Open Mathematical Problems in Dimension Reduction Methods and Energy Landscapes

Antonio Auffinger

Mathematics of Cancer: Open Mathematical Problems

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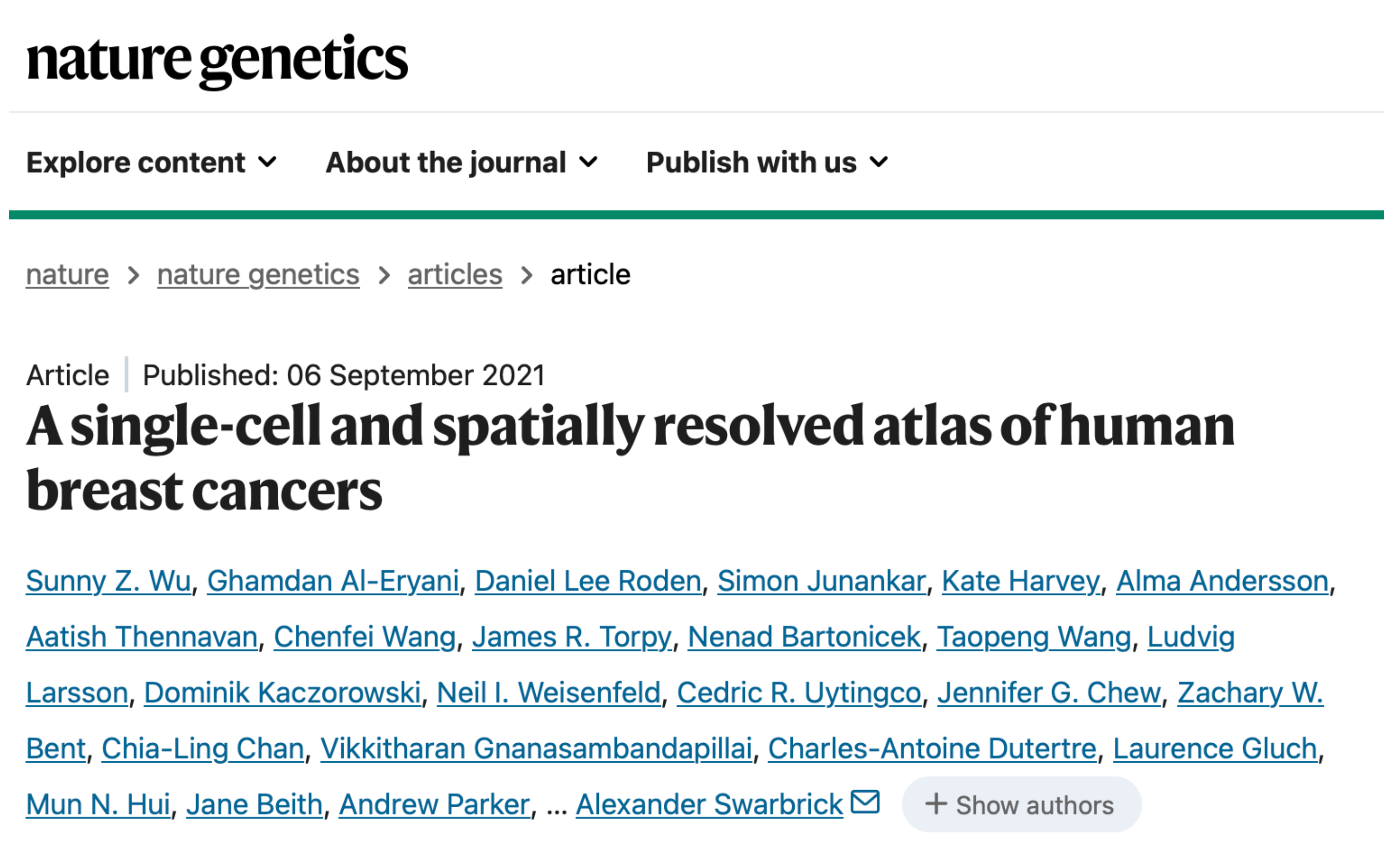

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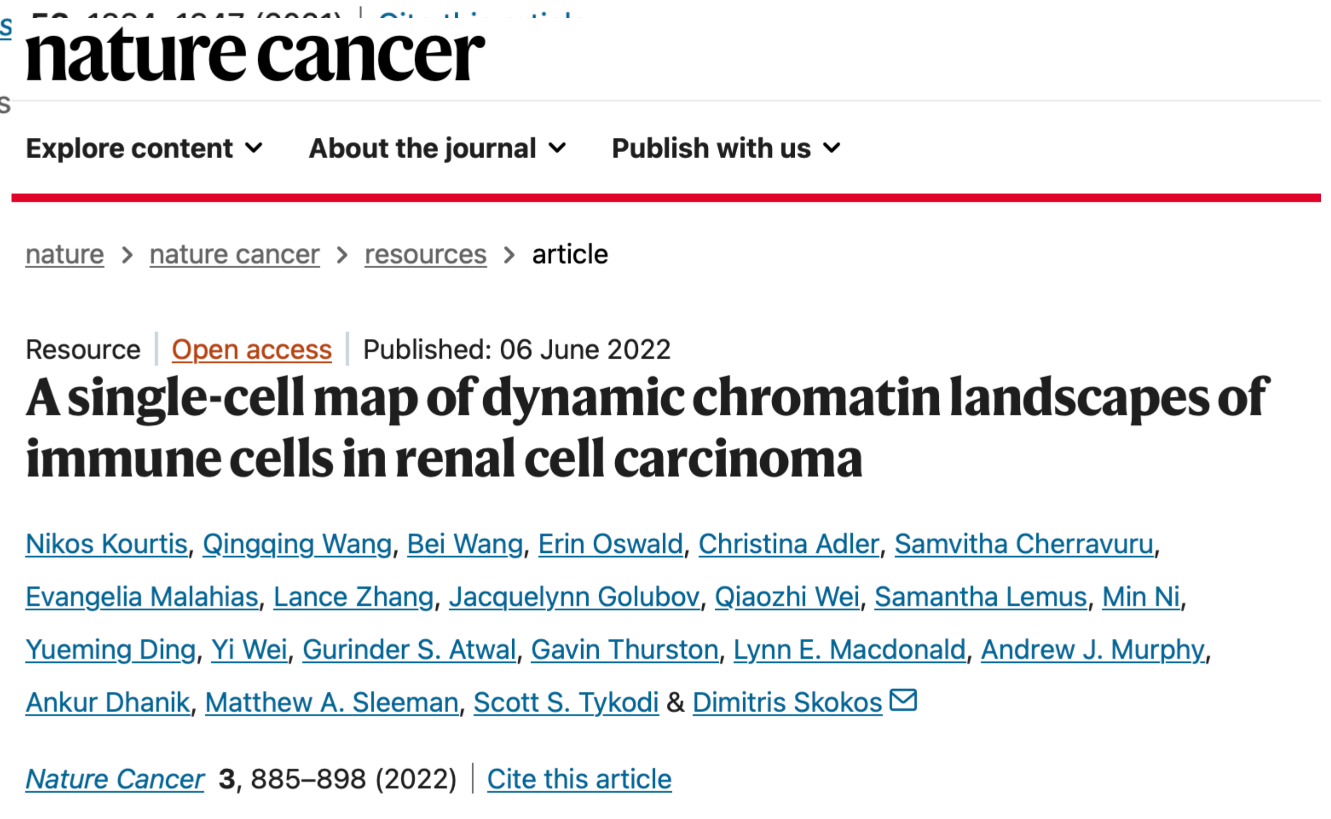

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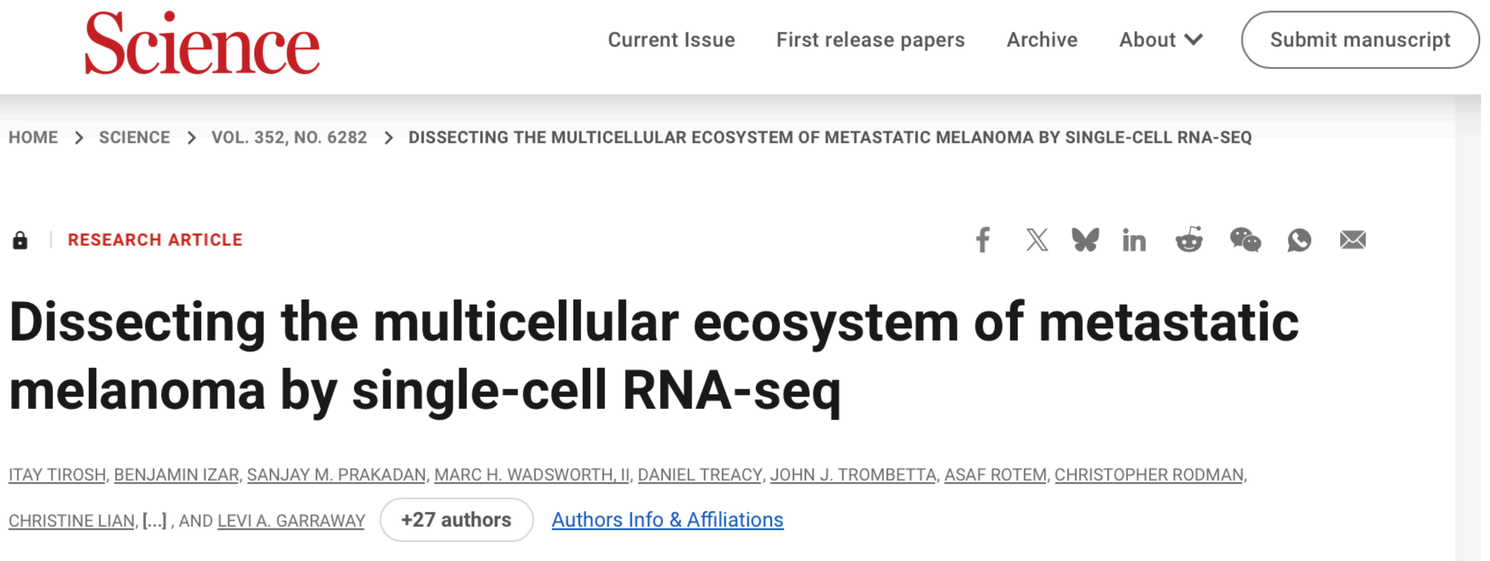

