Image Denoising Using Mean Curvature of Image Surface

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Joint work with Wei ZHU (U. Alabama) &
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In honor of Russ Caflisch's 60 birthday

IPAM April 25, 2014

My Relationship with Russ

- His college room-mate was my graduate student office-mate (mid 70’s?)
- He was related to Gil Strang!
- Stanford (late 70’s?)
- UCLA colleague (late 80’s – 2009)
- IPAM
- Yet to write a joint paper!

Outline

- Problem
- Related Work
- Our Model
- Fast Algorithm Using Augmented Lagrangian Method
- Numerical Experiments
- Summary and Future Work

Our related publications:

- *Zhu and Chan,*

  Image denoising using mean curvature of image surface, SIIMS 2012.

- *Zhu, Tai and Chan,*

  Augmented Lagrangian method for a mean curvature based image denoising model, Inverse Probl Imag, In Press*, 2013.*

- *Zhu, Tai and Chan,*

  Image Segmentation Using Euler's Elastica as the Regularization, J. Scientific Computing, 2013.

- *Zhu, Tai and Chan,*

  A fast algorithm for a mean curvature based image denoising model using augmented Lagrangian method, To appear in LNCS 2014, in "Efficient Algorithms for Global Optimisation Problems in Computer Vision".

Typical Methods of Image Denoising

- Variational method, PDE-based method, statistical method and many other ones

- Variational method

$$f = u + n$$

Given image **Desired clean image** **Noise**

$$f : \Omega \to R^1$$

How to decompose the given noisy image using appropriate regularizers?

Classical Variational Models

- Mumford-Shah (89)

$$E(u, K) = \int_{\Omega} (f - u)^2 + \lambda \int_{\Omega \setminus K} |\nabla u|^2 + \mu H^1(K)$$

goal true image goal boundary positive parameters

- Rudin-Osher-Fatemi (92)

$$E(u) = \lambda \int_{\Omega} |\nabla u| + \int_{\Omega} (f - u)^2, \quad \lambda > 0$$

  - Powerful & popular, excellent analytical properties
  - Preserve edges and sweep noise very efficiently
  - Cannot preserve corner & image contrast
  - Suffers from the staircase effect

Related high-order models for image denoising

- Euler's Elastica: C-Kang-Shen (2002), Ambrosio-Masnou-Morel (2003)

$$E(u) = \int_\Omega \left[a + b\left(\nabla \cdot \frac{\nabla u}{|\nabla u|} \right)^2 \right] |\nabla u| + \frac{1}{2}\int_\Omega (f-u)^2$$

  - Originally proposed for the disocclusion problem
  - Noise removal efficiently, no staircase effect
  - Need to solve a fourth-order PDE

- Lysaker-Lundervold-Tai (LLT)(2003)

$$L(u,\lambda) = \lambda \int_\Omega \sqrt{u_{xx}^2 + u_{xy}^2 + u_{yx}^2 + u_{yy}^2} + \frac{1}{2}\int_\Omega (f-u)^2$$

  - Excellent noise suppression, no staircase effect
  - Need to solve a fourth-order PDE

Mean curvature of image surface

- Give an image :

$$f : \Omega \rightarrow R^1, \quad \Omega \subset R^2$$

- Consider the function :

$$\Phi(x,z) = z - f(x), \quad x \in \Omega$$

Its zero level set corresponds to the image surface $(x, f(x))$, whose mean curvature reads:

$$\frac{1}{2}\nabla_{(x,z)} \cdot \left(\frac{\nabla_{(x,z)}\Phi}{\left|\nabla_{(x,z)}\Phi\right|} \right) = \frac{1}{2}\nabla_{(x,z)} \cdot \left(\frac{(\nabla_x f, -1)}{\left|(\nabla_x f, -1)\right|} \right) = \frac{1}{2}\nabla_x \cdot \left(\frac{\nabla_x f}{\sqrt{1 + \left|\nabla_x f\right|^2}} \right) = H_f$$

Our Model (Zhu, Chan *SIIMS* 2012)

- Energy:

$$E(u) = \lambda \int_\Omega |H_u| + \frac{1}{2}\int_\Omega (f-u)^2$$

$$= \frac{\lambda}{2}\int_\Omega \left|\nabla \cdot \left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)\right| + \frac{1}{2}\int_\Omega (f-u)^2$$

- Gradient Descent Equation:

$$\frac{\partial u}{\partial t} = -\lambda \nabla \cdot \left[\frac{1}{\sqrt{1+|\nabla u|^2}}(I-P)\nabla(\Phi'(H_u))\right] + (f-u)$$

The two operators $I, P : R^2 \to R^2$ are defined as

$$I(\vec{v}) = \vec{v}, \quad P(\vec{v}) = \left(\vec{v} \cdot \frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}, \quad \Phi(x) = |x|$$

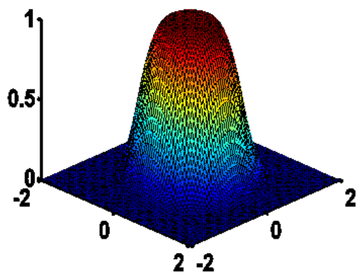
- If $|\nabla u| << 1$, $\frac{\partial u}{\partial t} \approx -\lambda \Delta^2 u + (f-u)$, the bi-harmonic equation, explaining why small oscillation part can be removed effectively.

