An Enriched Category Theory of Language

Tai-Danae Bradley

SandboxAQ / The Master's University / Nov. 5, 2024

a mathematical framework for language
(inspired by LLMs)

category theory

Why? It allows us to bring in *few* assumptions.

a **mathematical** framework for language (inspired by LLMs)

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language

It's a little different than fitting a model to language.
Instead, we're just seeing "what's there."

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a mathematical framework for language (inspired by LLMs)

AN ENRICHED CATEGORY THEORY OF LANGUAGE: FROM SYNTAX TO SEMANTICS

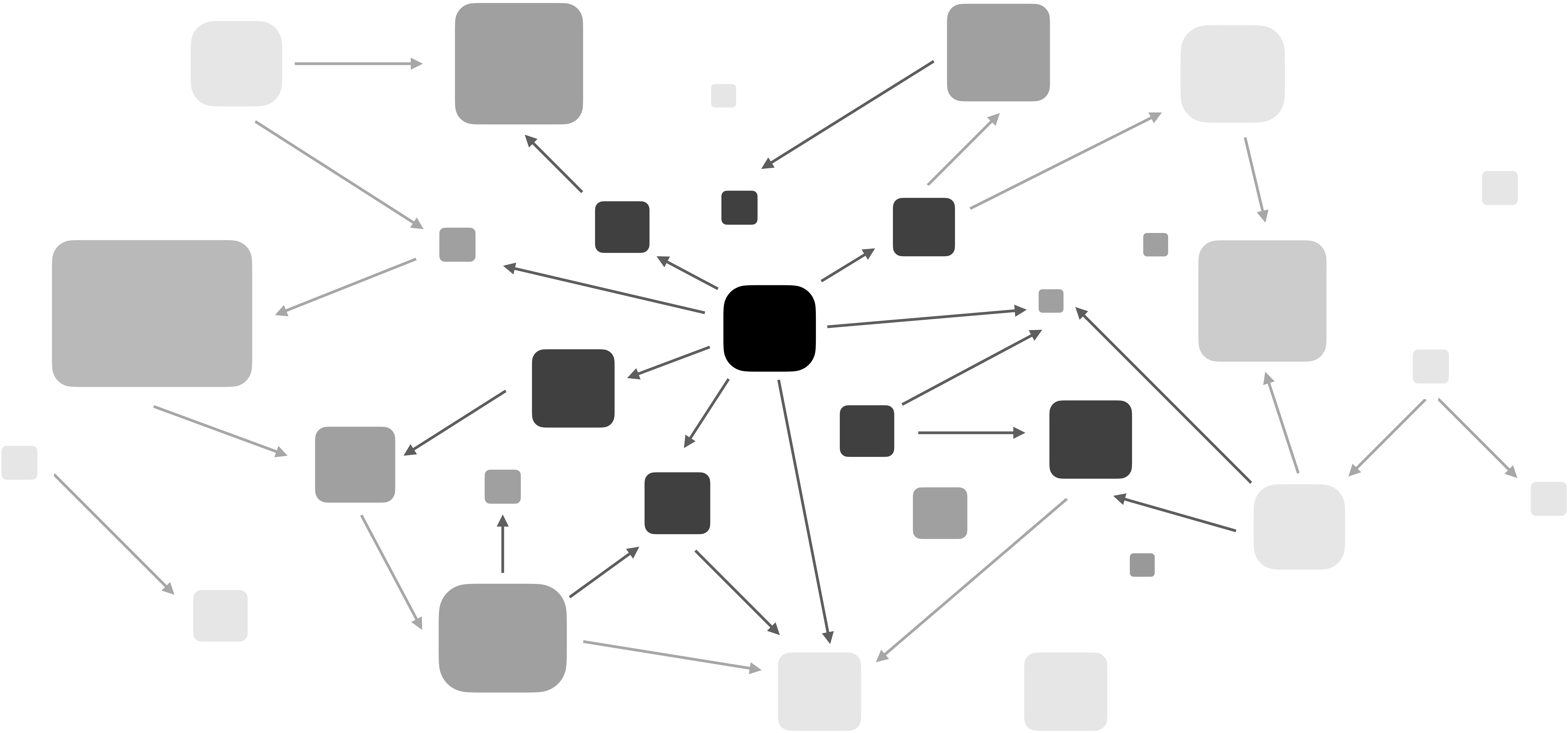
TAI-DANAE BRADLEY[1], JOHN TERILLA[2], AND YIANNIS VLASSOPOULOS[3]

La Matematica, vol. 1, pp. 551-580 (2022)

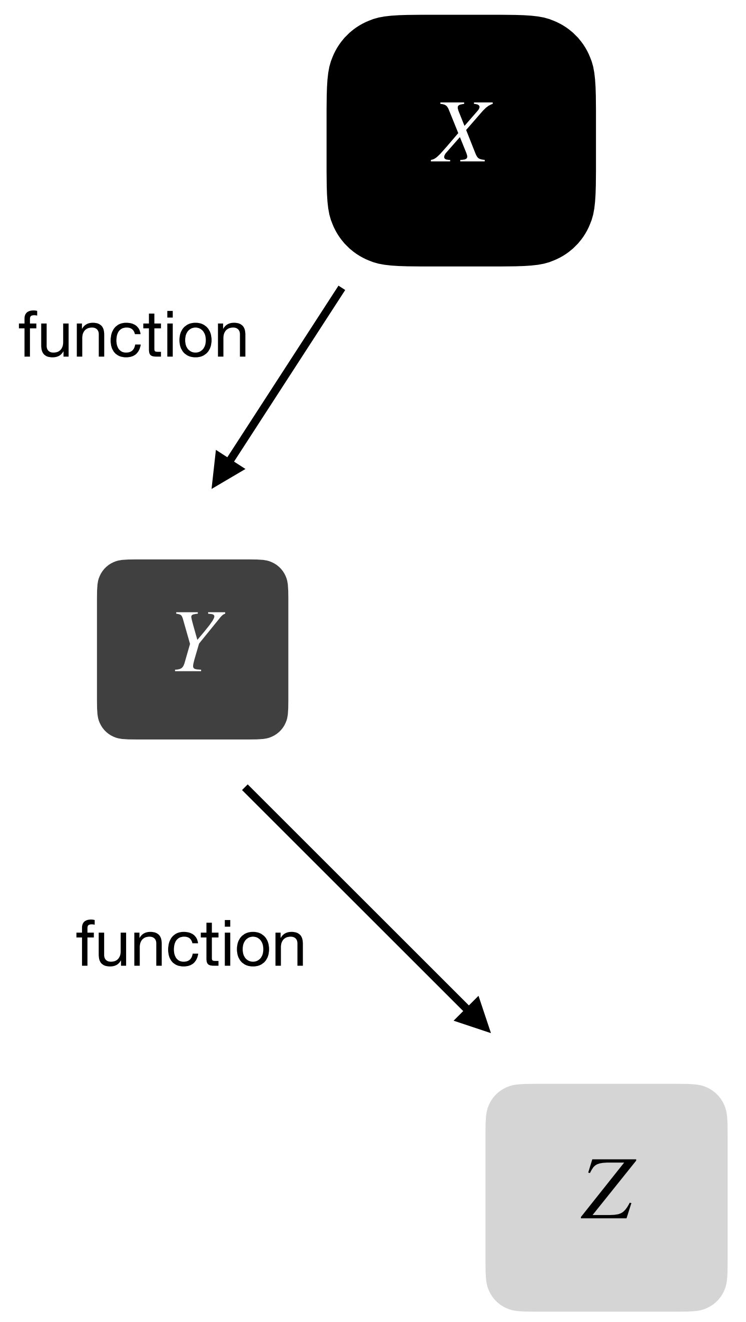
enriched

The statistics of texts can be captured neatly using
a "richer" version of category theory.

1. introduce category theory
2. describe language as a category
3. see what we can do



