Kiiking and other pumping sports

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What is kiiking?

Height record (Bastian Kurtz):
http://www.youtube.com/watch?v=3Z3hmtt1ldQ

Speed record (Manuel Helster):
http://www.youtube.com/watch?v=SIUO6ILSkW0

Skateboarding:
http://www.youtube.com/watch?v=r1S7zwU0uCA

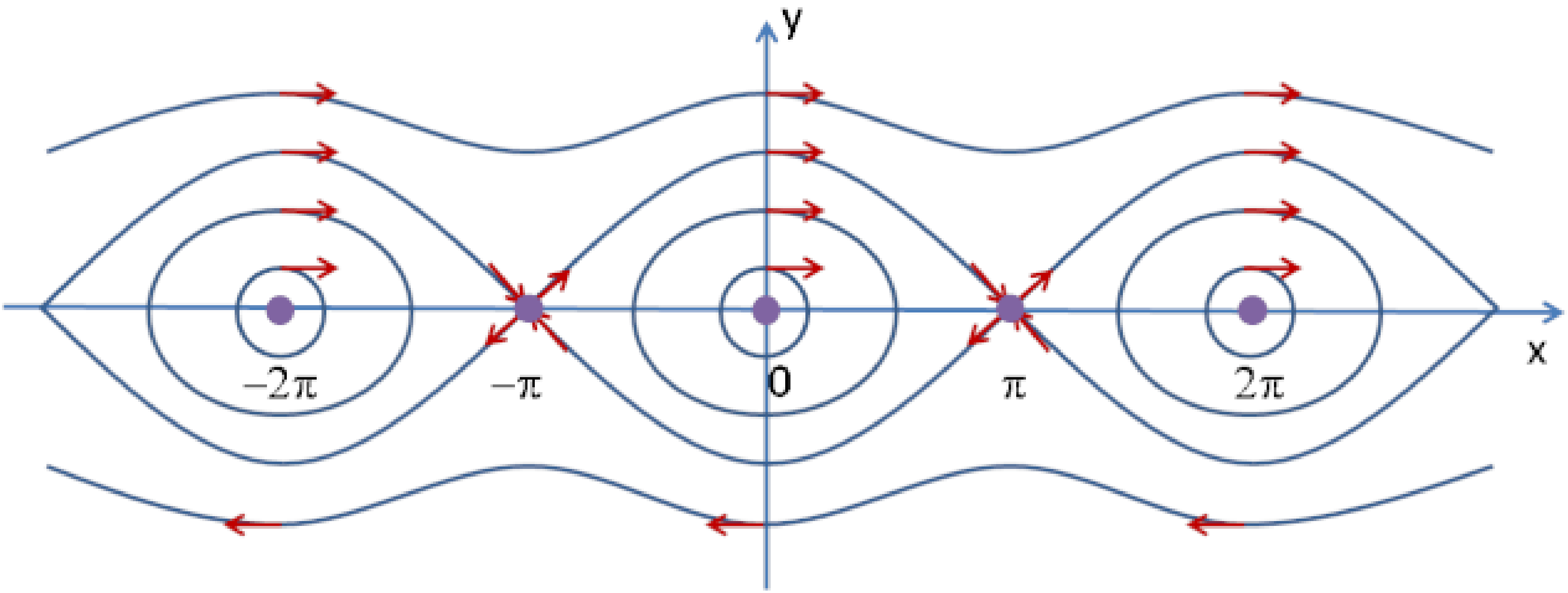
Model 1

Key is to get energy into the system by standing up and sitting down at the right times.

Treat as a simple pendulum with lengths L_1 and L_2 where

- L_1 is standing up
- L_2 is sitting down

so $L_1 < L_2$.

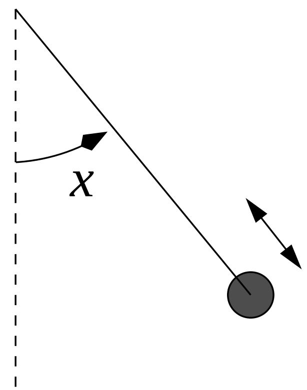


We want to work outwards from the equilibrium point.

Recall the equation of motion (while L is constant)

$$L\ddot{x} + g\sin x = 0$$

where x is the angle from the downwards vertical.



There is a first integral

$$E = \frac{1}{2}L^2\dot{x}^2 + gL(1 - \cos x) = \text{constant on trajectories}$$

which is the energy per unit mass. Note $E = 0$ at the equilibrium point $x = 0$, $\dot{x} = 0$.

