Approximate Separability of Green’s Function for Helmholtz Equation in High Frequency Limit

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**Approximate separability of Green’s function**

**Green’s function**: given a linear differential operator $L$, $G(x, y)$ is the fundamental solution to

$$\begin{cases} 
L_x G(x, y) = \delta(x - y), & x, y \in \Omega \subseteq \mathbb{R}^n \\
+ \text{boundary condition}
\end{cases}$$

**Approximate separability of $G(x, y)$**: given two domains $X, Y \subseteq \Omega \subseteq \mathbb{R}^n$, $\forall \epsilon > 0$, there is a smallest $N^\epsilon$ s.t. there are $f_p(x), g_p(y), p = 1, 2, \ldots, N^\epsilon$

$$\left\| G(x, y) - \sum_{p=1}^{N^\epsilon} f_p(x) g_p(y) \right\|_{X \times Y} \leq \epsilon, \quad x \in X, y \in Y.$$
Approximate separability of Green's function

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The key issue: How does $N^\epsilon$ depend on $\epsilon$?
Implication of approximate separability

- The dependence of $N^\epsilon$ on $\epsilon$ manifests the complexity of the PDE within $\epsilon$-approximation

  $L_x u = h \Rightarrow u(x) = \int G(x, y) h(y) dy \approx \sum_{p=1}^{N^\epsilon} f_p(x) \int g_p(y) h(y) dy$

- View $G(x, y)$ as a family of functions on $X$ parametrized by $y \in Y$, the Kolmogorov $n$-width $^1$ is $\epsilon$ when $n = N^\epsilon$.

  $d(G(\cdot, y), L_n) \leq \epsilon, \ y \in Y, \quad L_n = \text{span}\{f_p(\cdot)\}_{p=1}^{N^\epsilon}$

---

$^1$Let $W$ be a set in a normed space $S$, the Kolmogorov $n$-width $d_n(W, S)$ is

$$d_n(W, S) := \inf_{L_n} d(W, L_n)$$

where $L_n \subset S$ is a linear subspace of dimension $n$.
Importance for developing fast algorithms

High separability $\implies$ existence of low rank approximation for the discretized system which is crucial for developing fast numerical algorithms.

- Fast (dense) matrix vector multiplication, e.g., for fast multipole methods, boundary integral methods, Fourier integral operators, ...

$L u = f$ after discretization $\implies A x = b$

Each columns of $A^{-1}$ is a numerical approximation of Green’s function. Low rank structure for off-diagonal submatrices of $A^{-1}$ can be explored to develop efficient numerical for solving the linear system such as hierarchical matrix method and structured inverse method.
Some intuition for approximate separability

- $X, Y$ are separated and compact, $G(x, y)$ is continuous. There exist polynomial approximations by Weierstrass Theorem.
- Fourier series type of expansion of $G(x, y)$.
- $X, Y$ are separated and compact, $\dim(X) \geq \dim(Y) = d$. If $G(x, y)$ is $C^m(X \times Y)$

$$N^\epsilon \leq O(\epsilon^{-\frac{d}{m}})$$

Proof: Lay down a grid $y_j, j = 1, 2, \ldots J = O(\epsilon^{-\frac{d}{m}})$ with grid size $h = O(\epsilon^{\frac{1}{m}})$. Use interpolation of $m-1$ order.

$$|G(x, y) - \sum a_j(y)G(x, y_j)| \leq C\|D^m_y G(x, y)\|h^m$$
Previous work on approximate separability

Mostly about showing upper bounds $N^\epsilon \leq O(|\log \epsilon|^q)$ for high separable cases due to regularity or special setup of $X, Y$.

- Construct separable approximation using explicit expression of $G(x, y)$ and asymptotic expansions with fast convergence.

- M. Bebendorf and W. Hackbusch’03 proved approximate separability in $L_2$ norm for Green’s function of strict elliptic operator with $L^\infty$ coefficients, $Lu = \sum \partial_j(c_{ij}\partial_i u)$ (+ low order perturbation), on two separated compact sets $X, Y$

  $$N^\epsilon \leq C|\log \epsilon|^{d+1}$$

Key estimate: Caccioppoli inequality for $\|\nabla u\|_2$ in term of $\|u\|_2$ on a little larger domain.