sets



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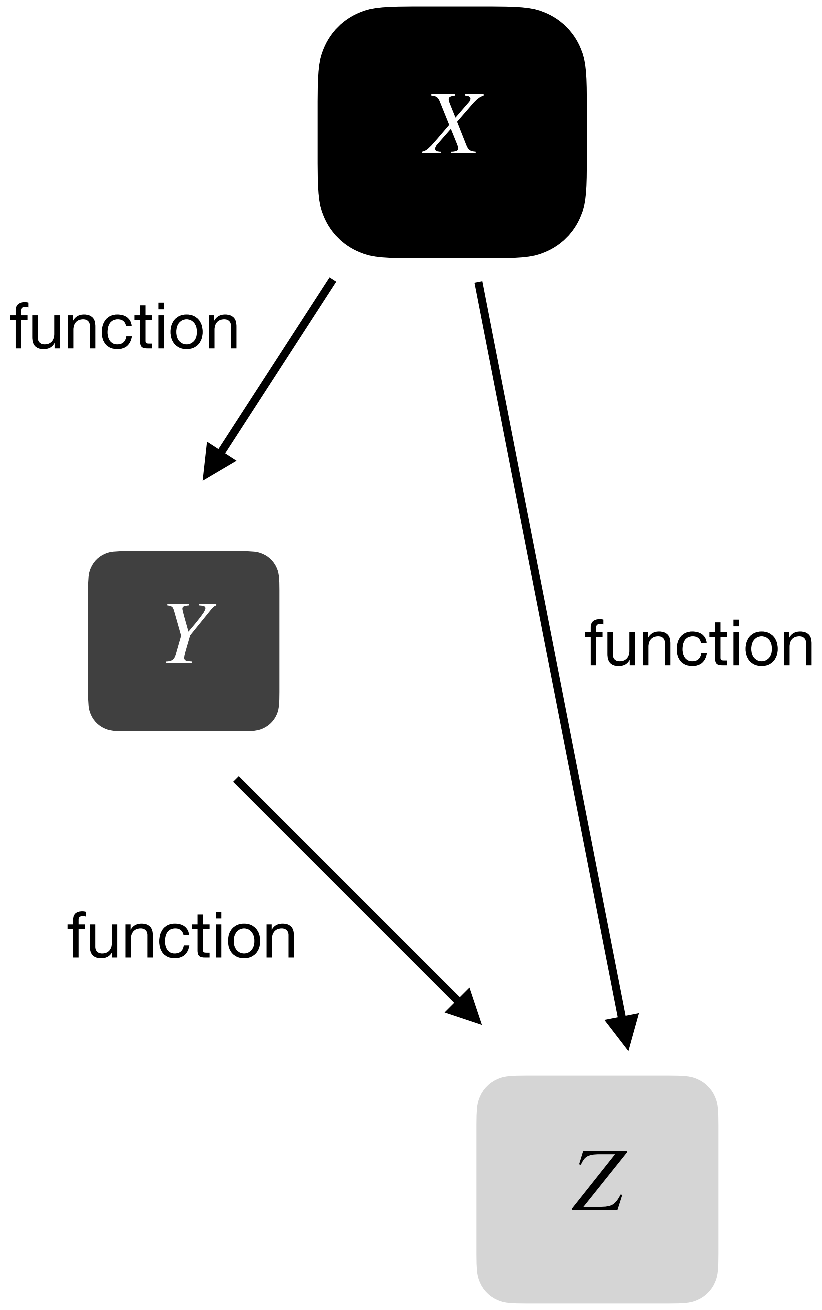
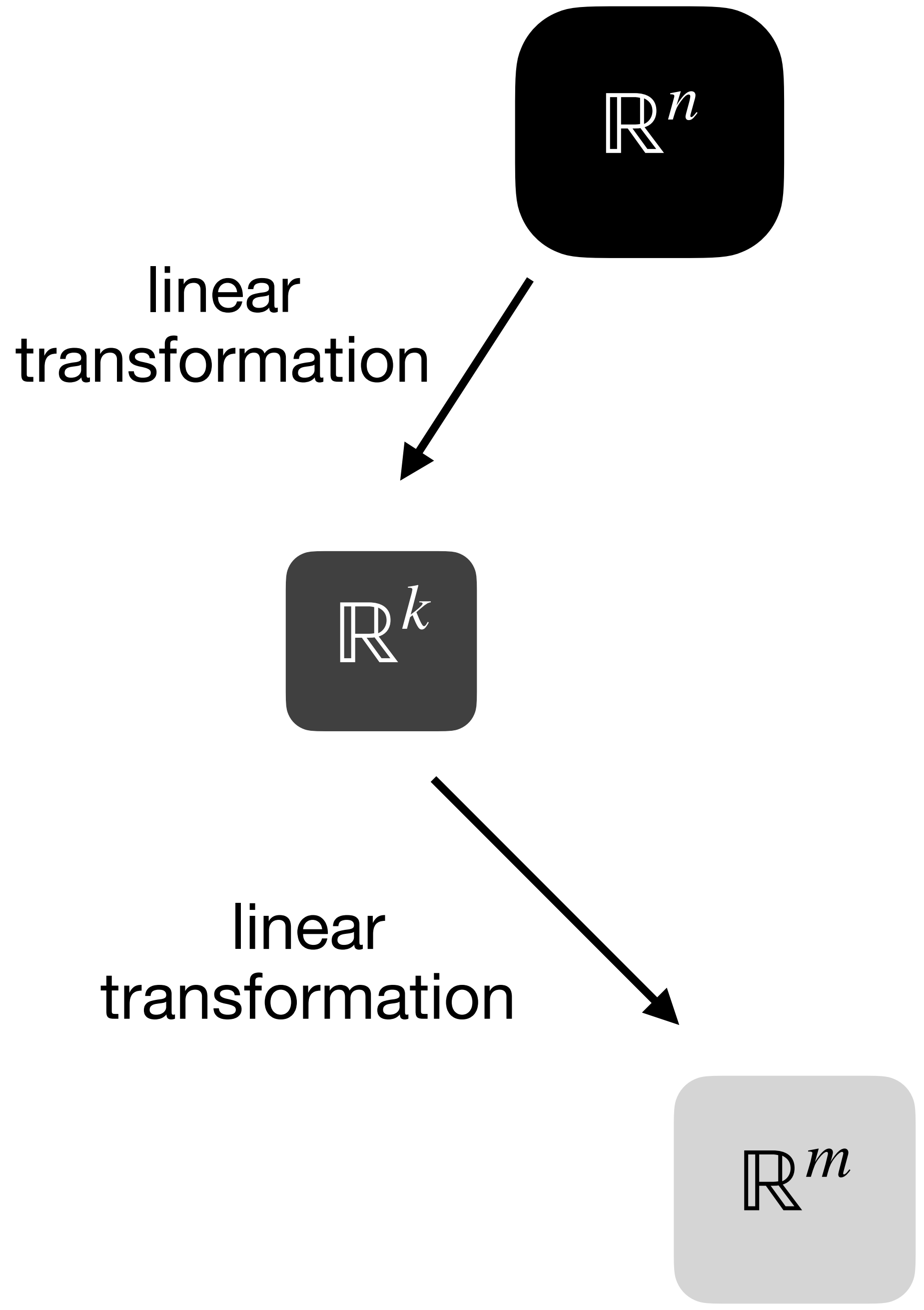


vector spaces



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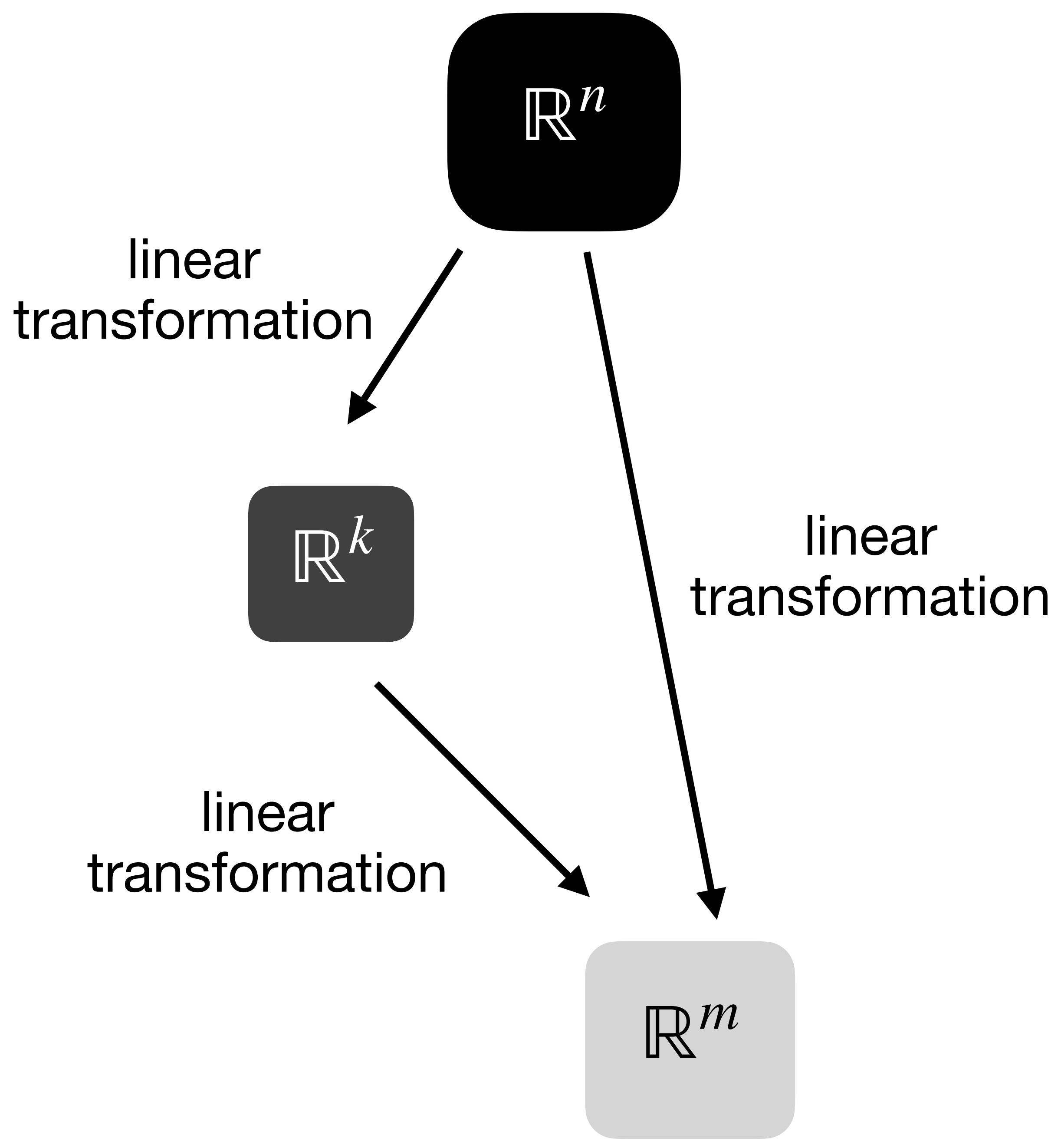
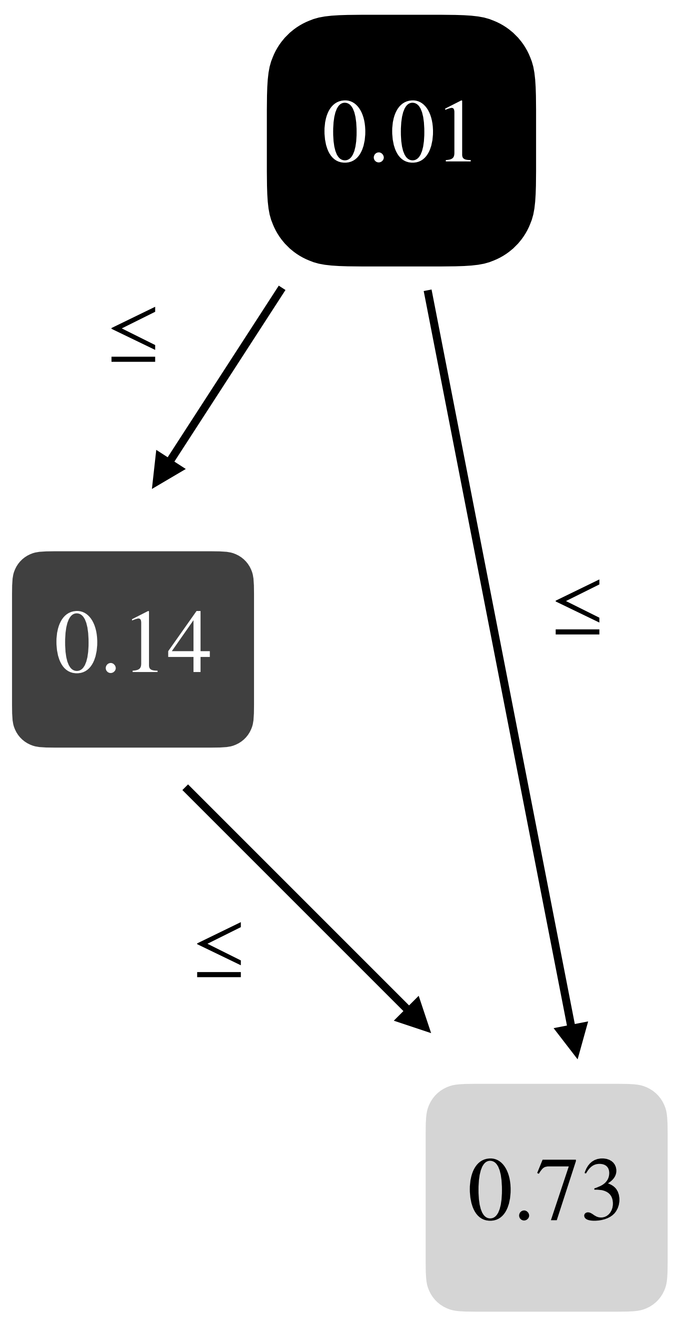