Suppose we stand/sit, ie change from L_i to L_j ($i = 1$ and $j = 2$ or vice versa) at an angle x. Then x is constant across the transition but

$$L_i^2 \dot{x}_i = L_j^2 \dot{x}_j$$

by angular momentum.

Then using this to eliminate $\dot{x}$ terms,

$$E_j - E_i = \left[(L_i^2 - L_j^2) E_i - g(1 - \cos x)(L_i^3 - L_j^3) \right] / L_j^2.$$

With

$$\Delta E = E_j - E_i = \left[(L_i^2 - L_j^2)E_i - g(1 - \cos x)(L_i^3 - L_j^3)\right] / L_j^2$$

suppose we stand up from L_2 to $L_1 < L_2$, so $i = 2$, $j = 1$. We maximise ΔE when $x = 0$. Conversely minimise ΔE when x is as close to π as possible.

- Stand up at the lowest point

- Sit down at the highest point

Round-trip energy budget

While doing less than whole rotations, easily find that a half cycle

$$\text{stand at } x = 0 \rightarrow \text{sit at } x = x_{max} \rightarrow \text{return to } x = 0$$

takes the initial energy E_0 to $(L_2/L_1)^3 E_0$. Pretty efficient. Eg start at rest at an angle of 10°, so $E_0 \approx 0.015 g L_2$, and $L_2/L_1 = 1.2$. We need to get to $E = 2gL_2$ which takes 6 of these cycles.

When doing 360° rotations we find

$$\Delta E = 2g(L_2^3 - L_1^3)/L_2^2$$

so we switch from multiplicative to additive.

Energy in the phase plane

To compare swings of different lengths on the same phase plane we scale time with $\sqrt{L/g}$ in each phase and use

$$e = \tfrac{1}{2}\dot{x}^2 + 1 - \cos x$$

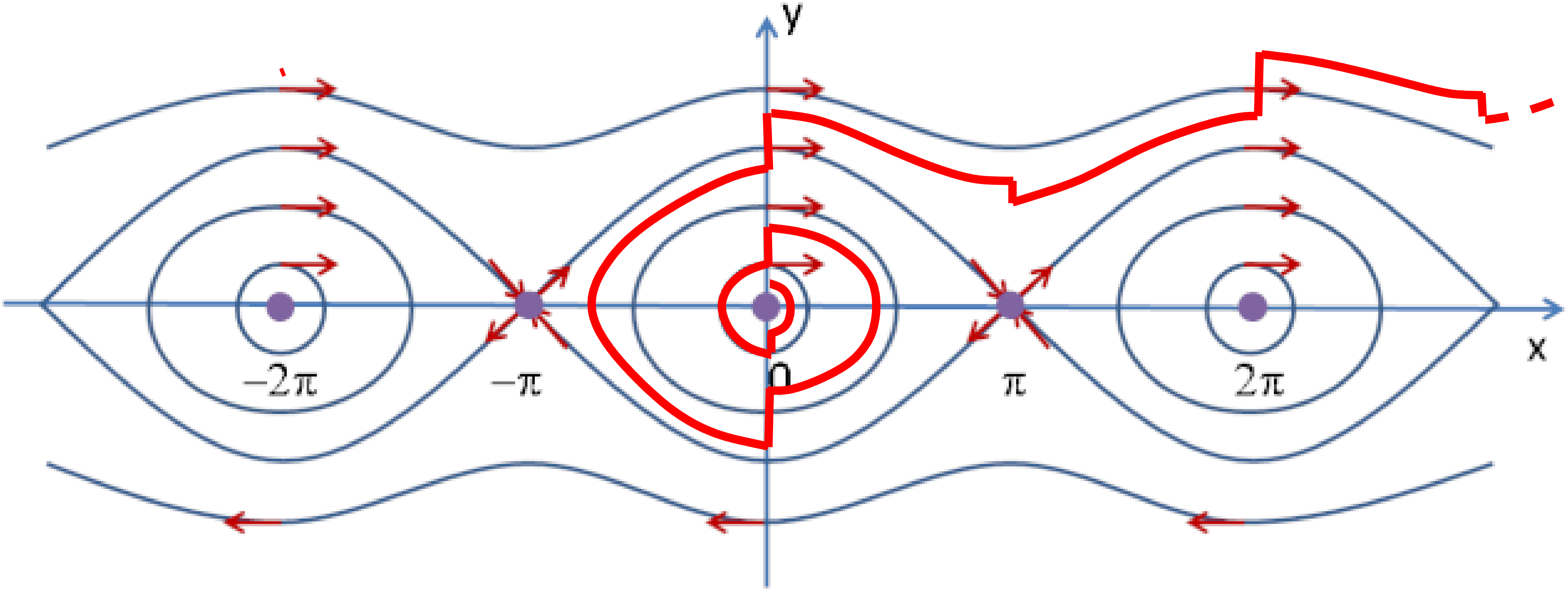
and note that this does not change if we stand/sit when $\dot{x} = 0$.

