Dynamical Systems

Any system that evolves (changes) in time according to some rules

Often EQUATIONS ARE UNKNOWN or partially known:

- Model discovery with machine learning

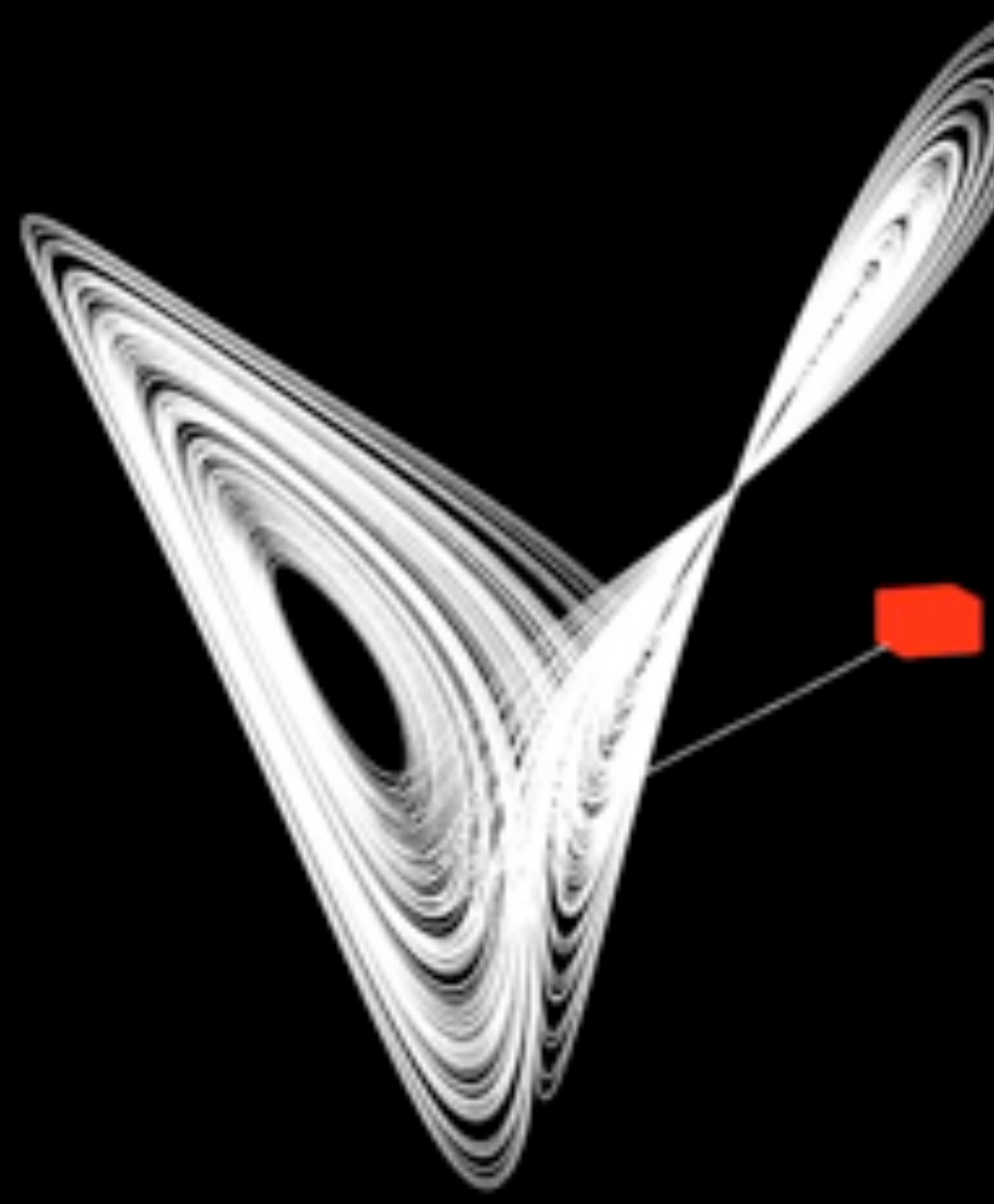
NONLINEAR dynamics are still poorly understood:

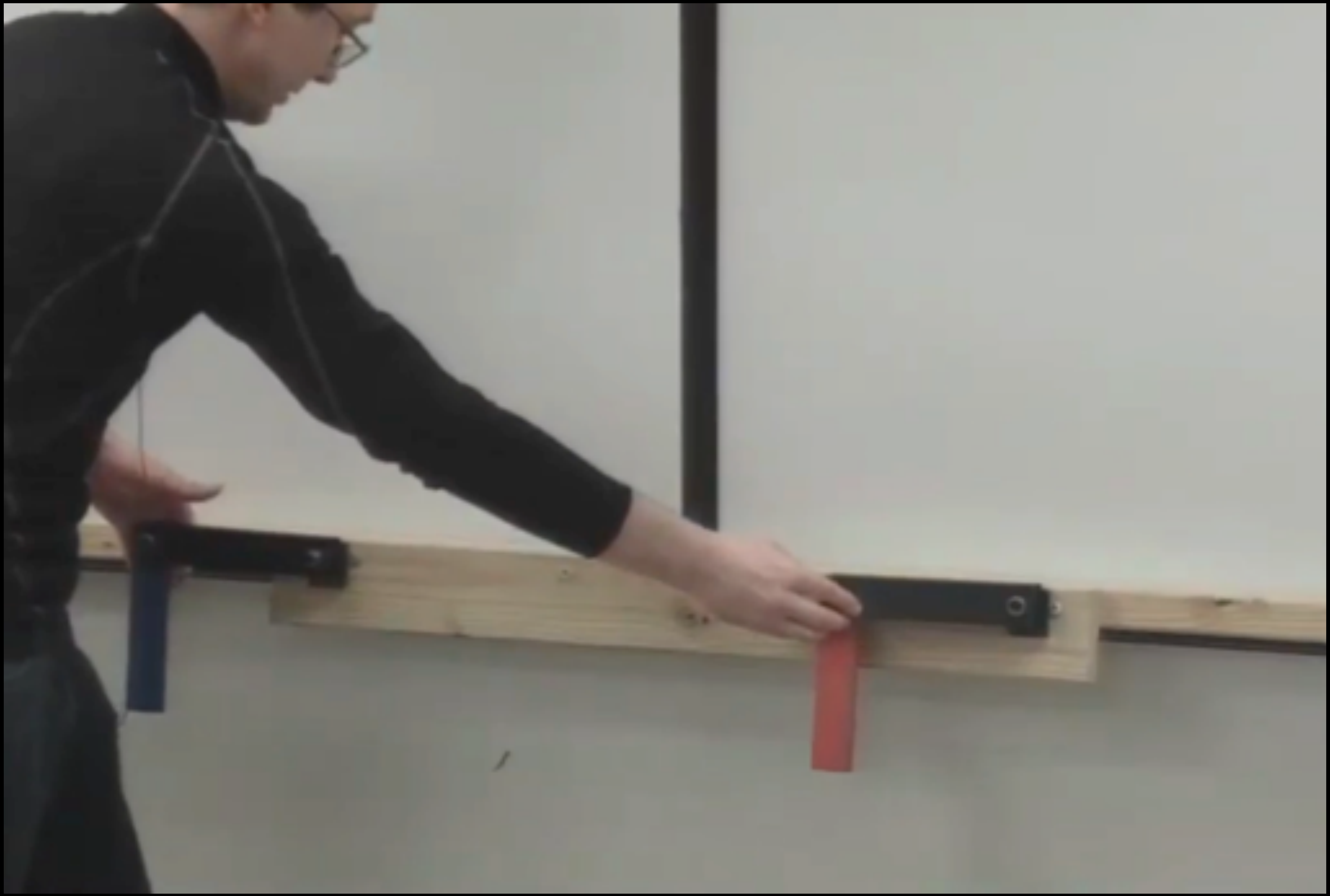
- Coordinate transformations to linearize dynamics

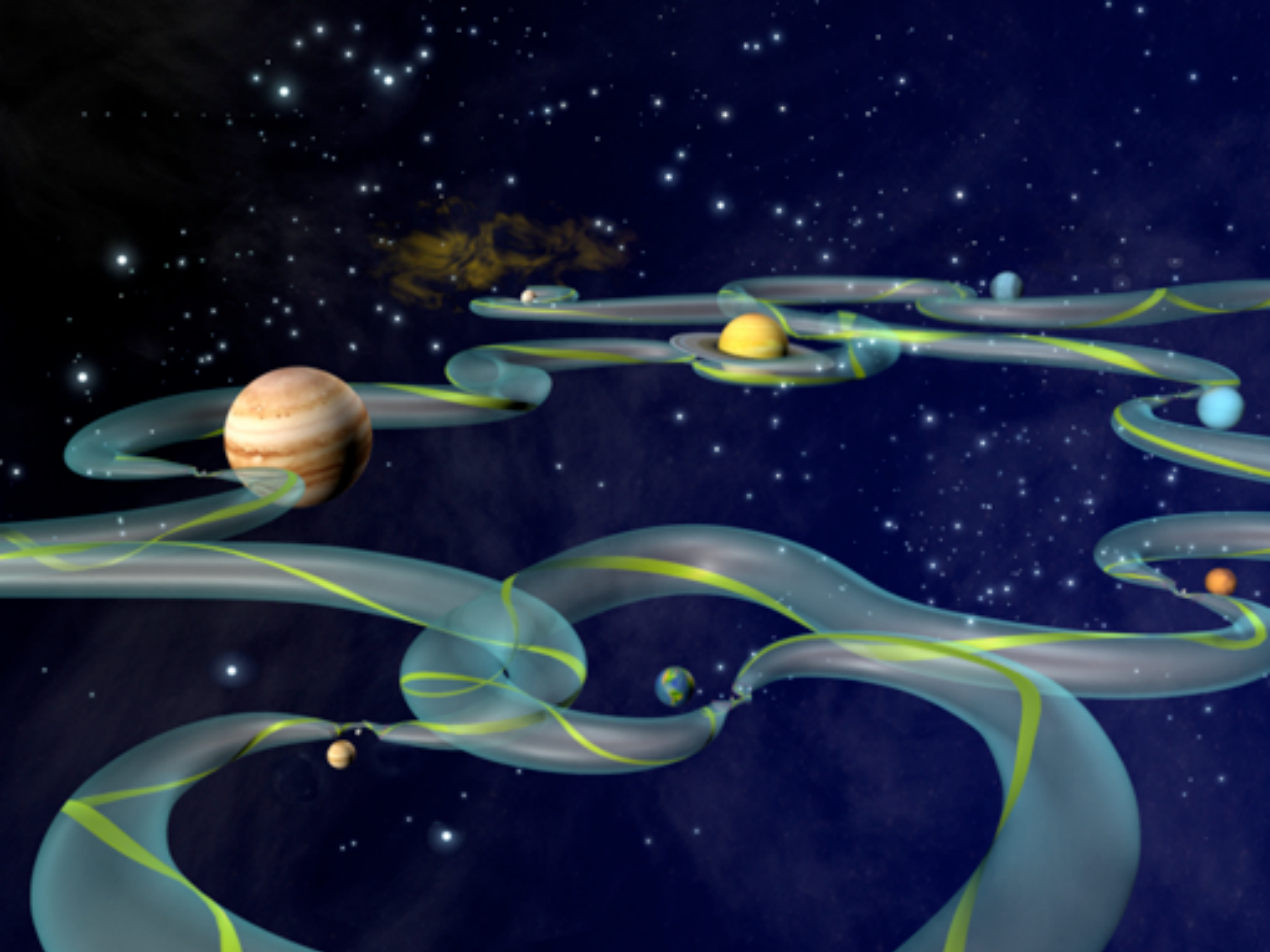
HIGH-DIMENSIONALITY often obscures dynamics:

- Patterns exist, facilitating reduction

Chaos

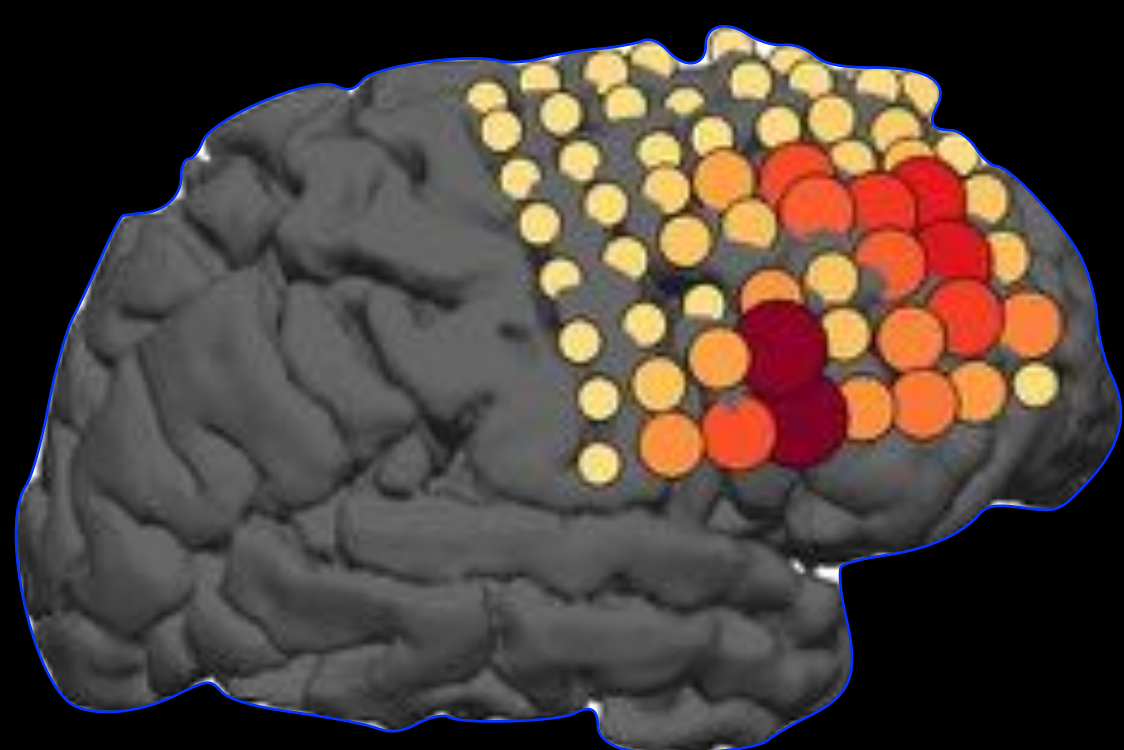


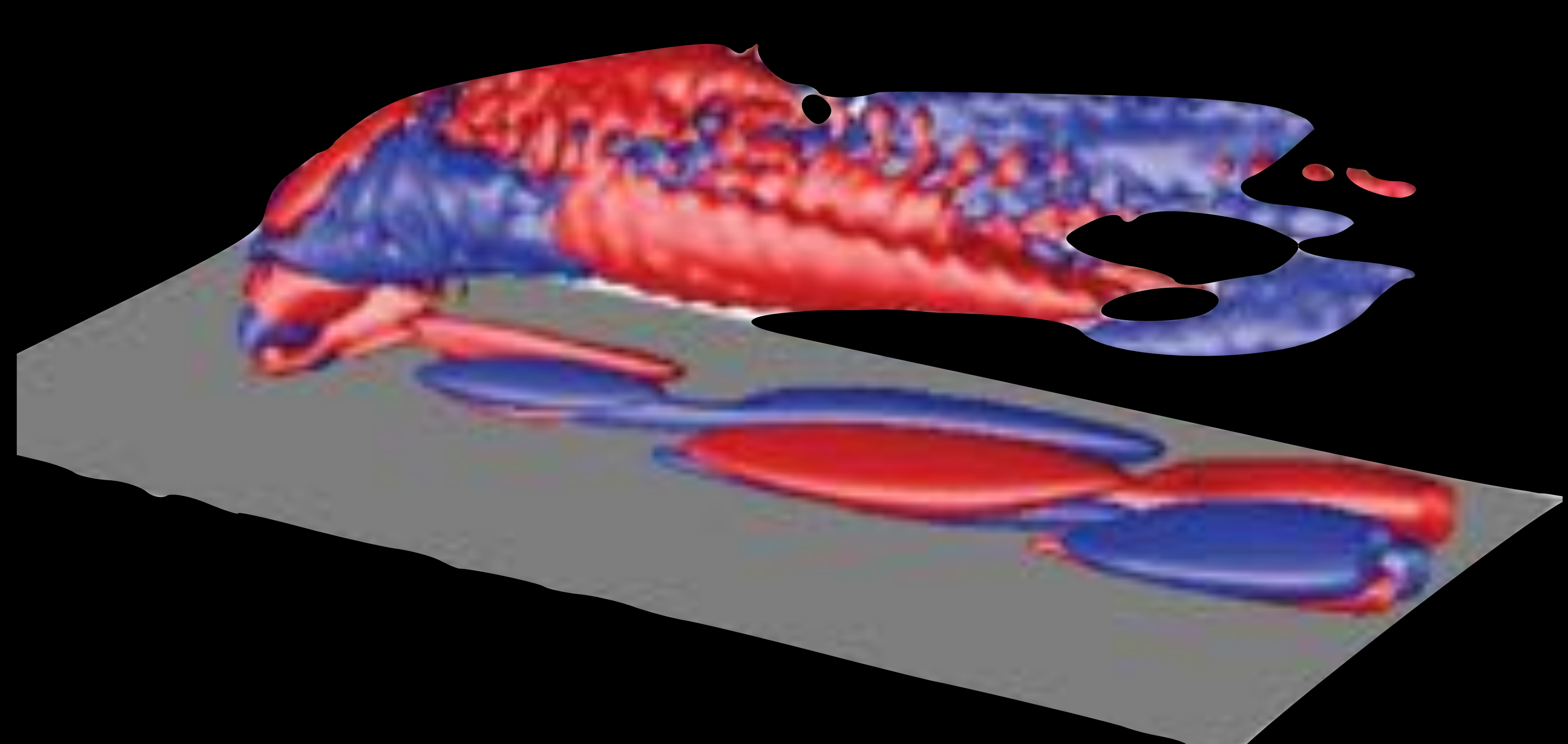