real numbers in $[0,1]$



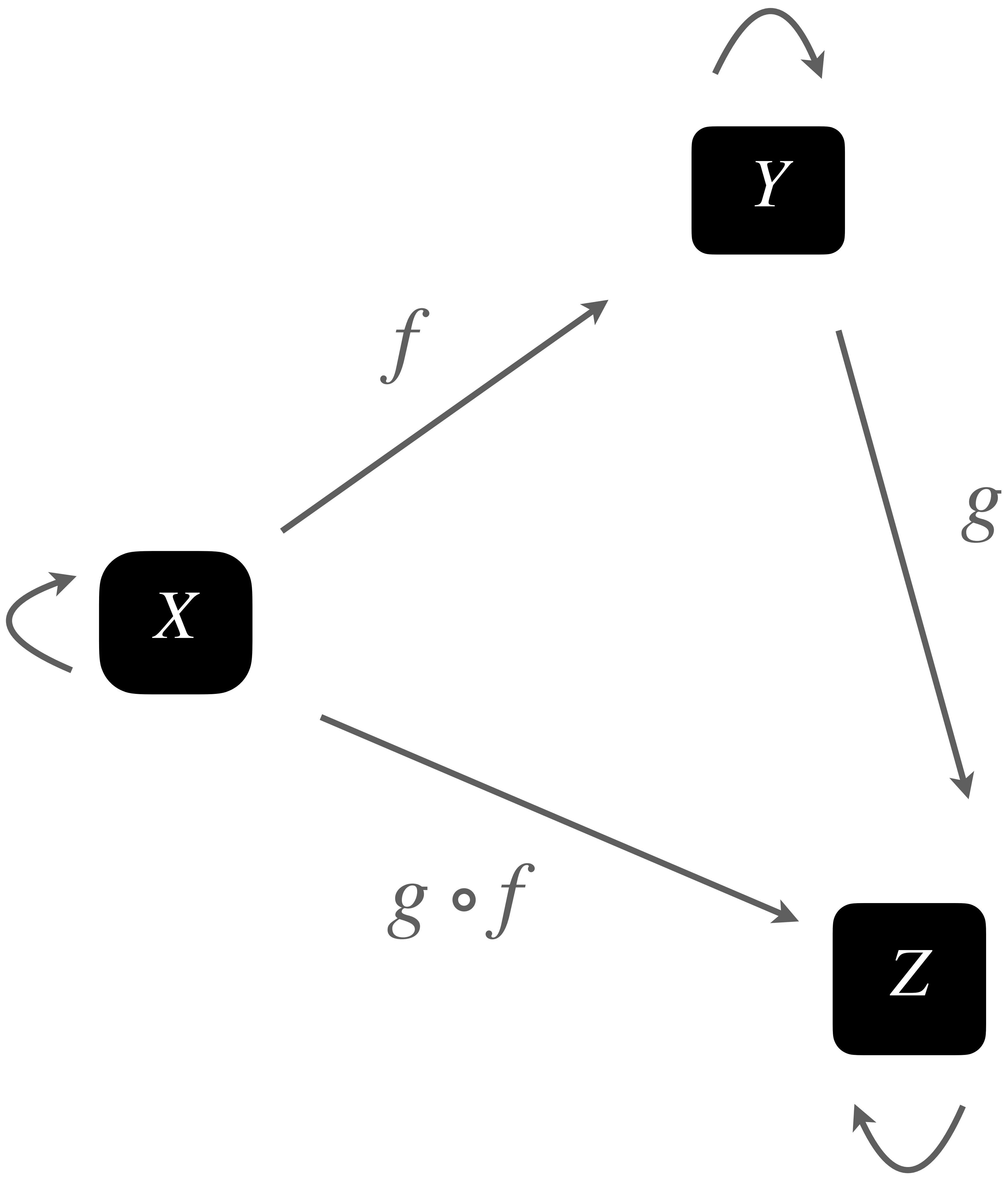
A Category

Loose Definition

A **category** C consists of

- **objects** $X, Y, \ldots$
- **morphisms** (i.e. arrows) between them
- a **composition rule**

that satisfy some reasonable axioms.



categories	language

categories

OBJECTS

$$X, Y, \ldots$$

language

categories

OBJECTS

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language

STRINGS

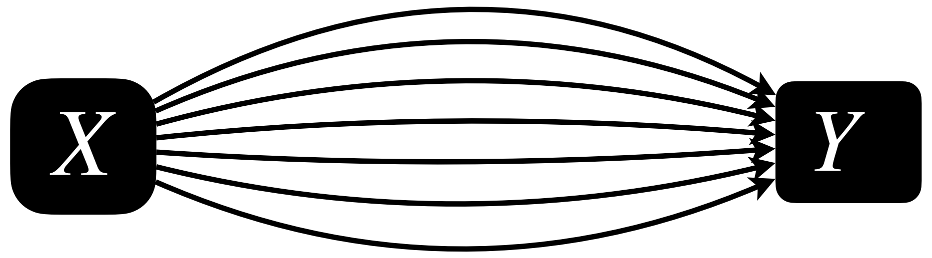
$x, y, \ldots$

categories

OBJECTS

$$X, Y, \ldots$$

A SET



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categories

OBJECTS

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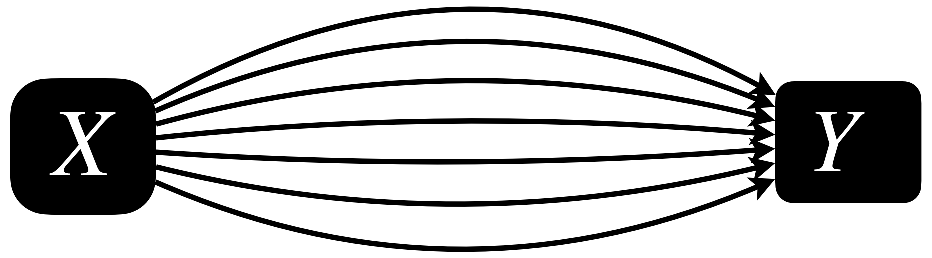

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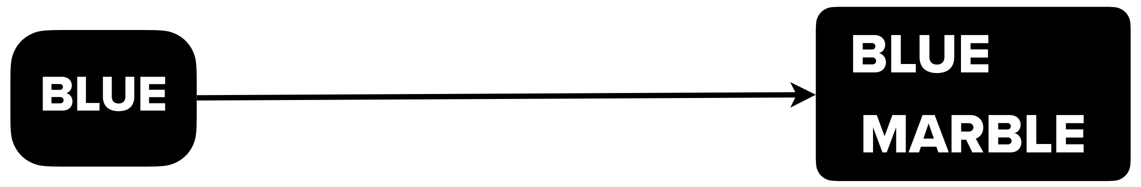

categories

OBJECTS

$$X, Y, \ldots$$

A SET

$$\mathsf{C}(X, Y)$$

language

STRINGS

$$x, y, \ldots$$

A SET

$$\text{BLUE} \longrightarrow \text{BLUE MARBLE}$$

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language

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A NUMBER

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$$\mathsf{C}(X, Y) \times \mathsf{C}(Y, Z) \to \mathsf{C}(X, Z)$$

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A NUMBER

$$\pi(y \mid x)$$

$$\boxed{?}$$

Language as a Category, L

Summary so far…

Consider all strings $x, y, ...$ from a finite set of atomic symbols. (*Think: expressions in a language.*)