- Chandrasekaran’10 et al showed low rank estimate for off-diagonal blocks of matrices after finite difference discretization of elliptic operator.
Helmholtz equation in high frequency limit

Helmholtz equation (HE):

\[ \Delta \! \! \times G(\mathbf{x}, \mathbf{y}) + k^2 n^2(\mathbf{x}) G(\mathbf{x}, \mathbf{y}) = \delta(\mathbf{x} - \mathbf{y}) + \text{b.c.} \]

Free space \((n(\mathbf{x}) \equiv 1)\) Green's function,

\[ G_0(\mathbf{x}, \mathbf{y}) = \frac{1}{4\pi} \frac{e^{ik|\mathbf{x} - \mathbf{y}|}}{|\mathbf{x} - \mathbf{y}|} \]

In high frequency regime, \(k \gg 1\), HE is notoriously difficult to solve numerically.

▶ Solution is highly oscillatory \(\Rightarrow\) derivatives are unbounded a.e. as \(k \to \infty\).

▶ Singularity propagates along rays.

▶ Fine grids and large degrees of freedom are needed.

▶ The discretized system is indefinite. No effective iterative solvers.
Our main results

We give a sharp lower bound for the approximate separability for Green’s function of the HE in high frequency limit

\[ N_k^\epsilon \geq O(k^\rho), \quad \rho = \rho(d, X, Y) > 0, \quad \text{as } k \to \infty. \]

which characterizes intrinsic complexity for high frequency waves mathematically.

Key ideas:

▶ a geometric characterization of the relation between two Green’s functions with sources separated in term of wavelength,

\[ \left| \langle \hat{G}(\cdot, y_1), \hat{G}(\cdot, y_2) \rangle \right| \lesssim (k|y_1 - y_2|)^{-\alpha}, \quad \alpha = \alpha(d, X, Y) > 0 \]

where \( \hat{G}(x, y) = \frac{G(x,y)}{\|G(\cdot,y)\|_{2(X)}}. \)

▶ a sharp dimension estimate for approximating a set of almost orthogonal vectors
Almost orthogonality between two Green’s functions

**Theorem 1**

Assume $X \subset \mathbb{R}^n$, $n = 2, 3$ is a compact domain with piecewise smooth boundary. Depending on the positions of $y_1, y_2$ relative to $X$ and its boundary, there is some $\alpha > 0$ such that

$$
\left| \langle \hat{G}_0(\cdot, y_1), \hat{G}_0(\cdot, y_2) \rangle \right| \lesssim (k|y_1 - y_2|)^{-\alpha}, \quad \min\{1, \frac{n-1}{2}\} \leq \alpha \leq \frac{n+1}{2}
$$

as $k|y_1 - y_2| \to \infty$, where $\hat{G}_0(x, y) = \frac{G_0(x, y)}{\|G_0(\cdot, y)\|_{L^2(X)}}, x \in X$.

[Diagram: case 1, case 2, case 3]
Sketch of Proof

**Case 1:** no stationary point in $X$.

\[ \tilde{k} = k|y_1 - y_2|, \quad \phi(x) = |y_1 - y_2|^{-1}(|x - y_1| - |x - y_2|), \quad u(x) = \frac{1}{|x - y_1||x - y_2|} \]

\[ \Rightarrow \]

\[ \langle \hat{G}_0(\cdot, y_1), \hat{G}_0(\cdot, y_2) \rangle = \int_X e^{i\tilde{k}\phi(x)}u(x)dx \]

\[ = O(\tilde{k}^{-\alpha}), \quad 1 \leq \alpha \leq \frac{n+1}{2} \]

by integration by part and using stationary phase theory to the boundary integral on $\partial X$. 

\[ y_1 \quad \hat{X} \quad y_2 \]
Case 2: all points on the line $l_{y_1}^{y_2}$ and inside $X$ are stationary points.

$$< \hat{G}_0(\cdot, y_1), \hat{G}_0(\cdot, y_2) > = \int_{r_1}^{r_2} \int_{(\xi, \eta)} e^{i\tilde{k}\phi(r, \xi, \eta)} u(r, \xi, \eta) d\xi d\eta dr$$

$$= \int_{r_1}^{r_2} 2\pi i \tilde{k}^{-1} e^{i\tilde{k}} u(r, 0, 0) [1 + O(\tilde{k}^{-1})] dr \lesssim \tilde{k}^{-\frac{n-1}{2}}$$

by using stationary phase on $\xi - \eta$ plane for each $r$, since

$$\phi(r, 0, 0) = 1, \quad D_{\xi, \eta}^2 \phi(r, 0, 0) = \frac{1}{r(|y_1 - y_2| + r)} l,$$
Sketch of Proof (continued)