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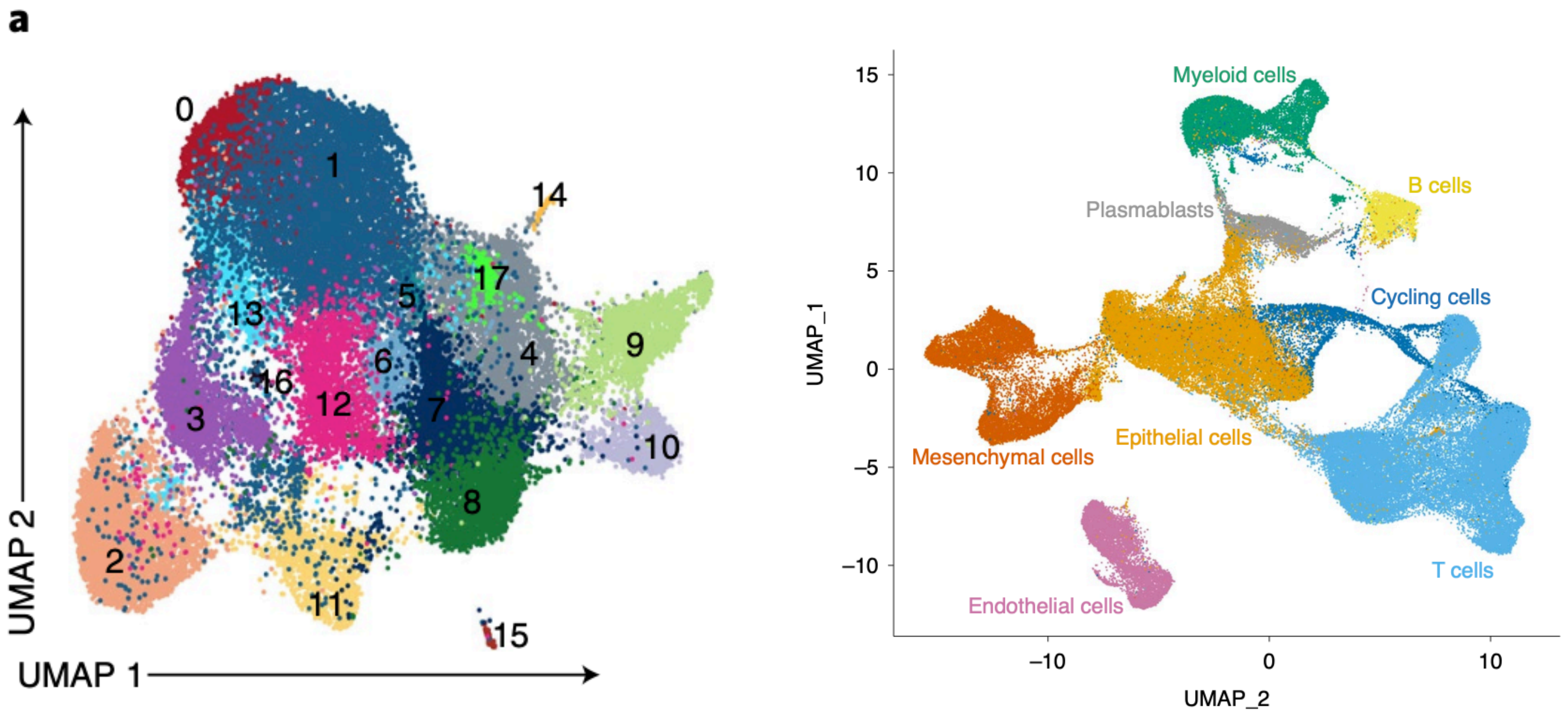
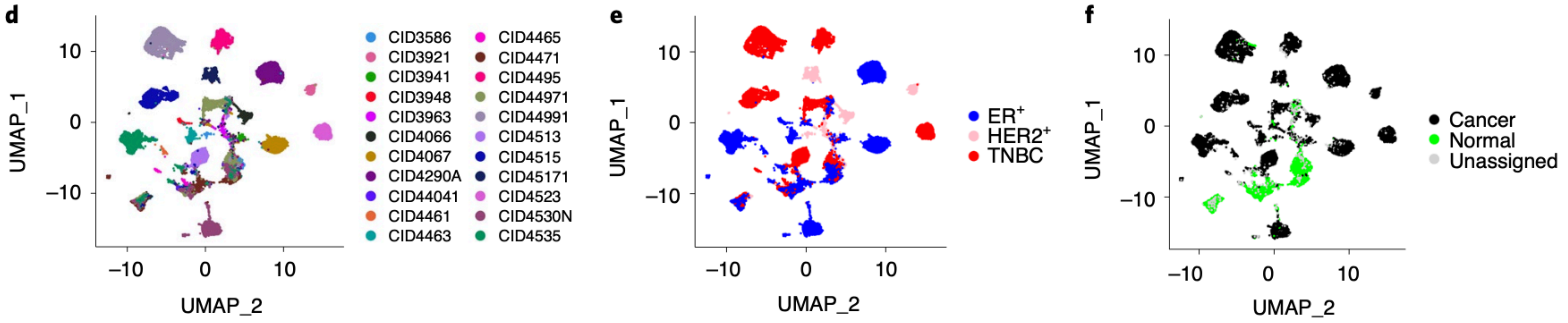
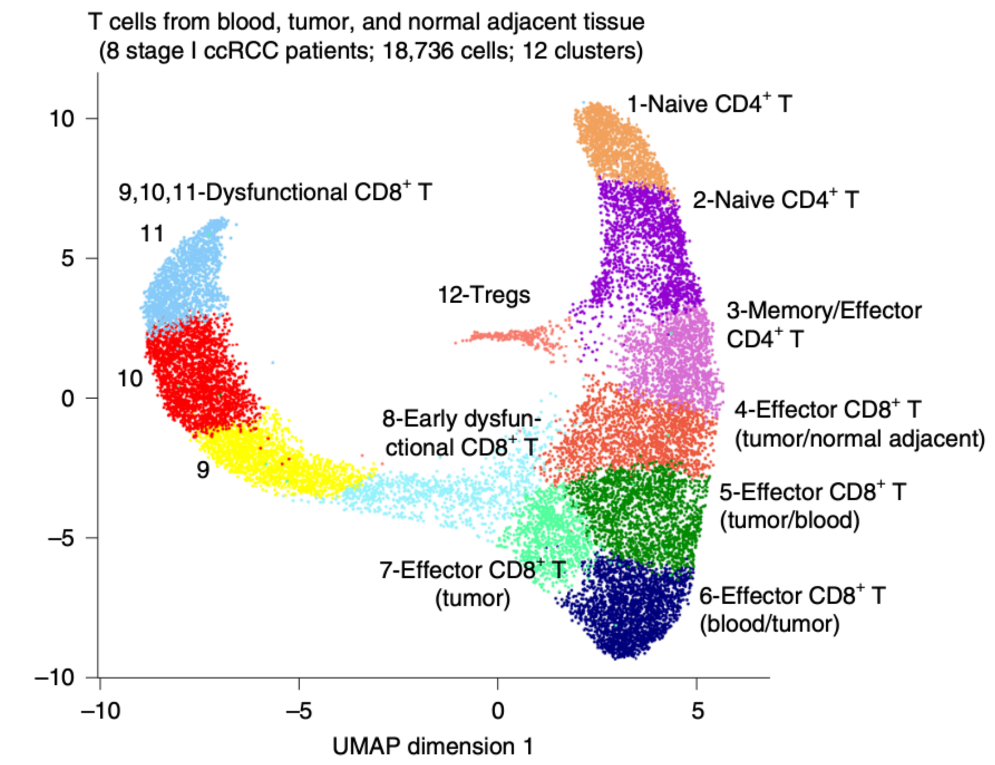
RESEARCH ARTICLE

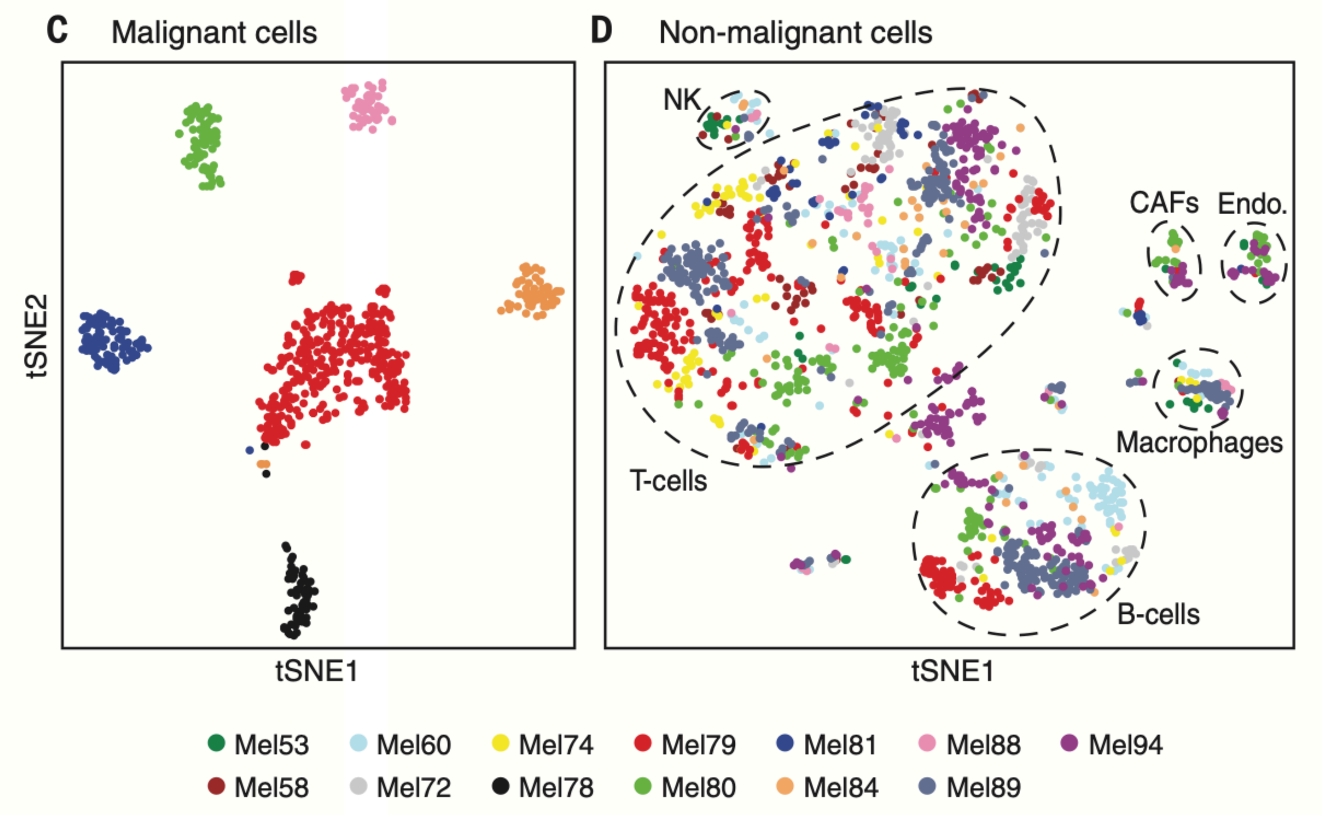
Dissecting the multicellular ecosystem of metastatic melanoma by single-cell RNA-seq

ITAY TIROSH, BENJAMIN IZAR, SANJAY M. PRAKADAN, MARC H. WADSWORTH, II, DANIEL TREACY, JOHN J. TROMBETTA, ASAF ROTEM, CHRISTOPHER RODMAN, CHRISTINE LIAN, [...], AND LEVI A. GARRAWAY +27 authors Authors Info & Affiliations









Dimension Reduction Problem

Main problem: Given a set of n high-dimensional points

$$X_1, \ldots, X_n \in \mathbb{R}^d$$

Find a low dimensional collection of points

$$Y_1, \ldots, Y_n \in \mathbb{R}^s, \quad s < d$$

that captures most of the geometry of $\{X_i : i = 1, \ldots, n\}$.

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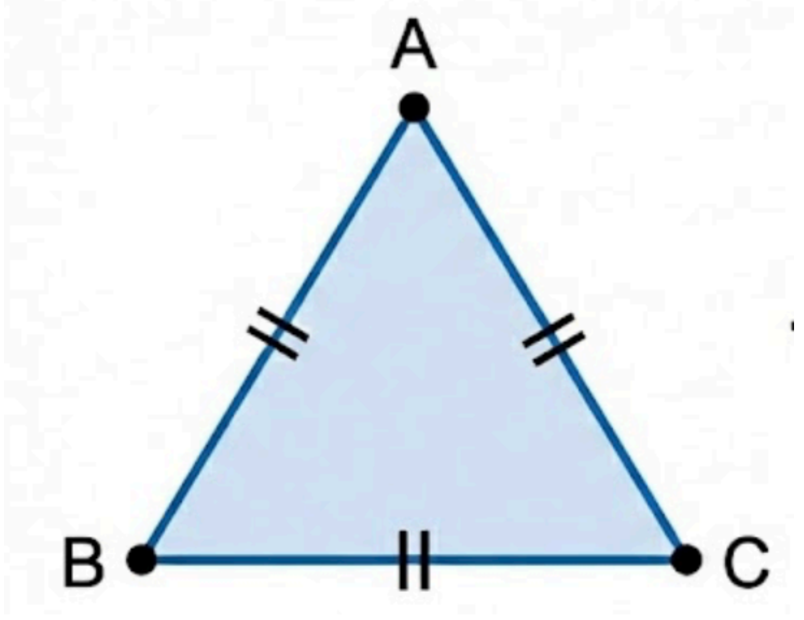

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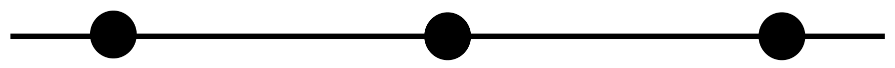


that captures most of the geometry of $\{X_i : i = 1, \ldots, n\}$.

