_2[t] + z[t]$$

- Linear signal interaction model.
- Transmit power constraint: $\mathbb{E}[|x_i|^2] \leq P_i, \; i = 1, 2$
- Gaussian channel due to noise $z[t] \sim \mathcal{N}(0, 1)$.

$$R_1 \leq \frac{1}{2}\log(1 + |h_1|^2 P_1) \qquad R_2 \leq \frac{1}{2}\log(1 + |h_2|^2 P_2)$$

$$R_1 + R_2 \leq \frac{1}{2}\log(1 + |h_1|^2 P_1 + |h_2|^2 P_2)$$

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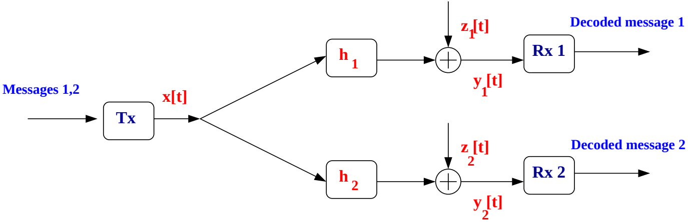
Gaussian broadcast channel



BROADCAST CHANNEL: Cover (1972)

$$y_1[t] = h_1 x[t] + z_1[t], \qquad y_2[t] = h_2 x[t] + z_2[t]$$

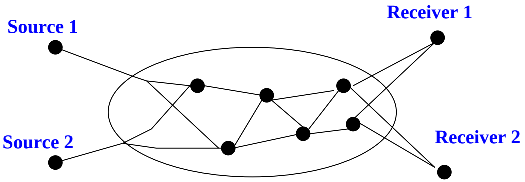
- Common transmit signal $x[t]$, different channels.
- Transmit power constraint: $\mathbb{E}[|x|^2] \leq P$.

$$R_1 \leq \frac{1}{2}\log(1 + \frac{|h_1|^2\theta P}{1+|h_1|^2(1-\theta)P})$$

$$R_2 \leq \frac{1}{2}\log(1 + |h_2|^2(1-\theta)P)$$

Rate region region for $|h_1| \geq h_2|$

Signal interaction: Gaussian wireless networks



$$\mathbf{y}_j[t] = \sum_i \mathbf{H}_{ij}\mathbf{x}_i[t] + \mathbf{z}_j[t]$$

- **Broadcast** because transmission $\mathbf{x}_i$ is heard by all receivers.
- **Multiple access** because transmitted signals from all nodes mix linearly at the receiver j.
- **Dynamic range** depends on relative "strengths" of $\mathbf{H}_{ij}$.

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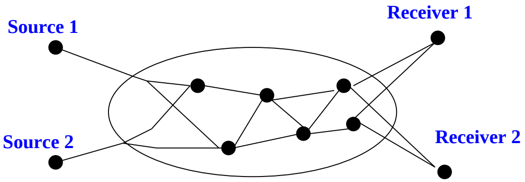

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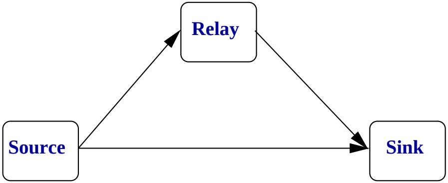
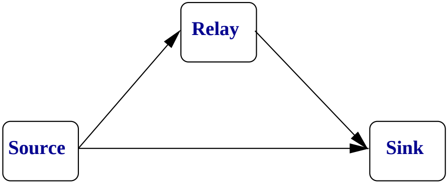
Question: Can we characterize capacity of such networks?

Gaussian network capacity: unresolved



RELAY CHANNEL: Cover, El−Gamal (1979)

Gaussian network capacity: unresolved



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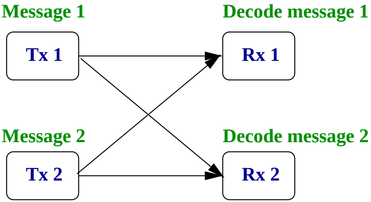
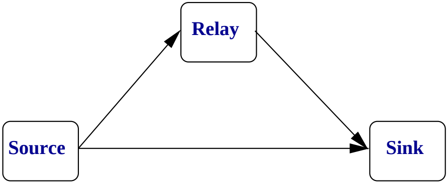


INTERFERENCE CHANNEL: Han−Kobayashi (1981)

Gaussian network capacity: unresolved



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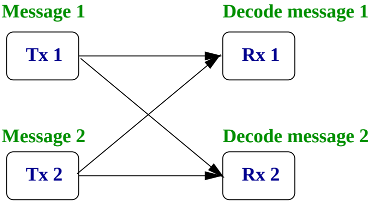

INTERFERENCE CHANNEL: Han−Kobayashi (1981)

Question: Thirty years have gone by... How can we make progress from here?

Simplify the model

Focus on signal interaction not noise

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- Many wireless systems are interference rather than noise limited.
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- Deterministic models more tractable.
- Use insight to obtain approximate characterizations for noisy (Gaussian) networks.

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Question: Can we develop relevant models and analyze networks with deterministic signal interactions to get the insights?

Approximate characterizations

Philosophy: Gain insight into central difficulties of problem by identifying underlying deterministic structures.

Goal: Use the insight of underlying problem to get (provable) approximate characterization for noisy problem.

- Underlying problem should be characterized exactly to give insight into solution structure for general case.
- **Universal approximation:** Approximation should depend *only* on the problem structure and *not* on parameters (like channel gains, SNRs etc.).

Question: Can we identify the appropriate underlying problems and use them to get provable (universal) approximations.

Overall agenda

- Introduce deterministic channel model.
- Motivate the utility of deterministic model with examples.
- Develop achievable rates for general deterministic relay networks
- Characterizations for linear finite field deterministic models.

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- Introduce deterministic channel model.
- Motivate the utility of deterministic model with examples.
- Develop achievable rates for general deterministic relay networks
- Characterizations for linear finite field deterministic models.
- **Connection to wireless networks:** Use insights on achievability of deterministic networks to obtain *approximate* characterization of noisy relay networks.

