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Improving the Efficiency of Kinetic Simulation of Plasmas*

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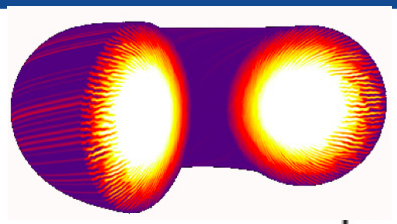
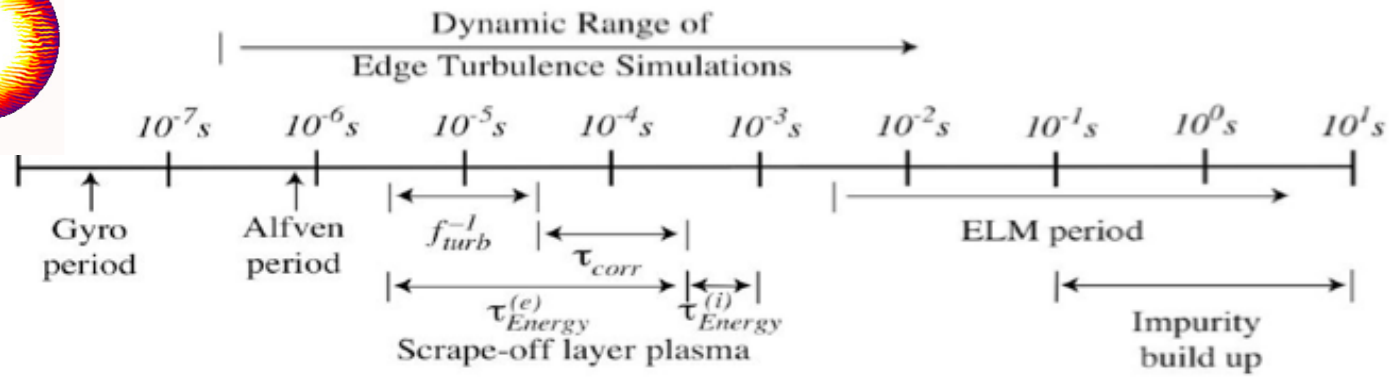
Multiscale Mathematics for Plasma Kinetics Spanning Multiple Collisionality Regimes



Tokamak plasma



- A very high fraction of the universe is in the plasma state
- Plasma phenomena are complex and highly nonlinear, and characterized by enormous ranges of time and space scales
- Plasmas exhibit a range of collisionality: when collisional mean-free-paths are short, plasmas behave as fluids; and when collisional mean-free paths are long, plasmas behave kinetically
- Coulomb collisions are a significant bottleneck in simulations, which motivates the development of advanced algorithms

Improving the Efficiency of Kinetic Simulation of Plasmas-- Outline

- Basic Monte Carlo algorithms for computing Coulomb collisions in particle simulation of plasmas
  - Binary collision methods
  - Langevin collision methods (stochastic differential equations)

- Fluid-kinetic methods using binary collisions methods
  - Fluid-particle representation of velocity distribution function
  - Collisions and thermalization/dethermalization
  - Relative entropy and thermalization criterion
  - Results for a test problem

- Higher-order methods for Langevin collision methods (SDEs)
  - Lowest order: Euler-Murayama
  - Higher order: Milstein
  - Higher order in multiple dimensions ➔Levy areas

- Multi-level Monte Carlo (MLMC) for Langevin methods (SDEs)
  - Combining solutions for different time steps
  - Using antithetic variables to remove the need for Levy areas
  - Results for test problems

- Summary

Basic Monte Carlo Algorithms for Computing Coulomb Collisions

- Coulomb collisions are computed in many different ways:
  - Binary: Takizuka & Abe '87, Nanbu '97, Dimits '09
    Velocities of pairs of particles are scattered through a small random angle with given variance in center of mass frame
  - Continuum representation: Abel '08, Xiong '08

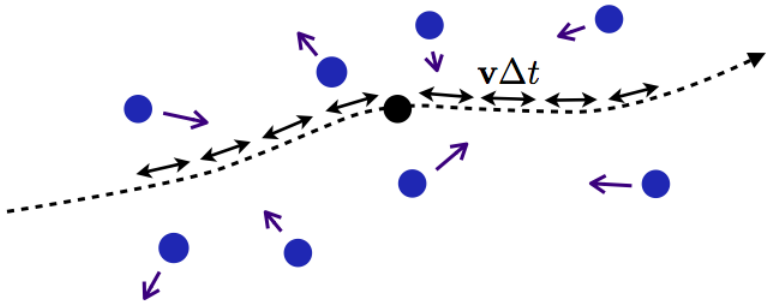
    Landau-Fokker-Planck PDE in velocity variables is solved

$$\partial_t f + \mathbf{v}\cdot\nabla_{\mathrm{x}} f + \mathbf{a}\cdot\nabla_{\mathbf{v}} f = C(f', f) = -\nabla_{\mathbf{v}}\cdot\mathbf{F}_d(f')f + \frac{\partial^2}{\partial\mathbf{v}\partial\mathbf{v}} : \mathbf{D}^2(f')f$$

  - Langevin equations (SDEs): Rognlien '80, Jones '96, Manheimer '97, Lemons '09, Cohen '06,'10