The budgets are

$$\Delta e = (L_2/L_1)^3 e$$

and

$$\Delta e = 2\left((L_2/L_1)^3 - 1\right)$$


y
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Limits on world records

Drag mostly from air resistance. Model resistive force per unit mass as

$$F = -\frac{1}{2} C_d \rho_a (L\dot{x})^2 A/M$$

for drag coefficient C_d, air density ρ_a and cross-section A; here M is kiiker's mass.

Energy loss from drag is balanced against input from standing/sitting.

With $E = \frac{1}{2}(L\dot{x})^2 + gL(1 - \cos x)$ we have $\mathrm{d}E/\mathrm{d}t = L|\dot{x}|F$ so

$$\begin{aligned} \mathrm{d}E/\mathrm{d}x &= LF \\ &= -\frac{1}{2} L C_d \rho_a (L\dot{x})^2 A/M \\ &= -\delta(L\dot{x})^2 \quad \text{where } \delta = \frac{1}{2} L C_d \rho_a A/M \\ &= -2\delta\left(E - gL(1 - \cos x)\right). \end{aligned}$$

So the crucial parameter is δ.

With SI values $C_d \approx 1.3$, $\rho_a \approx 1.2$, $A \approx 1$ and $M \approx 80$, $\delta \approx L/100$ is small. So the change in E over a half cycle $0 < \theta < \pi$ is approximately

$$-2\delta(E_0 - gL) \times \pi.$$

Height record: To just get to 360° needs $E_0 = 2gL$ before drag losses. The gain is approximately $\left[(L_2/L_1)^3 - 1\right] E_0$. So the drag-induced limit on the height record should be

$$\left[(L_2/L_1)^3 - 1\right] \times 2gL \approx 2\delta\pi(2gL - gL)$$

where L could be L_1 or L_2. With $L_2 = L_1 + \Delta L$, this gives, roughly,

$$\Delta L/L \approx \pi\delta/3 \approx L/100 \quad (\delta \approx L/100, \pi \approx 3\ldots)$$

so we predict

$$L_{max}^2 \approx 100\Delta L \quad (\text{in metres}^2).$$

Suppose ΔL is 0.4m; then $L_{max} \approx 6.3$ m which is a little more than both Bastian's German record and the current world record of 7.02 m (remember this is from pivot to feet but L is to the centre of mass).

Speed record: Here we know L and we balance drag losses against the fixed energy input per lap to find E_0 (and hence the initial speed). The drag loss is roughly

$$-2\delta(E_0 - gL) \times 2\pi.$$

and the energy input is

$$\Delta E = 2g(L_2^3 - L_1^3)/L_2^2 \approx 6g\Delta L.$$

So

$$E_0 = \frac{1}{2}L^2\dot{x}_0^2 \approx gL + 6g\Delta L/4\pi\delta$$

Take $L = 2.5$ m and then

$$\frac{1}{2}\dot{x}_0^2 \approx g/L + 6g\Delta L/4\pi\delta L^2 \approx 30$$

so $\dot{x} \approx 7.7$ rad/sec which is quite a bit faster than the current record $30 * \times 2\pi/60 = 4.2$ rad/sec.

g-forces

What is the rotational acceleration at the lowest point?

For a 360° swing that just makes it (height record),

$$\tfrac{1}{2}L^2\dot{x}^2_{max} = 2gL$$

so the peak acceleration is $L\dot{x}^2_{max} = 4g$ independently of L.

For the speed record, we use $E = \tfrac{1}{2}L^2\dot{x}^2 + gL(1 - \cos x)$ so $L\dot{x}^2_{max} = 2E/L$. A reasonable estimate at current record speeds gives accelerations of $5g$ so the kiiker has to lift 6 times body weight.

I think air drag limits the height record (stand sideways, wear lycra) but strength limits the speed record (take steroids).

Model 2: $L(x, \dot{x})$

Natural to try $L = L(x, y)$ where $y = \dot{x}$ (NB *not* $L = L(t)$).
Assume $L(x, y)$ is smooth.