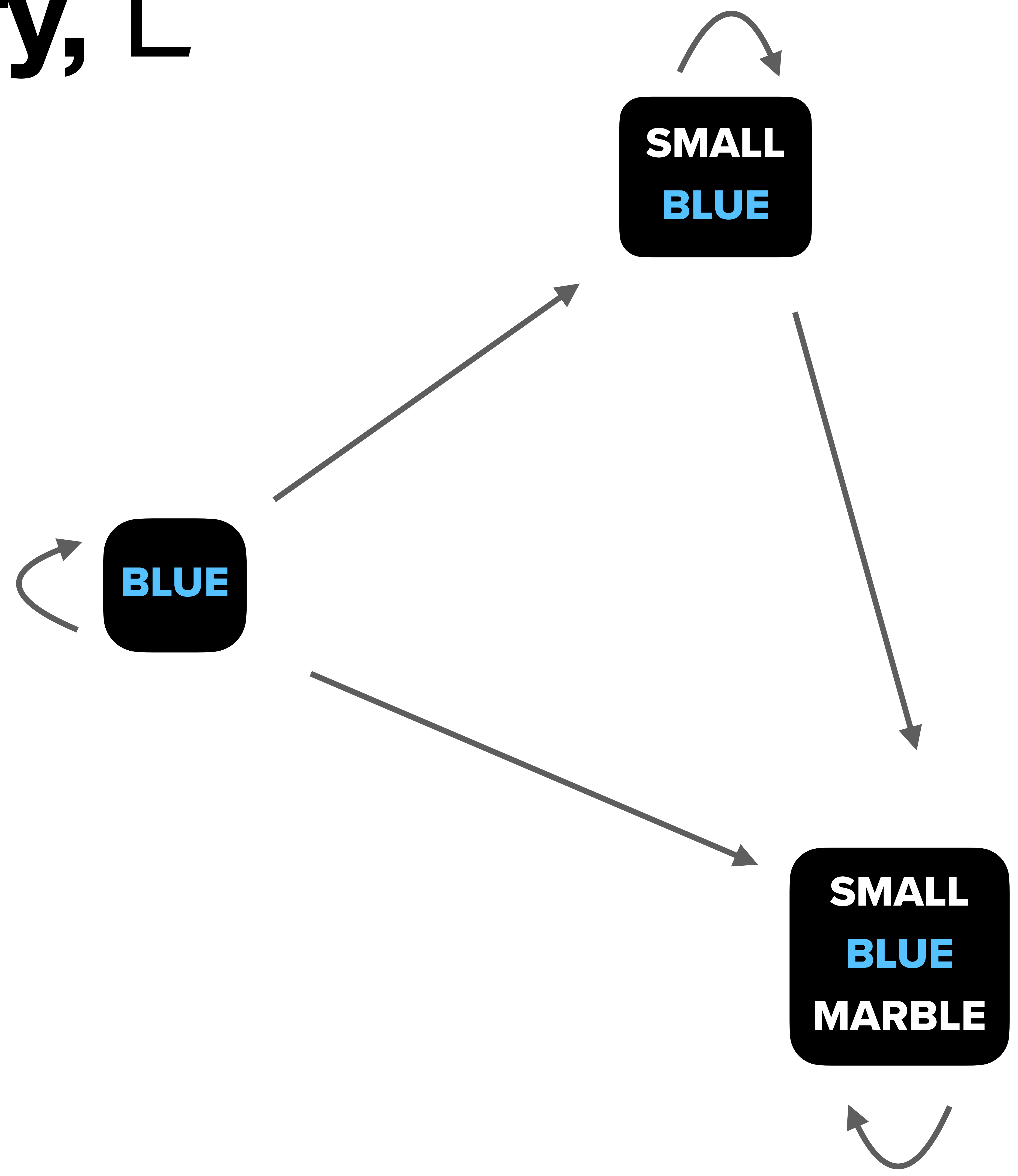
Write $x \to y$ to indicate **substring containment.**

- Arrows compose:

$$\text{if } x \to y \to z, \text{ then } x \to z$$

- Each string contains itself:

$$x \to x$$



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Meanings

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Representations

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Probabilities

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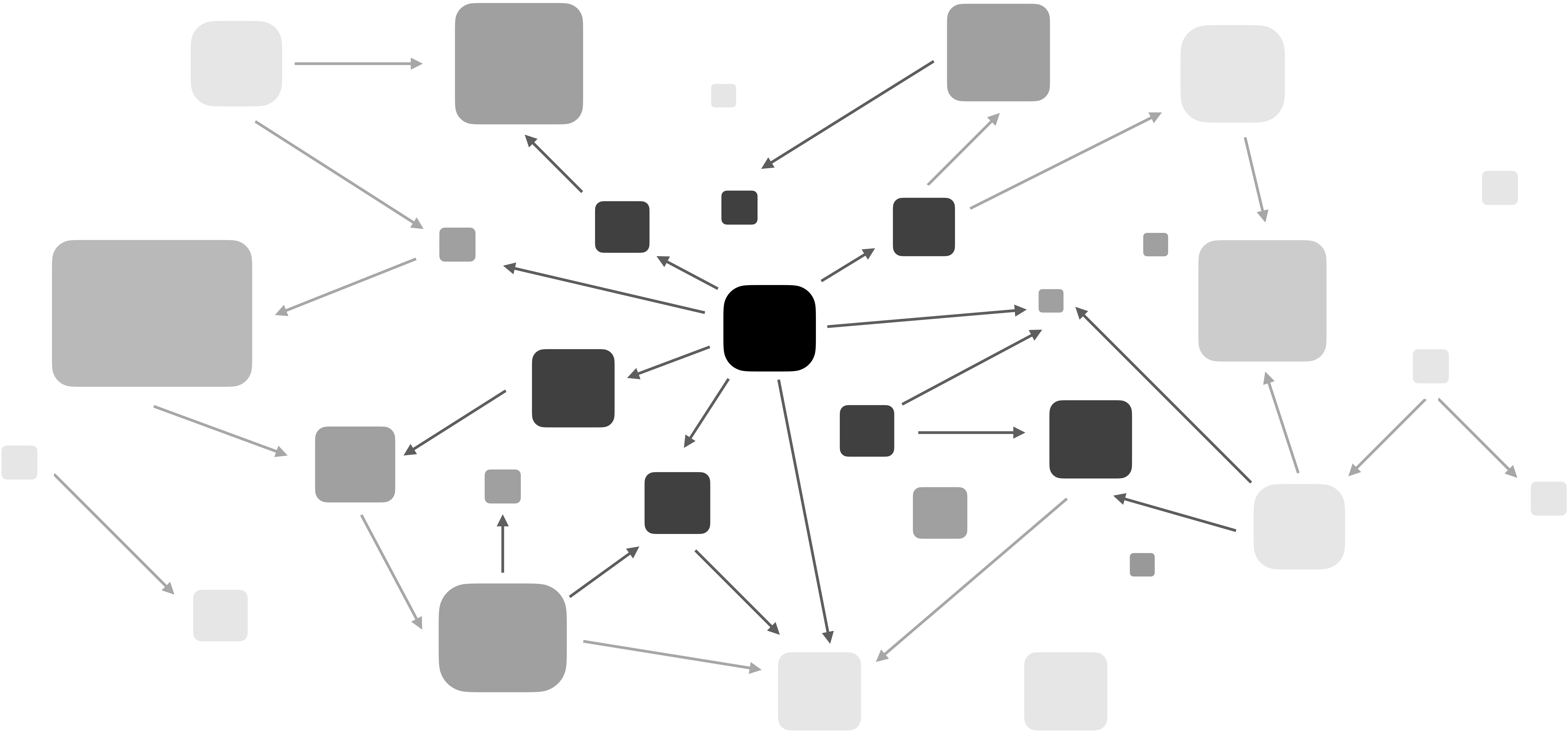

$$\mathrm{C}(X, -)$$

$$\mathsf{C}(X, -)$$

$$\mathsf{C} \xrightarrow[\text{functor}]{\mathsf{C}(X,\, -)} \mathsf{Set}$$

$$\mathsf{C}(X, -)$$

$$\mathsf{C} \xrightarrow{\mathsf{C}(X,\, -)} \mathsf{Set}$$

functor

$$X \cong Y \quad \text{iff} \quad \mathsf{C}(X, -) \cong \mathsf{C}(Y, -)$$

Yoneda Lemma

(or rather, a corollary of it)

Idea: Given a “prompt” (i.e. expression) x, consider the network of ways it fits into the language category L.