Our model preserves contrast with small regularization

- We can prove that:

  If E is an open set with C^2 boundary, and $f = h\chi_E$, then $\int_\Omega |H_f| = P(E,\Omega)$, the perimeter of set E inside the domain Ω (independent of h).

- These results suggest that the proposed model is able to preserve image contrasts, as the regularizer doesn't rely on the height of signal.



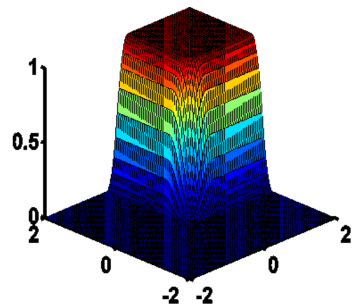
- Property of our model (contrast preservation):

Let $f = h\chi_{B(0,R)}$ be an image defined on $\Omega = (-2R, 2R) \times (-2R, 2R)$. Define $S = \{u \in C^2(R^2) : u(x,y) = g(\sqrt{x^2 + y^2}), \text{g takes the same type of profile as shown.}\}$, then there exists a constant $C > 0$, such that if $\lambda < C$, then the following holds:

$$E(f) = \inf\{E(u) : u \in S\}$$

This property shows that the model attains a minimum at f if λ is small enough, i.e. the model restores f exactly and thus preserves contrast.

Corner Preservation



Let $f = h\chi_{(0,R)\times(0,R)}$ be an image defined on $\Omega = (-R, R)\times(-R, R)$. Define $Q = \{u : \text{the surface of } z = u(x, y) \text{ is obtained by rotating the generatrix along the orbit.}\}$, then there exists a constant $C > 0$, such that if $\lambda < C$, then the following holds

$$E(f) = \inf\{E(u) : u \in Q\}$$

For small enough regularization (e.g. low noise level), our model can preserve corners.

Summary of our model:

- Using L1 norm of mean curvature of image surface as regularization
- Regularization does not penalize contrast or discontinuties
- For small regularization, can preserve contrast, edges and corners.
- Complete theory still lacking

Augmented Lagrangian Method

- Related functionals

$$E(u) = \lambda \int_{\Omega} |\nabla u| + \int_{\Omega} (f - u)^2$$

- non-differentiable
- nonlinear

$$E(u) = \int_{\Omega} \left[a + b \left(\nabla \cdot \frac{\nabla u}{|\nabla u|} \right)^2 \right] |\nabla u| + \frac{1}{2} \int_{\Omega} (f - u)^2$$

- high order
- non-differentiable
- nonlinear

- Augmented Lagrangian method (ALM) has been successfully applied to the minimization of the above functionals by Tai et al. (*SIIMS 2010 & 2011*)
  - ➢ convert the original minimization of those functionals to be constrained optimization problems
  - ➢ search for saddle points of the resulting problem by solving several associated subproblems

- Key of ALM: whether the subproblems can be solved efficiently

Review of ALM for Euler's Elastica Denoising (Tai,Hahn,Chung, *SIIMS*,2011)

- Tai et al. applied ALM to minimize the following functional for image denoising through minimization of Euler's Elastica

$$E(u) = \int_\Omega \left[a + b\left(\nabla \cdot \frac{\nabla u}{|\nabla u|} \right)^2 \right] |\nabla u| + \frac{1}{2}\int_\Omega (f-u)^2$$

- Introducing new variables for the gradient and the unit normal vector

$$p = \nabla u, \quad n = \frac{p}{|p|},$$

- The problem can be casted as a constrained minimization problem with new variables

$$\min_{u,p,n} \int_\Omega \left[a + b(\nabla \cdot n)^2 \right] |p| + \frac{1}{2}\int_\Omega (f-u)^2$$

$$\text{subject to } p = \nabla u, \; n|p| = p$$

- The last constraint is difficult to handle. Needed a new idea.

A new constraint

- In (Tai et al SIIMS11)

$$\text{If } n \neq 0, p \neq 0, \textit{ and } |n| \leq 1, \textit{ then}$$

$$|p| = n \cdot p \longleftrightarrow n = p / |p|$$

- A new constraint problem is to solve:

$$\min_{u,p,n} \int_\Omega \left[a + b(\nabla \cdot n)^2 \right] |p| + \frac{1}{2} \int_\Omega (f-u)^2$$

$$\text{subject to } p = \nabla u, |p| = p \cdot n, |n| \leq 1$$

- The minimization variables are: u, p, n. When two of them are fixed and we just need to minimize with one of them, the problem is convex

Fast Augmented Lagrangian

- Augmented Lagrangian method (ALM) has been used to solve:

$$\min_{u,p,n} \int_\Omega \left[a + b(\nabla \cdot n)^2\right] |p| + \frac{1}{2}\int_\Omega (f-u)^2$$

$$\text{subject to } p = \nabla u, |p| = p \cdot n, |n| \le 1$$

Features of the ALM in Tai et al SIIMS11:

- ALM with L2 penalization is used to handle: $p = \nabla u$
- ALM with L1 penelization is used to handle: $|p| = n \cdot p$
- All the subproblems either has closed form solutions or can be solved by fast solvers like FFT.
- Need few iterations (total). Around 100-200. Makes this algorithm very fast.

Mean curvature minization (Zhu, Tai, Chan *IPI* 2013)

- How to obtain fast algorithm to minimize:

$$E(u) = \lambda \int_\Omega \left| \nabla \cdot \left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) \right| + \frac{1}{2} \int_\Omega (f - u)^2$$

- Can introduce new variables and consider:

$$\min_{u,p,q,n} \lambda \int_\Omega |q| + \frac{1}{2} \int_\Omega (f - u)^2$$

$$\text{subject to} \quad p = \nabla u, \quad n = \nabla u / \sqrt{1 + |\nabla u|^2}, \quad q = \nabla \cdot n$$

- It is very difficult to handle:

$$n = p / \sqrt{1 + |p|^2}$$

.