**Case 3:** $y_1, y_2$ are inside $X$. Main contributions come from stationary line and singularity. Let $I(k) = \int e^{ik\phi(x)}u(x)$, by applying a general stationary phase result

$$|I(k) - (\frac{2\pi}{k})^{d/2} \sum_{m=1}^{M} \frac{e^{ik\phi(x_m)}}{|\det[D^2\phi(x_m)]|^{1/2}} e^{i\frac{\pi}{4} \text{sgn}(D^2\phi(x_m))} u(x_m)|$$

$$\leq Ck^{-d/2-1} |\det[D^2\phi(x_m)]|^{-1/2} \|D^2\phi(x_m)^{-1}\| \sum_{\beta \leq s+2} \|D^\beta u\|_{L^2},$$

where $s > d/2$ and $C$ is a universal constant, to

$$\int_{\partial B(y_1,r)} e^{i\tilde{k}\phi(x)}u(x)$$

$$= 2\pi ir^2 \tilde{k}^{-\frac{n-1}{2}} [-e^{i\tilde{k}}u(r, \theta, 0) + \frac{e^{i\tilde{k}|y_1-y_2|^{-1}(2r-|y_1-y_2|)}u(r,\theta,\pi)}{|y_1-y_2|^{-2r}}] + O(\tilde{k}^{-\frac{n+1}{2}r})$$

$$\Rightarrow \int_X e^{i\tilde{k}\phi(x)}u(x)dx = \int_0^r \int_{\partial B(y_1,r)} e^{i\tilde{k}\phi(x)}u(x)dsdr \lesssim \tilde{k}^{-\frac{n-1}{2}}$$

![Diagram](image)
2D case

Green’s function for 2D in free space is

\[ G_0(x, y) = -\frac{i}{4} H_0^{(1)}(k|x - y|) = -\frac{1}{2\pi} \int_0^\infty e^{ik|x-y|} \cos \theta \, d\theta, \]

where \( H_0^{(1)}(r) \) is the Hankel function which has the following asymptotic behavior

\[ \lim_{r \to 0^+} H_0^{(1)}(r) = \frac{2i}{\pi} \log r, \]

and

\[ H_0^{(1)}(r) = \sqrt{\frac{2}{\pi r}} \left[ 1 - \frac{8i}{r} - \frac{9}{128r^2} + \cdots + \frac{(2n-1)!!}{(8i)^n n!} \frac{1}{r^n} + \cdots \right] e^{i(r-\pi/4)} \]

\[ = \sqrt{\frac{2}{\pi r}} e^{i(r-\pi/4)} + O(r^{-3/2}), \quad \text{if} \quad r \gg 1. \]
Green’s function in heterogeneous medium

Use geometric optics ansatz \( G(x, y) = e^{ik\phi(x,y)} \sum_{m=0}^{\infty} a_m(x, y)(ik)^{-m}. \)

\[
< \hat{G}(\cdot, y_1), \hat{G}(\cdot, y_2) > = \int_{\mathbb{R}} e^{ik\phi(x)} u(x) dx + O(k^{-(M+1)}),
\]

where

\[
\tilde{k} = k\phi(y_1, y_2), \quad \phi(x) = \phi^{-1}(y_1, y_2)(\phi(x, y_1) - \phi(x, y_2)).
\]

If rays do not cross, i.e., no caustics, \( \phi(x, y) \) is the shortest travel time along the unique ray connecting \( x, y. \)

\[
|\phi(x, y_1) - \phi(x, y_2)| \leq \phi(y_2, y_1)
\]

The phase function \( \phi(x; y_1, y_2) \) attains the global maximum or minimum \( \pm 1 \) on the ray \( \Gamma_{y_1}^{y_2} \) outside the interval between \( y_1 \) and \( y_2. \)
Embedding a set of almost orthogonal unit vectors

The question: the minimum dimension $d$ of a linear subspace that can contain a set of almost orthogonal unit vectors $\hat{v}_n \in \mathbb{R}^m$, $n = 1, 2, \ldots, N$. Define $A = (a_{ij})_{n \times n}$, $a_{ij} = \langle \hat{v}_i, \hat{v}_j \rangle$. Then $a_{ii} = 1$, $|a_{ij}| < \epsilon$, $i \neq j$, one has the following results (N. Alon’03).