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$$x \quad \mapsto \quad \mathsf{L}(x, -)$$

$$\mathsf{L}(\text{blue}, \text{blue marble}) = \{ \to \}$$

$$\mathsf{L}(\text{blue}, \text{curiosity killed the cat}) = \varnothing$$

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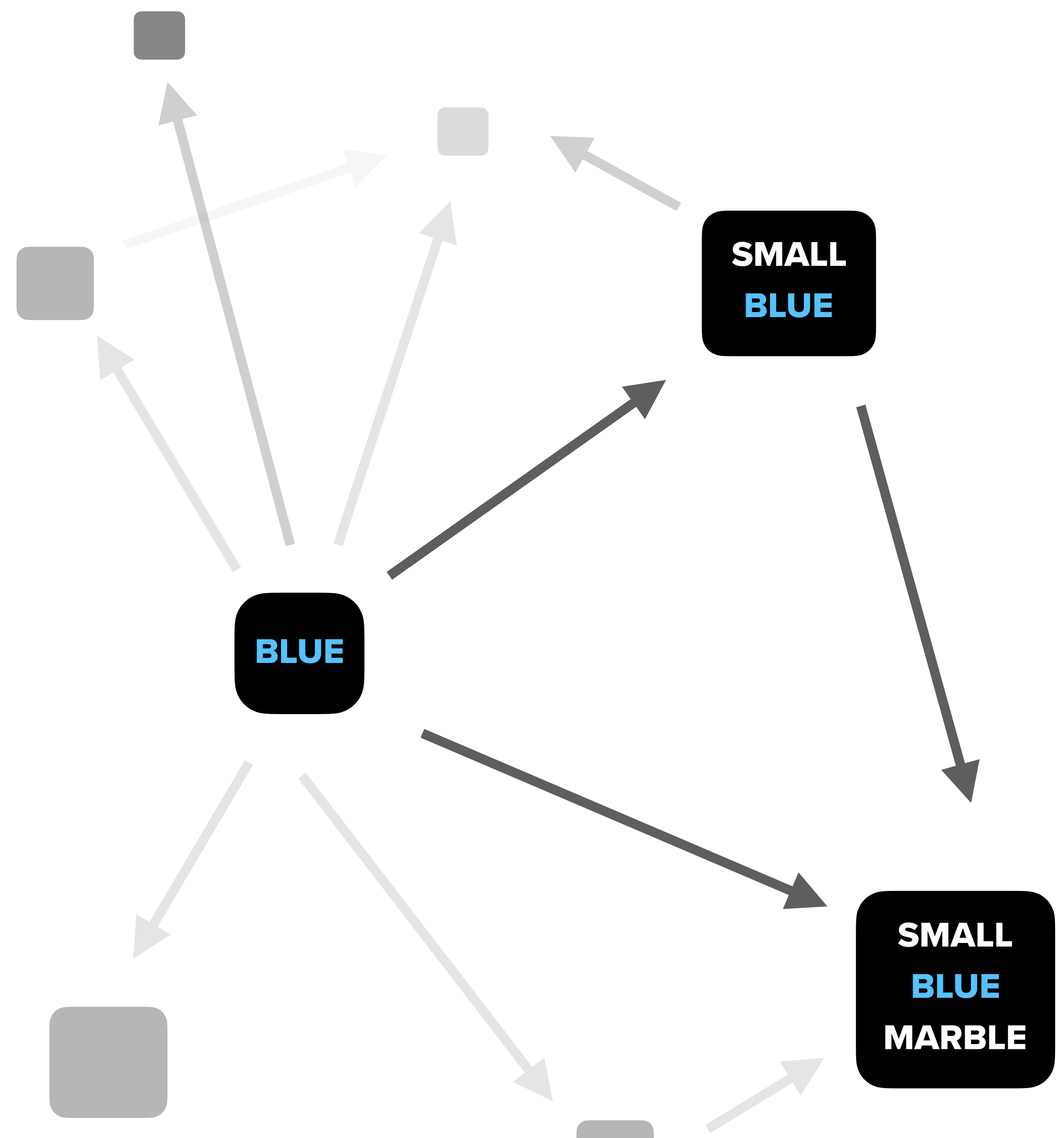
Upshot: The Yoneda Lemma motivates us to consider the functor

$$\mathsf{L}(x, -): \mathsf{L} \to \mathsf{Set}$$

which "represents" (or is a rough approximation to) the meaning of x.

Example:

$$\mathsf{L}(\text{blue}, y) = \begin{cases} \{\to\} & \text{if blue} \leq y \\ \varnothing & \text{otherwise} \end{cases}$$

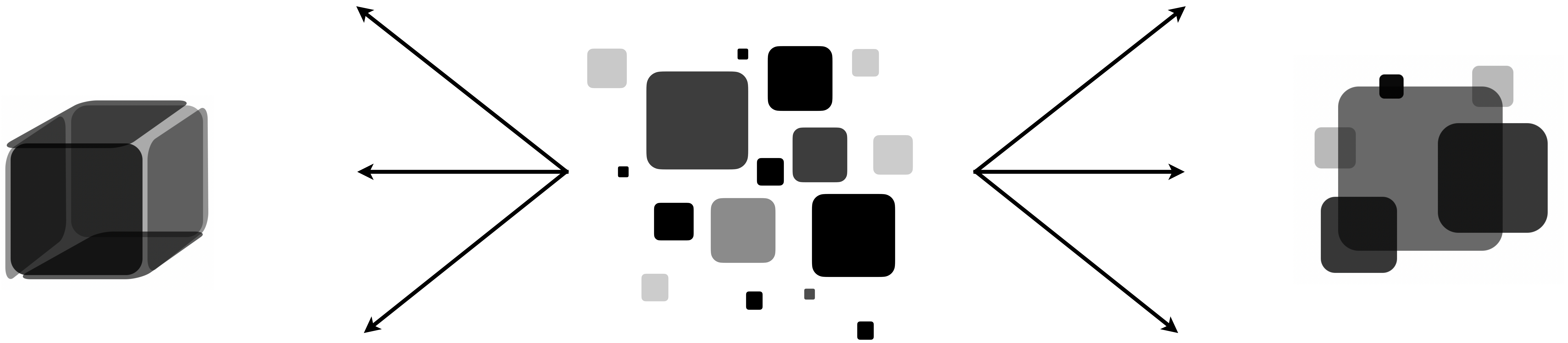


$$L(\textbf{blue}, -) \overset{\text{SORT OF}}{=} \begin{array}{cl} \begin{bmatrix} \varnothing \\ * \\ * \\ \varnothing \\ * \\ \varnothing \\ \vdots \end{bmatrix} & \begin{array}{l} \text{deep red Bing cherries} \\ \text{small } \textbf{blue} \text{ marble} \\ \text{beautiful } \textbf{blue} \text{ ocean} \\ \text{did you put the kettle on} \\ \text{red and } \textbf{blue} \text{ fireworks} \\ \text{Sencha green tea} \\ \vdots \end{array} \end{array}$$

1. Represent Meanings	**2. Operate on Representations**	3. Incorporate Probabilities
4. Represent Enriched Meanings	5. Operate on Enriched Representations	6. Adopt a Geometric Perspective

limits

colimits



intersections, products, direct sums, meets, greatest common divisors, kernels,…

unions, coproducts, direct sums, joins, least common multiples, cokernels,…

All functors $\mathsf{L} \to \mathsf{Set}$ form a new category $\mathsf{Set}^{\mathsf{L}}$ that has *lots* of structure, which it inherits from the category of sets. Just as we can combine sets in many ways (intersections, unions, etc.) we can now combine *functors* in many ways.

Practically speaking, this means we have notions of conjunction, **disjunction**, and implication. (Formally speaking, $\mathsf{Set}^{\mathsf{L}}$ has "all limits, colimits, and is Cartesian closed.")