- *Ill-posed.*
- *Input* $d \gg 1$, *Output* $s = 2$.

Johnson-Lindenstrauss's Lemma

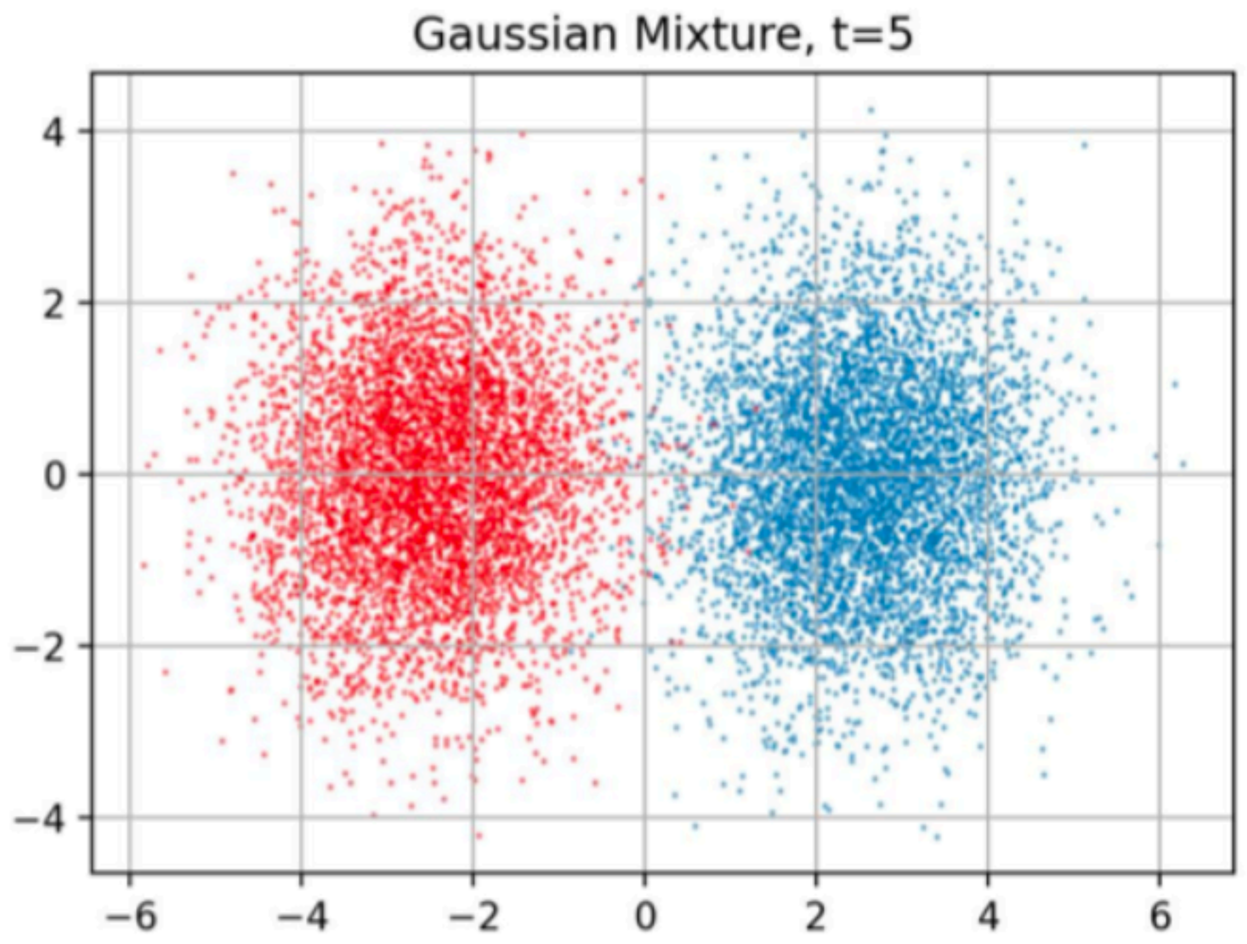
Theorem (Johnson–Lindenstrauss, 1984)

Let $X \subset \mathbb{R}^d$ be a set of n points. There exists some universal constant $C > 0$ such that, given $\varepsilon > 0$, for $s \geq \frac{C \log n}{\varepsilon^2}$ there exists a linear map $f : \mathbb{R}^d \to \mathbb{R}^s$ such that, for all $x, x' \in X$

$$(1-\varepsilon)\|x - x'\|^2 \leq \|f(x) - f(x')\|^2 \leq (1+\varepsilon)\|x - x'\|^2.$$

- $n = 10,000$, $\varepsilon = 1/10$ leads to $s \approx 250,000$.
- Probabilistic method. Solution: a $d \times s$ random matrix works with high probability.

Two illustrative examples: MNIST and Gaussian noise



MNIST

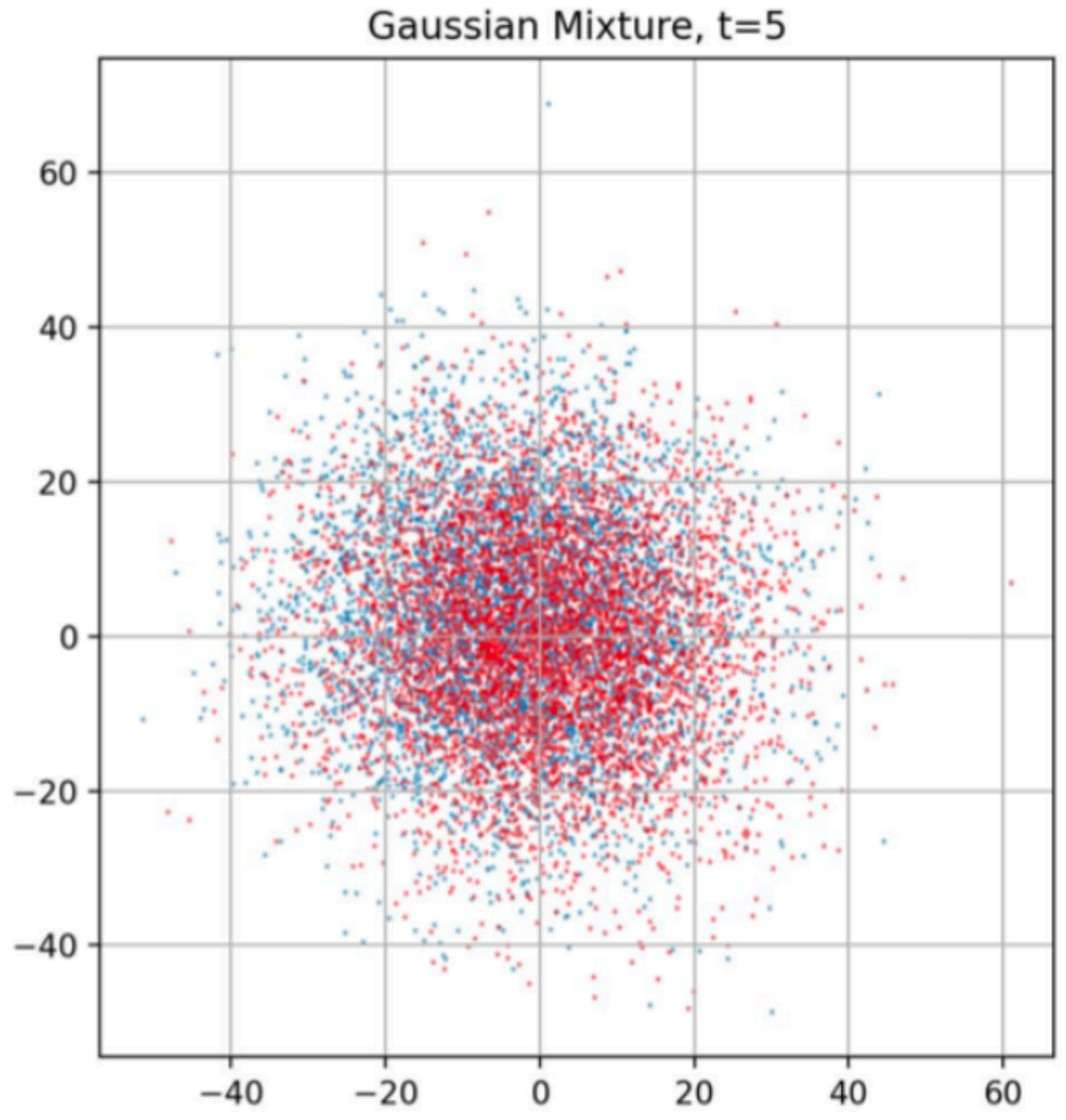
Figure: 28 × 28 pixel boxes.

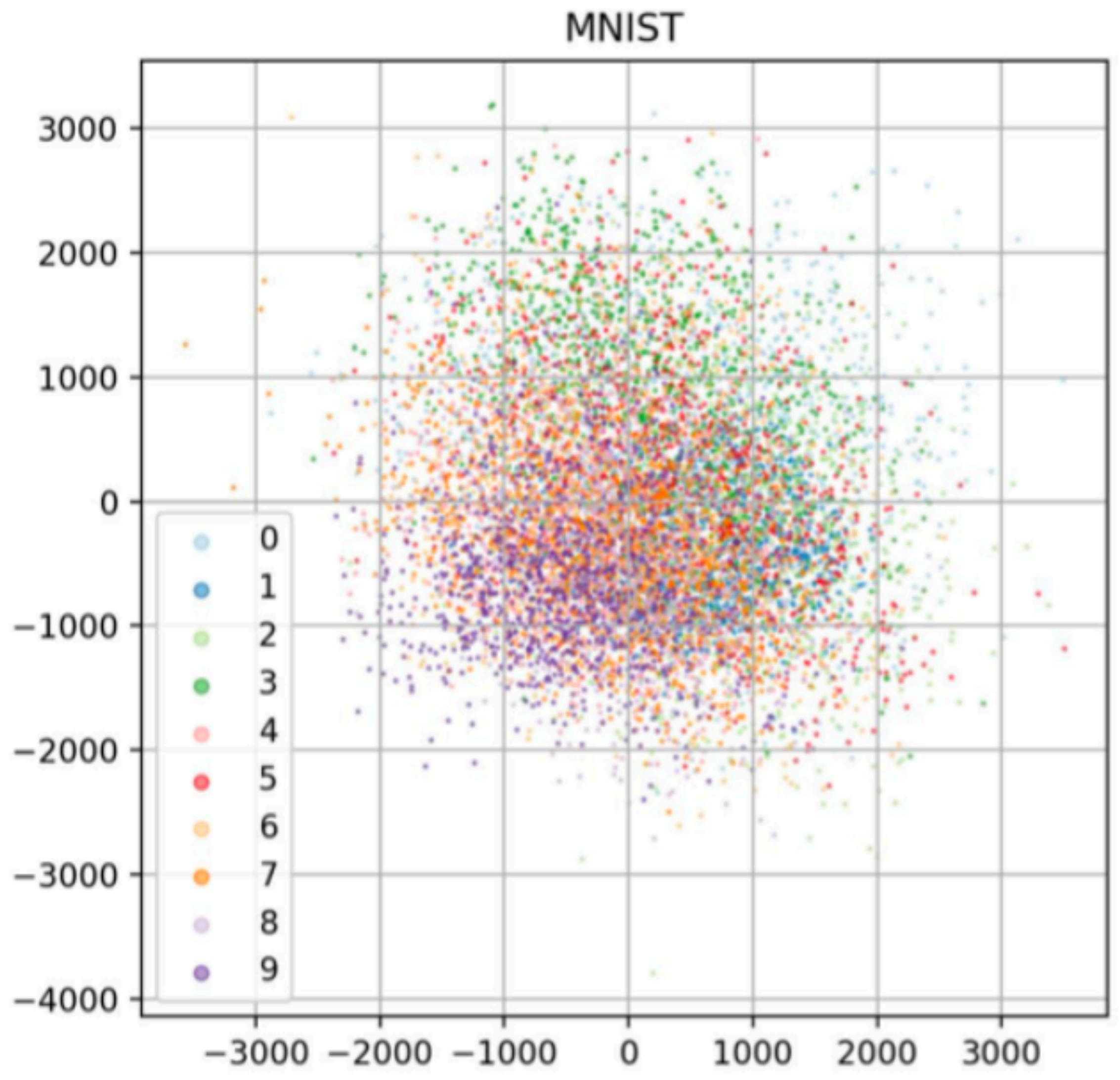
- MNIST: Each number corresponds to a vector in $\mathbb{R}^d$ with $d = 28 \times 28 = 784$.
- Gaussian Noise: $5,000$ iid standard Gaussians in dimension 784 and $5,000$ iid Gaussian centered at te_1, $t > 0$.
- Visualize different clusters.