$$d = \text{rank}(A) \geq \frac{N}{1 + (N - 1)\epsilon^2}$$

Proof: Let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_d > \lambda_{d+1} = \ldots = \lambda_N = 0$ be the eigenvalues of $A$.

$$\frac{N^2}{d} = d \left( \sum_{n=1}^{N} \frac{\lambda_n}{d} \right)^2 \leq \sum_{n=1}^{N} \lambda_n^2 = \text{tr}(A^T A) = \sum_{i,j=1}^{N} a_{ij}^2 \leq N + N(N - 1)\epsilon^2.$$  

If $|a_{ij}| < \epsilon$, $i \neq j$ for $\frac{1}{\sqrt{N}} \leq \epsilon < \frac{1}{2}$, then

$$d \gtrsim \frac{1}{\epsilon^2 |\log \epsilon|} \log N$$

- The above estimate shows sharpness of Johnson-Lindenstrauss Lemma.
- Direct application of the above results to our problem (1) renders suboptimal bounds, and (2) can not deal with approximate separability.
Approximation of a set of vectors

Denote $V = [v_1, v_2, \ldots, v_N]$ and $A = [a_{mn}]_{N \times N} = V^T V$. Let $\lambda_1^2 \geq \ldots \geq \lambda_N^2$ be the eigenvalues of $A$, then

$$tr(A) = \sum_{m=1}^{N} \lambda_m^2 = \sum_{m} ||v_m||_2^2$$

$$\sum_{m} ||v_m - P_{S_i} v_m||_2^2 = \min_{S_i, \dim(S_i)=l} \sum_{m} ||v_m - P_{S_i} v_m||_2^2 = \sum_{m=l+1}^{N} \lambda_m^2,$$

where $P_{S_i} v$ denotes projection of $v$ in $S_i$.

**Def.** Given $1 \geq \epsilon > 0$, $N^\epsilon = \min M$, s.t. $\sum_{m=M+1}^{N} \lambda_m^2 \leq \epsilon^2 \sum_{m=1}^{N} \lambda_m^2$.

Assume $0 < c < ||v_m||_2 < C < \infty$, $\forall m$, if a linear subspace $S^\epsilon$ satisfies

$$\sqrt{\frac{\sum_{m} ||v_m - P_{S^\epsilon} v_m||_2^2}{N}} \leq c\epsilon \quad \Rightarrow \quad \frac{\sum_{m} ||v_m - P_{S^\epsilon} v_m||_2^2}{\sum_{m} ||v_m||_2^2} \leq \epsilon^2$$

then $\dim(S^\epsilon) \geq N^\epsilon$. 
Lemma 2 (key lemma)

Let $X, Y$ be two compact manifolds in $\mathbb{R}^n$ and $\dim(X) \geq \dim(Y) = d$. For $y_1, y_2 \in Y$, assume

$$| \langle \hat{G}(\cdot, y_1), \hat{G}(\cdot, y_2) \rangle | \lesssim (k|y_1 - y_2|)^{-\alpha} \quad \text{as } k|y_1 - y_2| \to \infty$$

there are points $y_m \in Y, m = 1, 2, \ldots, N_{\beta}^d = O(k^{d\beta})$, such that for the set of Green’s functions $\{G(x, y_m)\}_{m=1}^{N_{\beta}^d} \subset L_2(X)$ and matrix $A = \langle G(\cdot, y_m), G(\cdot, y_n) \rangle$

$$N_k^\epsilon \gtrsim \begin{cases} (1 - \epsilon^2)^2 k^{2\alpha}, & \alpha < \frac{d}{2}, \\ (1 - \epsilon^2)^2 k^{d\beta}, & \alpha \geq \frac{d}{2}, \end{cases} \quad \text{as } k \to \infty.$$
Sketch of proof

Lay down a uniform grid $y_m$, $m = 1, 2, \ldots, n_k^h = O(k^{d\beta})$ with grid size $h = k^{-\beta}, 0 < \beta < 1$. Let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{n_k^h} \geq 0$ be the eigenvalues of $A = \langle \hat{G}(\cdot, y_m), \hat{G}(\cdot, y_n) \rangle$.

$$N_{\epsilon_k}^\epsilon = \min M, \ s.t. \sum_{m=M+1}^{n_k^h} \lambda_m \leq \epsilon^2 \sum_{m=1}^{n_k^h} \lambda_m = \epsilon^2 n_k^h$$