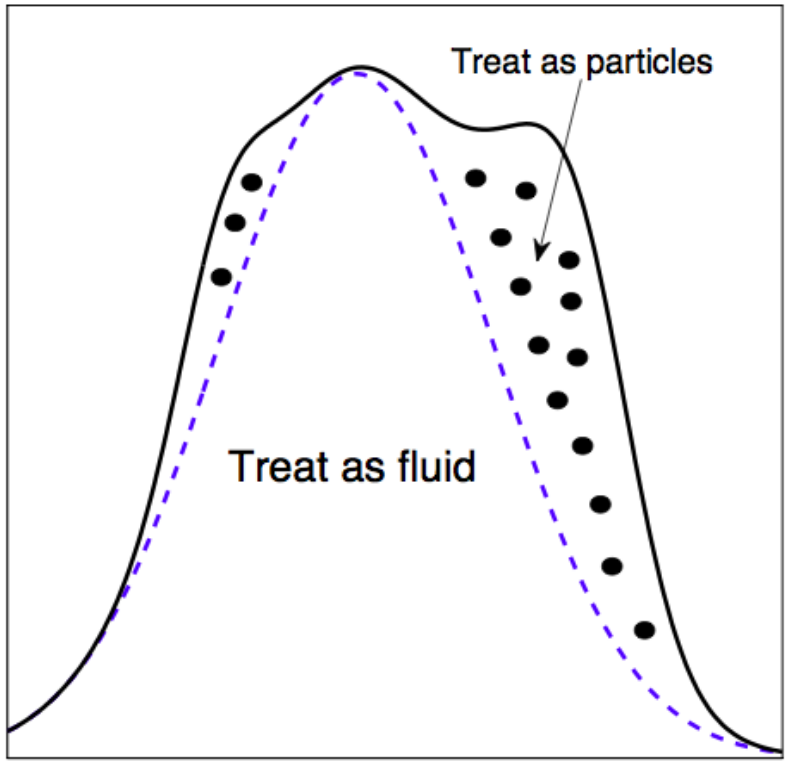
    SDEs are solved for test particles with drag and diffusion determined from moments of the field particles computed on a spatial grid

$$\Delta\mathbf{v}(t) = \mathbf{F}_d(\mathbf{v})dt + \mathbf{D}(\mathbf{v})\cdot d\mathbf{W}(t), \mathbf{F}_d = \text{deterministic drag}, d\mathbf{W}(t) = \text{stochastic}$$



Hybrid Algorithm for Binary Collisions

Combine fluid and particle simulation methods[1]:



- Separate f into Maxwellian and non-Maxwellian components: $f = m + k$
- Treat m as fluid $\rightarrow$ solves Euler equations
- Simulate k by Monte Carlo algorithm
- Interaction of m and k is the key step

[1]R. Caflisch et al., Multiscale Model. Simul. **7**, 865 (2008)

Interaction of m and k: thermalization and dethermalization

Two steps

- Collisions
  - Choose a particle from k and sample a particle from M
  - Perform collision as in Monte Carlo algorithm
- Thermalization/dethermalization
  - Collisions drive particles into equilibrium
  - Move particles from k to M when they have collided enough
  - Move sampled particles from M into k if the collision is strong enough
- (De)Thermalization criterion using entropy[2]
  - Alternative criterion based on scattering angle[3]

[2] Ricketson et. al, preprint, 2013
[3] Dimits et. al., private communication

Theorem (Boltzmann H-theorem)

If f solves the kinetic equation, and

$$H = -\int f \log f \, d\vec{v}$$

then $\partial_t H \geq 0$, with equality achieved iff f is Maxwellian.

Lesser known theorem about relative entropy:

Theorem (Relative Entropy Decay)

If f solves $\partial_t f + \vec{v} \cdot \nabla f = C(f, m)$ with m a fixed Maxwellian, and

$$H_{rel}(f, m) = \int f \log\left(\frac{f}{m}\right) d\vec{v}$$

then $\partial_t H_{rel} \leq 0$, with equality achieved iff $f = c(\vec{x})m$.

Hybrid Method: Thermalization/dethermalization Algorithm (cont'd)

Reinterpreting f

- The velocity space dependence of f is usually interpreted as an ensemble average over many particle velocities.
- We may also assign a velocity space distribution f_p to a single particle, which is interpreted as the probability density of that particle's velocity
- Relative entropy decay theorem ensures that $H_{rel}(f_p, m) \to 0$ through collisions with the fluid component of the scheme
- **Idea**: Track H_{rel} of each simulation particle, thermalize when it falls below some threshold[4]

[4] Ricketson et. al, preprint, 2013

Approximation by Maxwellian

To track H_{rel} exactly, we need to track f_p, which is computationally infeasible

- **Simplifying (but reasonable) assumption**: Approximate f_p by a Maxwellian
- In this case,

$$H_{rel} = \frac{3}{2}\left[\frac{T_p - T_m}{T_m} + \log\left(\frac{T_m}{T_p}\right)\right] + \frac{|\vec{u}_p - \vec{u}_m|^2}{v_{tm}^2} \qquad (8)$$

- So, tracking H_{rel} reduces to tracking T_p and $\vec{u}_p$
- Efficient and accurate method developed for this[5]