Johnson-Lindenstrauss's Lemma

- What happens if we use Johnson-Lindenstrauss's result with output dimension 2?

Using a 2 × 758 random matrix A with *iid* Bernoulli ± 1 entries. $Y = AX$.

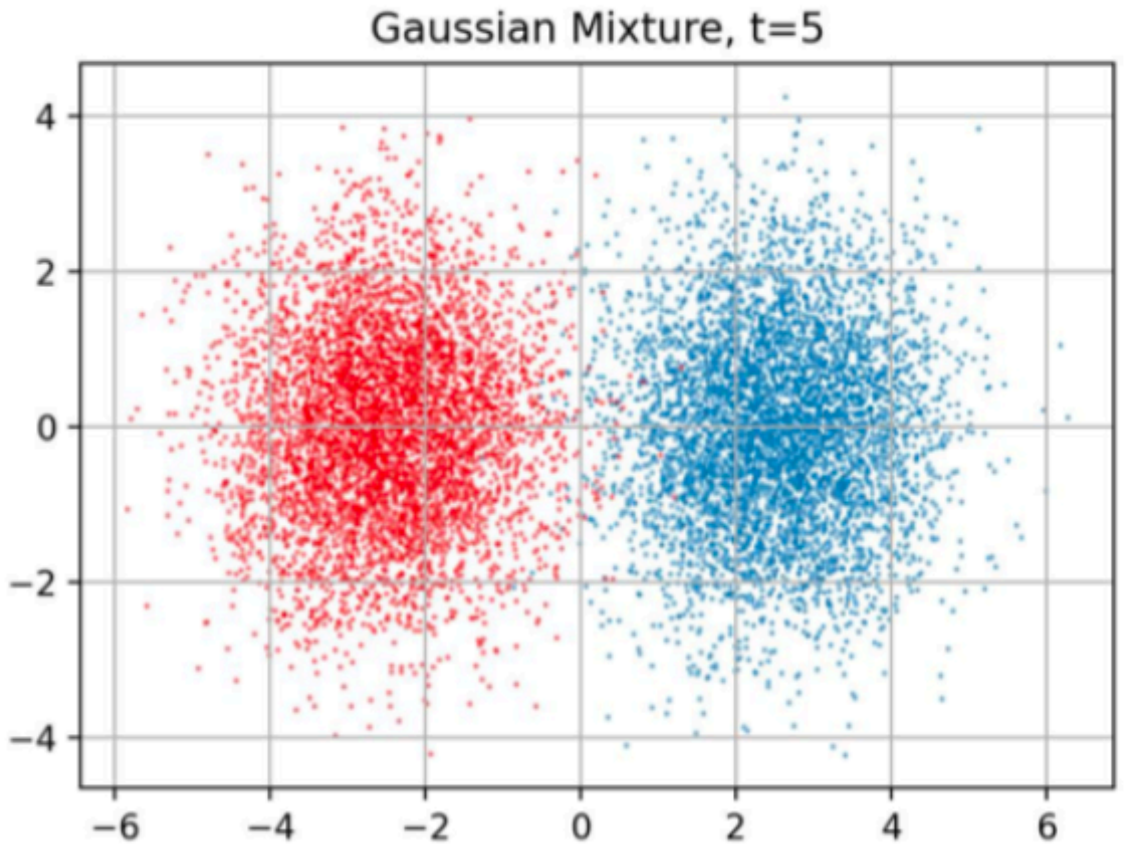


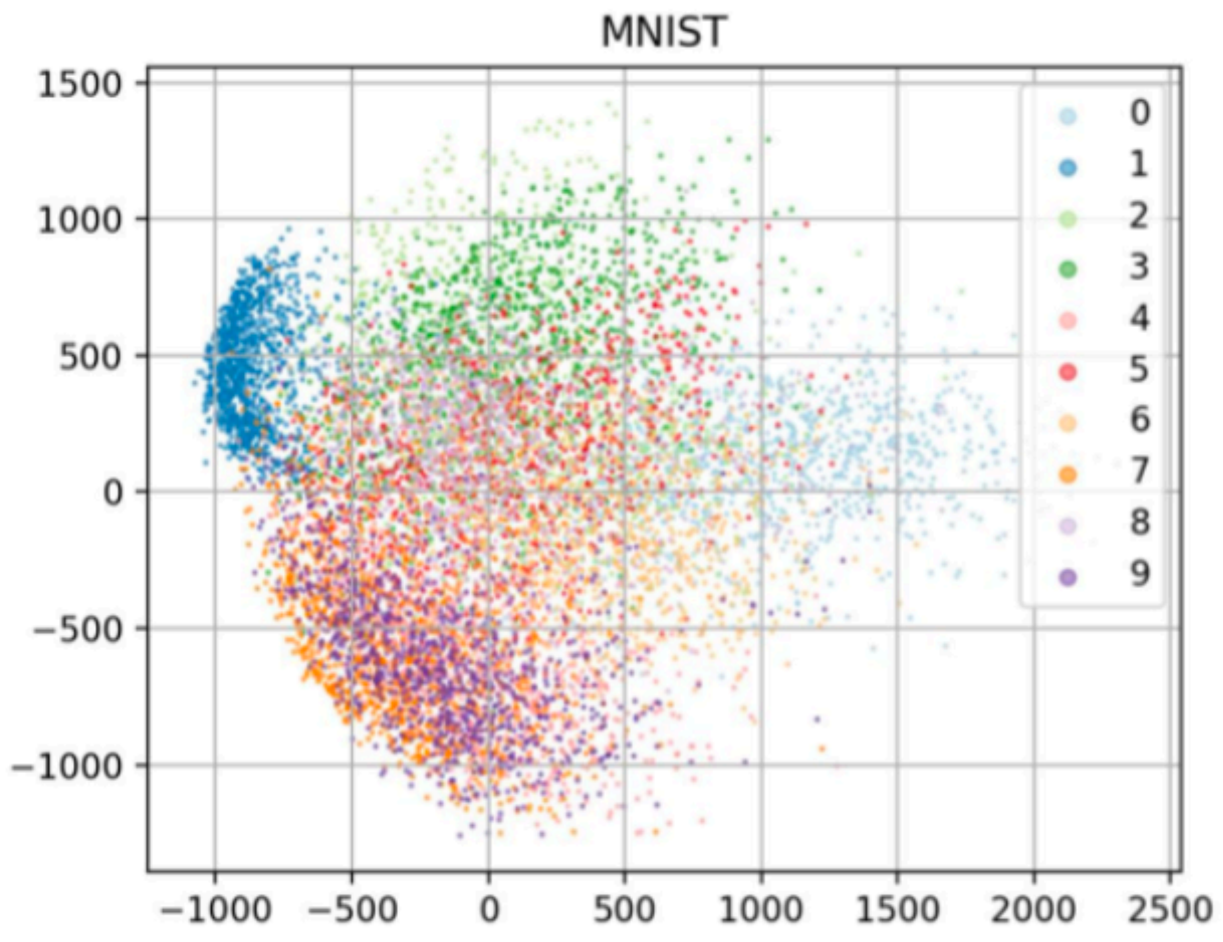


Poor performance as expected.

Principal Component Analysis (Pearson 1901)

- Write your input as a $n \times d$ matrix X.
- Construct the matrix $C = XX^t$.
- Project the input data X onto the subspace spanned by the top 2 eigenvectors of XX^T.



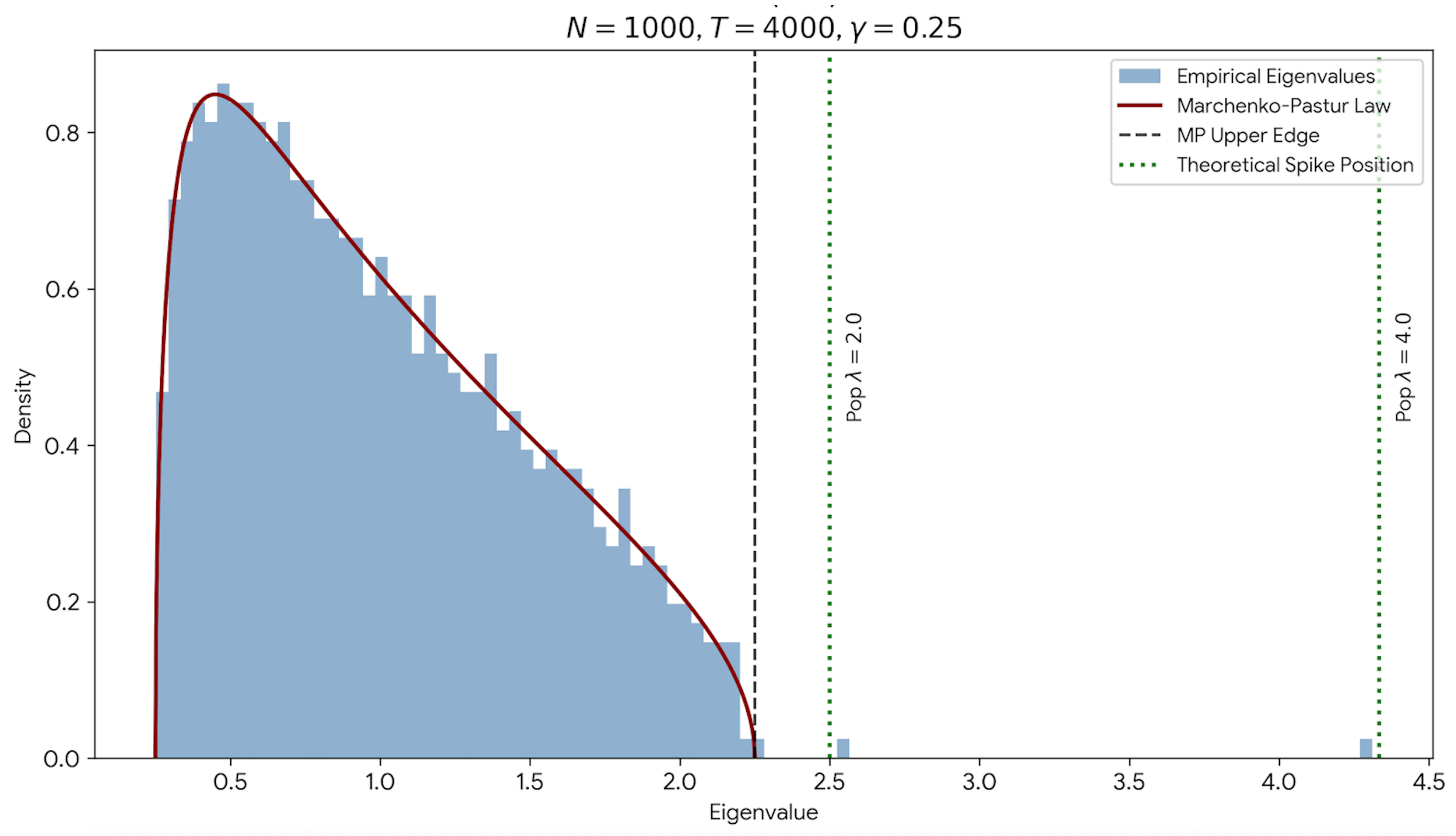


- Top eigenvectors are called the principal components.