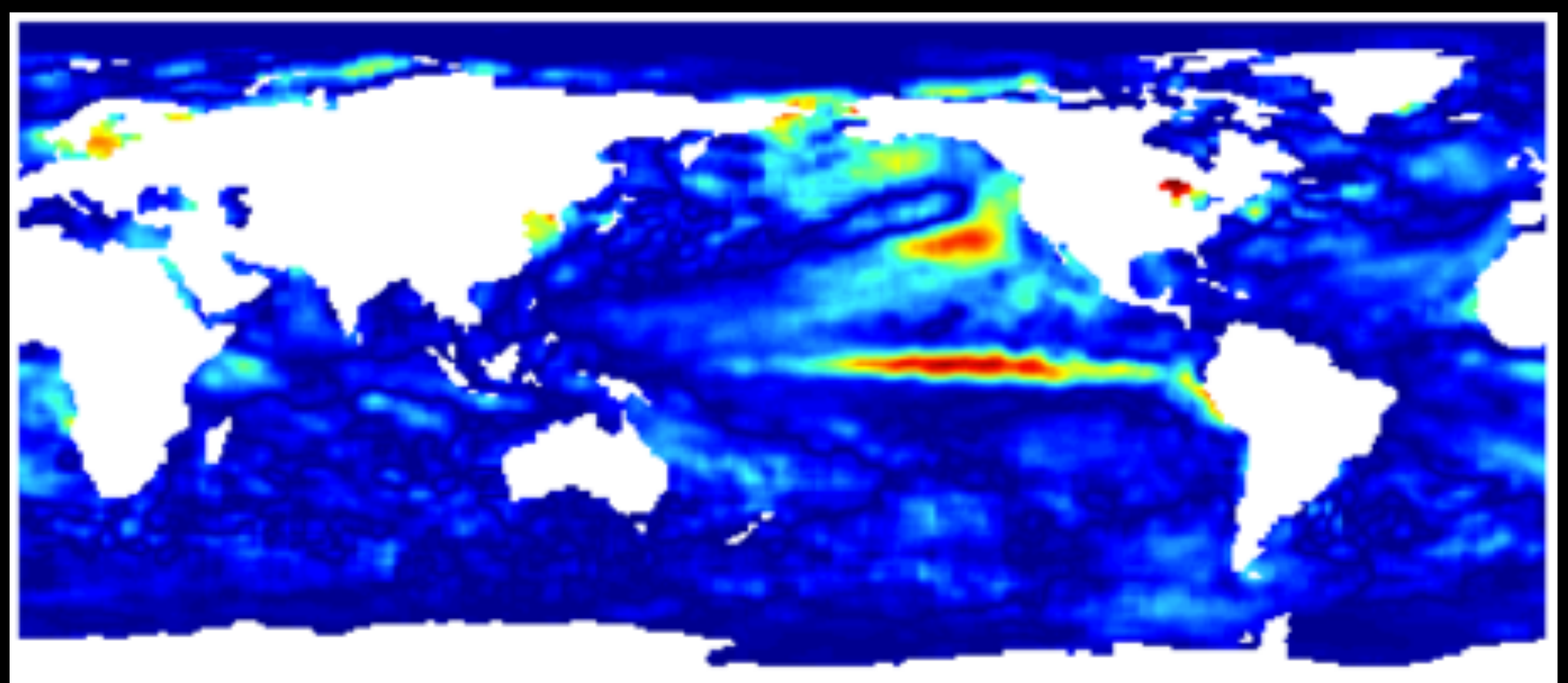


https://www.youtube.com/watch?v=5v5eBf2KwF8

Data-Driven Dynamical Systems I