[5] Ricketson et. al, preprint, 2013

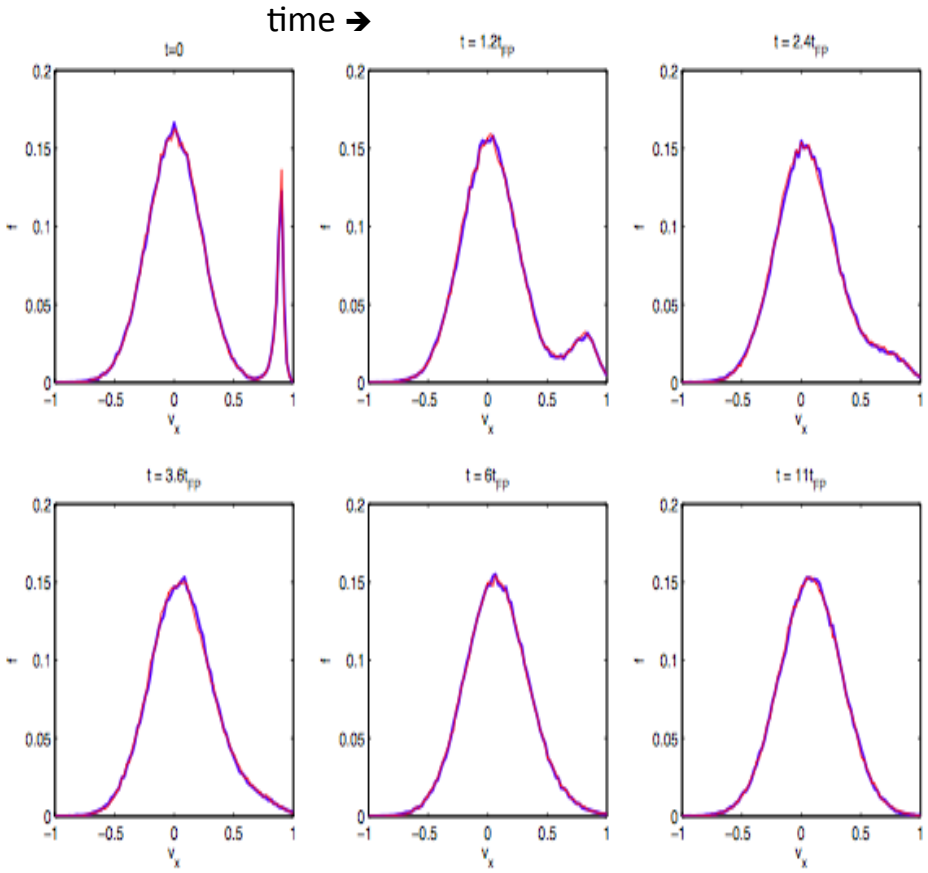
Hybrid Method: Thermalization/dethermalization Algorithm (cont'd)

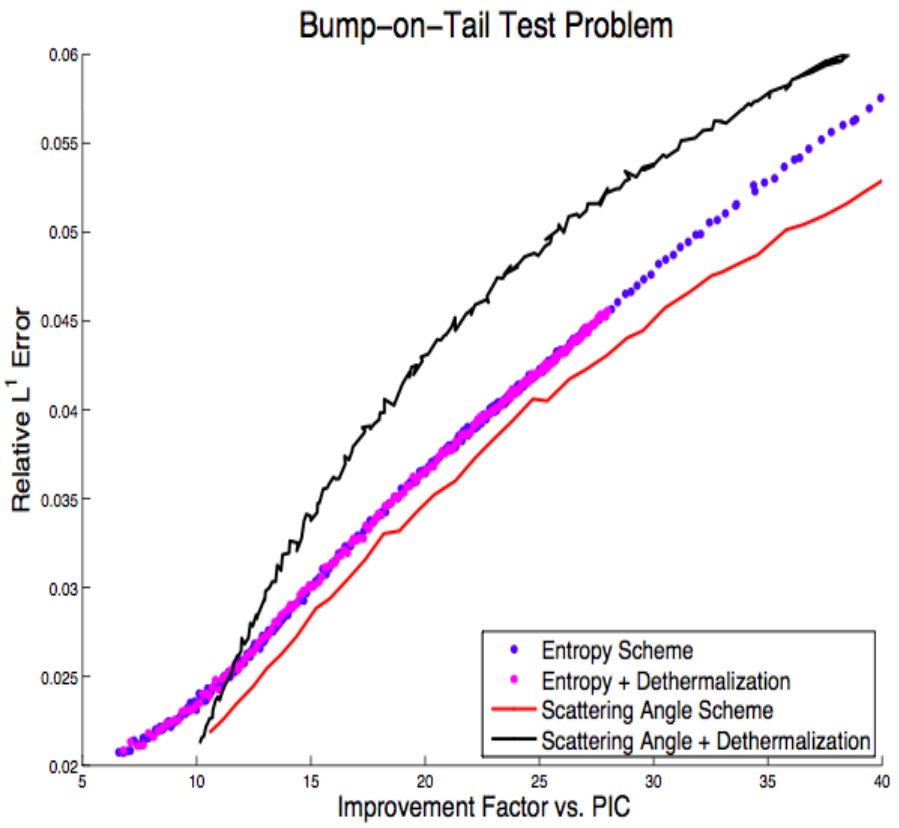
Algorithm Summary

First, fix a value $H_c > 0$

- Simulate collisions using Monte-Carlo algorithm for Fokker-Planck equation (see Takizuka-Abe or Nanbu). Sample particles from fluid portion of scheme where necessary, assigning them $T_p = T_m$, $\vec{u}_p = \vec{u}_m$
- Evolve $\vec{u}_p$ and T_p according to relevant ODEs, using parameters of collision partner as input
- Loop over all kinetic particles: thermalize if $H_{rel} < H_c$
- Loop over all sampled fluid particles: dethermalize if $H_{rel} \geq H_c$

Example Computation: Collisional Relaxation of a Bump on Tail





- The hybrid scheme achieves improvements of 5-40x over full PIC depending on the level of accuracy desired.