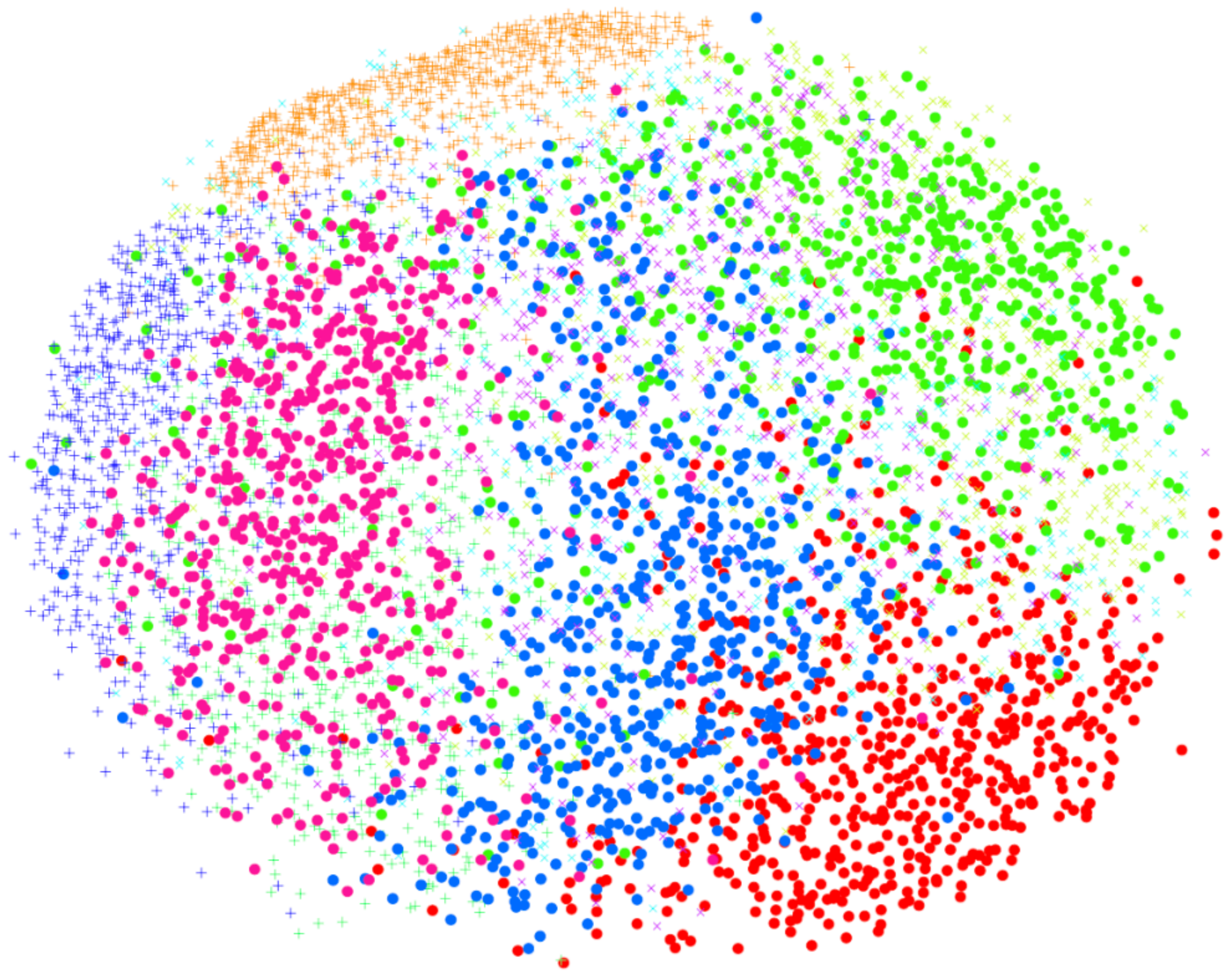
- Why and when does it work?
- Let X be a matrix of independent random variables (say Gaussian).
- (Baik-Ben Arous-Péché, 2002) The top eigenvector v_1 of the matrix $Z = XX^T + te_1e_1^T$ goes through a phase transition.

$$\langle v_1, e_1 \rangle \to \begin{cases} 0, & t < 1 \\ c(\gamma) > 0, & t > 1. \end{cases}$$

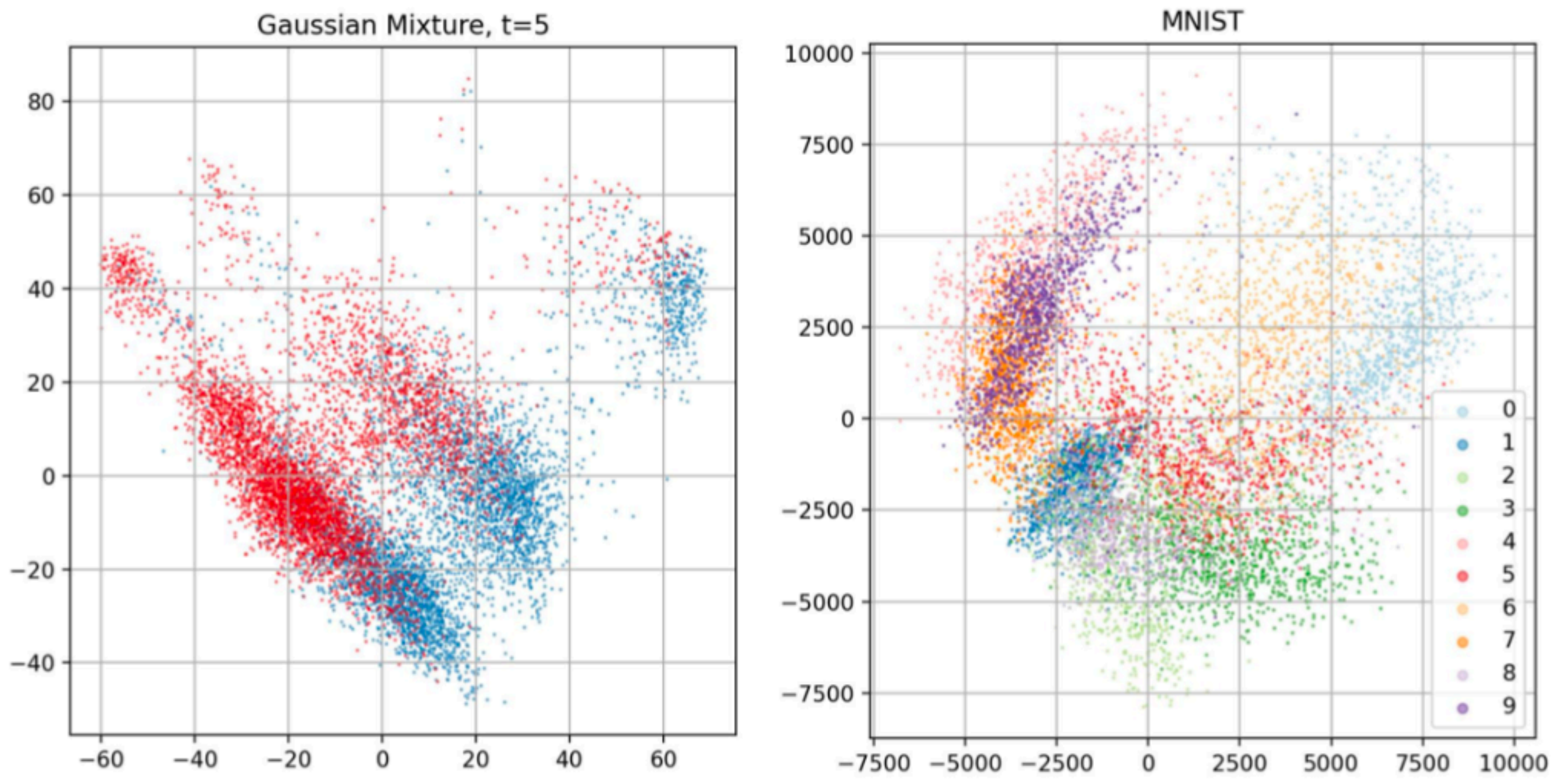
as $n \to \infty$ where $\gamma = d/n \in (0, 1)$.



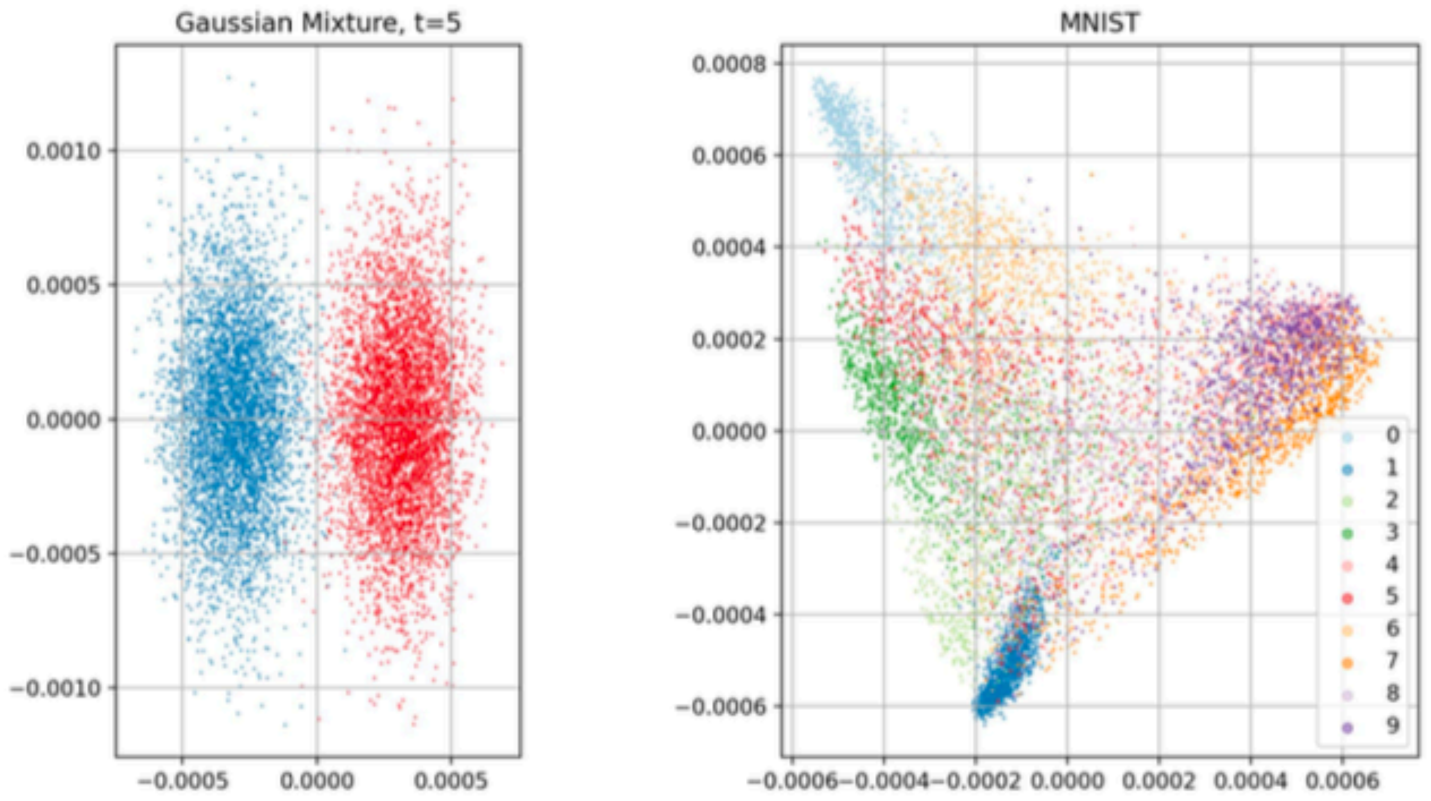
Sammon (1969)



Isomaps (Tenenbaum, de Silva, Langford 2000)