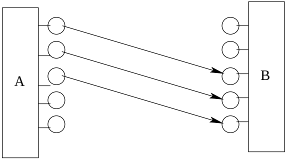
Example 1: Point-to-point link

Gaussian

$$y = 2^{\alpha/2} x + z$$

Capacity is $\log(1 + 2^{\alpha}) \approx \alpha \log 2$ assuming unit variance noise.

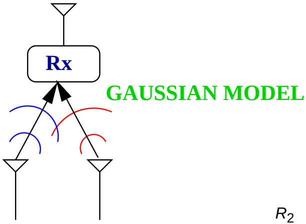
Deterministic



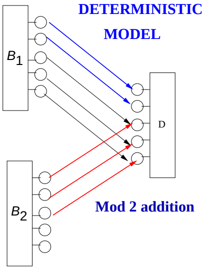
Receiver observes α most significant bits of transmitted signal.

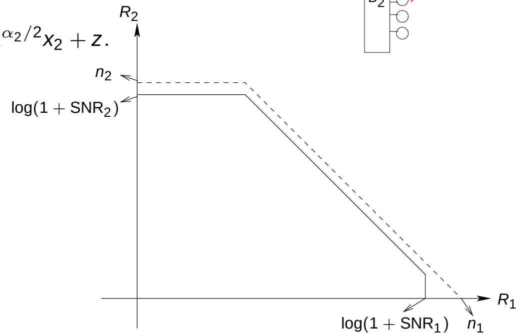
- Number of levels received shows scale of channel strength.
- Scale important when signals interact in broadcast and multiple access.

Example 2: Multiple access channel

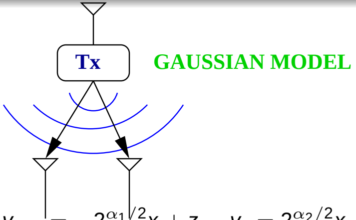


$$y = 2^{\alpha_1/2} x_1 + 2^{\alpha_2/2} x_2 + z.$$

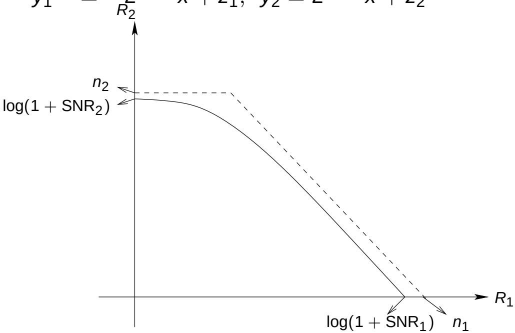


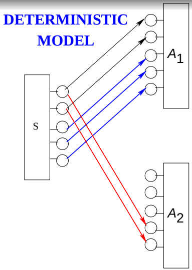


Example 3: Scalar broadcast channel



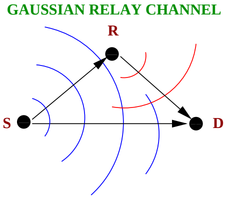
$$y_1 = 2^{\alpha_1/2}x + z_1, \quad y_2 = 2^{\alpha_2/2}x + z_2$$





Approximation of 1 bit

Relay channel: deterministic approximation



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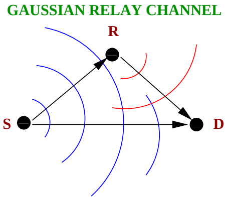

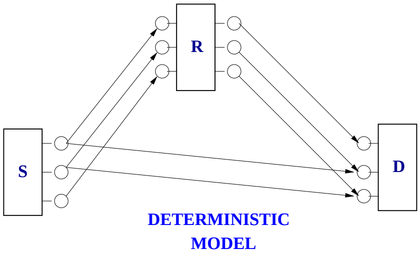

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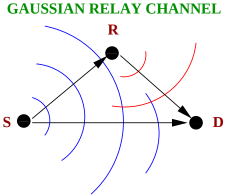

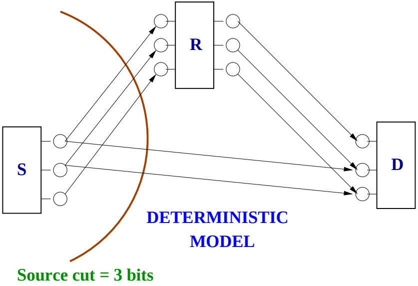

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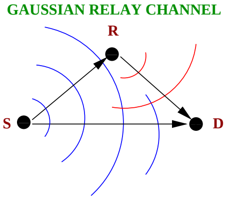

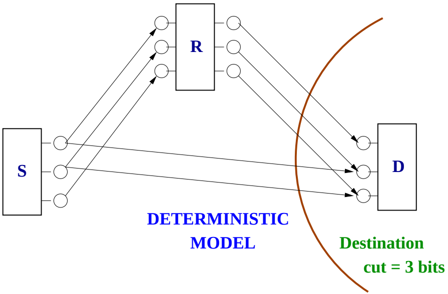

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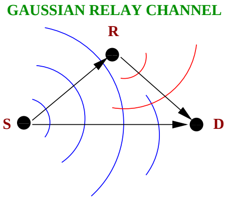

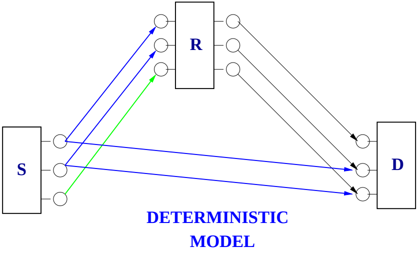

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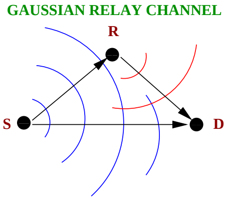