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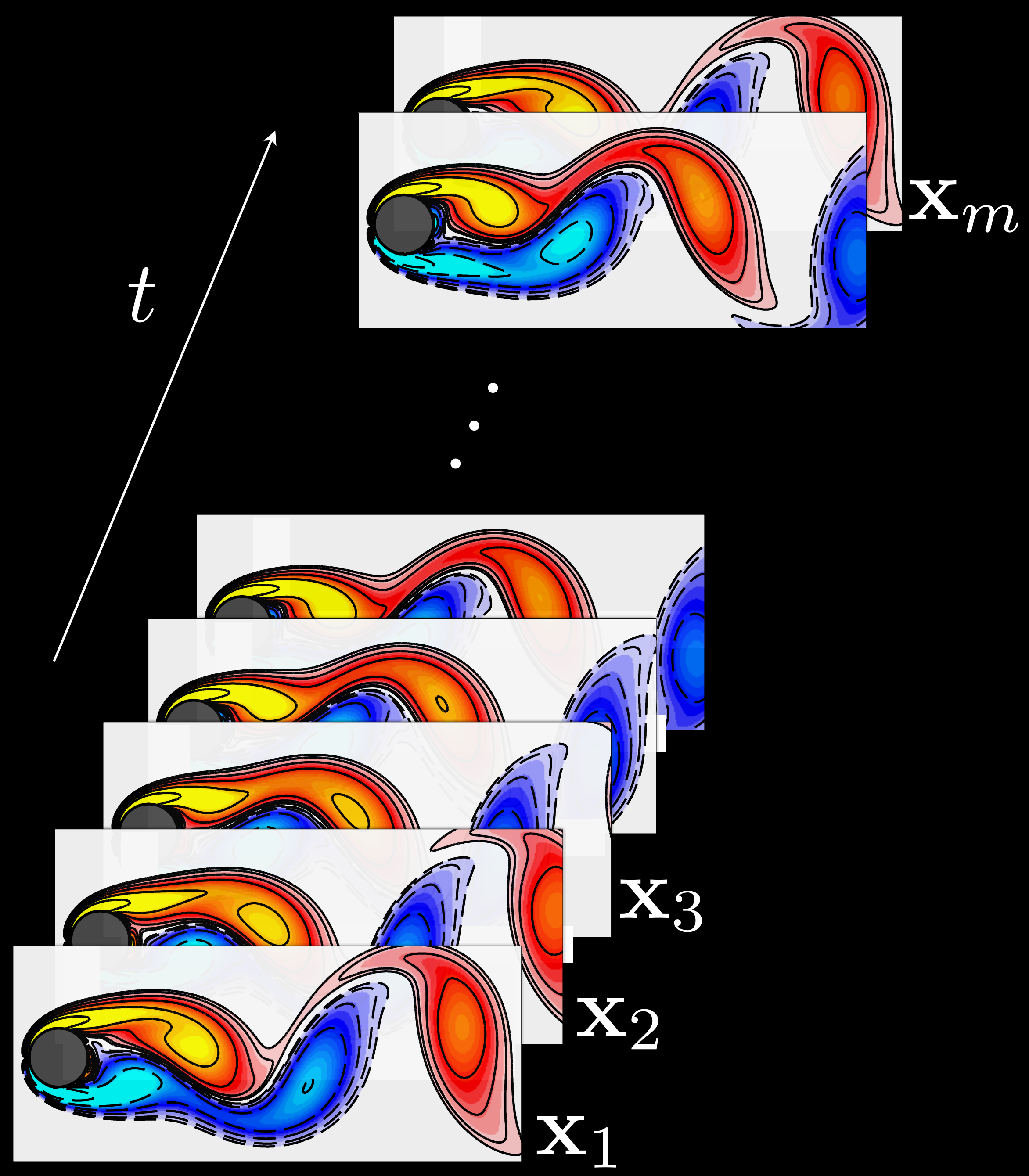
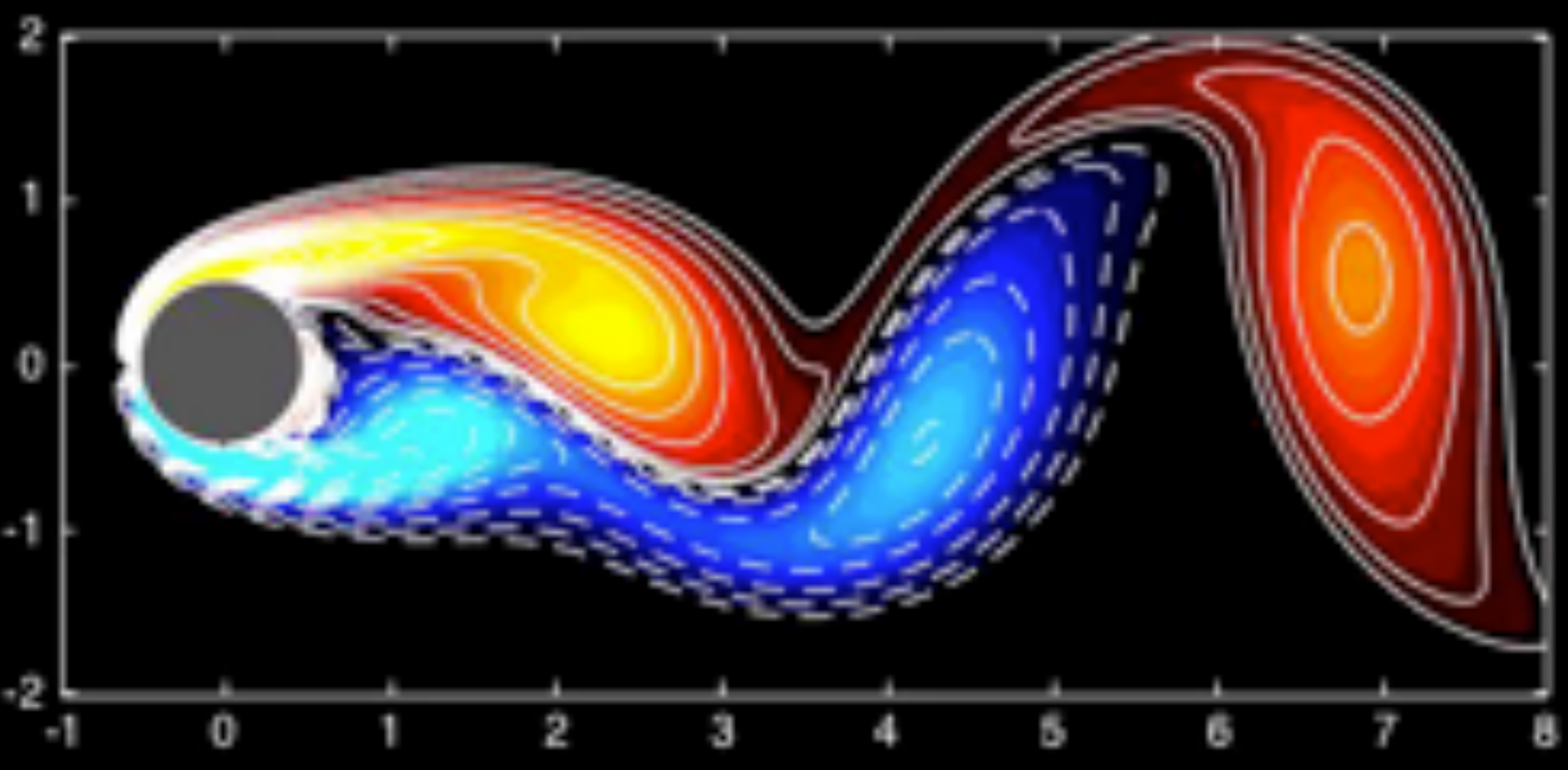
Dynamic Mode Decomposition (DMD)