Higher-order Methods for Langevin Collision Methods (SDEs)

Near Equilibrium

Linear Landau-Fokker-Planck (LFP) equation:

$$\partial_t f = \frac{1}{\mathrm{Kn}} C(M, f) \tag{9}$$

with linear LFP collision operator

$$C(M, f) \equiv -\frac{\partial}{\partial \mathbf{v}} \cdot \mathbf{F} f + \frac{\partial^2}{\partial \mathbf{v} \partial \mathbf{v}} : \mathbf{D}^2 f \tag{10}$$

in which $F = \mathbf{A}(M)$ and $D^2 = \mathbf{B}(M)$.

Langevin Formulation

Linear LFP equation for $f(v, t)$ is in exact correspondence with the Langevin equation (SDE) for $v(t)$

$$dv_i = F_i \, dt + D_{ij} \, dW_j, \tag{11}$$

where f is probability density of $\mathbf{v}$ and i, j are component indices

- $W = W(t)$ is Brownian motion in velocity
- dW is white noise in velocity
- Direct extension to spatial dependence
- Valid for nonlinear LFP, if $\mathbf{F}$ and $\mathbf{D}$ are updated as needed

Objective is an average of f:

$$\frac{1}{\rho} \int P(\mathbf{v}) f(\mathbf{v}, t) \, d\mathbf{v} \equiv \mathbb{E}[P(\mathbf{v}(t))] \tag{12}$$

Lowest-order Methods for Langevin Collision Methods

Discretization of SDEs

Euler-Maruyama discretization in time:

$$v_{i,n+1} = v_{i,n} + F_{i,n}\Delta t + D_{ij,n}\,\Delta W_{j,n}, \quad (13)$$

$$\Delta \mathbf{W}_n = \mathbf{W}_{n+1} - \mathbf{W}_n \quad (14)$$

in which $v_{i,n} = v_i(t_n)$ and $\mathbf{F}_n = \mathbf{F}(\mathbf{v}_n)$

- Choose N Brownian paths to get N values of $P(\mathbf{v}(T))$
- Average to approximate $\mathbb{E}[P(\mathbf{v}(T))]$

Computational cost vs. Error ε:

- Statistical error is $O(N^{-1/2})$
- Δt error is $O(\Delta t)$, since $\Delta W = O(\sqrt{\Delta t})$ and random
- Optimal choice is $\varepsilon = N^{-1/2} = \Delta t$
- Cost $= N\Delta t^{-1} = \varepsilon^{-3}$
- From $\varepsilon^2 \geq MSE \sim c_1^2 \Delta t_l^{2\alpha} + \frac{Var[P_l^i]}{N_l}$, $\alpha = 1$, solve for optimal Δt_l and N_l

by minimizing computational cost $K = N_l / \Delta t_l$ using Lagrange multipliers

Higher-order Methods for Langevin Collision Methods (cont'd)

Higher Order Discretization

Milstein discretization in time:

$$v_{i,n+1} = v_{i,n} + F_{i,n}\Delta t + D_{ij,n}\,\Delta W_{j,n} + G_{ijk,n} I_{jk,n} \quad (15)$$

$$I_{jk,n} = \int_0^{\Delta t}\int_0^{s} dW_j(t_n + s')dW_k(t_n + s) \quad (16)$$

in which $v_{i,n} = v_i(t_n)$ and $\mathbf{F}_n = \mathbf{F}(\mathbf{v}_n)$

- **G** depends on **D** and its derivatives
- Milstein is tractable in 1D
  - Only requires diagonal term $I_{11} = ((\Delta W_1)^2 - \Delta t)/2$
- Milstein is intractable in 3D and higher
  - Off-diagonal I_{jk}'s involve "Levy areas" which are intractable
- Milstein is tractable in 2D
  - Special methods for calculating a single Levy area[6]

[6] Dimits et. al., JCP, 2013

Approximation of Milstein in 2D

Off-diagonal Milstein term includes Levy area L_{12}:

$$I_{12} = \frac{1}{2}\Delta W_1 \Delta W_2 + \frac{1}{2} L_{12} \quad (17)$$

$$L_{12} = \int_0^{\Delta t}\int_0^{s} dW_1(s')dW_2(s) - dW_2(s')dW_1(s) \quad (18)$$

Requires conditional probability distribution function

$$P(L_{12}|\Delta W_1, \Delta W_2) = \hat{P}(L_{12}|\sqrt{\Delta W_1^2 + \Delta W_2^2})^7 \quad (19)$$

- Dimits found a simple approximation of $\hat{P}(L_{12}|R_{12})$[8]
  - Numerical values given through a 1D lookup table
  - Related to earlier work[9]
- Generalization to $d > 2$ is possible but difficult

[7] Levy, 2nd Berkeley Symp Pob Stat, 1951
[8] Dimits et. al., JCP, 2013
[9] Gaines & Lyons , SIAM J Appl Math, 1997

Whether Higher-order Discretizations Are Useful Is Related to Strong and Weak Convergence

Weak and Strong Convergence

Weak convergence of time discretization:

$$|\mathbb{E}[P(\mathbf{v})] - \mathbb{E}[P(\mathbf{v}_{\Delta t})]| = O(\Delta t) \quad \text{for Euler-Maruyama} \quad (20)$$
$$|\mathbb{E}[P(\mathbf{v})] - \mathbb{E}[P(\mathbf{v}_{\Delta t})]| = O(\Delta t) \quad \text{for Milstein} \quad (21)$$

- Weak convergence implies convergence of distributions
- Milstein is no better than Euler-Maruyama in weak sense

Strong convergence of time discretization:

$$\mathbb{E}[|\mathbf{v} - \mathbf{v}_{\Delta t}|] = O(\sqrt{\Delta t}) \quad \text{for Euler-Maruyama} \quad (22)$$
$$\mathbb{E}[|\mathbf{v} - \mathbf{v}_{\Delta t}|] = O(\Delta t) \quad \text{for Milstein} \quad (23)$$