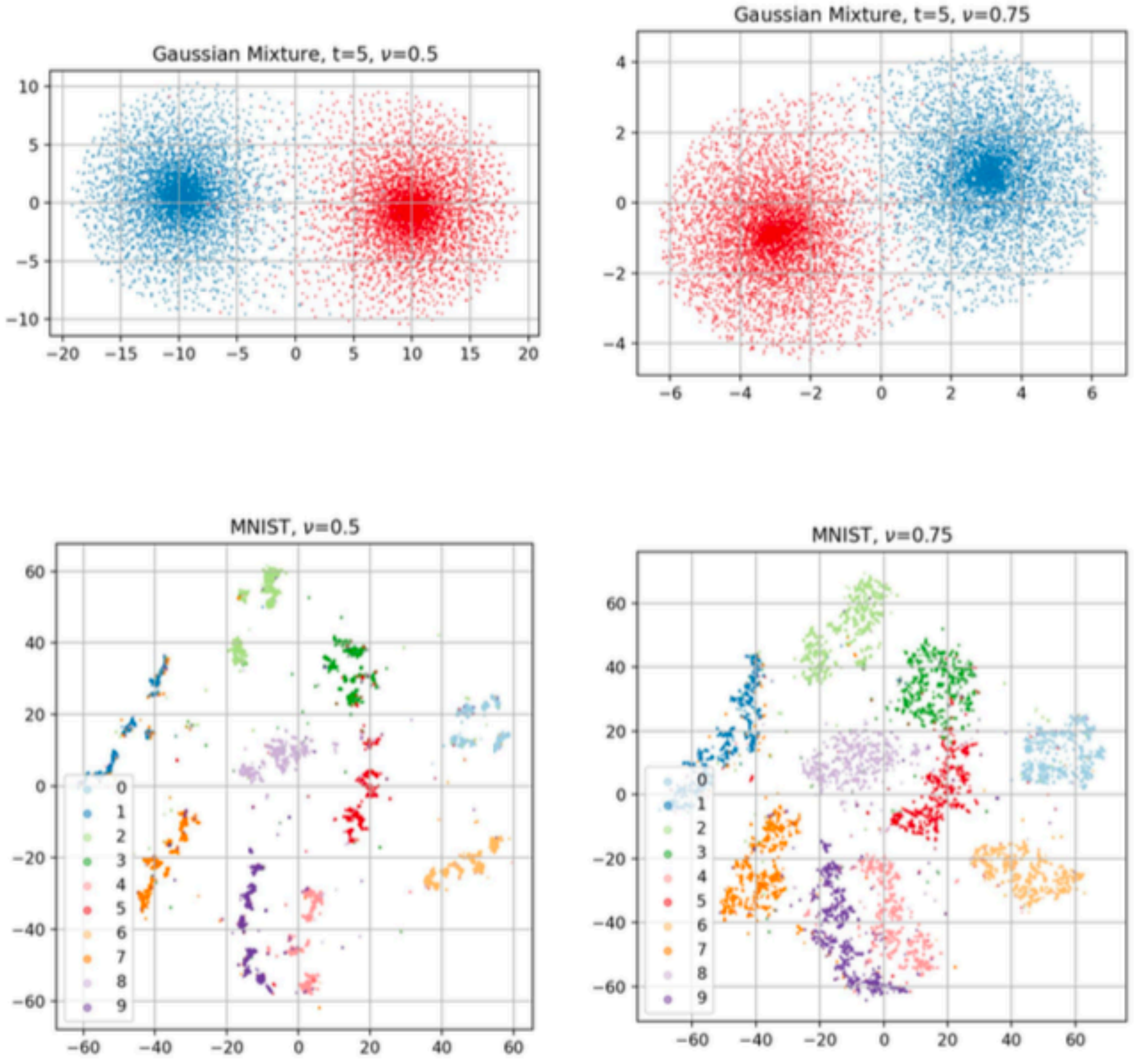


Laplacian Eigenmaps (2003)

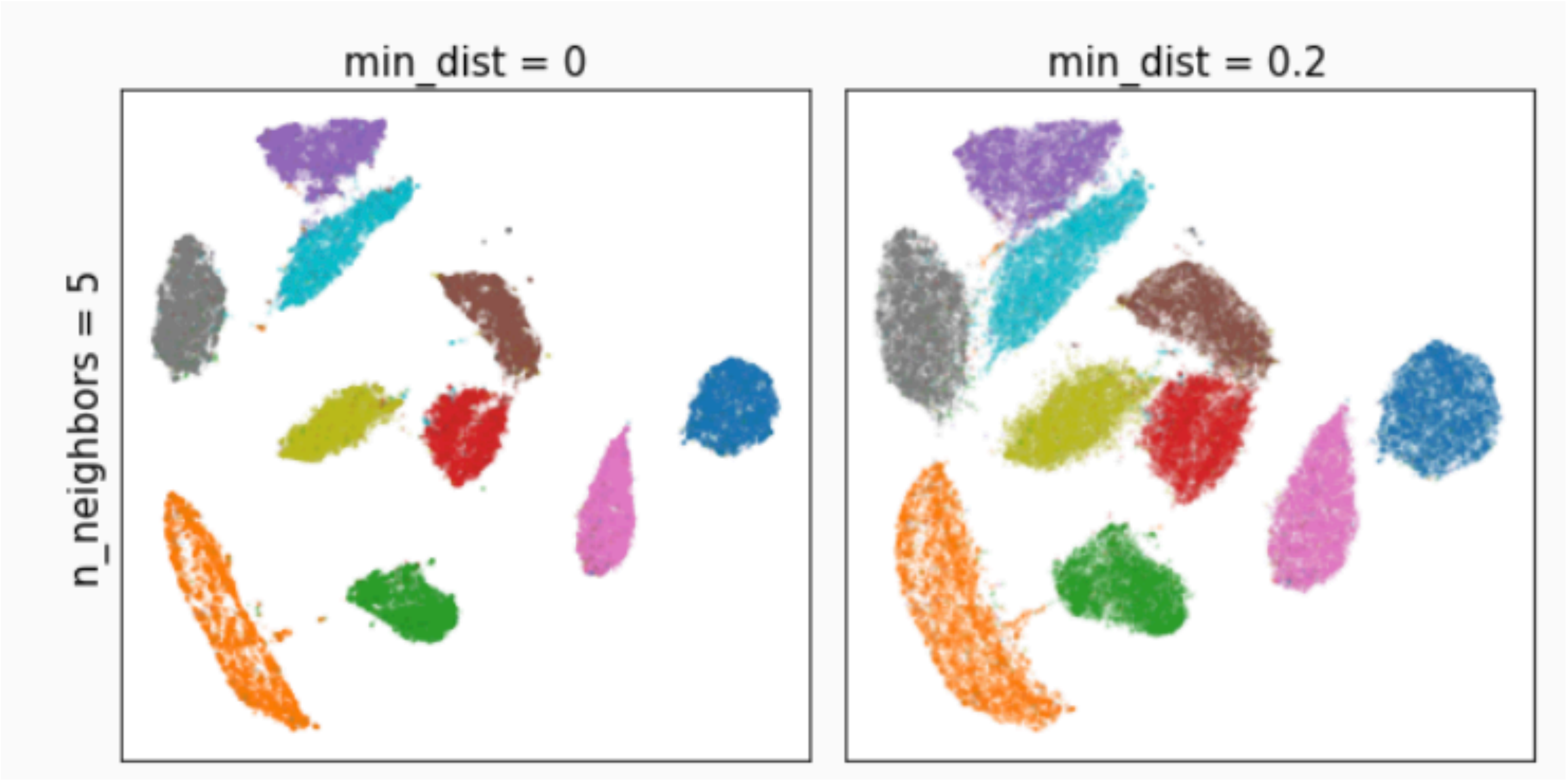


t-Stochastic Neighbor Embedding

(van Der Maaten & Hinton, 2008)



UMAP (McInnes & Healy, 2018)



What is t-SNE?

- **Main idea.** Turn a configuration of points into probability distributions and force these distributions to be similar.

- The collection $X = \{X_1, \ldots, X_n\}$ is mapped to a sequence of probability measures

$$\mu_X = (\mu_1, \ldots, \mu_n).$$

- The measure μ_i is supported on $\{x_j\}, j \neq i$ and represents the likelihood that point i will choose point j (or the similarity between i and j).

- Do the same for $(Y_1, \ldots, Y_n)$ and create $\mu_Y = (\nu_1, \ldots, \nu_n)$.

What is t-SNE?

- Given μ_X optimize the choice of $(Y_1, \ldots, Y_n)$ by minimizing the Kullbeck-Leibler divergence (or relative entropy) between μ_X and μ_Y.

$$d(\mu_X, \mu_Y) = \sum_{i=1}^{n} D_{KL}(\mu_i, \nu_i), \quad D_{KL}(\mu, \nu) = \int \log\left(\frac{d\mu}{d\nu}\right) d\nu.$$

KL Divergence (Kullback & Leibler, 1951)

- $D_{KL}(\mu_i, \nu_i) \geq 0$.
- $D_{KL}(\mu_i, \nu_j) = \sum_j p_{ij} \log\left(\frac{p_{ij}}{q_{ij}}\right)$.
- $D_{KL}(\mu_i, \nu_i) = 0$ if and only if $\mu_i = \nu_i$.

How to choose the measures?

- Gaussian kernel for the input

$$\mu_i(x_j) = p_{j|i} = \frac{\exp(-\|x_i - x_j\|^2/2\sigma_i^2)}{\sum_{j\neq i} \exp(-\|x_i - x_j\|^2/2\sigma_i^2)}$$

  with $p_{i|i} = 0$ and set $p_{ij} = (p_{j|i} + p_{i|j})/2$.

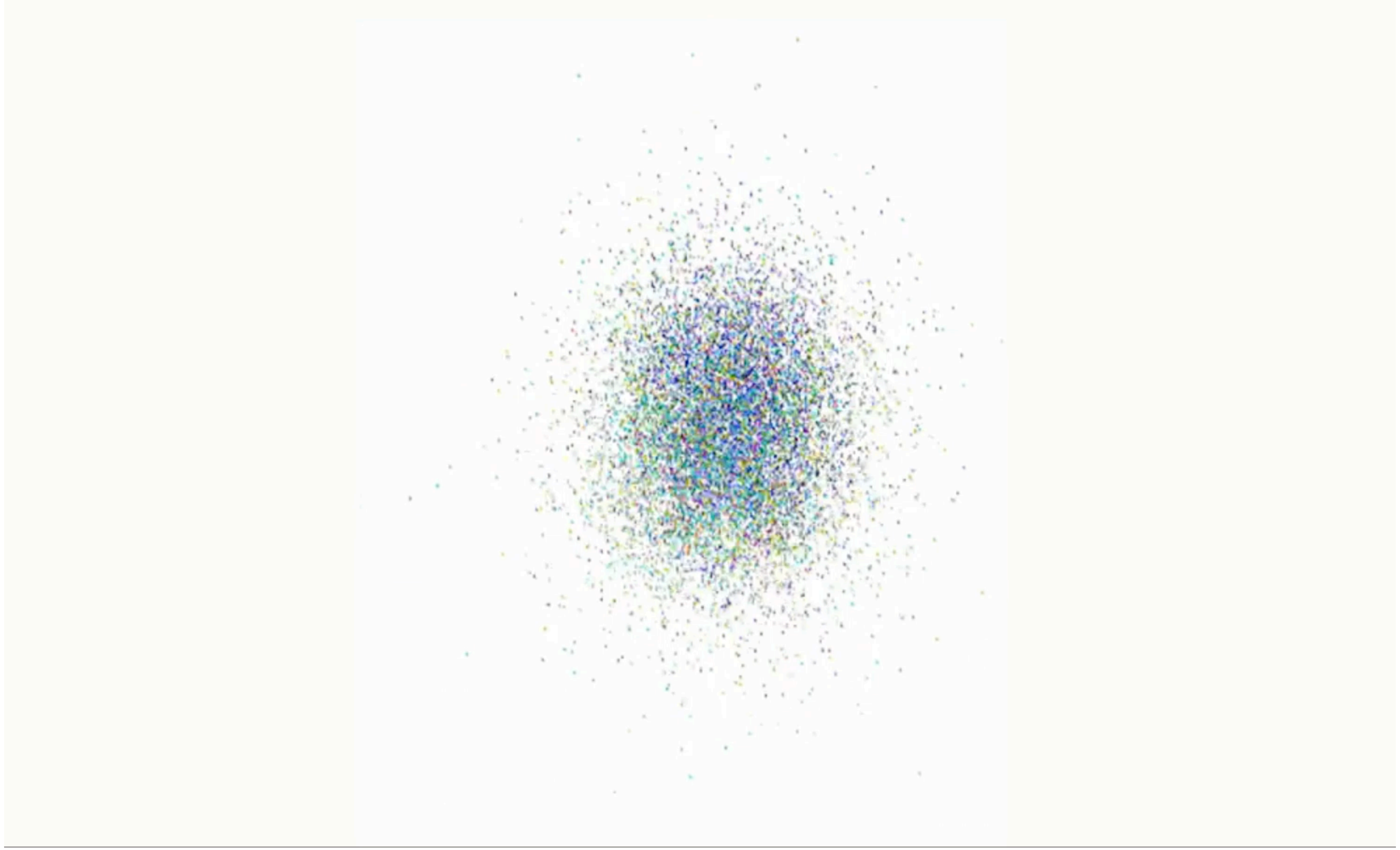
- A t-distribution for the output variables

$$\nu_i(y_j) = q_{ij} = \frac{(1 + \|y_i - y_j\|^2)^{-1}}{\sum_{j\neq i}(1 + \|y_i - y_j\|^2)^{-1}}$$

so we want to minimize

$$L_n(X,Y) = L_n(Y|X) = \sum_{i\neq j} p_{ij} \log\left(\frac{p_{ij}}{q_{ij}}\right)$$

over the choice of q_{ij} coming from $Y \in (R^2)^n$ (non-linear constraint).