AND

$\mathsf{L}(\text{red}, -)$ **OR** $\mathsf{L}(\text{blue}, -)$

IMPLIES

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$$
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\begin{array}{cl}
\begin{bmatrix} 0 \\ .68 \\ .01 \\ 0 \\ .17 \\ 0 \\ \vdots \end{bmatrix} &
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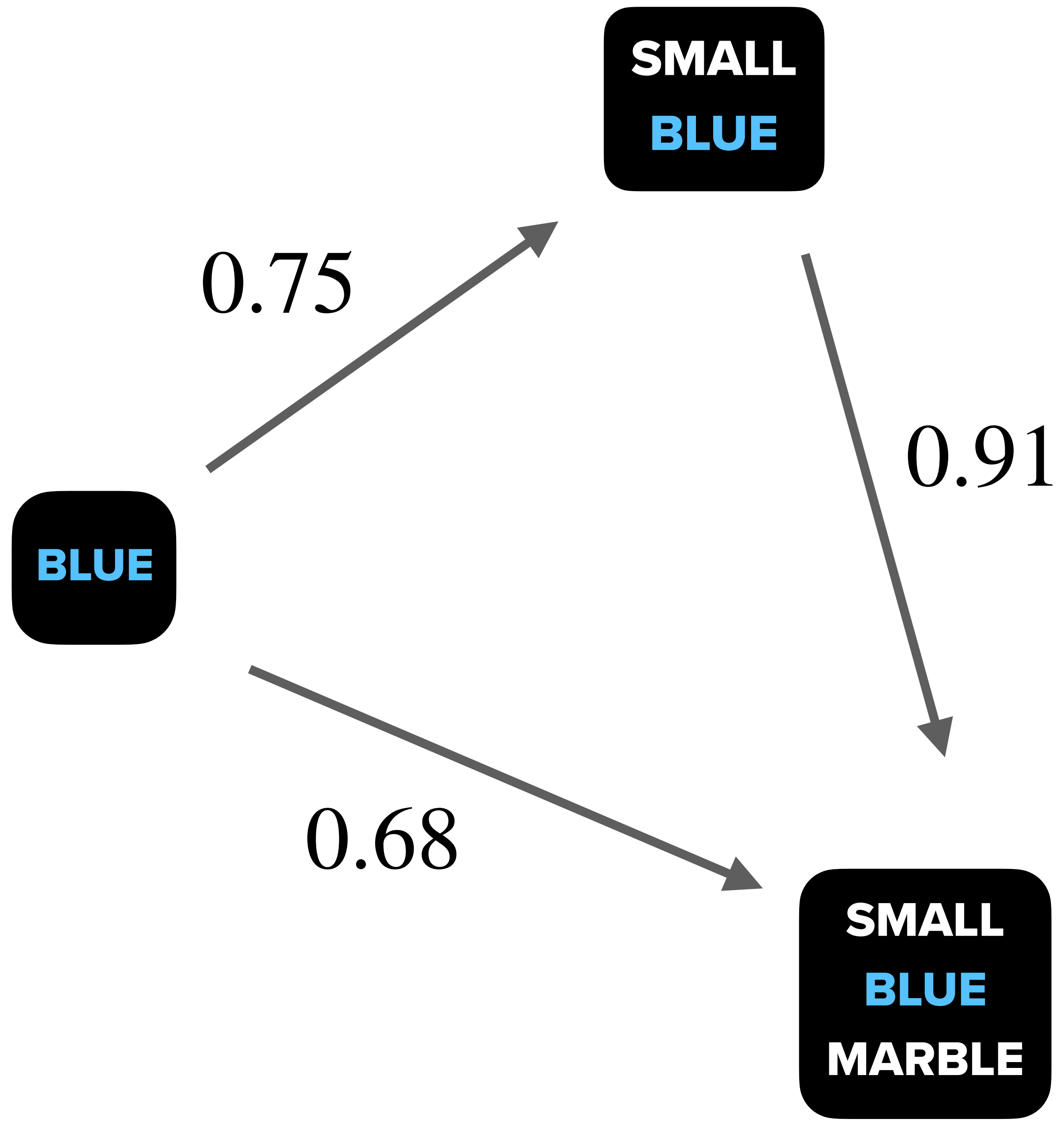
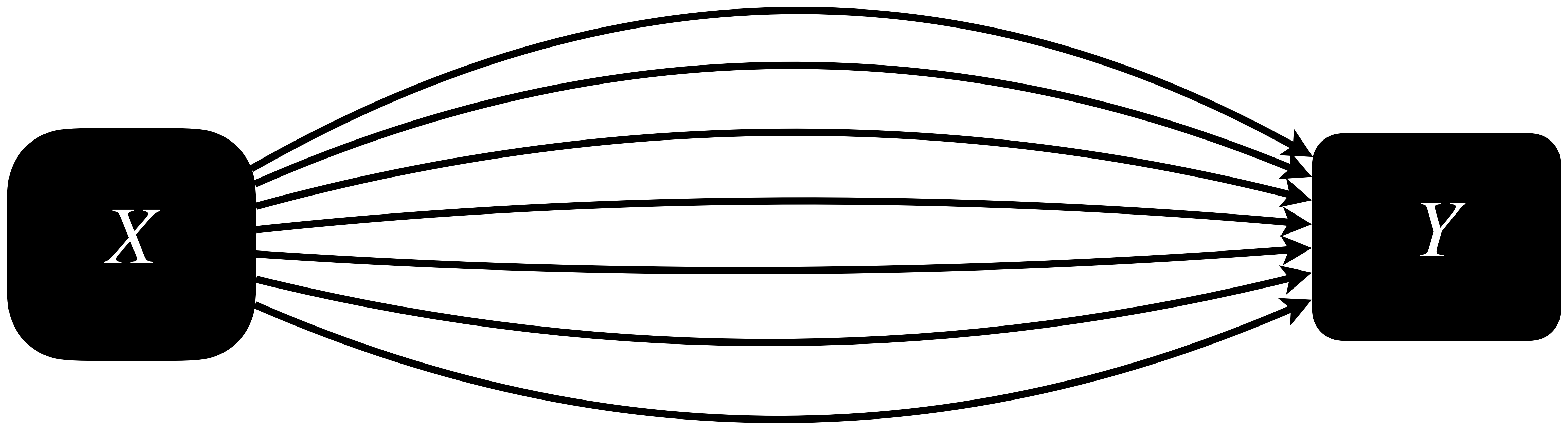
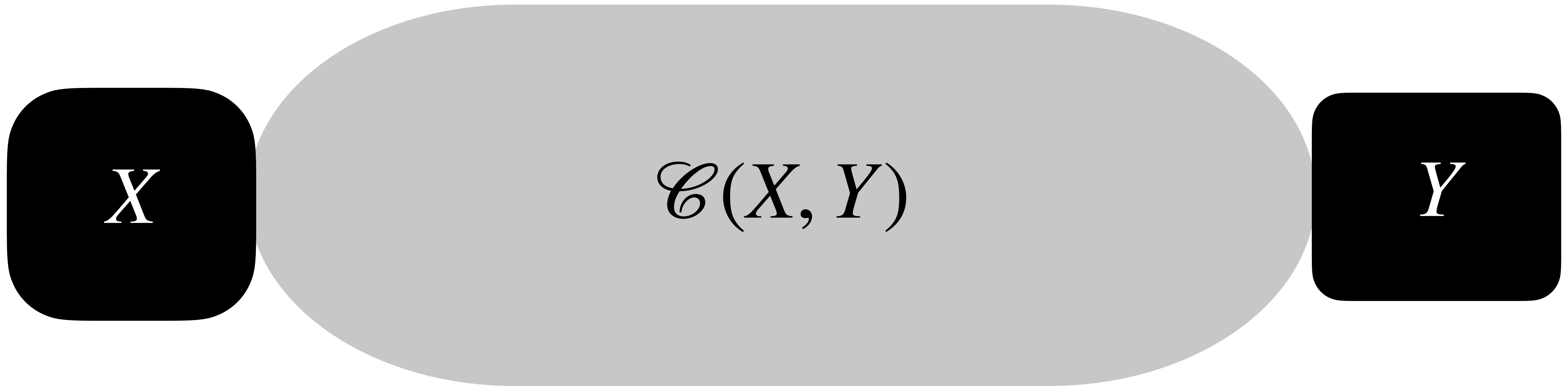
$$





In (ordinary) **category theory**, each pair of objects has an associated **set**

$$\mathrm{C}(X, Y)$$



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In **enriched category theory**, each pair of objects has an associated **object**

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in some sufficiently nice category

$$\text{BLUE} \xrightarrow{0.12} \text{BLUE MARBLE}$$

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In **enriched category theory**, each pair of objects has an associated **object**

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in some sufficiently nice category

like $[0,1]$, as hinted earlier

An Enriched Category

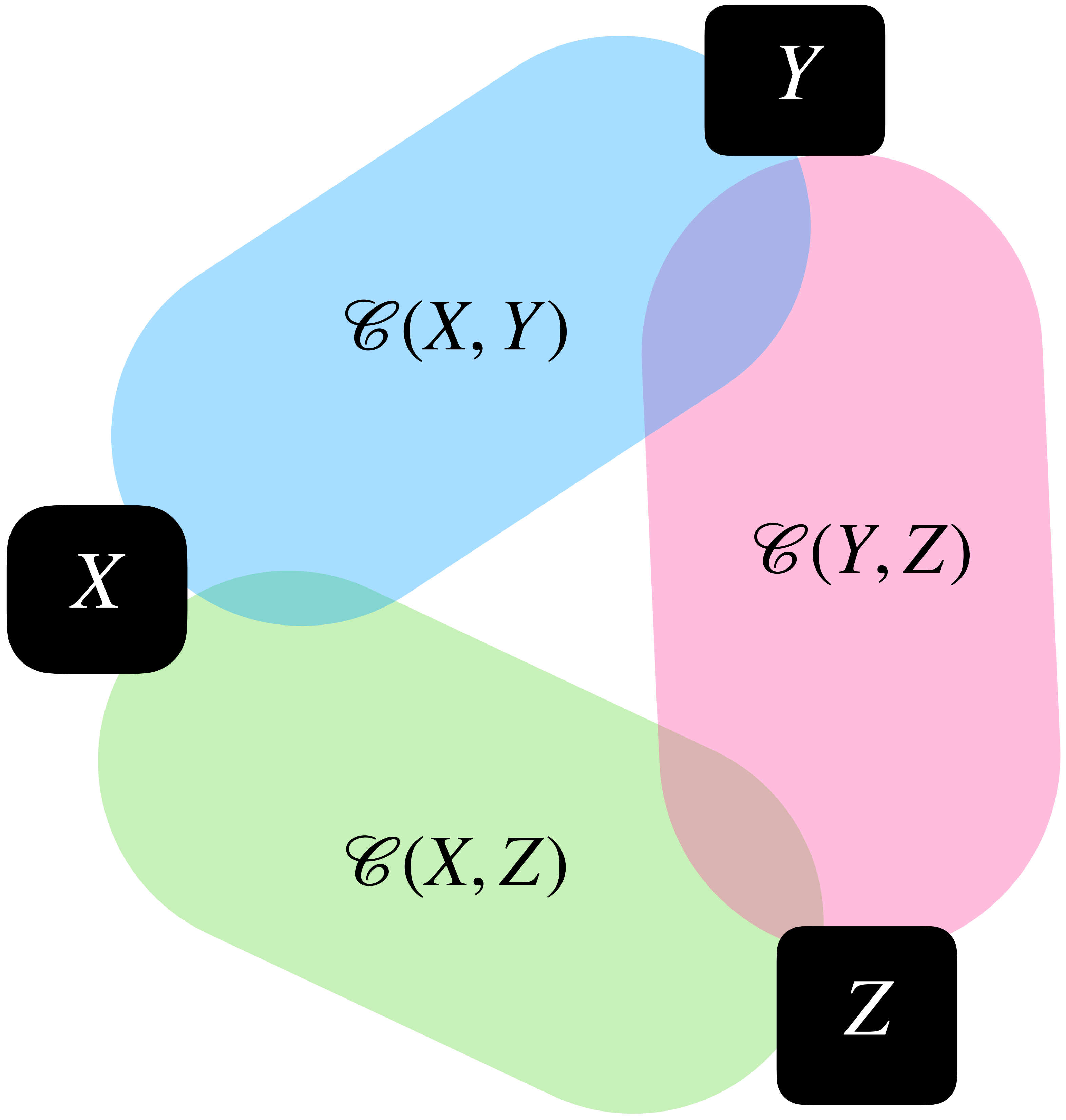
Very Loose Definition

Given a sufficiently nice category $\mathscr{V}$, a $\mathscr{V}-$**enriched category** $\mathscr{C}$ has

- **objects** $X, Y, \ldots$
- an **object** $\mathscr{C}(X, Y)$ in $\mathscr{V}$
- a "**composition rule**"

$$\mathscr{C}(X,Y) \otimes \mathscr{C}(Y,Z) \to \mathscr{C}(X,Z)$$

that satisfy reasonable axioms.