- Strong convergence implies convergence for each realization
- Milstein is better than Euler-Maruyama in strong sense

Usefulness of Milstein

- Monte Carlo mostly aimed at computation of $\mathbb{E}[P(\mathbf{v})]$
  - Milstein offers no advantage over Euler-Maruyama
- Multilevel Monte Carlo (MLMC) leverages strong convergence to accelerate computation of $\mathbb{E}[P(\mathbf{v})]$ [10]
  - Milstein superior to Euler-Maruyama
  - Previous uses of MLMC mostly confined to finance
  - Our application of MLMC to plasma simulation is its first use for SDEs from physics

[10] Giles, *Operations Research*, 56(3):607, 2008

Application of Milstein Scheme to MC Coulomb Collisions for Spherical Coordinates in a Fixed Frame

Coulomb test-particle problem as SDE's for spherical coordinates wrt a fixed frame

- Write as Ito form drag-diffusion (forward Kolmogorov) equation:

$$\left(\frac{\partial \hat{f}_t}{\partial \hat{t}}\right)_c = -\frac{\partial}{\partial v}\left[F_d(v)\,\hat{f}_t\right] + \frac{\partial^2}{\partial v^2}\left[D_v(v)\,\hat{f}_t\right] + \frac{\partial}{\partial \mu}\left[2D_a(v)\mu\,\hat{f}_t\right] + \frac{\partial^2}{\partial \mu^2}\left[D_a(v)\left(1-\mu^2\right)\,\hat{f}_t\right] + \frac{\partial^2}{\partial \phi^2}\left[\frac{D_a(v)}{\left(1-\mu^2\right)}\,\hat{f}_t\right],$$

where $\hat{f}_t = 2\pi v^2 f_t$

- Corresponding Ito-Langevin equations:

$$\begin{aligned} dv(t) &= F_d(v)\,dt + \sqrt{2D_v(v)}\,dW_v(t), \\ d\mu(t) &= -2D_a(v)\mu dt + \sqrt{2D_a(v)\left(1-\mu^2\right)}\,dW_\mu(t), \\ d\phi(t) &= \sqrt{\frac{2D_a(v)}{\left(1-\mu^2\right)}}\,dW_\phi(t). \end{aligned}$$

Milstein scheme for Coulomb test-particle problem

$$\begin{aligned} \Delta v &= F_{d0}\Delta t + \sqrt{2D_{v0}}\Delta W_v + \kappa_{\mathrm{M}} D'_{v0}\frac{1}{2}\left(\Delta W_v^2 - \Delta t\right), \\ \Delta \mu &= -2D_{a0}\mu_0\Delta t + \sqrt{2D_{a0}\left(1-\mu_0^2\right)}\Delta W_\mu, \\ &\quad + \kappa_{\mathrm{M}}\left[-2D_{a0}\mu_0\frac{1}{2}\left(\Delta W_\mu^2 - \Delta t\right) + \sqrt{\frac{D_{v0}}{D_{a0}}}\sqrt{\left(1-\mu_0^2\right)}D'_{a0}A_{v\mu}\right], \\ \Delta \phi &= \sqrt{\frac{2D_a(v)}{1-\mu_0^2}}\Delta W_\phi + \kappa_{\mathrm{M}}\left[\sqrt{\frac{D_{v0}}{D_{a0}}}\frac{D'_{a0}}{\sqrt{1-\mu_0^2}}A_{v\phi} + \frac{2D_{a0}\mu_0}{1-\mu_0^2}A_{\mu\phi}\right], \end{aligned}$$

$$\begin{aligned} \Delta\psi &= \psi(t_{i+1}) - \psi(t_i), \\ \psi_0 &= \psi(t_i), \\ A_{kl} &= \int_{t_i}^{t_{i+1}} dW_l(s)\int_{t_i}^{s} dW_k(\xi), \end{aligned}$$

Accurate Evaluation of 2D Area Integrals Involved in Multi-dimensional Milstein Method

Theory and numerical implementations exist for the sampling of the stochastic integral terms

$$\int_0^{\Delta t} dW^i(t_n+s)\int_0^s dW^j(t_n+\eta) = \begin{cases} \frac{1}{2}\left[\left(\Delta W_n^i\right)^2 - \Delta t\right], & i=j \\ \frac{1}{2}\left[\Delta W_n^i \Delta W_n^j + L_n^{i,j}\right], & i\neq j \end{cases}$$

- Levy, '51

$$\begin{aligned} P_{cL}\left(L_n^{i,j}|\Delta W_n^i, \Delta W_n^j\right) &= \hat{P}_{cL}\left(L_n^{i,j}|R_n^{i,j}\right) \\ R_n^{i,j} &= \sqrt{\left(\Delta W_n^i\right)^2 + \left(\Delta W_n^j\right)^2} \\ \phi_{cL}\left(k|R\right) &\equiv \left\langle \exp\left(-ikL\right)\right\rangle|_R \\ &= \frac{k/2}{\sinh\left(k/2\right)} \exp\left\{\frac{R^2}{2}\left[1 - \frac{\left(k/2\right)\cosh\left(k/2\right)}{\sinh\left(k/2\right)}\right]\right\}. \end{aligned}$$