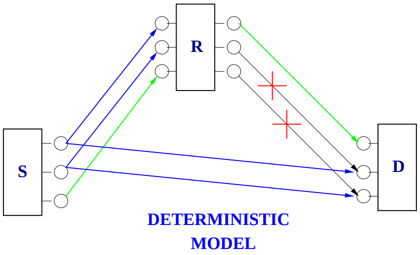

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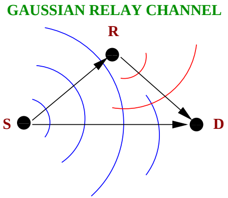
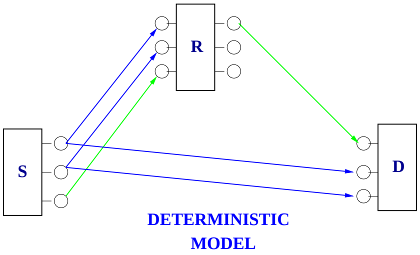
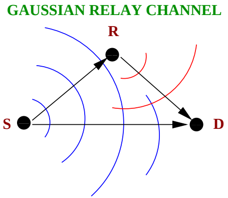
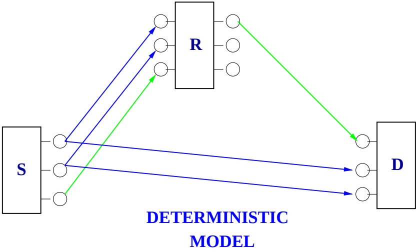




Relay channel: deterministic approximation



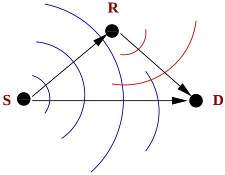


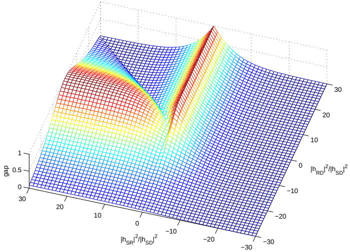
Cut-set bound achievable.

Decode and forward is optimal.

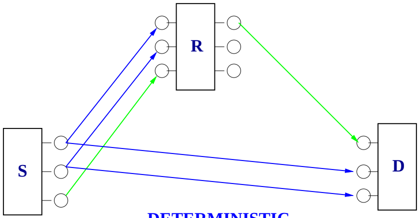
Relay channel: deterministic approximation

GAUSSIAN RELAY CHANNEL





Result: Gap from cut-set less than 1 bit, on average much less.

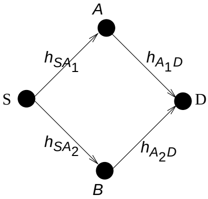


DETERMINISTIC MODEL

Cut-set bound achievable.
Decode and forward is optimal.

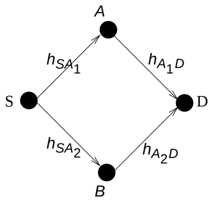
Diamond network

Gaussian



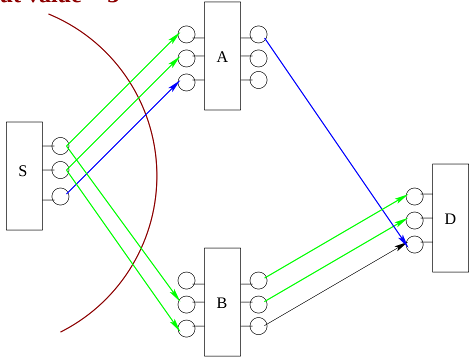
Diamond network

Gaussian



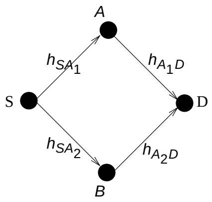
Deterministic

Cut value = 3

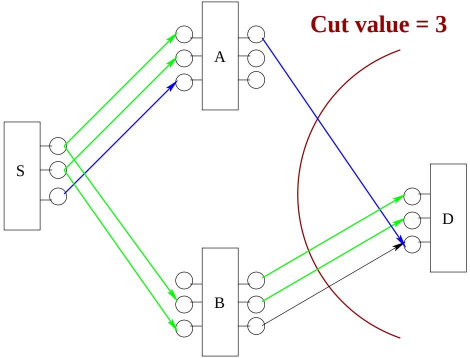


Diamond network

Gaussian

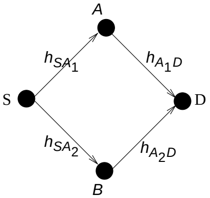


Deterministic



Diamond network

Gaussian



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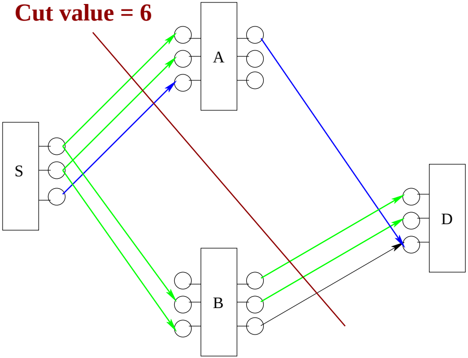

Diamond network

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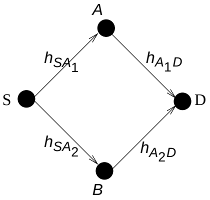

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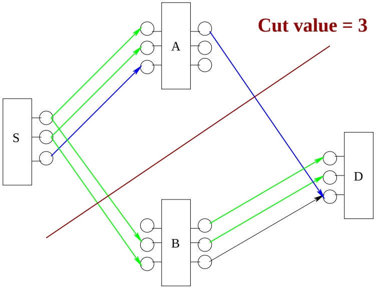

Diamond network

Deterministic

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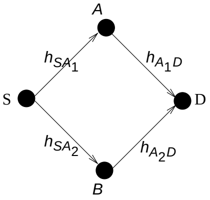
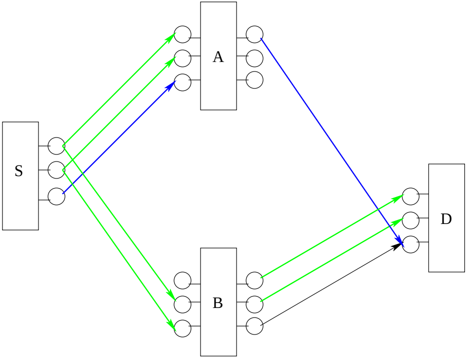


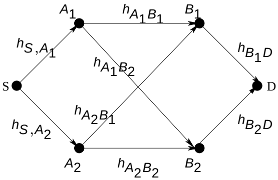


Result: Gap from cut-set less 1 bit.

Cut-set bound achievable.
Partial decode-forward is optimal.