A new constraint

- We introduce the following new variables

$$p = \langle \nabla u, 1 \rangle,\ n = \langle \nabla u, 1 \rangle / \left| \langle \nabla u, 1 \rangle \right|,\ q = \nabla \cdot n$$

- The original minimization problem is reformulated as

$$\min_{u,p,q,n} \lambda \int_\Omega |q| + \frac{1}{2} \int_\Omega (f-u)^2$$

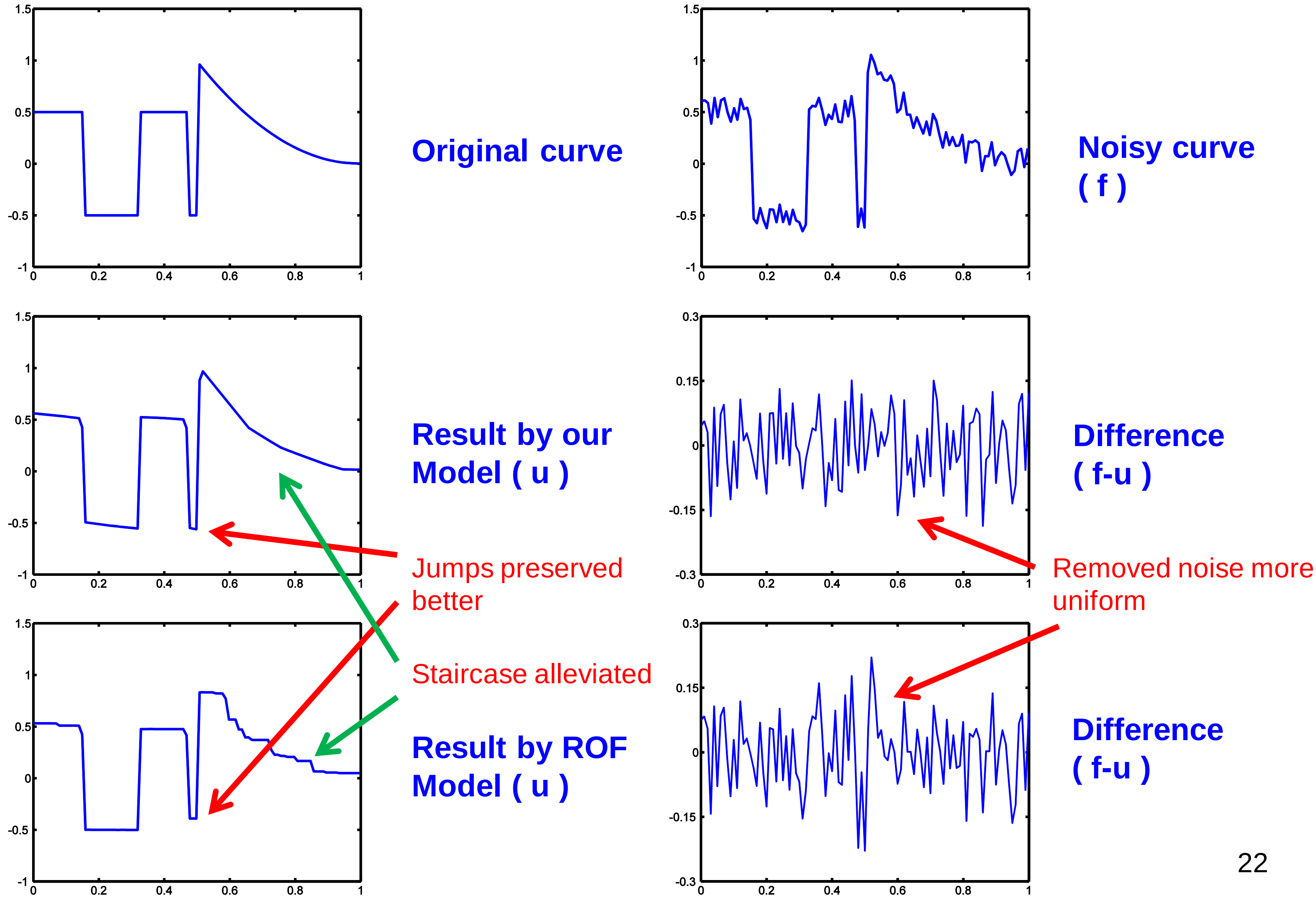
$$\text{subject to} \quad q = \nabla \cdot \langle n_1, n_2 \rangle,\ \ n = \langle n_1, n_2, n_3 \rangle = p/|p|,\ p = \langle \nabla u, 1 \rangle$$

- Same idea: the following two are equivalent:

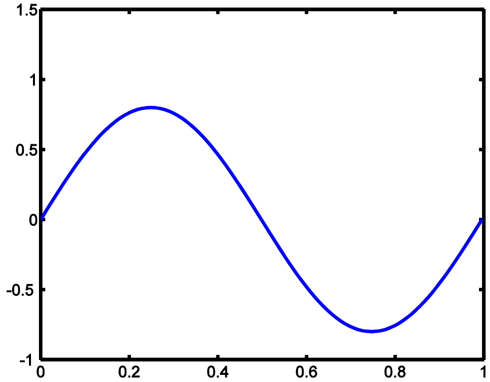
$$|n| \le 1,\ \ |p| = n \cdot p \longleftrightarrow n = p / |p|$$