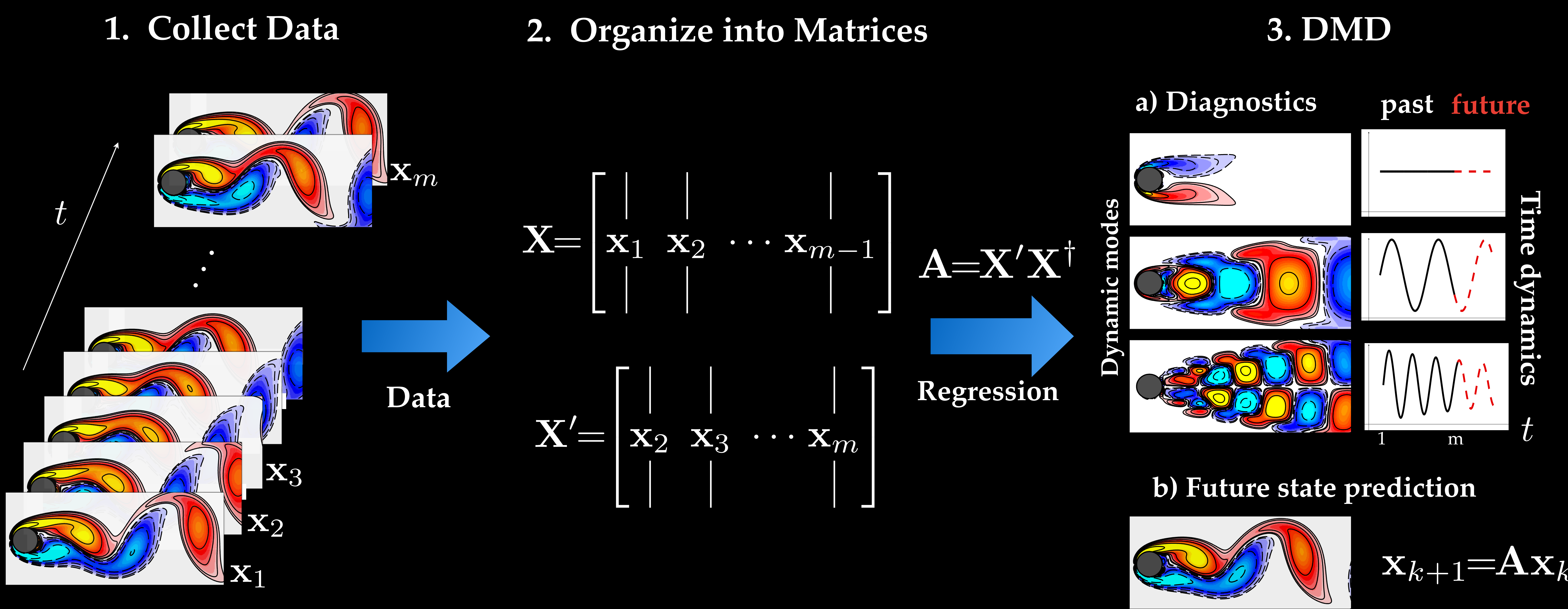
1. Collect Data

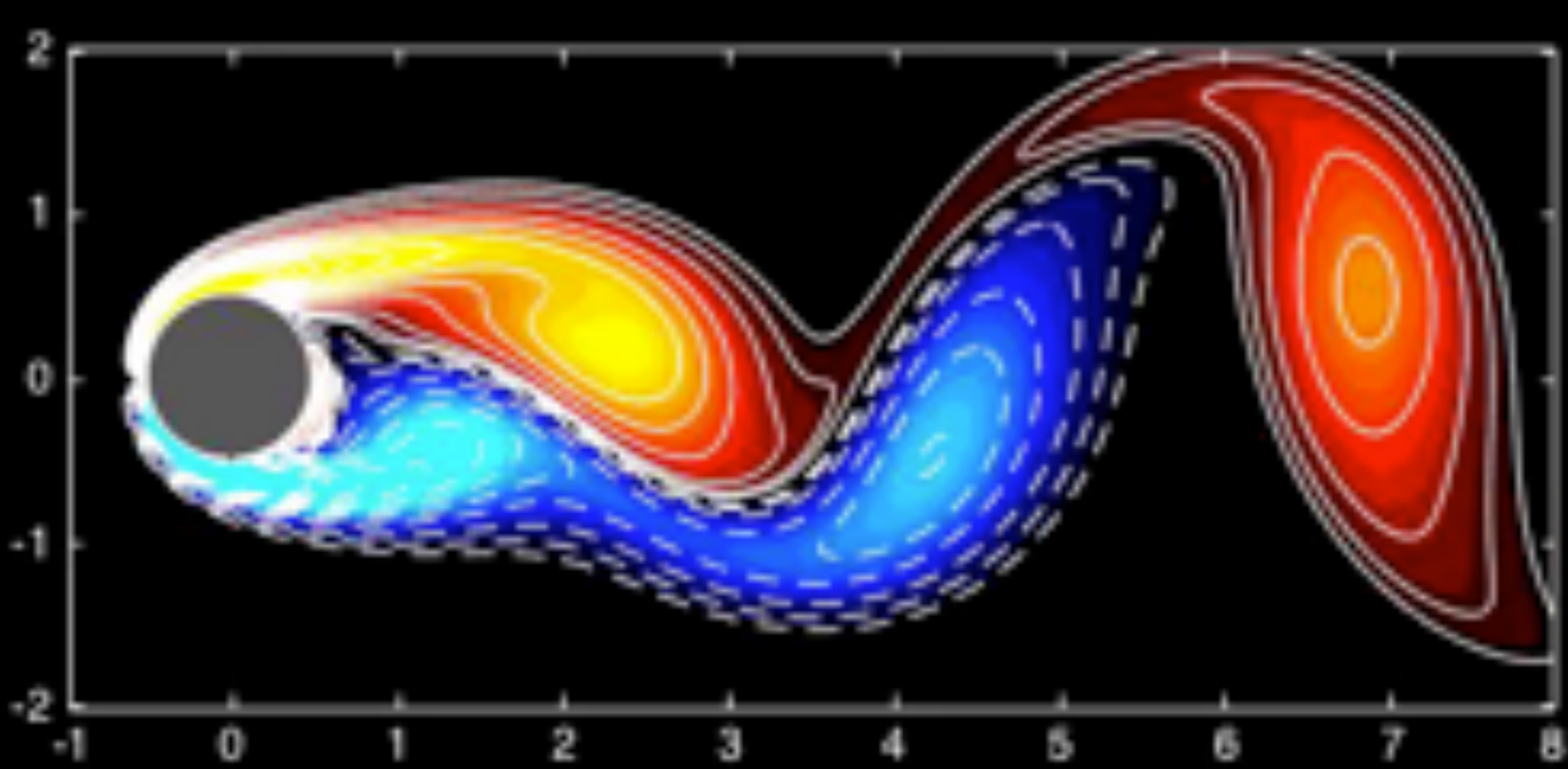




Schmid, *JFM* 2010.
Rowley, Mezic, Bagheri, Schlatter, Henningson, *JFM* 2009.
Tu, Rowley, Luchtenburg, Brunton, Kutz, *JCD* 2014.