Two-layer network

Gaussian



Deterministic

Two-layer network

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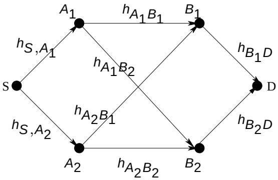
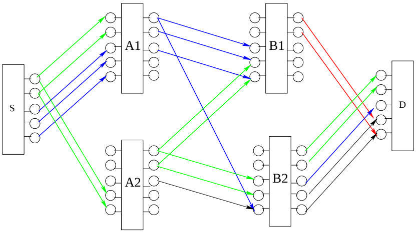


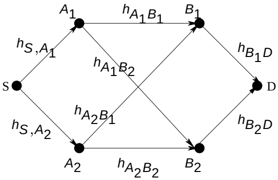
Deterministic



Cut-set bound achievable.
Linear map and forward is optimal.

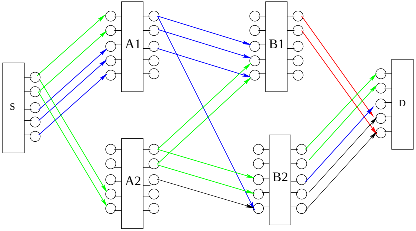
Two-layer network

Gaussian



Result: Gap from cut-set less than constant number of bits.

Deterministic



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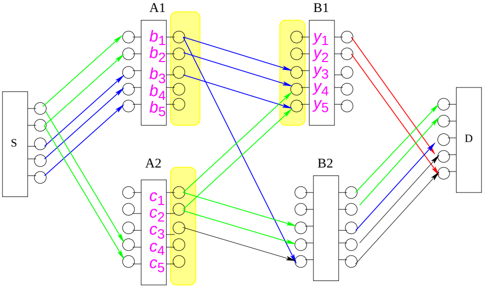
- What is the structure of the optimal strategy?
- Can we use insight from deterministic analysis to get approximately optimal strategy for Gaussian networks?

Algebraic representation



$$\mathbf{S} = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \end{bmatrix}$$

$\mathbf{S}$ is shift matrix of size $q = \max_{i,j} n_{i,j}$.

$$\mathbf{y}_{B_1} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix} \oplus \begin{bmatrix} 0 \\ 0 \\ 0 \\ c_1 \\ c_2 \end{bmatrix} = \mathbf{S}^{5-3}\mathbf{x}_{A_1} \oplus \mathbf{S}^{5-2}\mathbf{x}_{A_2} = \mathbf{S}^{5-3}\mathbf{b} \oplus \mathbf{S}^{5-2}\mathbf{c}$$

Generalizations

Linear finite field model

- Channel from i to j is described by channel matrix $\mathbf{G}_{ij}$ operating over $\mathbb{F}_2$.
- Received signal at node j:

$$\mathbf{y}_j[t] = \sum_{i=1}^{N} \mathbf{G}_{ij}\mathbf{x}_i[t]$$

- Special case: our model given in examples

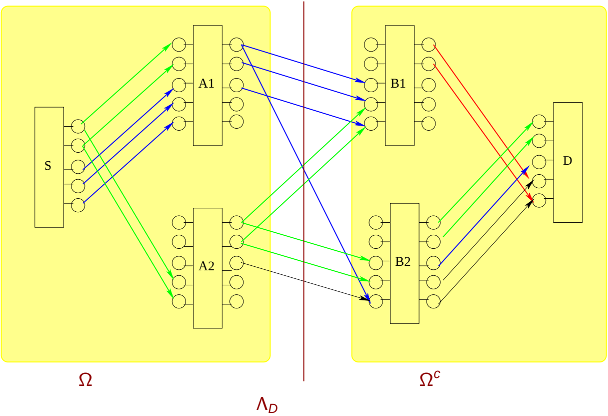
$$\mathbf{G}_{ij} = \mathbf{S}^{q-\alpha_{ij}}$$

General deterministic network:

$$\mathbf{y}[t] = \mathbf{G}(\mathbf{x}_1[t], \ldots, \mathbf{x}_N[t])$$

Observation: Wireline networks are a special case.

Information-theoretic cut-set



Cut: Separates S from D

Cut transfer matrix $\mathbf{G}_{\Omega,\Omega^c}$: Transfer function from nodes in Ω to Ω^c.

Cutset upper bound

General relay network:

$$C_{\text{relay}} \leq \bar{C} = \max_{p(\mathbf{X}_1,\ldots,\mathbf{X}_N)} \min_{\Omega} I(\mathbf{X}_{\Omega};\mathbf{Y}_{\Omega^c}|\mathbf{X}_{\Omega^c})$$

Cutset upper bound

General relay network:

$$C_{\text{relay}} \leq \bar{C} = \max_{p(\mathbf{X}_1,\ldots,\mathbf{X}_N)} \min_{\Omega} I(\mathbf{X}_{\Omega}; \mathbf{Y}_{\Omega^c} | \mathbf{X}_{\Omega^c})$$

General deterministic relay network:

$$C_{\text{relay}} \leq \bar{C} = \max_{p(\mathbf{X}_1,\ldots,\mathbf{X}_N)} \min_{\Omega} H(\mathbf{Y}_{\Omega^c} | \mathbf{X}_{\Omega^c})$$

Cutset upper bound

General relay network:

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General deterministic relay network:

$$C_{\mathrm{relay}} \leq \bar{C} = \max_{p(\mathbf{X}_1,\ldots,\mathbf{X}_N)} \min_{\Omega} H(\mathbf{Y}_{\Omega^c} | \mathbf{X}_{\Omega^c})$$

Linear finite field network: Optimal input distribution $\mathbf{x}_1, \ldots, \mathbf{x}_N$ independent and uniform

$$C_{\mathrm{relay}} \leq \bar{C} = \min_{\Omega} \mathrm{rank}(\mathbf{G}_{\Omega,\Omega^c})$$

where $\mathbf{G}_{\Omega,\Omega^c}$ is the transfer matrix $\mathbf{X}_\Omega \rightarrow \mathbf{Y}_{\Omega^c}$.