- All ALM subproblems can be solved using FFT or thresholding

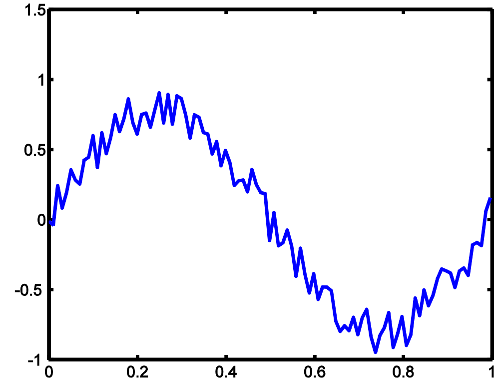
Experiments (1D)



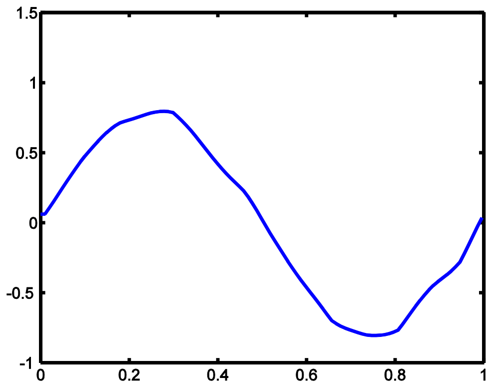
Experiments (1D)



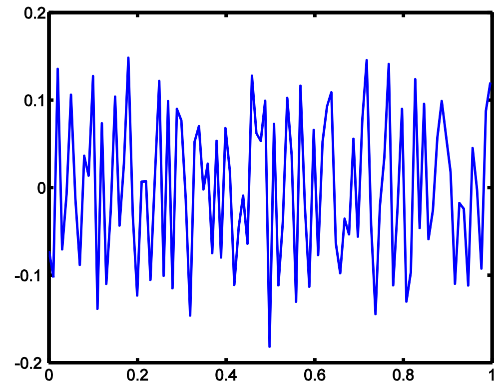
Original curve



Noisy curve (f)



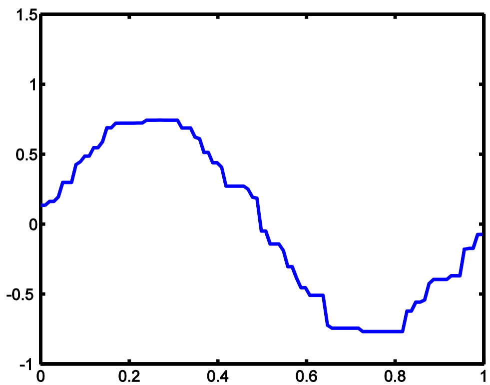
Result by our Model (u)



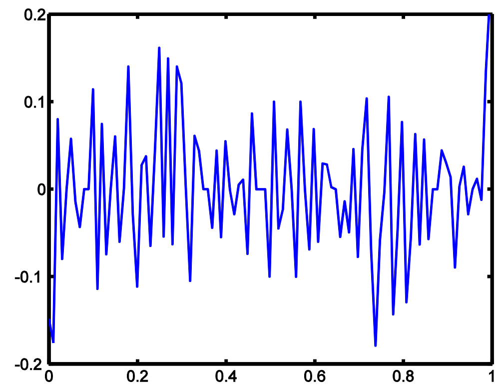
Difference (f-u)

Removed noise more uniform

Staircase alleviated

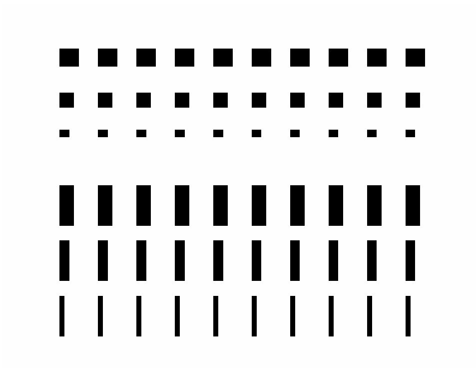


Result by ROF Model (u)

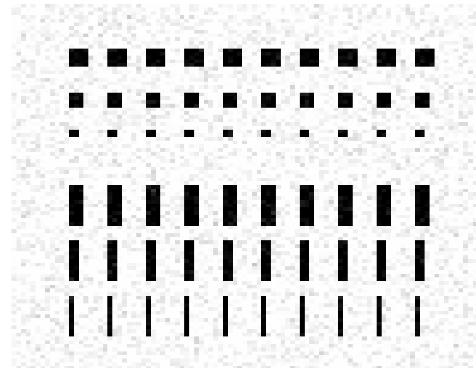


Difference (f-u)

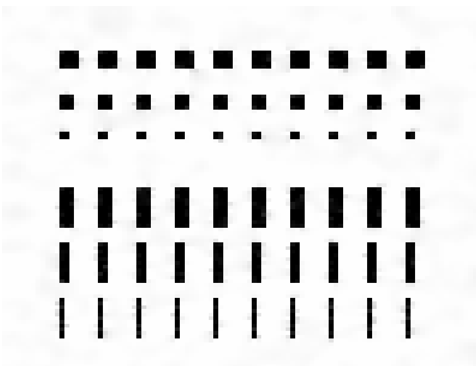
Experiments (2D)