For each $y_m$, divide all other points into 1st square neighbors, 2nd square neighbors, $\ldots$, $j$-th square neighbors $S_j$ centered at $y_m$ with length $2jh$, $j = 1, 2, \ldots, J = O(h^{-1} = k^{\beta})$. Each $S_j$ contains at most $8j$ grid points.

$$\Rightarrow a_{m,n_j} \lesssim (kjh)^{-\alpha} = j^{-\alpha} k^{\alpha(\beta-1)}, \ y_{n_j} \in S_j.$$
Sketch of the proof

Let \( \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{n^h_k} \geq 0 \) be the eigenvalues of \( A \). \( n^h_k = O(k^{d\beta}) \).

key inequality:

\[
\sum_{m=1}^{n^h_k} \sum_{n=1}^{n^h_k} a_{mn}^2 = \text{tr}(A^T A) = \sum_{m=1}^{n^h_k} \lambda_m^2 \geq \sum_{m=1}^{N^e_k} \lambda_m^2 \geq N^e_k \left[ \frac{(1 - \epsilon^2) n^h_k}{N^e_k} \right]^2 = \left[ \frac{(1 - \epsilon^2) n^h_k}{N^e_k} \right]^2
\]

key estimate:

sum of each row

\[
\sum_{n=1}^{n^h_k} a_{mn}^2 = 1 + \sum_{j=1}^{J} \sum_{n_j} a_{m,n_j}^2 = \begin{cases} O(k^{d\beta - 2\alpha}) & \alpha < \frac{d}{2} \\ O(1) & \alpha \geq \frac{d}{2} \end{cases}
\]

for \( \beta \) arbitrary close to 1

\[
\Rightarrow \sum_{m=1}^{n^h_k} \sum_{n=1}^{n^h_k} a_{mn}^2 = \begin{cases} O(k^{2(d\beta - \alpha)}) & \alpha < \frac{d}{2} \\ O(k^{d\beta}) & \alpha \geq \frac{d}{2} \end{cases} \Rightarrow N^e_k \sim \begin{cases} (1 - \epsilon^2)^2 k^{2\alpha} & \alpha < \frac{d}{2} \\ (1 - \epsilon^2)^2 k^{d\beta} & \alpha \geq \frac{d}{2} \end{cases}
\]
Lower bound estimate for approximate separability

Theorem 3 (main theorem)

Let $X, Y$ be two compact manifolds in $\mathbb{R}^n$ and $\dim(X) \geq \dim(Y) = d$. For $y_1, y_2 \in Y$, assume

$$| \langle \hat{G}(\cdot, y_1), \hat{G}(\cdot, y_2) \rangle | \lesssim (k|y_1 - y_2|)^{-\alpha}, \quad \text{as } k|y_1 - y_2| \to \infty.$$ 

If there are $f_p(x) \in L^2(X), g_p(y) \in L^2(Y), p = 1, 2, \ldots, N^\epsilon_k$ such that

$$\left\| G(x, y) - \sum_{p=1}^{N^\epsilon_k} f_p(x)g_p(y) \right\|_{L^2(X \times Y)} \leq \epsilon,$$

then

$$N^\epsilon_k \geq \begin{cases} 
  c_\epsilon k^{2\alpha}, & \alpha < \frac{d}{2}, \\
  c_\epsilon k^{d\beta}, & \alpha \geq \frac{d}{2}, 
\end{cases} \quad \text{as } k \to \infty$$

for any $\beta < 1$ and arbitrary close to 1, where $c_\epsilon \geq c(1 - (C_\epsilon)^2)^2$ for some positive constants $c$ and $C$ that only depend on $X, Y$ and $n(x)$.
Sketch of the proof

Lay down a two scale grid with coarse cell $\Omega_m, m = 1, 2, \ldots, N_k^h = O(k^{d\beta})$ of size $\bar{h} = k^{-\beta}, 0 < \beta < 1$ and subdivide each coarse cell into finer cells $\Omega_{m,n}, n = 1, 2, \ldots, N_k^h = (h/h)^d$ of size $h < k^{-\gamma}, \gamma > 1$.

Define piecewise constant $G_h(x, y) = G(x, y_{m,n})$ for $y \in \Omega_{m,n}$.