We have developed a simple accurate method for sampling area integrals

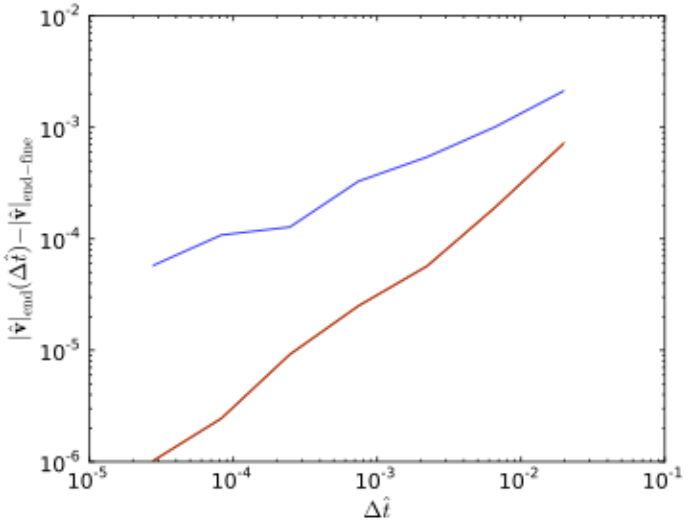
- Existing methods
  - Interpolation from 2D table based on Levy's results (Gaines and Lyons '94)
    - ★ accurate and efficient
    - ★ somewhat involved
    - ★ challenging for conditional sampling - adaptive integration
  - Discrete approximations (Clark and Cameron '80; Kloeden and Platen '92; Gaines and Lyons '97)
    - ★ simple to implement
    - ★ straightforward for adaptive integration
    - ★ expensive for good accuracy (many random numbers per L sample)
- Our method is a simplification of that of Gaines and Lyons '94
  - based on an accurate approximation to Levy's PDF
  - can implement with 1D tables or analytical functions
  - can be used to significantly reduce memory and computation requirements for conditional sampling

Application of Milstein Scheme to MC Coulomb Collisions – Test Problem Results

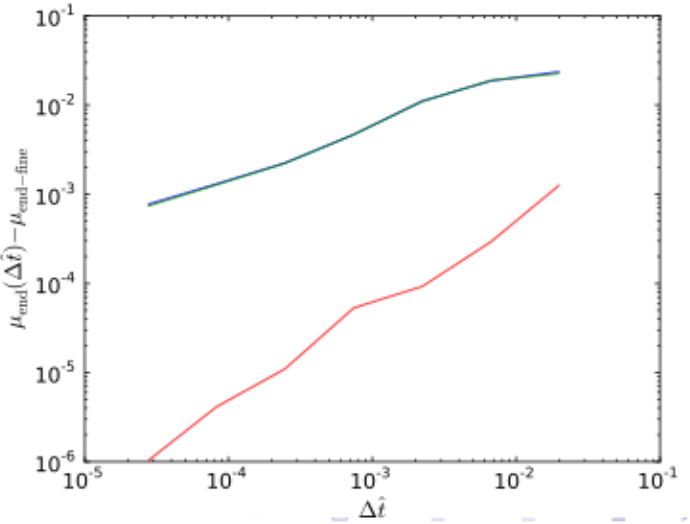
Approach 2 achieves $O(\Delta t)$ strong convergence for v and for angular component

- v evolution unaffected by angular evolution, and ∴ by area terms
- Angular evolution has poor convergence without area terms
- 16 realizations; time step range = 3^8; end time $\nu(v_{\rm th})\, t_{\rm end} = 0.1$
- Blue-Euler, Green-Milstein diagonal, Red-full Milstein

$$||v_{\rm end}(\Delta t)| - |v_{\rm end-fine}||$$



$$|\mu_{\rm end}(\Delta t) - \mu_{\rm end-fine}|$$



Multi-level Monte Carlo (MLMC) for Langevin Methods

Standard Monte Carlo

Normally, to estimate $\mathbb{E}[P(\mathbf{v}(T))]$, discretize in time, e.g. Euler-Maruyama:

$$v_{i,n+1} = v_{i,n} + F_{d,i}(\mathbf{v}_n)\Delta t + D_{ij}(\mathbf{v}_n)\,\Delta W_{j,n}, \qquad (14)$$

then choose N different Brownian paths to get N values of $P(\mathbf{v}(T))$, and average.

Note: $\Delta W_{j,n}$ is a normal r.v. w/ mean 0, variance Δt, so is characteristically $O(\sqrt{\Delta t})$.

$\varepsilon \sim N^{-1/2}$ and $\varepsilon \sim \Delta t^p$. Cost $\sim N\Delta t^{-1} \implies$ Cost $\sim \varepsilon^{-(2+1/p)}$

Idea of MLMC

Standard MC leverages weak convergence of time discretization:

$$\varepsilon_w = |\mathbb{E}[P(\mathbf{v})] - \mathbb{E}[P(\mathbf{v}_{\Delta t})]| = O(\Delta t^p). \qquad (15)$$

Standard discretizations also converge in the strong sense:

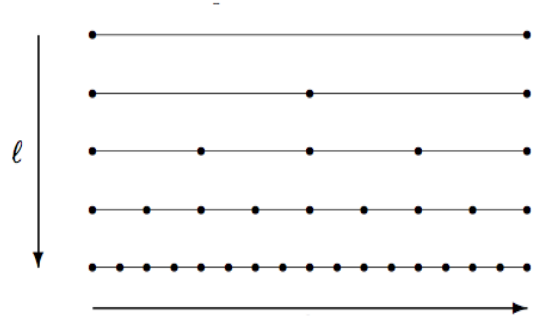
$$\varepsilon_s = \mathbb{E}[|\mathbf{v} - \mathbf{v}_{\Delta t}|] = O(\Delta t^q). \qquad (16)$$

Can we leverage strong convergence to improve performance? MLMC answers yes.