An Enriched Category

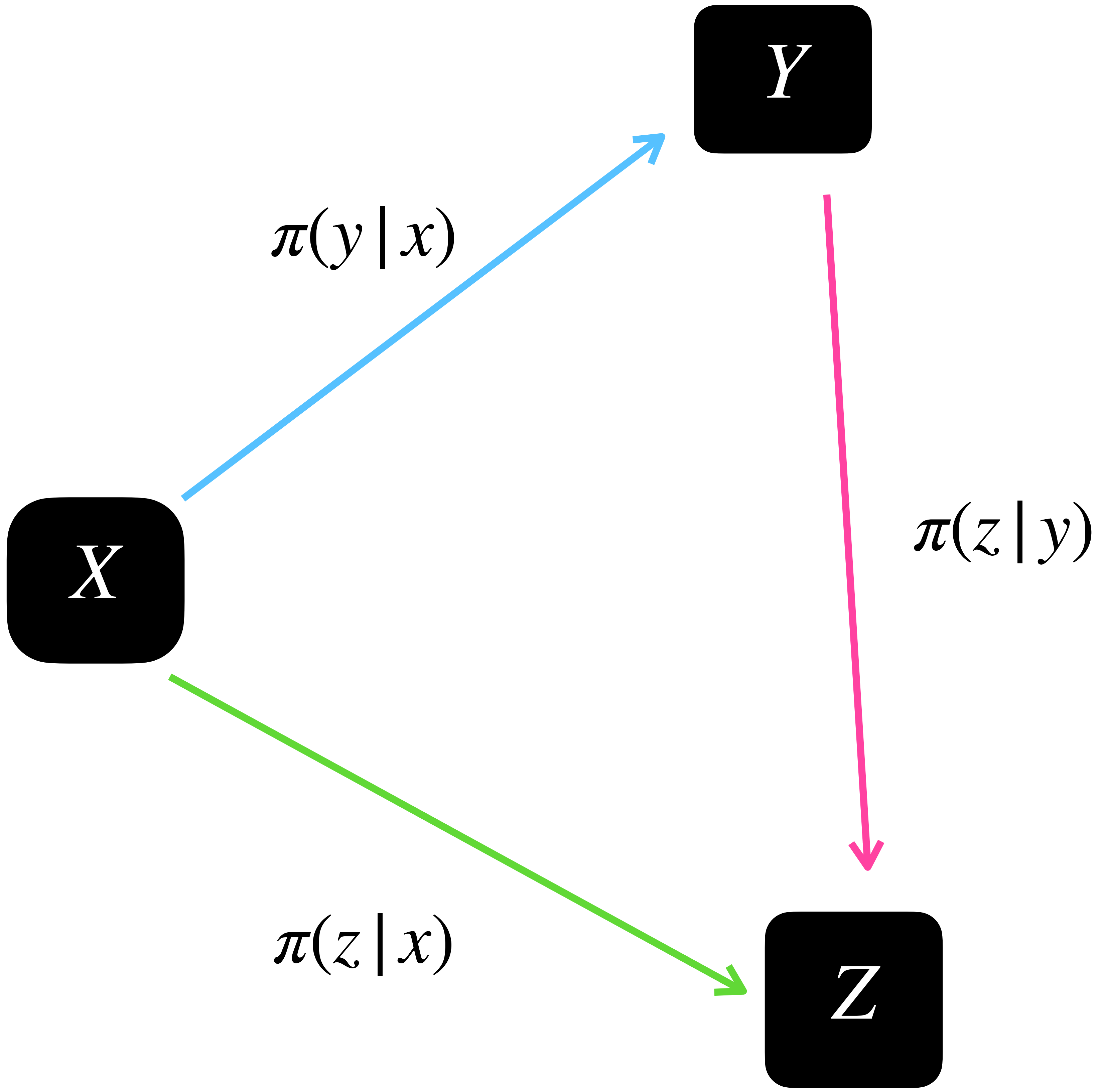
Very Loose Definition

We are interested in the case when the base category is the unit interval $[0,1]$.

We want:

- strings $x, y, \ldots$
- a probability of continuation $\pi(y \mid x)$
- But do we have this inequality?

$$\pi(y \mid x) \cdot \pi(z \mid y) \leq \pi(z \mid x)$$



Yes, suppose we have an LLM.

For any prompt x, the LLM gives a probability distribution $p(- \mid x)$ on the set of tokens.

These probabilities multiply in the following sense:

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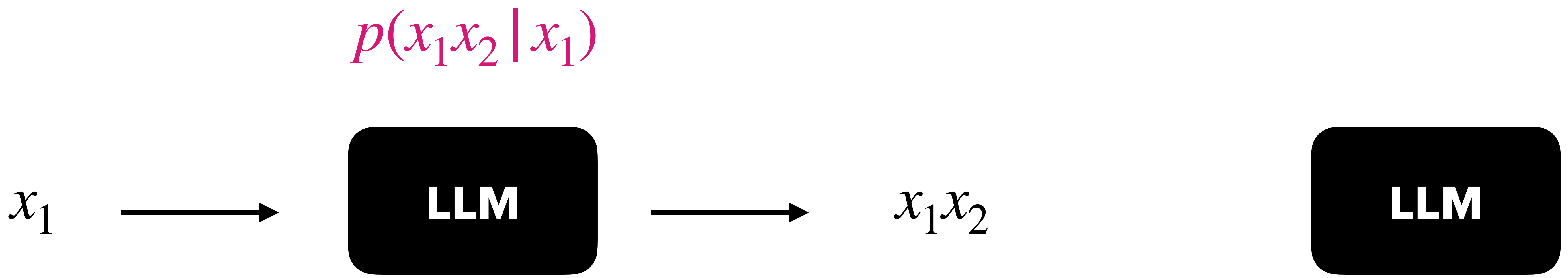

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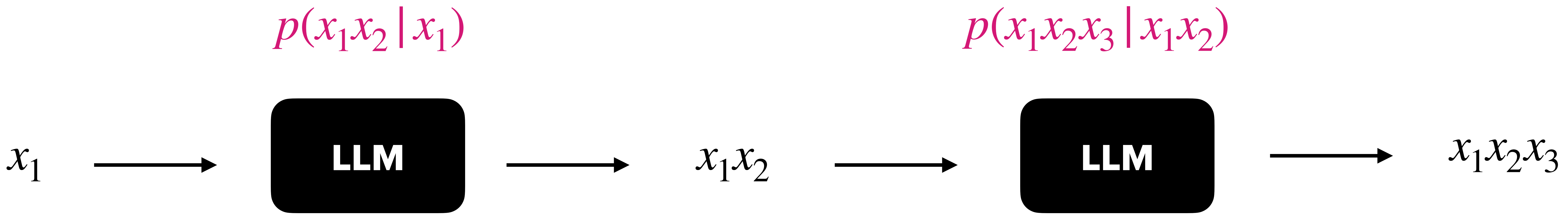

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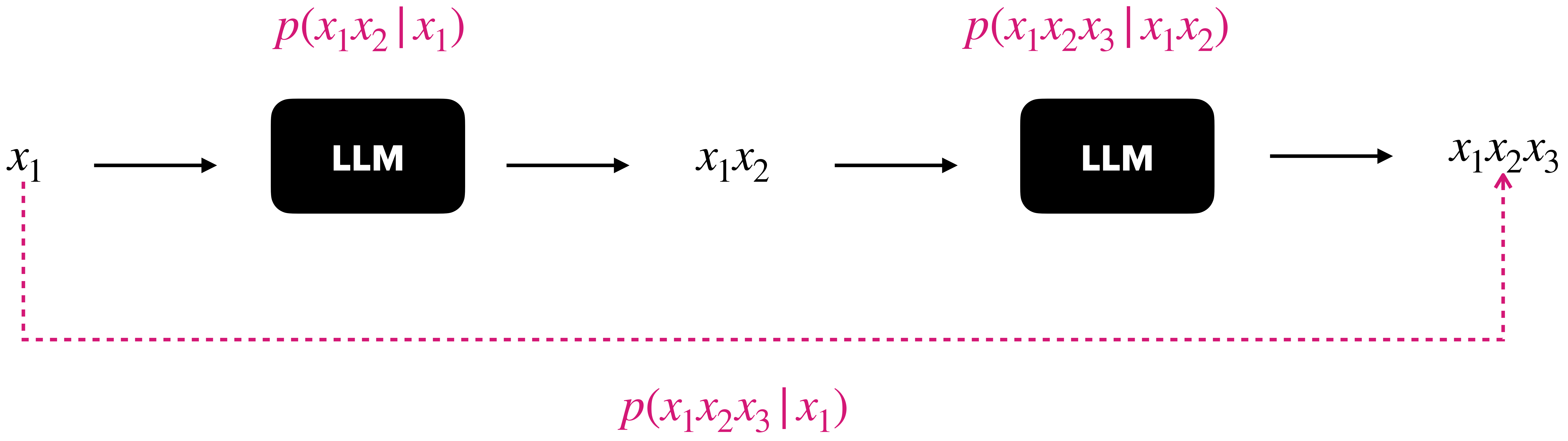