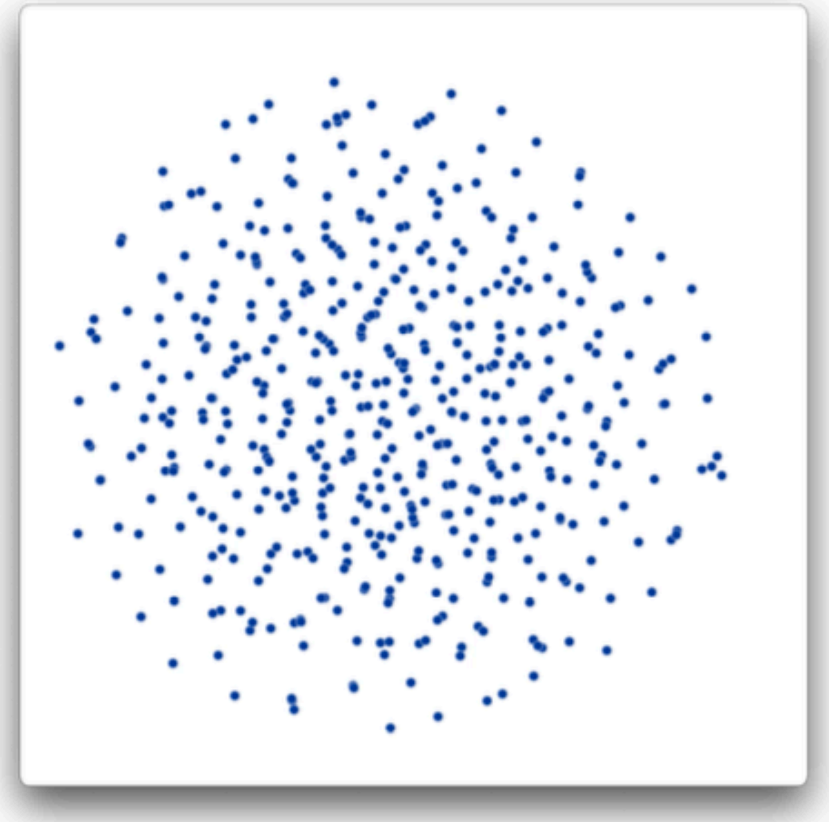
Remarks/Previous work

- SNE - Hinton & Roweis 2002
- van de Maaten & Hinton, 2008 (t-SNE)
- 56k+ citations. Wide range of use.

In mathematics:

- Linderman & Steinerberger, 2017
- Arora, Hu & Kothari, 2018
- Cai & Ma, 2022
- A. & Fletcher, 2023, A. & Gu 2025
- Jeong & Wu, Weinkove, 2024
- Bergam, Snoeck, & Verma, 2025

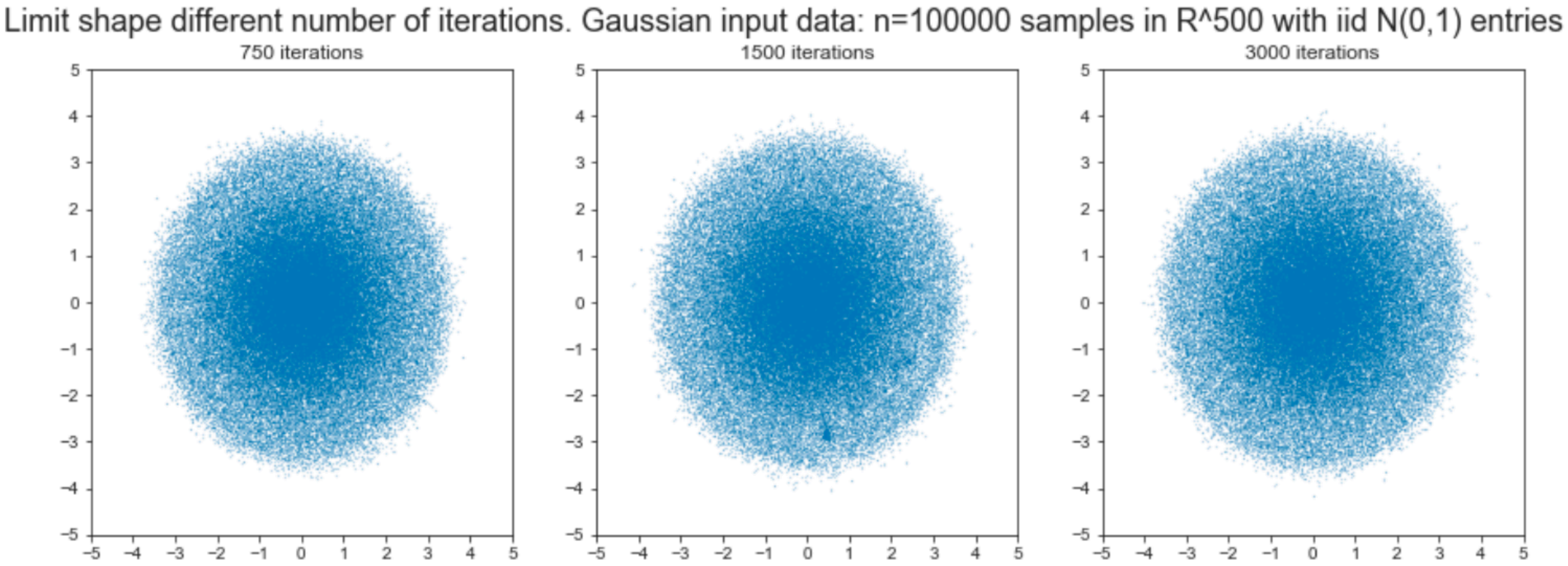
Question 1: Pure noise $\xrightarrow{\text{t-SNE}}$???



Question 2: Signal + noise – how strong should be a signal so we can detect? Recovery?

Pure noise simulations

- Set (X_i) to be independent identically distributed standard Gaussians.



Limit shape different number of iterations. Gaussian input data: n=100000 samples in R^500 with iid N(0,1) entries

750 iterations

1500 iterations

3000 iterations

Back to t-SNE

Input: $X \in (\mathbb{R}^d)^N$, Output: $Y \in (\mathbb{R}^2)^N$

$$\text{Goal:} \quad \min_Y \sum_{i \neq j} p_{ij}(X) \log \left(\frac{p_{ij}(X)}{q_{ij}(Y)} \right)$$

Perplexity

-
$$p_{j|i} = \frac{\exp(-\|x_i - x_j\|^2 / 2\sigma_i^2)}{\sum_{j \neq i} \exp(-\|x_i - x_j\|^2 / 2{\sigma_i}^2)}$$

- The choice of σ_i balances local/large scale density of the original dataset.
- Choose σ_i so that

$$-\sum_{j \neq i} p_{j|i} \log p_{j|i} = \mathrm{Ent}(\mu_i) = \log \mathrm{Perp}.$$

- Perp is a parameter of the model called Perplexity.

Theorem (A.-Fletcher ’23, A.-Gu ’25)

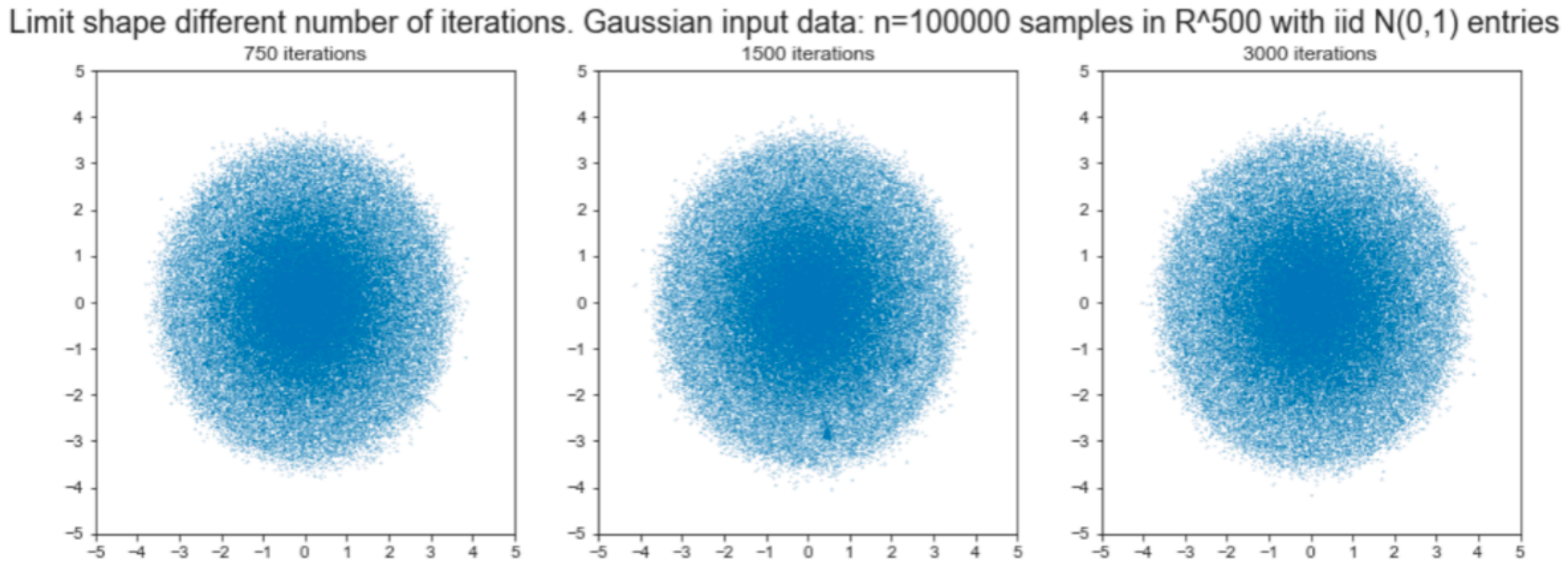
Let $X_i \in \mathbb{R}^d$ be independent, identically distributed r.v. with distribution μ_X with finite exponential moments. For $\rho \in (0,1)$, let $Y_i \in \mathbb{R}^s$ be the output of t-SNE with Perp $= n\rho$. Then