Given $\epsilon > 0$, choose $h$ small enough

$$\int_Y \int_X |G(x, y) - G_h(x, y)|^2 = \sum_{m=1}^{N_k^h} \sum_{n=1}^{N_k^h} \int_{\Omega_{m,n}} \int_X |G(x, y) - G(x, y_{m,n})|^2 \lesssim \epsilon^2$$
Sketch of the proof

Define \( G_h(x, y) = G(x, y_{m,n}) \) for \( y \in \Omega_{m,n}^Y \). Given \( \epsilon > 0 \), choose \( h \) small enough

\[
\int_Y \int_X |G(x, y) - G_h(x, y)|^2 = \sum_{m=1}^{\tilde{N}_k^h} \sum_{n=1}^{N_k^h} \int_{\Omega_{m,n}^Y} \int_X |G(x, y) - G(x, y_{m,n})|^2 \lesssim \epsilon^2
\]

\[
\Rightarrow \epsilon^2 \gtrsim \int_{\Omega^Y} \| (I - P_{S_X}) G_h(x, y) \|^2_{L^2(X)}
\]

\[
\gtrsim \frac{1}{N_k^h} \sum_{n=1}^{N_k^h} \frac{1}{N_k^h} \sum_{m=1}^{\tilde{N}_k^h} \| G(x, y_{m,n}) - P_{S_X} G(x, y_{m,n}) \|^2_{L^2(X)}
\]

\[
\Rightarrow \min_n \frac{1}{N_k^h} \sum_{m=1}^{\tilde{N}_k^h} \| G(x, y_{m,n}) - P_{S_X} G(x, y_{m,n}) \|^2_{L^2(X)} \lesssim \epsilon^2,
\]

where \( S_X = \text{span}\{f_p(x)\}_{p=1}^{N_k^}\).
Theorem 4
Let $X, Y$ be two compact manifolds embedded in $\mathbb{R}^n$ and $\dim(X) \geq \dim(Y) = d, d \leq n$. For any $\epsilon > 0$, there are $f_p(x) \in L_2(X), g_p(y) \in L_2(Y), p = 1, 2, \ldots, N_k^\epsilon \lesssim k^{d\gamma}$ such that

$$\left\| G(x, y) - \sum_{p=1}^{N_k^\epsilon} f_p(x)g_p(y) \right\|_{L_2(X \times Y)} \leq \epsilon$$

for any $\gamma > 1$ and arbitrary close to 1 as $k \to \infty$. 
Sketch of proof

Denote $S_X = \text{span}\{G(x, y_m)\}_{m=1}^{N^h_k} \subset L_2(X)$.

- Lay down a uniform grid in $\mathbb{R}^3$ that covers $Y$ with grid size $h = k^{-\gamma}$.
- If $X$ and $Y$ are separated, uniformly in $X$ and $Y$

$$|\nabla_y G(x, y)| \lesssim k, \quad \|D^2_y G(x, y)\| \lesssim k^2.$$

- Use linear interpolation

$$\sqrt{\int_Y \|G(x, y) - P_{S_X} G(x, y)\|_{L_2(X)}^2 dy} \leq \epsilon$$
Lower bound in $L_\infty$

**Theorem 5**
Let $X, Y$ be two separated compact manifolds embedded in $\mathbb{R}^n$ and $\dim(X) \geq \dim(Y) = d, d \leq n$. For $y_1, y_2 \in Y$, assume

$$| \langle \hat{G}(\cdot, y_1), \hat{G}(\cdot, y_2) \rangle | \lesssim (k|y_1 - y_2|)^{-\alpha} \quad \text{as } k|y_1 - y_2| \to \infty.$$

If there are $f_p(x) \in L_\infty(X), g_p(y) \in L_\infty(Y), p = 1, 2, \ldots, N^c_k$ such that

$$\left\| G(x, y) - \sum_{p=1}^{N^c_k} f_p(x)g_p(y) \right\|_{L_2(X \times Y)} \leq \epsilon,$$

then

$$N^c_k \geq \begin{cases} c_\epsilon k^{2\alpha}, & \alpha < \frac{d}{2}, \\ c_\epsilon k^{d\beta}, & \alpha \geq \frac{d}{2} \end{cases} \quad \text{as } k \to \infty$$

for any $\beta < 1$ and arbitrary close to 1, where $c_\epsilon \geq c(1 - (C\epsilon)^2)^2$ for some positive constants $c$ and $C$ that only depend on $X$, $Y$ and $n(x)$. 
Upper bound in $L_\infty$