Main results: Deterministic relay networks

Theorem (Avestimehr, Diggavi and Tse, 2007)

Given a general deterministic relay network (with broadcast and multiple access), we can achieve all rates R upto

$$\max_{\prod_i p(\mathbf{X}_i)} \min_{\Omega} H(\mathbf{Y}_{\Omega^c} | \mathbf{X}_{\Omega^c})$$

Multicast information flow:

Theorem (Avestimehr, Diggavi and Tse, 2007)

Given a general deterministic relay network (with broadcast and multiple access), we can achieve all rates R from S multicasting to all destinations $D \in \mathcal{D}$ up to,

$$\max_{\prod_{i \in \mathcal{V}} p(x_i)} \min_{D \in \mathcal{D}} \min_{\Omega \in \Lambda_D} H(Y_{\Omega^c} | X_{\Omega^c})$$

Application
Linear deterministic models

Corollary (Avestimehr, Diggavi and Tse, 2007)

Given a linear finite-field relay network (with broadcast and multiple access), the capacity C of such a relay network is given by,

$$C = \min_{\Omega \in \Lambda_D} \operatorname{rank}(\mathbf{G}_{\Omega,\Omega^c}).$$

Multicast information flow:

Corollary (Avestimehr, Diggavi and Tse, 2007)

Given a linear finite-field relay network (with broadcast and multiple access), the multicast capacity C of such a relay network is given by,

$$C = \min_{D \in \mathcal{D}} \min_{\Omega \in \Lambda_D} \operatorname{rank}(\mathbf{G}_{\Omega,\Omega^c}).$$

Consequences: Deterministic Relay Networks

General deterministic networks: Cutset upper bound was $C_{\text{relay}} \leq \max_{p(\mathbf{x}_1,\ldots,\mathbf{x}_N)} \min_{\Omega} H(\mathbf{Y}_{\Omega^c}|\mathbf{X}_{\Omega^c}) \Longrightarrow$ achievable if optimum was product distribution.

Linear finite field model: Cutset upper bound was $C_{\text{relay}} \leq \min_{\Omega} \operatorname{rank}(\mathbf{G}_{\Omega,\Omega^c}) \Longrightarrow$ cutset bound achievable

For wireline graph model $\operatorname{rank}(\mathbf{G}_{\Omega,\Omega^c})$ is number of links crossing the cut.

Observation: We have a generalization of Ford-Fulkerson max-flow min-cut theorem to linear finite field relay networks with broadcast and multiple access.

Main results: Gaussian relay networks

Theorem (Avestimehr, Diggavi and Tse, 2007)

Given a Gaussian relay network, $\mathcal{G}$, we can achieve all rates R up to $\overline{C} - \kappa$. Therefore the capacity of this network satisfies

$$\overline{C} - \kappa \leq C \leq \overline{C},$$

where $\overline{C}$ is the cut-set upper bound on the capacity of $\mathcal{G}$, and κ is a constant independent of channel gains.

Theorem (Multicast information flow)

Given a Gaussian relay network, $\mathcal{G}$, we can achieve all multicast rates R up to $\overline{C}_{mcast} - \kappa$, i.e., *for $\overline{C}_{mcast} = \min_{D \in \mathcal{D}} \overline{C}_D$,*

$$\overline{C}_{mcast} - \kappa \leq C \leq \overline{C}_{mcast}$$

Ingredients and insights

Main steps: Gaussian strategy

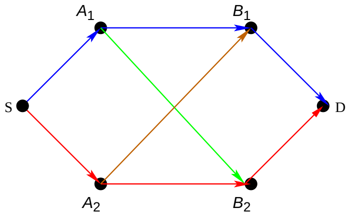
- **Relay operation:** Quantize received signal at noise-level.
- **Relay function:** Random mapping from received quantized signal to transmitted signal.
- Handle unequal (multiple) paths between nodes like “inter-symbol interference”.

Consequences:

- With probabilistic method we demonstrate min-cut achievability for linear deterministic networks.
- Gaussian networks constant gap independent of SNR operating point.
- Engineering insight of (almost) optimal coding strategies.

Achievability program: Deterministic networks

Layered (equal path) networks

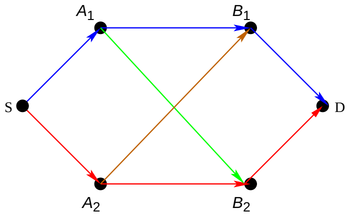


Lengths of ALL paths from source to destination are the same.

Broadcast and multiple access for general deterministic functions.

Achievability program: Deterministic networks

Layered (equal path) networks



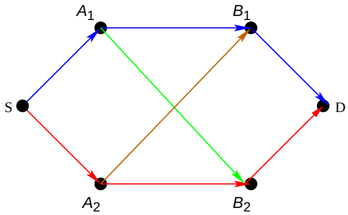
Lengths of ALL paths from source to destination are the same.

Broadcast and multiple access for general deterministic functions.

- Illustrate analysis through equal path network.

Achievability program: Deterministic networks

Layered (equal path) networks



Lengths of ALL paths from source to destination are the same.

Broadcast and multiple access for general deterministic functions.

- Illustrate analysis through equal path network.
- Extend to unequal path networks through time-expansion.

Achievability: Deterministic networks

- Map each message into random codeword of length T symbols.
- Each relay randomly independently maps its received signal onto transmit codewords $\Longrightarrow$ transmit distributions independent across relays.

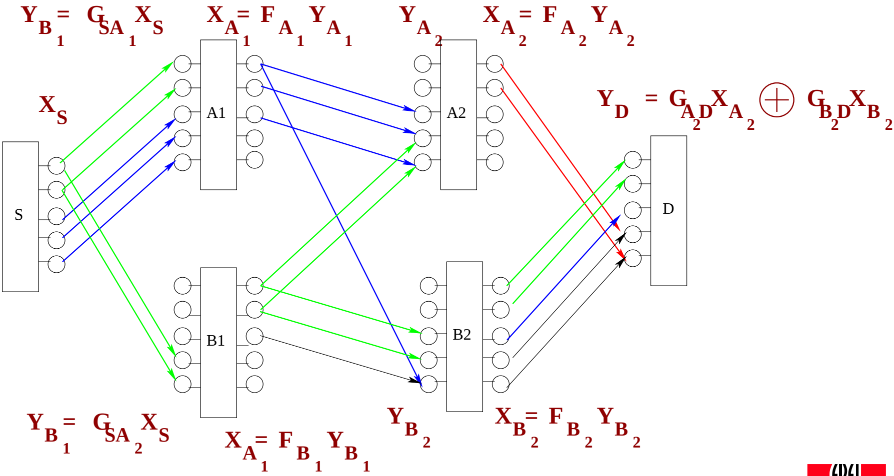
$$\mathbf{x}_j = \mathbf{f}_j(\mathbf{y}_j)$$

- Strategy similar to network coding for wireline graphs (Ahlswede *et al* 2000).
- For linear deterministic network, simplification in relay function:

$$\mathbf{x}_j = \mathbf{F}_j \mathbf{y}_j$$

where $\mathbf{F}_j$ is randomly chosen matrix.