$$\lim_{n\to\infty} \inf_Y L_n(X,Y) = \inf_{\mu\in\mathcal{P}_X} I_\rho(\mu).$$

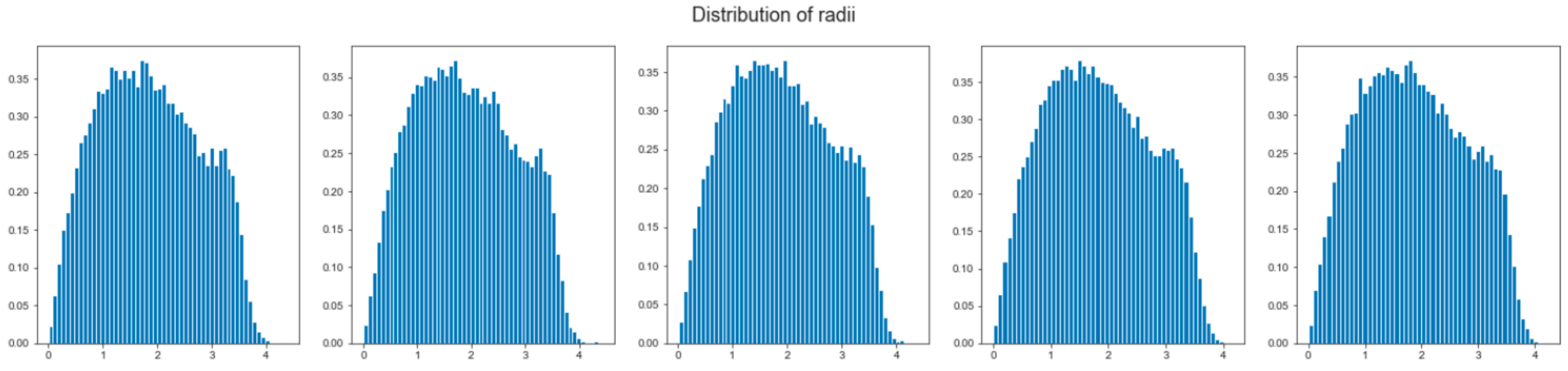
Moreover, there exists a sub-sequence n_k such that m_{n_k} converges weakly to μ^* and

$$I_\rho(\mu^*) = \inf_{\mu\in\mathcal{P}_X} I_\rho(\mu).$$

If μ_X has compact support, then μ^* has compact support.



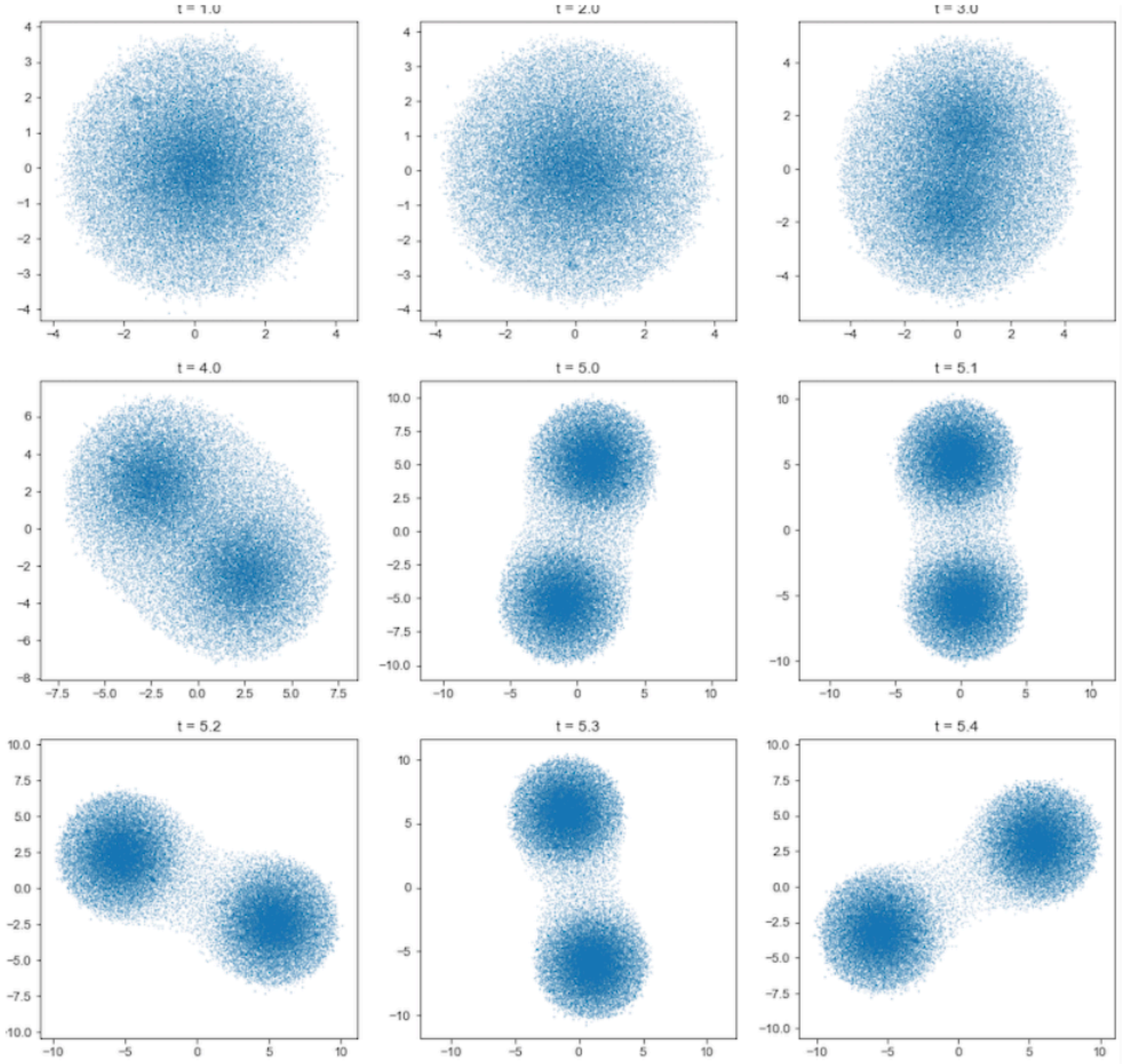
Limit shape different number of iterations. Gaussian input data: n=100000 samples in R^500 with iid N(0,1) entries

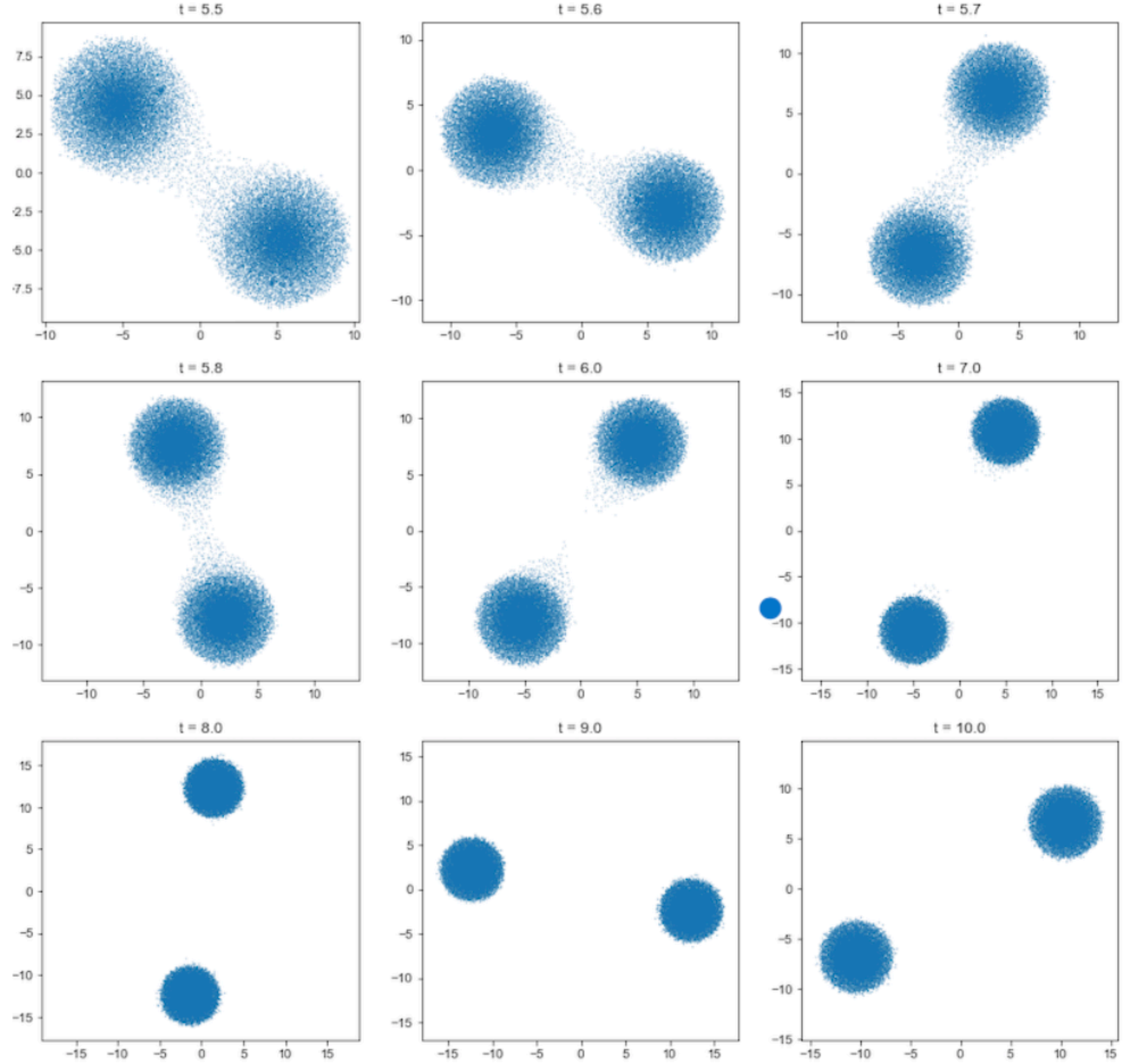


Distribution of radii

Adding a signal

Take $\mu_{X_i} \sim \frac{1}{2}\delta_Z + \frac{1}{2}\delta_{W+te_1}$, Z, W i.i.d. Gaussian.





There exists $t_c = t_c(d, \rho)$ such that $\mu^*(t)$ is connected if and only if $t \leq t_c$.

A continuous problem - limiting perplexity

Let $\mu \in \mathcal{M}(\mathbb{R}^d)$. For $\sigma \in \mathbb{R}$, $\rho \in (0,1)$ as

$$
\begin{aligned}
&F_{\rho,\mu}(x,\sigma) \\
&= \int \frac{p_\sigma(x,x')}{\int p_\sigma(x,x')\,d\mu(x')} \log\left(\frac{p_\sigma(x,x')}{\int p_\sigma(x,x')\,d\mu(x')}\right) d\mu(x') + \log \rho.
\end{aligned}
$$

with $p_\sigma(x,x') = \exp(-\|x-x'\|^2/2\sigma)$.

Let $\sigma^*(x)$ be defined as the unique solution of

$$F_{\rho,\mu}(x, \sigma^*_{\rho,\mu}(x)) = 0.$$

Now, given a function $\psi : \mathbb{R}^d \to \mathbb{R}$, define the function $p_\psi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ as

$$P_\mu(x, x') = \frac{1}{2}\left(\frac{p_{\sigma^*(x)}(x, x')}{\int p_{\sigma^*(x)}(x, x') d\mu(x')} + \frac{p_{\sigma^*(x')}(x, x')}{\int p_{\sigma^*(x')}(x, x')\, d\mu(x)}\right)$$

and also let $q : \mathbb{R}^s \times \mathbb{R}^s \to \mathbb{R}$ be given by

$$q(y, y') = \frac{(1 + \|y - y'\|^2)^{-1}}{\iint (1 + \|y - y'\|^2)^{-1}\, d\mu(y) d\mu(y')}.$$

Set

$$I_\rho(\mu) = \iint P_\mu(x, x') \log\left(\frac{P_\mu(x, x')}{q(y, y')}\right)\, d\mu(x, y) d\mu(x', y')$$

Some Open Challenges

1) Highly non-convex: when/where to stop?

2) Quantitative, error estimates

3) Different Signals; how to choose Perplexity

4) Noise dependence? Universality class?

5) UMAP: no mathematical results