**Theorem 6**

Let $X, Y$ be two separated compact manifolds embedded in $R^n$ and $\dim(X) \geq \dim(Y) = d$, $d \leq n$. For any $\epsilon > 0$, there are $f_p(x) \in L_\infty(X), g_p(y) \in L_\infty(Y), p = 1, 2, \ldots, N_k^\epsilon \leq Ck^{d\gamma}$ such that

$$\left| G(x, y) - \sum_{p=1}^{N_k^\epsilon} f_p(x)g_p(y) \right| \leq \epsilon, \quad \forall x \in X, \forall y \in Y,$$

(1)

for any $\gamma > 1$ and arbitrary close to 1 as $k \to \infty$, where $C > 0$ is some constant that depends on $X$, $Y$ and $n(x)$. 
Upper bound using Weyl’s formula

Let \( u_m(x), \|u_m\|_{L^2(\Omega)} = 1, m = 1, 2, \ldots \) be the eigenfunctions for

\[
\Delta u_m(x) = \lambda u_m(x), \quad x \in \Omega, \quad u_m(x) = 0, x \in \partial \Omega
\]

with eigenvalues \( 0 > \lambda_1 \geq \lambda_2 \geq \ldots \). The Weyl’s asymptotic formula

\[
|\lambda_m| \approx \frac{4\pi^2 m^2/d}{(C_d|\Omega|)^{2/d}}
\]

Then \( u_m(x) \) are also the eigenfunctions for the homogeneous Helmholtz operator with eigenvalues \( \lambda_m + k^2 \). Assuming \( \Omega \) is not resonant

\[
G(x, y) = \sum_{m=1}^{\infty} (\lambda_m + k^2)^{-1} u_m(y)u_m(x).
\]

Choose \( N_k^\epsilon = M = O(k^{d+\delta}), \delta > 0, \) so \( |\lambda_m + k^2|^{-1} \lesssim |\lambda_m|^{-1}, m > M, \)

\[
\int_\Omega \int_\Omega |G(x, y) - \sum_{m=1}^{M} (\lambda_m + k^2)^{-1} u_m(y)u_m(x)|^2 = O(\sum_{m=M+1}^{\infty} m^{-\frac{4}{d}}) = O(M^{1-\frac{4}{d}}) < \epsilon^2
\]

for any \( \epsilon > 0 \) when \( k \) is large enough.
Let $X$, $Y$ be two separated compact manifolds embedded in $\mathbb{R}^n$ and $\dim(X) \geq \dim(Y) = d$, $d \leq n$.

- When Green's functions at two points become almost orthogonal or decorrelate fast, $|<G(\cdot, y_1), G(\cdot, y_2)>| \lesssim (k|y_1 - y_2|)^{-\alpha}$ with $\alpha \geq \frac{d}{2}$, Our estimates are sharp

$$k^{d - \delta} \lesssim N_k^\epsilon \lesssim k^{d + \delta}, \quad \forall \delta > 0.$$

- The sharpness of the lower bound implies sharpness of the following

$$\sum_{m=1}^{N_k^\epsilon} \chi_m^2 \geq N_k^\epsilon \left[ \frac{(1 - \epsilon^2)n_k^h}{N_k^\epsilon} \right]^2 = \left[ \frac{(1 - \epsilon^2)n_k^h}{N_k^\epsilon} \right]^2$$

$\Rightarrow$ flatness of the leading spectrum of $A = <G(\cdot, y), G(\cdot, y)>$. 
Practical cases when the bounds are sharp

Assume $|<G(\cdot, y_1), G(\cdot, y_2)>| \lesssim (k|y_1 - y_2|)^{-\alpha}$ for some $\alpha > 0$.

- $X, Y$ are two surfaces in 3D, e.g., (a) boundary of two scatterers, or (b) the case for multi-frontal type algorithms,
  \[ \alpha = 1 \implies k^{2-\delta} \lesssim N_k^e \lesssim k^{2+\delta}, \quad \forall \delta > 0. \]

- $X, Y$ are two curves,
  \[ \alpha = \frac{1}{2} \implies k^{1-\delta} \lesssim N_k^e \lesssim k^{1+\delta}, \quad \forall \delta > 0. \]

- dim$(X) = 3$, dim$(Y) = d \leq 2$,
  \[ \alpha = 1 \implies k^{d-\delta} \lesssim N_k^e \lesssim k^{d+\delta}, \quad \forall \delta > 0. \]
Practical cases when the bounds are not sharp

Assume $| \langle G(\cdot, y_1), G(\cdot, y_2) \rangle | \lesssim (k|y_1 - y_2|)^{-\alpha}$ for some $\alpha > 0$.