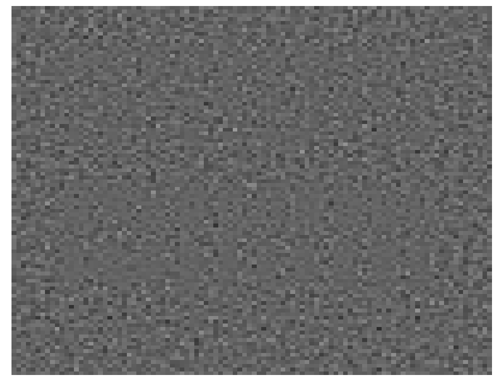
Original “Bars”



Noisy “Bars” (f)

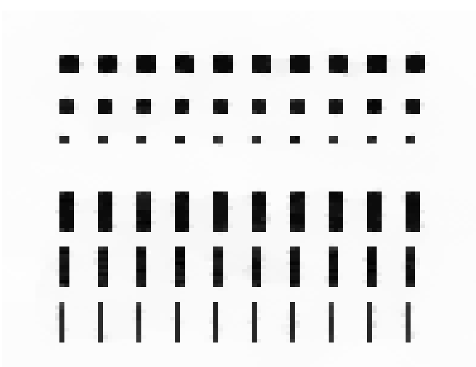


Result by our Model (u)

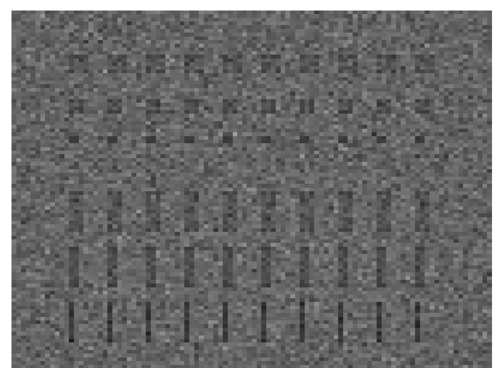


Difference (f-u)

Contrast preserved better

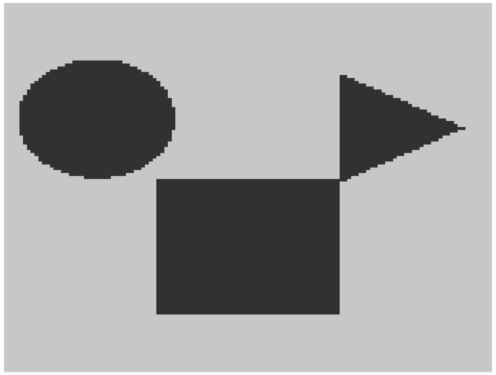


Result by ROF Model (u)

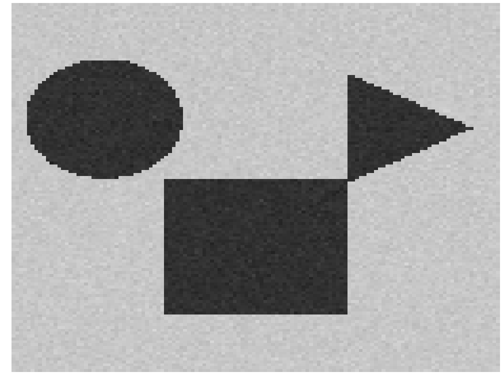


Difference (f-u)

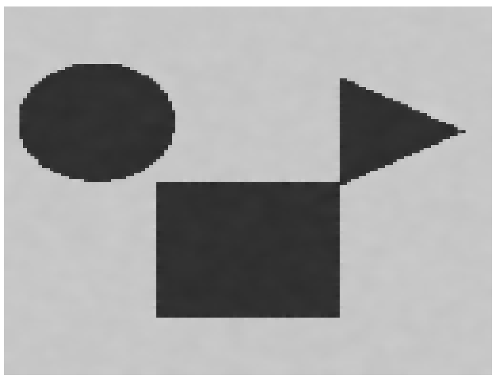
Experiments (2D)



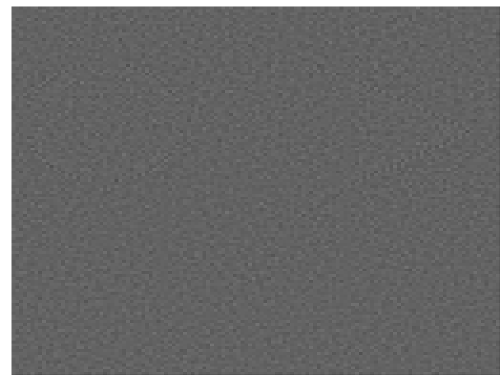
Original "Shapes"



Noisy "Shapes" (f)

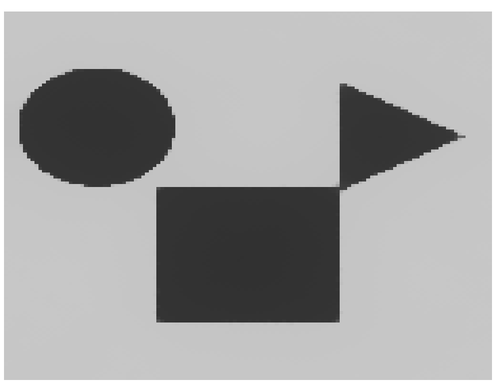


Result by our Model (u)