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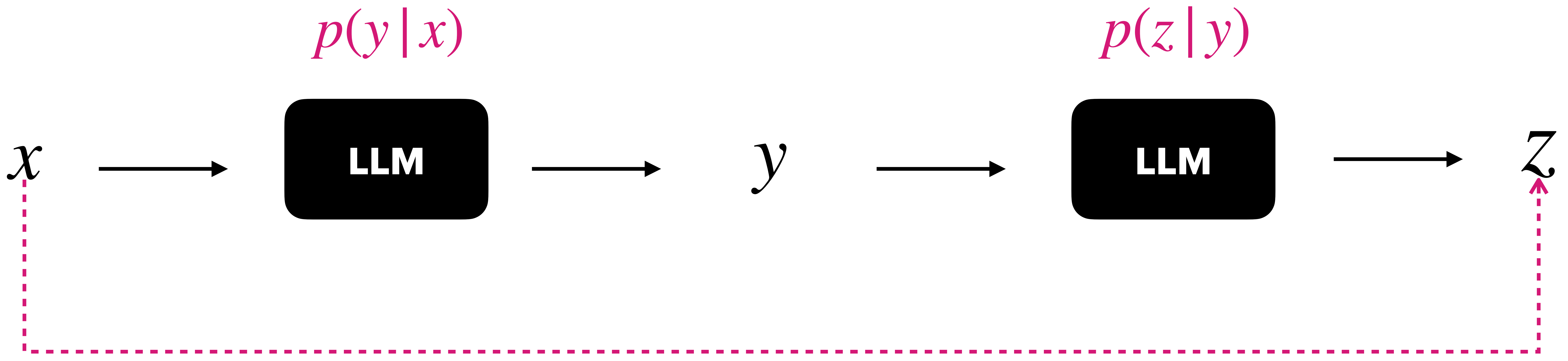

$$p(z\,|\,x) = p(y\,|\,x)p(z\,|\,y)$$

(so we get an *equality,* in fact)

Language as an Enriched Category, $\mathscr{L}$

Over the Unit Interval

Given an LLM and strings $x \to y$, define* the number $\pi(y\,|\,x)$ as a product of the successive probabilities used to obtain y from x one token at a time:

$$\pi(y\,|\,x) := \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \nrightarrow y \\ \displaystyle\prod_{i=1}^{k(y)} p(x_{t+i}\,|\,x_{<t+i}) & \text{if } x \to y \end{cases}$$

* Thanks to Juan Pablo Vigneaux for this observation. In this definition, we write $x \to y$ whenever y extends x on the right.

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This number is an object in the category $[0,1]$, and it satisfies the "**composition rule**"

$$\pi(y\,|\,x) \cdot \pi(z\,|\,y) = \pi(z\,|\,x).$$

for all strings x, y, z.

So, we view **language as a category $\mathscr{L}$ enriched over** $[0,1]$.

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categories

OBJECTS

$$X, Y, \ldots$$

A SET

$$\mathsf{C}(X, Y)$$

A COMPOSITION RULE

$$\mathsf{C}(X, Y) \times \mathsf{C}(Y, Z) \to \mathsf{C}(X, Z)$$

language

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$$\pi(y \mid x)$$

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4. Represent
Enriched Meanings

Consider enriched functors $\mathscr{L} \to [0,1]$ associated to expressions. These contain the same information as before, plus probabilities.

Ex: The functor $\mathscr{L}(\text{blue}, -)$ is supported on all texts that contain “blue.”

$$\mathscr{L}(\textbf{blue}, -) \overset{\text{SORT OF}}{=} \begin{bmatrix} 0 \\ .22 \\ .73 \\ 0 \\ .07 \\ \boxed{0} \\ \vdots \end{bmatrix} \begin{array}{l} \text{deep red Bing cherries} \\ \text{small } \textbf{blue} \text{ marble} \\ \text{beautiful } \textbf{blue} \text{ ocean} \\ \text{did you put the kettle on} \\ \text{red and } \textbf{blue} \text{ fireworks} \\ \text{Sencha green tea} \\ \vdots \end{array}$$

$\pi(\text{Sencha green tea} \,|\, \text{blue})$

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$\pi(\text{Sencha green tea} \,|\, \text{blue})$

5. Operate on
Enriched Representations

The enriched functor category $[0,1]^{\mathscr{L}}$ has rich structure, including the *enriched* versions of limits, colimits, and Cartesian closure.

So, we can again make sense of logical operations like conjunction, disjunction, and implication.

$$\mathscr{L}(\textbf{red}, -) \sqcup \mathscr{L}(\textbf{blue}, -) \overset{\text{SORT OF}}{=} \left[\begin{array}{c} .10 \\ .22 \\ \boxed{.73} \\ 0 \\ .59 \\ 0 \\ \vdots \end{array}\right]$$

$\max\{\mathscr{L}(\textbf{red}, y), \mathscr{L}(\textbf{blue}, y)\}$

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We can work with *distances* instead of *probabilities by* considering the function

$$-\ln : [0,1] \to [0,\infty] \, .$$

The **distance** between expressions x and y is defined by

$$d(x, y) = -\ln \pi(y \,|\, x).$$

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The **distance** between expressions x and y is defined by

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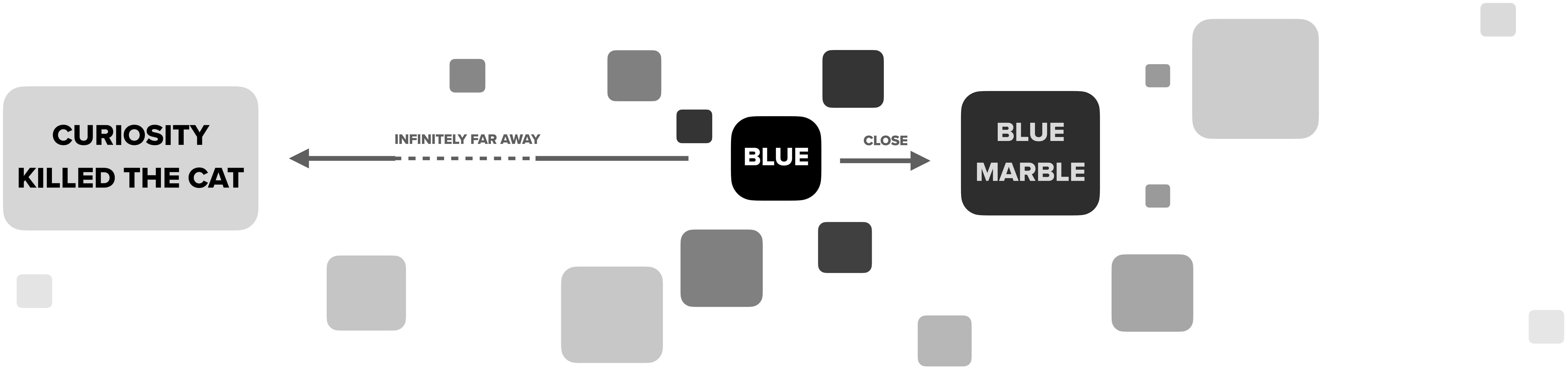
Likely continuations of a text x are **close** to it. Other texts that are not continuations are **infinitely far away**.



Repeat the story all over again.

Distances satisfy a "**composition rule**."
In fact, it is enriched category theory all over again!

$$d(x, y) + d(y, z) \geq d(x, z)$$

A $[0,\infty]$-enriched category is also called a **generalized metric space**, and we can compute the versions of the previous constructions:

- **represent** meanings as "vectors" (i.e. enriched functors)
- **combine** those representations using enriched categorical operations

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What *else* do we gain from a geometric perspective?

- Stéphane Gaubert (INRIA) and Yiannis Vlassopoulos (IHES, ARC) recently interpreted this generalized metric space through the lens of **tropical geometry.**

The $[0,\infty]$-category of language can be viewed as a polyhedron, with a geometric interpretation of the "meaning" of texts as generating this polyhedron.

DIRECTED METRIC STRUCTURES ARISING IN LARGE LANGUAGE MODELS

STÉPHANE GAUBERT AND YIANNIS VLASSOPOULOS

arXiv: 2405.12264

Repeat the story all over again.

Distances satisfy a "**composition rule**." In fact, it is enriched category theory all over again!

$$d(x,y) + d(y,z) \geq d(x,z)$$

A $[0,\infty]$-enriched category is also called a **generalized metric space**, and we can compute the versions of the previous constructions:

- **represent** meanings as "vectors" (i.e. enriched functors)
- **combine** those representations using enriched categorical operations

What *else* do we gain from a geometric perspective?

- Juan Pablo Vigneaux (Caltech) recently computed the **magnitude** of (a finite version of) this generalized metric space.

  Magnitude is a numerical invariant for finite enriched categories.

  *You can rescale via a parameter t to obtain a **magnitude function**, which is even more interesting.*

  For our $[0,\infty]$-enriched category of language, the magnitude function is a sum over prompts of Tsallis entropies:

$$|\, t\mathscr{L} \,| = (t-1) \sum_{x \in \mathit{ob}(\mathscr{L})} H_t(p(-\,|\,x))$$

T.-D. B. and Juan Pablo Vigneaux, in preparation (2024)

$$
\begin{array}{cc}
 & y \\
x & \begin{bmatrix} & \vdots & \\ \cdots & e^{-td(x,y)} & \\ & & \end{bmatrix}
\end{array}
$$

used for
magnitude

$$\begin{array}{cc} & y \\ x & \begin{bmatrix} & \vdots & \\ \cdots & e^{-td(x,y)} & \\ & & \end{bmatrix} \end{array}$$

used for
magnitude

$$\begin{array}{cc} & y \\ x & \begin{bmatrix} & \vdots & \\ \cdots & \mathrm{pmi}(x,y) & \\ & & \end{bmatrix} \end{array}$$

used with
SVD

$$\begin{array}{cc} & y \\ x & \begin{bmatrix} & \vdots & \\ \cdots & e^{-td(x,y)} & \\ & & \end{bmatrix} \end{array}$$

used for
magnitude

$$\begin{array}{cc} & y \\ x & \begin{bmatrix} & \vdots & \\ \cdots & \mathrm{pmi}(x,y) & \\ & & \end{bmatrix} \end{array}$$

used with
SVD

$$\begin{array}{cc} & y \\ x & \begin{bmatrix} & \vdots & \\ \cdots & 0,1 & \\ & & \end{bmatrix} \end{array}$$

used in
formal concept analysis

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Thanks

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