Achievability: Deterministic networks



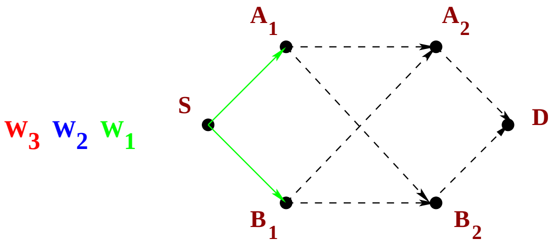
Linear deterministic networks: relay strategy

Key simplification for staged networks: In equal path networks all nodes in a stage are transmitting information about *same* message.

Linear deterministic networks: relay strategy

Key simplification for staged networks: In equal path networks all nodes in a stage are transmitting information about *same* message.

Time Block 1



Linear deterministic networks: relay strategy

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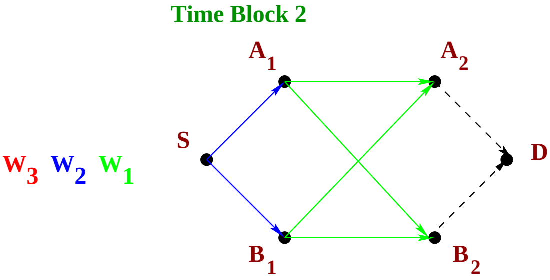
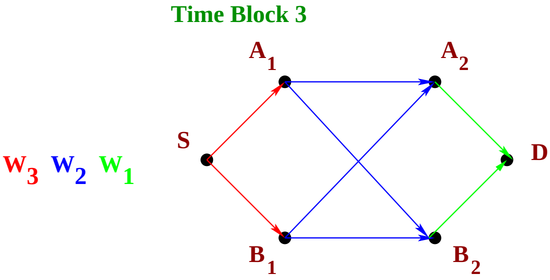


Linear deterministic networks: relay strategy

Key simplification for staged networks: In equal path networks all nodes in a stage are transmitting information about *same* message.



Implication: Focus on message $\mathbf{w} = \mathbf{W}_1$, which passes through layer l at block time l.

Analysis: Layered (equal path) deterministic networks

Focus on one message w of RT bits.

$$\mathbb{P}\left\{\text{error}\right\} \leq 2^{RT}\,\mathbb{P}\left\{w \to w'\right\}$$

Distinguishablity: Nodes that receive distinct signals under w and w' can disambiguate between them $\Longrightarrow$ received signals under two message distinct or $\mathbf{y}_j \neq \mathbf{y}'_j$ when j can distinguish.

$$\mathbb{P}\left\{w \to w'\right\} = \sum_{\Omega} \underbrace{\mathbb{P}\left\{\text{Nodes in } \Omega \text{ can distinguish } w, w' \text{ and nodes in } \Omega^c \text{ cannot}\right\}}_{\mathcal{P}}$$

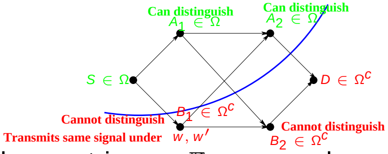
where Ω is a source-destination separation cut.

Linear deterministic networks: Confusability analysis

Key idea: Nodes in $\Omega^c(w, w')$, will transmit *same* codeword/signal under both w, w'.

Consequence: Error event being analyzed when $\mathbf{y}_{\Omega^c} = \mathbf{y}'_{\Omega^c}$, *i.e.*,

$$\begin{aligned} \mathbf{G}_{\Omega,\Omega^c}\mathbf{x}_\Omega &= \mathbf{G}_{\Omega,\Omega^c}\mathbf{x}'_\Omega \\ \mathbf{G}_{\Omega,\Omega^c}\left(\mathbf{x}_\Omega - \mathbf{x}'_\Omega\right) &= \mathbf{0} \end{aligned}$$



For $\mathbf{x}_j = \mathbf{F}_j\mathbf{y}_j$ with $\mathbf{F}_j$ uniform i.i.d. random matrix over $\mathbb{F}_2$, we need,

$$\mathbf{G}_{\Omega,\Omega^c}\begin{bmatrix} \vdots \\ \underbrace{\mathbf{F}_j\left(\mathbf{y}_j - \mathbf{y}'_j\right)}_{\mathbf{z}_j} \\ \vdots \end{bmatrix} = \mathbf{G}_{\Omega,\Omega^c}\mathbf{z} = \mathbf{0} \qquad j \in \Omega.$$

Confusability analysis: Linear deterministic networks

Since $\mathbf{F}_j$ is a uniform i.i.d. matrix over $\mathbb{F}_2$, $\mathbf{z}_j = \mathbf{F}_j \left(\mathbf{y}_j - \mathbf{y}'_j\right)$ is uniform vector over $\mathbb{F}_2^q$, due to $\mathbf{y}_j \neq \mathbf{y}'_j$ for $j \in \Omega$.

Therefore $\mathbf{z}$ is a uniform vector over $\mathbb{F}_2^{q|\Omega|}$ and we are calculating its probability of being in null space of $\mathbf{G}_{\Omega,\Omega^c}$.

This probability is: $2^{-T \operatorname{rank}(\mathbf{G}_{\Omega,\Omega^c})}$, *i.e.,*

$$\mathbb{P}\left\{\Omega(w,w') = \Omega\right\} = 2^{-T \operatorname{rank}(\mathbf{G}_{\Omega,\Omega^c})}$$

Hence for linear deterministic layered networks taking union bound,

$$\mathbb{P}\left\{\text{error}\right\} \leq 2^{RT} \sum_{\Omega} 2^{-T \operatorname{rank}(\mathbf{G}_{\Omega,\Omega^c})}$$

Implication: $R < \min_\Omega \operatorname{rank}(\mathbf{G}_{\Omega,\Omega^c})$ is achievable.