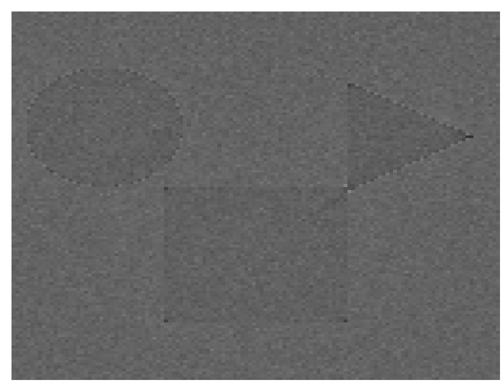


Difference (f-u)

As indicated in f-u, Contrast and corners Preserved better



Result by ROF Model (u)



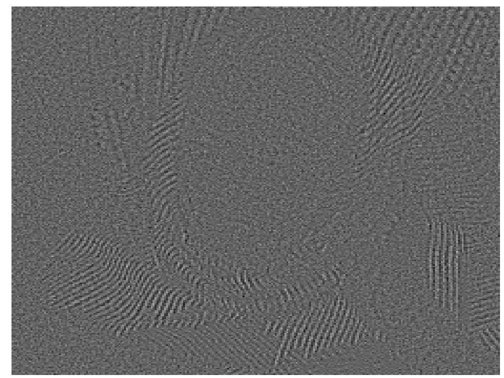
Difference (f-u)

Experiments (2D)

Original “Barbara”

Noisy “Barbara” (f)

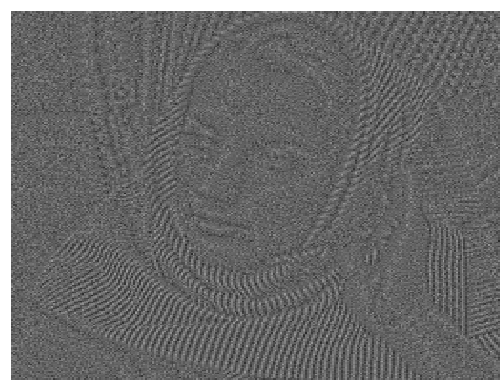
Result by our Model (u)



Difference (f-u)

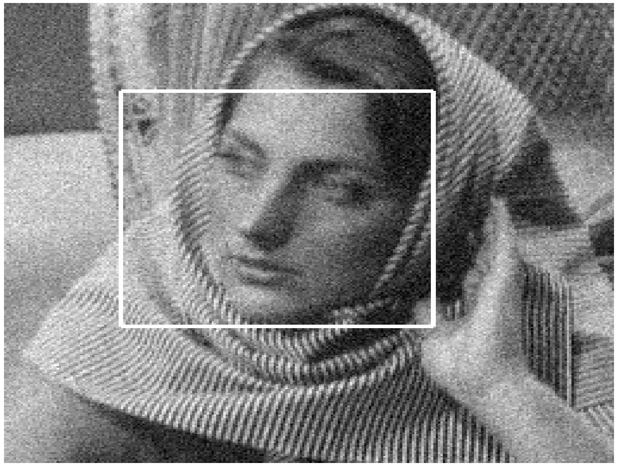
Large scale signal, such as face preserved better

Result by ROF Model (u)



Difference (f-u)

Experiments (2D)



Original "Barbara"

Local patch

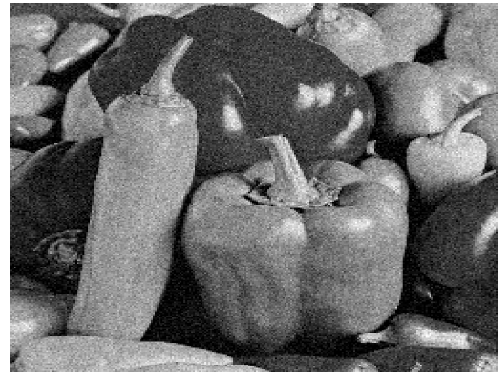
By our model

Staircase effect alleviated

By ROF model

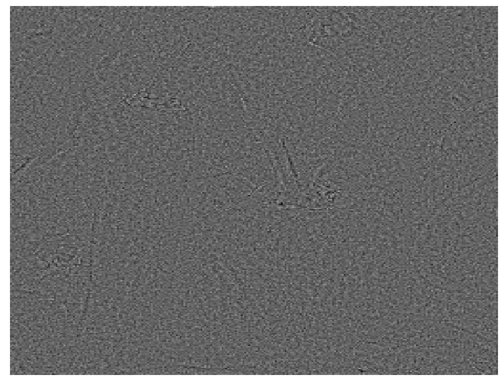
Experiments (2D)

Original “Peppers”



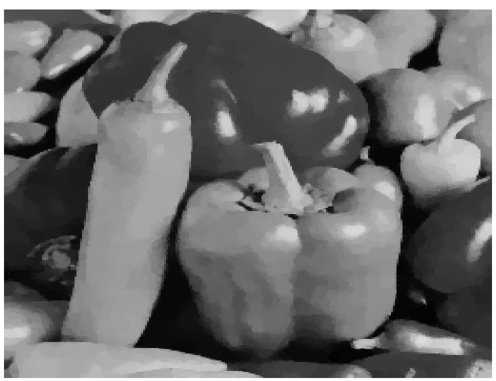
Noisy “Peppers” (f)

Result by our Model (u)