- Two domains in $R^3$ and $\dim(X) = \dim(Y) = 3$,
  $\alpha = 1 \implies k^2 \lesssim N_k^\epsilon \lesssim k^{3+\delta}, \quad \forall \delta > 0.$

- Two domains in $R^2$ and $\dim(X) = \dim(Y) = 2$,
  $\alpha = \frac{1}{2} \implies k \lesssim N_k^\epsilon \lesssim k^{2+\delta}, \quad \forall \delta > 0.$
Special setups where high separability can be achieved

**Key:** Fast oscillation in the phase is not felt.
Assume $G(x, y) = A(x, y)e^{ik\phi(x,y)} \Rightarrow$ Find $\phi_1(x)$ and $\phi_2(y)$ s.t.
$k(\phi(x, y) - \phi_1(x) - \phi_2(y))$ is uniformly bounded with respect to $x \in X, y \in Y$ and $k$. So the phase difference

$$k(\phi(x, y_1) - \phi(x, y_2))$$

$$= k(\phi_2(y_1) - \phi_2(y_2)) + k[ \phi(x, y_1) - \phi_1(x) - \phi_2(y_1) - (\phi(x, y_2) - \phi_1(x) - \phi_2(y_2))]$$
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Example 1: Two collinear line segments, $\phi_1(x) = -r_x, \phi_2(y) = r_y$.

$$< G_0(\cdot, y_1), G_0(\cdot, y_2) > = e^{ik(y_2-y_1)} \int_X \frac{1}{|x-y_1||x-y_2|} dx = O(1).$$

$$G_0(x, y) = \frac{1}{4\pi} e^{-ikr_x} e^{ikr_y} \frac{1}{r_y - r_x} = \frac{1}{4\pi} e^{-ikr_x} e^{ikr_y} r_y^{-1} \sum_{m=0}^{\infty} \left(\frac{r_x}{r_y}\right)^m.$$

$$\Rightarrow N_k^\epsilon \leq (\log \frac{l_X}{l_X + d})^{-1} \log(4\pi d\epsilon)$$
Special setups where high separability can be achieved

Example 2: $X$ and $Y$ are two collinear separated narrow tubes (similar to a 2D case by Martinsson-Rokhlin’07).

Let $\rho = \inf_{x \in X, y \in Y} (r_x - r_y)$ and $\tau = \sup_{x \in X, y \in Y} \sqrt{\xi^2 + \eta^2}$. Assume $k\tau < \frac{1}{2}$, $\mu = \frac{\tau}{\rho} < \frac{1}{2}$. Take $\phi_1(x) = -r_x, \phi_2(y) = r_y$

\[
k|\phi(x, y) - \phi_1(x) - \phi_2(y)| = k(|x - y| - (r_y - r_x)) < 2k\tau = 1
\]

\[
\Rightarrow N_k^\epsilon \lesssim |\log \epsilon|^{12}
\]
Special setups where high separability can be achieved

Example 3: Fast butterfly algorithms for computing highly oscillatory Fourier integral operators (Candes et al’ 09) and boundary integrals for HE (Luo-Qian’14). A hierarchical decomposition and pairing of domains, $X, Y, |X||Y| \leq 1/k$.

Key observation: $k|\phi(x, y) - \phi(x_0, y) - \phi(x, y_0) + \phi(x_0, y_0)|$ is uniformly bounded for all $k, x \in X, y \in Y$, where $x_0, y_0$ are the centers of $X, Y$ respectively. $\Rightarrow$ low rank approximation $O(|\log \epsilon|^4)$
Special setups where high separability can be achieved

Example 3: Fast butterfly algorithms for computing highly oscillatory Fourier integral operators (Candes et al’ 09) and boundary integrals for HE (Luo-Qian’14). A hierarchical decomposition and pairing of domains, $X, Y, |X||Y| \leq 1/k$.

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Example 4: Sweeping preconditioner for HE (Engquist-Ying’11).

Then the Green’s function in for half space
$G_1(x, y) = G_0(x, y) - G_0(x, y)$. If the distance to the boundary is $\leq 1/k$, $G_1(x, y)$ is highly separable as in Example 2.
We developed a general approach to study approximate separability of Green’s function for Helmholtz equation in high frequency limit by

- showing a geometric characterization of relations between two Green’s function,
- deriving sharp lower and upper bounds for approximate separability,
- studying various setups of practical interest.