Equation of motion is

$$\frac{1}{L}\frac{\mathrm{d}}{\mathrm{d}t}\left(L^2\frac{\mathrm{d}x}{\mathrm{d}t}\right) + g\sin x = 0$$

which is

$$(L + 2\dot{x}L_{\dot{x}})\,\ddot{x} + 2L_x\dot{x}^2 + g\sin x = 0,$$

or the system

$$\dot{x} = y$$
$$\dot{y} = -\frac{g\sin x + 2y^2L_x}{L + 2yL_y}.$$

Linearised analysis for small x, y

The equilibrium $(0,0)$ is a centre (orbits are closed for the Hamiltonian system when L is constant). Expand L

$$L(x,y) \sim L_0 + xL_{0x} + yL_{0y} + \frac{1}{2}\left(x^2 L_{0xx} + 2xyL_{0xy} + y^2 L_{0yy}\right) + \cdots$$

Is there a standard classification of centres?

At quadratic order we'll get

$$\dot{x} = y, \quad \dot{y} \sim -gx/L_0 + (g/L_0)\left(2xyL_{0y}/L_0 - 2y^2 L_{0x}\right)$$

which seems unlikely to leave $(0,0)$ easily. So take

$$L_{0x} = 0, \quad L_{0y} = 0$$

and look at cubic order. We get (algebra)

$$\dot{x} = y, \quad (L_0/g)\dot{y} = -x + \alpha_1 x^3 + \alpha_2 x^2 y + \alpha_3 x y^2 + \alpha_4 y^3,$$

where

$$\alpha_1 = \frac{1}{6} + \frac{L_{0xx}}{2L_0}, \quad \alpha_2 = \frac{3L_{0xy}}{L_0}, \quad \alpha_3 = \frac{5L_{0yy}}{2L_0} - \frac{2L_{0xx}}{g}, \quad \alpha_4 = -\frac{2L_{0xy}}{g}.$$

So

$$\left(x - \alpha_1 x^3\right)\dot{x} + \left((L_0/g)y - \alpha_4 y^3\right)\dot{y} = \alpha_2 x^2 y^2 + \alpha_3 x y^3,$$

Small-amplitude kiiking joy if the RHS is positive definite so we take

$$\alpha_2 = \frac{3L_{0xy}}{L_0} > 0, \quad \alpha_3 = \frac{5L_{0yy}}{2L_0} - \frac{2L_{0xx}}{g} = 0,$$

for example $L = L_0 +$ positive constant $\times\, x\dot{x}$.

This is sufficient but not (?) necessary. Hm. Let's move on.

Small length changes

Consider small length changes ($\epsilon = \Delta L/L \ll 1$). Write

$$L(x,\dot{x}) = L_0(1+\epsilon\ell(x,\dot{x}))$$

and assume $0 \le \ell \le 1$. Also scale time with $\sqrt{L_0/g}$. Equation of motion now

$$(1+\epsilon\ell+2\epsilon\dot{x}\ell_y)\,\ddot{x}+2\epsilon\ell_x\dot{x}^2+\sin x = 0.$$

Energy is

$$\mathcal{E} = \tfrac{1}{2}(1+\epsilon\ell)^2\dot{x}^2+(1+\epsilon\ell)(1-\cos x)$$

and up to $O(\epsilon)$,

$$\mathrm{d}\mathcal{E}/\mathrm{d}t = \epsilon\,(\dot{x}\ell_x - \sin x\ell_y)\,(1-\cos x-\dot{x}^2)$$

We can expand $x(t) \sim x_0 + \epsilon x_1 + \cdots$, $\mathcal{E} \sim \mathcal{E}_0 + \epsilon \mathcal{E}_1 + \cdots$ and with an $O(\epsilon^2)$ error this is

$$\frac{\mathrm{d}\mathcal{E}_1}{\mathrm{d}t} = \left(\mathcal{E}_0 - \frac{3}{2}\dot{x_0}^2\right)\frac{\mathrm{d}\ell}{\mathrm{d}t}$$

where the d $/\mathrm{d}t$ are along the unperturbed trajectory. We want to maximise this over a trajectory. The first term integrates to zero and the second is linear in ℓ so solution is bang-bang. The factor 3 above agrees with $(L_2/L_1)^3$ amplification earlier.

Multiple scales

The discussion above suggests a multiple scale approach, because for small length changes the equation is of the form

$$\ddot{x} + \epsilon f(x, \dot{x}) + \sin x = 0,$$

but what to do about the nonlinearity in $\sin x$? Standard method is to put $\tau = \epsilon t$ and then $\mathrm{d}\,/\mathrm{d}t \mapsto \partial\,/\partial t + \epsilon\partial\,/\partial\tau$. Leading order term in a regular expansion contains an undetermined 'amplitude' which is found by eliminating secular terms at $O(\epsilon)$ by requiring RHS to be orthogonal to solution of a suitable homogeneous problem.