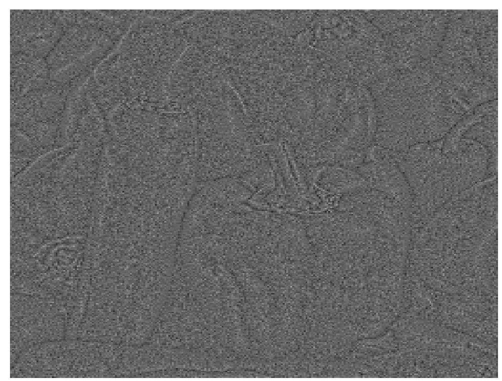


Difference (f-u)

Large scale signal, such as surface of pepper, preserved better

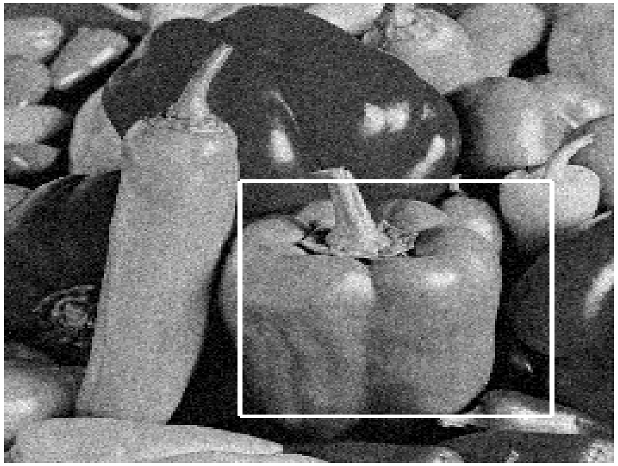


Result by ROF Model (u)

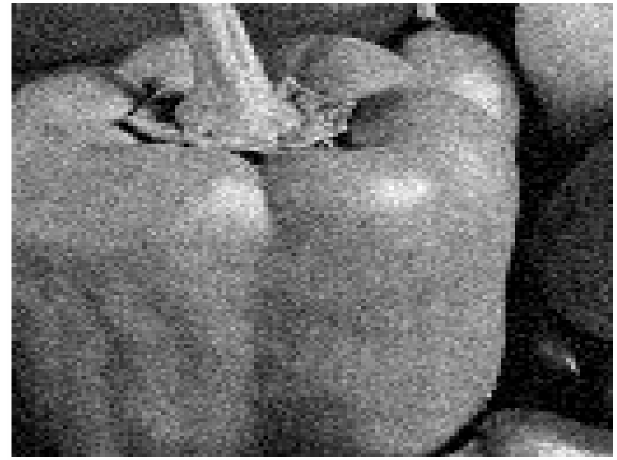


Difference (f-u)

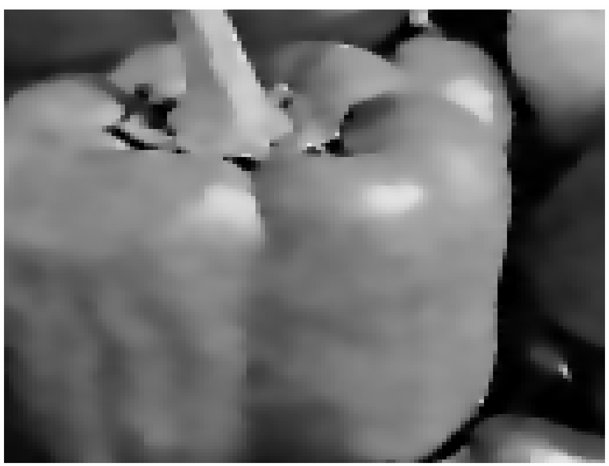
Experiments (2D)



Original “Peppers”