Large deviations result

When $\mathbf{y}_j = \mathbf{f}(\mathbf{x}_1, \ldots, \mathbf{x}_N)$ for general deterministic functions, we need a more sophisticated error calculation.

Basic large-deviations result: If a T-length sequence is generated i.i.d. according to probability law q the probability that its emperical behavior is like that of sequence generated as p is given by:

$$\mathbb{P}(q \rightarrow p) \doteq 2^{-TD(p||q)},$$

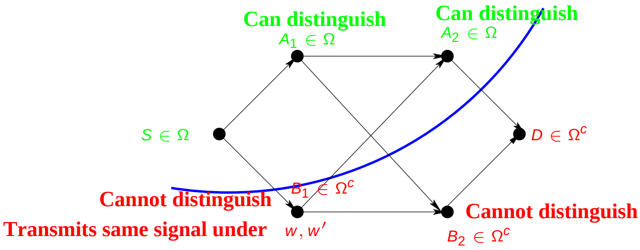
where $D(p||q)$ is the relative entropy given by:

$$D(p||q) = \sum_u p(u) \log \frac{p(u)}{q(u)}$$

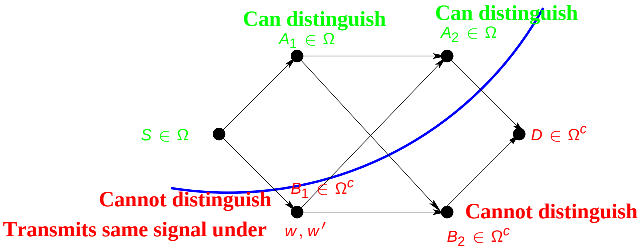
For $q = p(x)p(y), p = p(x, y)$ this probability is $2^{-TI(X;Y)}$ since

$$D(p||q) = \sum_u p(u) \log \frac{p(u)}{q(u)} = \sum_{x,y} p(x, y) \log \frac{p(x, y)}{p(x)p(y)} = \underbrace{I(X; Y)}_{\text{mutual information}}$$

Confusability analysis: Deterministic networks



Confusability analysis: Deterministic networks



Consequence: Error for *e.g.*, when w' codewords generated like $p(\mathbf{x}_{B_1}, \mathbf{y}_{B_2})p(\mathbf{x}_{A_1})$, are jointly typical with $\mathbf{y}_{B_2}$, occurs with probability $2^{-T\mathcal{E}}$

$$\begin{aligned} \mathcal{E} &= D(p(\mathbf{x}_{B_1}, \mathbf{y}_{B_2}, \mathbf{x}_{A_1}) \| p(\mathbf{x}_{B_1}, \mathbf{y}_{B_2})p(\mathbf{x}_{A_1})) \\ &= I(\mathbf{X}_{A_1}; \mathbf{Y}_{B_2}, \mathbf{X}_{B_1}) = \underbrace{I(\mathbf{X}_{A_1}; \mathbf{X}_{B_1})}_{0} + I(\mathbf{X}_{A_1}; \mathbf{Y}_{B_2} | \mathbf{X}_{B_1}) \end{aligned}$$

Confusability analysis: Deterministic networks

Continuing this way we get

$$\mathcal{P} \leq 2^{-T\left[I(\mathbf{X}_S;\mathbf{Y}_{B_1})+I(\mathbf{X}_{A_1};\mathbf{Y}_{B_2}|\mathbf{X}_{B_1})+I(\mathbf{X}_{A_2};\mathbf{Y}_D|\mathbf{X}_{B_2})\right]} = 2^{-TH(\mathbf{Y}_{\Omega^c}|\mathbf{X}_{\Omega^c})}$$

because we have a layered network and using Markov relationship.

Confusability analysis: Deterministic networks

Continuing this way we get

$$\mathcal{P} \leq 2^{-T\left[I(\mathbf{X}_S;\mathbf{Y}_{B_1})+I(\mathbf{X}_{A_1};\mathbf{Y}_{B_2}|\mathbf{X}_{B_1})+I(\mathbf{X}_{A_2};\mathbf{Y}_D|\mathbf{X}_{B_2})\right]} = 2^{-TH(\mathbf{Y}_{\Omega^c}|\mathbf{X}_{\Omega^c})}$$

because we have a layered network and using Markov relationship.

Implication: For layered networks an achievable rate is

$$R < \max_{\prod_i p(\mathbf{X}_i)} \min_{\Omega} H(\mathbf{Y}_{\Omega^c}|\mathbf{X}_{\Omega^c})$$

General deterministic networks

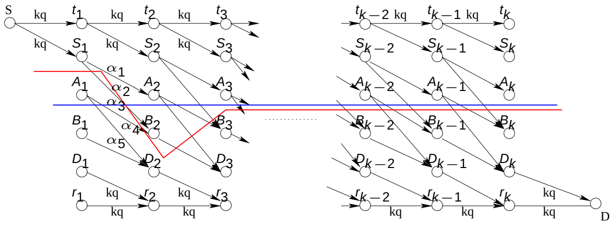
Unequal path networks $\Longrightarrow$ message synchronization lost.

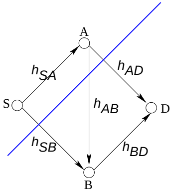
Approach:

- Consider time-expanded network over K blocks.
- **Observation:** Time-expanded network is an equal path network for (super) message of KRT bits.
- Convert multi-letter achievable rate to single-letter using submodularity of entropy.

Observation: Time-expansion was needed in network coding for cyclic networks, but in our case it is useful even in acyclic networks.

General networks: Time expansion





- Create a virtual node for every time-block $\Longrightarrow$ new network is layered $\Longrightarrow$ previous results apply.
- Many cuts in time-expanded network, not a cut in original network.
- Only horizontal cuts matter using submodularity of entropy.

Gaussian coding strategy

Encoding

- Each relay quantizes received signal to “noise-level” distortion.
- Each relay independently randomly maps quantized signal to a Gaussian transmit signal satisfying power constraints.
- **Caution:** This is *not* a compress-forward strategy since we are *not* trying to reconstruct *any* of the quantized outputs.

Decoding

- Destination D finds all the messages w that are jointly “typical” with received quantized sequence.
- **Note:** The relays do not decode any part of the message.