Multi-level Monte Carlo Basics

MLMC Basics

- Introduce time step levels, $\Delta t_\ell = T2^{-\ell}$, for $\ell = 0, ..., L$.



- Let $P_\ell = P(\mathbf{v}_{\Delta t_\ell})$. We have

$$\mathbb{E}[P_L] = \mathbb{E}[P_0] + \sum_{\ell=1}^{L} \mathbb{E}\left[P_\ell - P_{\ell-1}\right]. \qquad (17)$$

- When computed using same Brownian path, the variance of $(P_\ell - P_{\ell-1})$ is $O(\Delta t_\ell^{2q})$ by strong convergence.

- The number of samples N_l at each time level scales as $O(\Delta t)$ for Euler and $O(\Delta t^{3/2})$ including Milstein terms, i.e., fewer samples are computed as the time step is refined.

MLMC Scaling

A Lagrange multiplier argument gives the optimal number of samples N_ℓ used to compute each $\mathbb{E}[P_\ell - P_{\ell-1}]$, constrained by RMSE $< \varepsilon$. The complexity now scales like[3]

$$\text{Cost} = \begin{cases} O\left(\varepsilon^{-2}(\log\varepsilon)^2\right) & q = 1/2 \\ O\left(\varepsilon^{-2}\right) & q > 1/2 \end{cases} \qquad (18)$$

Notes:

- MLMC scales better than standard MC for any values of p, q.
- Milstein method ($q > 1/2$) is difficult to implement in $d > 1$, but possible thanks to Dimits et. al. '13.
- $q > 1/2$ is sufficient to get $O(\varepsilon^{-2})$ scaling, but not necessary[4] - can use antithetic sampling method.

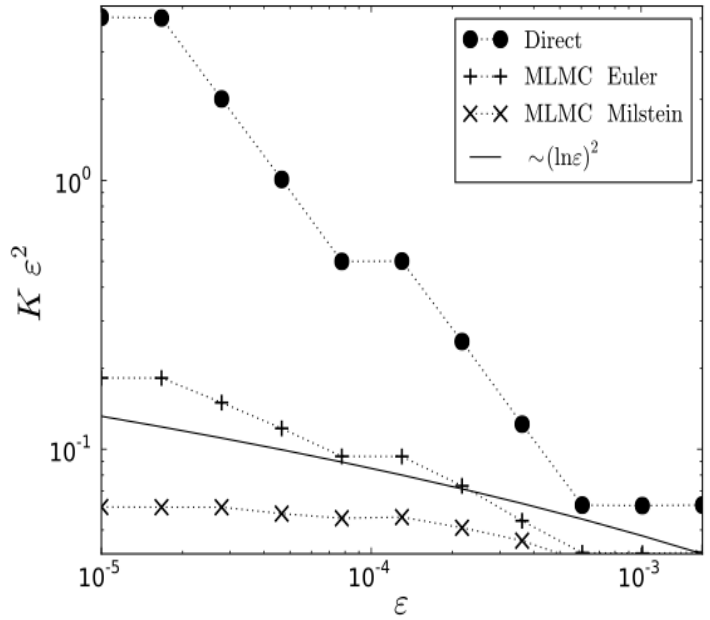
[3] Giles, *Operations Research*, 56(3):607, 2008
[4] Giles & Szpruch, *arXiv:1202.6283*, 2012

Comparison of Direct Euler, MLMC and MLMC with Milstein on a Test Problem

- A collisional relaxation initial-value problem was studied

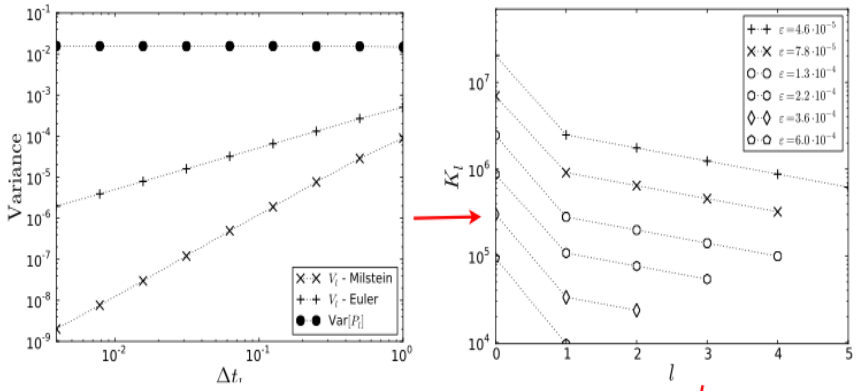
A Sample Plasma Problem

2D Coulomb Collisions



Rosin, LFR, et. al., submitted to JCP, 2013

Complexity analysis: MLMC



Operations K to ensure MSE < ϵ^2 is:

$$K = \sum K_l = \sum N_l \Delta t_l^{-1} \rightarrow K_{\mathrm{MLMC}} \sim \mathcal{O}(\epsilon^{-2})$$