Local patch

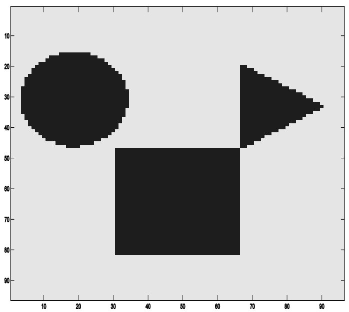


By our model

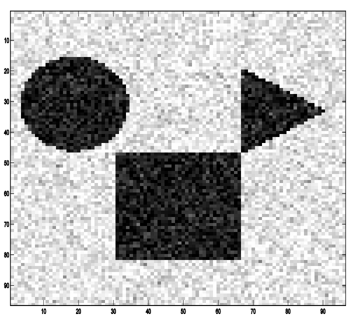
Staircase effect alleviated

By ROF model

Comparison with other high-order models

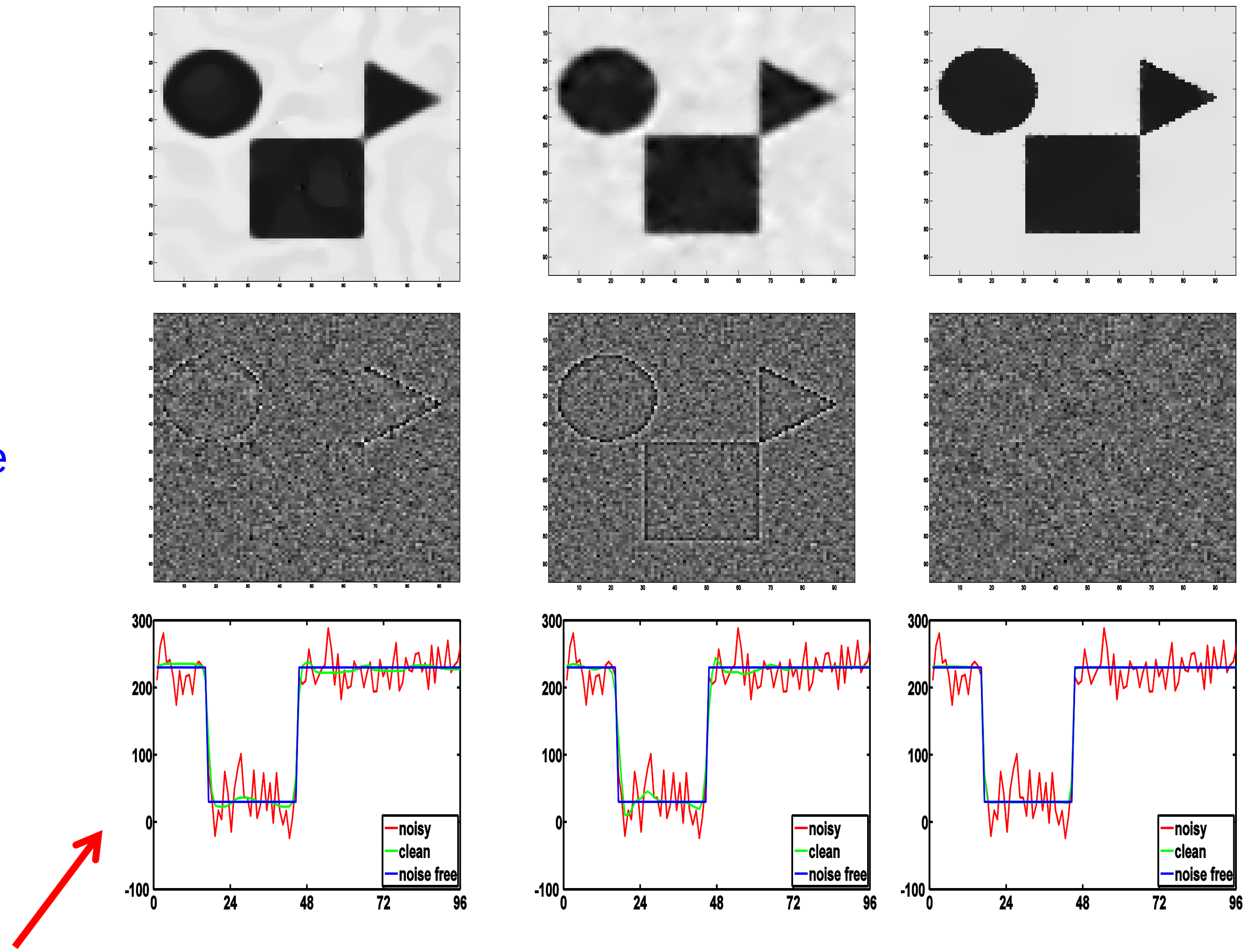


noise-free image



noisy image

A slice of the noise-free (B), noisy (R), and cleaned image (G)



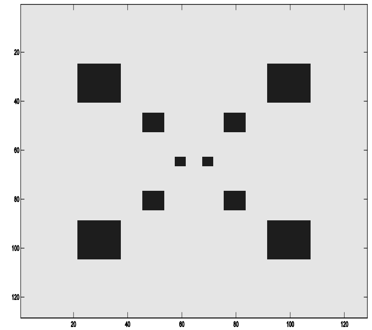
By Euler's elastica model

By the LLT model

By our model

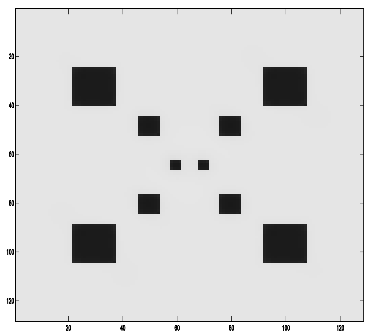
Contrast and corners preserved better than other models

Data-Driven Selection Property



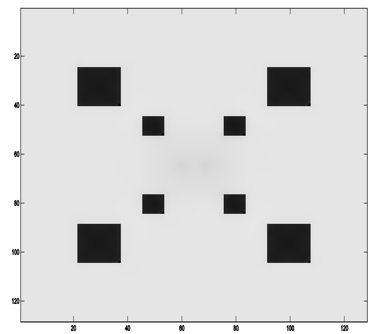
Original image f

When the regularization parameter increases, objects of small scales will be removed first and then the ones of relatively larger scales. But ultimately corners will be smeared.



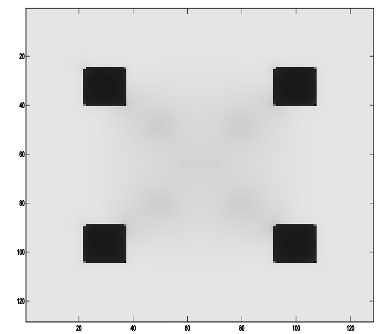
clean image u

$\lambda = 3.0 \times 10^3$



clean image u

$\lambda = 2.5 \times 10^4$



clean image u

$\lambda = 4.0 \times 10^4$

TV-L1 shares a similar property, but cannot preserve corners of objects

Summary and future work

- Summary of the proposed model
  - De-noise while keeping edges
  - preserve image contrast and corners, for small regularization
  - free of staircase effect
  - nonconvex
- Future work
  - Develop second order fast algorithm for the proposed model
  - apply the new regularizer for other image problems such as deblurring and inpainting

Happy Birthday, Russ!!!