Typicality and typical sequences

Important tool in information theory

Typical sequence: For i.i.d. generated sequence $u_1, \ldots, u_T$ is typical with respect to probability measure p if,

$$\frac{1}{T}\log p(u_1, \ldots, u_T) = \frac{1}{T}\sum_t \log p(u_t) \stackrel{T\to\infty}{\to} -H(U)$$

Jointly typical sequences: Sequences $\{(u_t, v_t\}$ generated i.i.d. are jointly typical if individually they are typical and

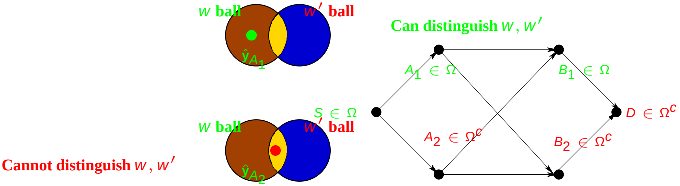
$$\frac{1}{T}\log p(u_1, \ldots, u_T, v_1, \ldots, v_T) = \frac{1}{T}\sum_t \log p(u_t, v_t) \stackrel{T\to\infty}{\to} -H(U, V)$$

Facts:

- For i.i.d. generated sequences, the probability of getting atypical sequence is exponentially small.
- All typical sequences are asymptotically equally likely.

Ingredients of analysis

Perturbation of deterministic case

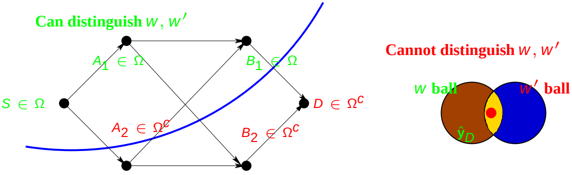


Typicality

- Message $w \to$ *multiple* transmitted signals.
- Message jointly typical (quantized) signal, $\to$ *any* plausible sequence that is typical with it.

Ingredients of analysis

Perturbation of deterministic case

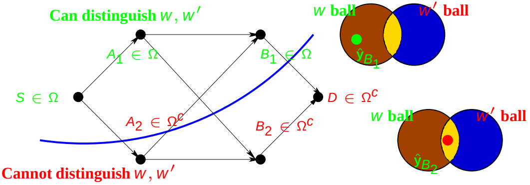


Error events

- Destination D quantized signal typical with w and w' $\Longrightarrow$ cannot distinguish between them. signals.
- **Distinguishablity:** Nodes that are not jointly typical with *both* w and w'.
- Divide network into nodes that can and cannot distinguish w, w' $\Longrightarrow$ defines a source-destination separation cut.

Ingredients of analysis

Perturbation of deterministic case



Putting it together:

- For particular plausible signal under w' is confusable with probability:

$$\mathcal{P} \leq 2^{-T\left[I(\mathbf{X}_\Omega;\hat{\mathbf{Y}}_{\Omega^c}|\mathbf{X}_{\Omega^c})\right]}$$

- Union bound:

$$\mathcal{P} \leq \underbrace{|\mathcal{X}(w')|}_{\leq 2^{T\gamma}} 2^{-T\left[I(\mathbf{X}_\Omega;\hat{\mathbf{Y}}_{\Omega^c}|\mathbf{X}_{\Omega^c})\right]} \leq 2^{-T\left[I(\mathbf{X}_\Omega;\hat{\mathbf{Y}}_{\Omega^c}|\mathbf{X}_{\Omega^c})-\gamma\right]}$$

Finishing touches

Components of gap: $\kappa = \beta + \gamma + \delta$

- Lose β bits due to noise-level quantization.
- Lose γ bits due to transmit list.
- Lose δ bits for independent distribution $\rightarrow$ beamforming loss.

Layered $\rightarrow$ general networks: Time expansion and single-letterization like in deterministic case.

Implication: For cut-set bound $\overline{C}$,

$$\overline{C} - \kappa \leq C \leq \overline{C}$$

Compound relay networks

Compound model: Channel realizations from a set $h_{i,j} \in \mathcal{H}_{i,j}$, unknown to sender.

Observations:

- Relay strategy does not depend on the channel realization.
- Overall network from source to destination behaves like a compound channel.
- Utilize point-to-point compound channel ideas get approximate characterization for compound network.

Theorem

Given a compound Gaussian relay network the capacity C_{cn} satisfies

$$\overline{C}_{cn} - \kappa \leq C_{cn} \leq \overline{C}_{cn},$$

where $\overline{C}_{cn} = \max_{p(\{x_j\}_{j\in\mathcal{V}})} \inf_{h\in\mathcal{H}} \min_{\Omega\in\Lambda_D} I(Y_{\Omega^c};X_\Omega|X_{\Omega^c})$.

Relay networks: Open questions and extensions

Extensions:

- Outage set behavior for full duplex networks.
- Analysis of half-duplex systems with fixed transmit fractions.
- Ergodic channel variations.

Open questions:

- D-M trade-off for channel dependent half-duplex systems.
- Tightening gap to cut-set bound.
- Use deterministic model directly to get Gaussian result.

Extensions of deterministic approach

- **Interference channel:** Successfully used to generate approximate characterization (Bresler and Tse, 2007),
- **K-user interference channel:** Used to demonstrate new phenomenon of *interference alignment* (Bresler-Tse, 2007, Jafar 2007).
- **Relay-interference networks:** Extension of multiple unicast to wireless networks (Mohajer, Diggavi, Fragouli and Tse, 2008).
- **Wireless network secrecy:** Used to demonstrate secrecy over networks (Diggavi, Perron and Telatar, 2008).
- **Network data compression:** Identify correct multi-terminal lossless structures to get approximation to multiple-description data compression (Tian, Mohajer and Diggavi, 2007).

Discussion

Program:

- Focus on underlying deterministic coding problem.
- Obtain exact characterization $\longrightarrow$ this is a central challenge.
- Use insight to obtain approximate characterization of noisy problem.

Hope:

- The program will yield insight to network flow problems.
- Exposes the central difficulties, solution insights and new schemes?
- Approximations may be sufficient for engineering practice.

In this talk: Obtained approximate max-flow min-cut characterization for noisy relay networks.

Papers/preprints: http://licos.epfl.ch