Kutz, Brunton, Brunton, Proctor, *SIAM* 2016.

Dynamic Mode Decomposition (DMD)





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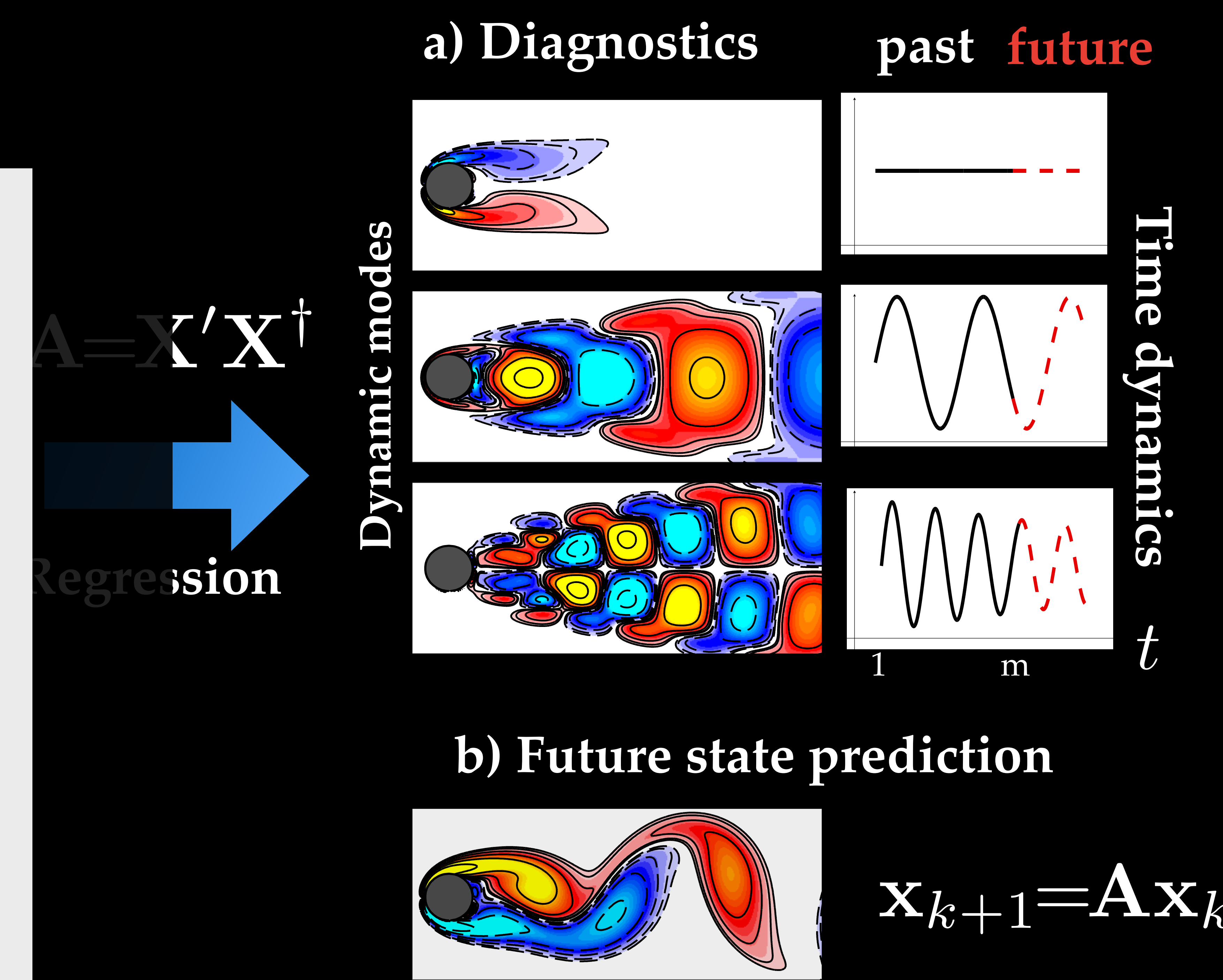
Kutz, Brunton, Brunton, Proctor, *SIAM* 2016.