UCLA

1. Giles in "Monte Carlo and Quasi-Monte Carlo Method", Springer-Verlag, (2006)

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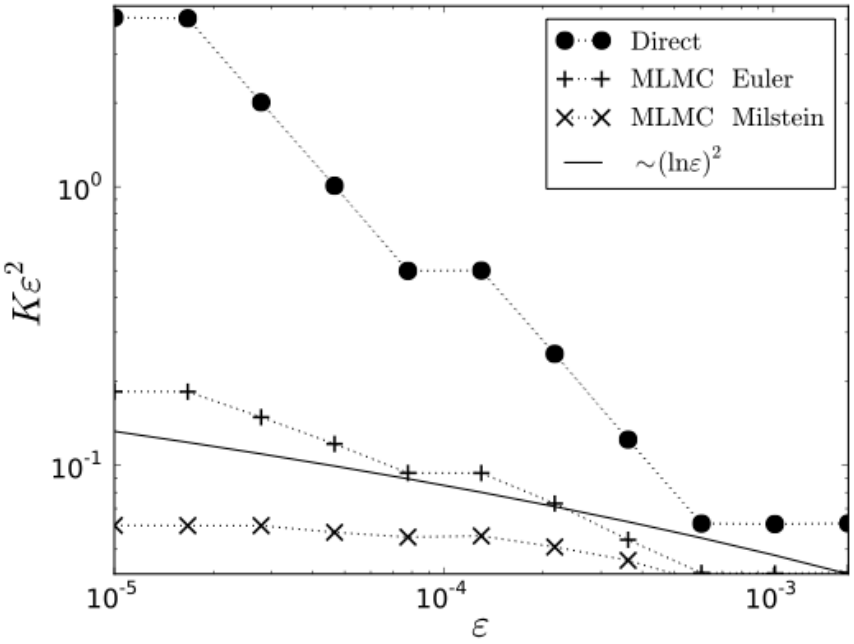
Complexity analysis: Comparison of all methods



Number of operations to bound error by ϵ

Error bound = ϵ

$K \sim \mathcal{O}(\epsilon^{-3})$ – Binary

$K \sim \mathcal{O}(\epsilon^{-3})$ – Direct

$K \sim \mathcal{O}(\epsilon^{-(2+1/\alpha)})$ – Order–α direct

$K \sim \mathcal{O}(\epsilon^{-D/\gamma-1/\alpha})$ – Continuum

$K \sim \mathcal{O}(\epsilon^{-2}(\ln\epsilon)^2)$ – MLMC Euler

$K \sim \mathcal{O}(\epsilon^{-2})$ – MLMC Milstein

MLMC Milstein is math. provably optimal Monte Carlo scheme. How good is it in practice?

1. Giles in "Monte Carlo and Quasi-Monte Carlo Method", Springer-Verlag, (2006)

Other Methods: Antithetic Sampling and Ito Linearization (ref. Ricketson)

Improving on MLMC through Antithetic Sampling

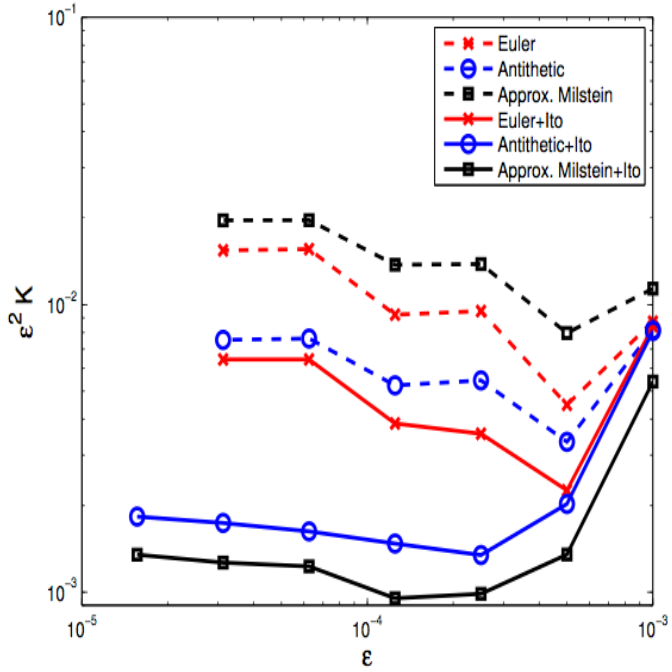
Antithetic sampling is a Monte Carlo variance reduction method

- For MC estimation of $E[f(x)]$ with normal random variable x
  - Standard estimator is $N^{-1}\sum f(x_i)$
  - Antithetic estimator is $(2N)^{-1}\sum (f(x_i)+f(-x_i))$
- Antithetic sampling for Milstein does not eliminate Levy areas
- Antithetic sampling for MLMC-Milstein achieves $O\left(\varepsilon^{-2}\right)$ without Levy areas! [13]

  The Levy area terms are anti-symmetric wrt to sign changes in the Brownian increments of the fine and antithetic paths, which are averaged, resulting in a cancellation of the Levy area contributions in the MLMC-Milstein computation.

[13] Giles & Szpruch, *arXiv:1202.6283*, 2012

Computational Results



Fusion Energy Sciences

Summary: Computational Complexity for Monte Carlo Simulation of Coulomb Collisions in Plasmas

- Two methods for Monte Carlo simulation of collisions:
  - Binary: Takizuka-Abe '77, Nanbu '97
  - Langevin: Jones '96, Manheimer '97
- Computation cost to achieve RMS error of size ε (Euler):
  - Binary: $O(\varepsilon^{-3})$ at best, $O(\varepsilon^{-4})$ at worst[15]
  - Langevin: $O(\varepsilon^{-3})$
- The **Hybrid Method** reduces the computational cost of the binary collision method.
- Higher order methods (Milstein or antithetic) are useful for **Multilevel Monte Carlo,** and the computational cost of the Langevin formulation can be reduced to $O(\varepsilon^{-2}(\log\varepsilon)^2)$ or even $O(\varepsilon^{-2})$, *e.g.*, 2 orders of magnitude acceleration in examples
- Future work needs to address the inclusion of both time-evolving electromagnetic fields and collisions that influence the plasma

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[15]Bobylev & Potapenko, J Comp Phys, 2013

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