Here we expand $x(t, \tau) \sim x_0 + \epsilon x_1 + \cdots$ and so

$$x_{0tt} + \sin x_0 = 0, \quad x_{1tt} + x_1 \cos x_0 = -f(x_0, x_{0t}) - 2x_{0t\tau}.$$

Difficulty: we can't solve $\ddot{x_1} + x_1 \cos x_0 = 0$.

Consider

$$\ddot{x} + \epsilon f(x, \dot{x}) + g(x) = 0,$$

written as

$$\dot{x} = y, \quad \dot{y} = -g(x) - \epsilon f(x, y).$$

Suppose $g(x) = G'(x)$ and $(0, 0)$ is a centre when $\epsilon = 0$ (Hamiltonian system) with closed orbits

$$\tfrac{1}{2}y^2 + G(x) = \text{constant}.$$

Perturb in ϵ: $x \sim x_0 + \epsilon x_1 + \cdots$, $y \sim y_0 + \epsilon y_1 + \cdots$. Then

$$\dot{x}_0 = y_0, \quad \dot{y}_0 = -g(x_0).$$

Orbits starting at $(0, \eta)$ are closed with period $T(\eta)$.

At $O(\epsilon)$,

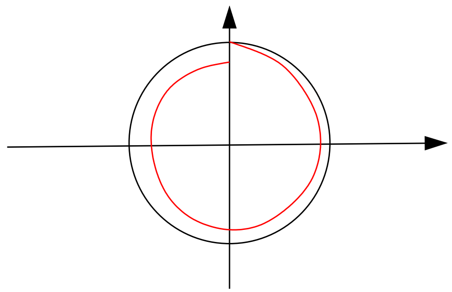
$$\dot{x}_1 = y_1, \quad \dot{y}_1 = -x_1 g'(x_0) - f(x_0, y_0).$$

Thus

$$\frac{\mathrm{d}}{\mathrm{d}t}(y_0 y_1 + x_1 g(x_0)) = -y_0 f(x_0, y_0),$$

which is an energy-change equation. For an orbit starting at $(0, \eta)$ and returning to $x = 0$,

$$[y_1]_0^{T(\eta)} = -\frac{1}{\eta}\int_0^{T(\eta)} y_0(t;\eta) f\Big(x_0(t;\eta), y_0(t;\eta)\Big)\,\mathrm{d}t.$$



Iterate this:

$$\eta_{n+1} - \eta_n = -\frac{\epsilon}{\eta_n} \int_0^{T(\eta_n)} y_0(t;\eta_n) f\Big(x_0(t;\eta_n), y_0(t;\eta_n)\Big)\,\mathrm{d}t.$$

Now think of η as a function of the multiple scales 'long' time $\tau = \epsilon t$: $\eta = \eta(\tau)$. Then

$$\begin{aligned} \eta_{n+1} - \eta_n &= \eta\Big(\tau + \epsilon T(\eta)\Big) \\ &\sim \epsilon T(\eta)\mathrm{d}\eta/\mathrm{d}\tau. \end{aligned}$$

So, the equation for the 'amplitude' parameter η is

$$\frac{\mathrm{d}\eta}{\mathrm{d}\tau} = -\frac{1}{T(\eta)} \int_0^{T(\eta)} \frac{y_0(t;\eta)}{\eta} f\Big(x_0(t;\eta), y_0(t;\eta)\Big)\,\mathrm{d}t.$$

and this gives the approximate solution $(x_0(t;\eta(\tau)), y_0(t;\eta(\tau)))$.

Recall the trivial example

$$\ddot{x} + \epsilon\dot{x} + x = 0, \quad x(0) = 0, \quad \dot{x}(0) = \eta$$

for which the solutions are combos of

$$\exp\left(-\epsilon/2 \pm \mathrm{i}(1-\epsilon^2/4)^{\frac{1}{2}}\right)t$$

Multiple scales gives solutions $A(\tau)\mathrm{e}^{\pm \mathrm{i}t}$ where $\mathrm{d}A/\mathrm{d}\tau = -\frac{1}{2}A$. In our notation

$$x_0(t) = \eta \sin t, \quad y_0(t) = \eta \cos t, \quad T(\eta) = 2\pi.$$

So

$$\frac{\mathrm{d}\eta}{\mathrm{d}\tau} = -\frac{1}{2\pi}\int_0^{2\pi} \frac{\eta^2}{\eta} \cos^2 t \, \mathrm{d}t = -\tfrac{1}{2}\eta$$

as expected.