Dynamic Mode Decomposition (DMD)

1. Collect Data
2. Organize into Matrices
3. DMD

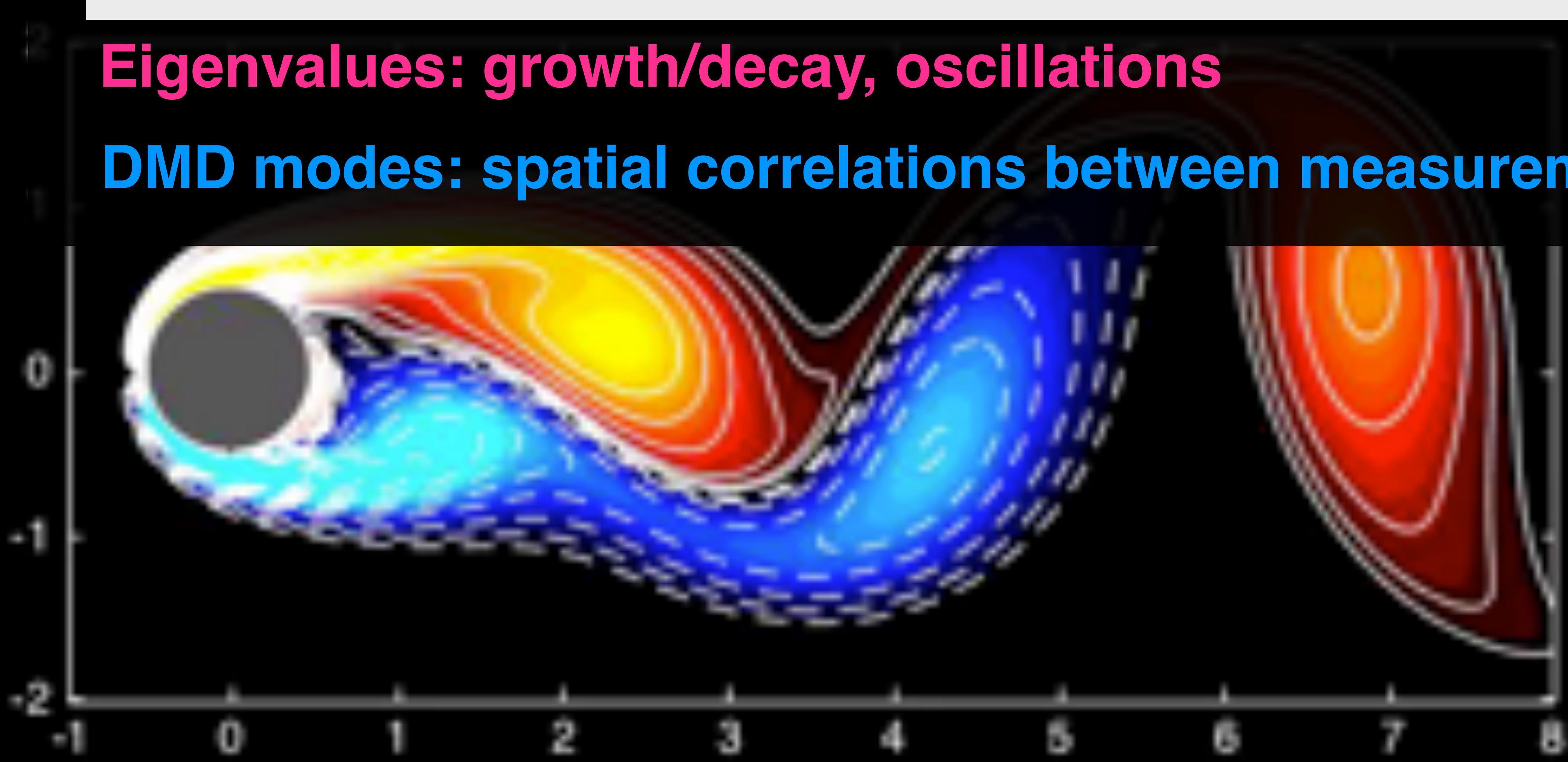
To compute DMD: $\hat{\mathbf{X}}(k\Delta t) = \boldsymbol{\Phi}\boldsymbol{\Lambda}^{t}\mathbf{b}_0$

svd	1. $\mathbf{X} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^* \qquad \mathbf{X}' = \mathbf{A}\mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^*$
	2.
*	3. $\mathbf{U}^*\mathbf{X}'\mathbf{V}\boldsymbol{\Sigma}^{-1} = \mathbf{U}^*\mathbf{A}\mathbf{U} = \tilde{\mathbf{A}}$
eig	4. $\tilde{\mathbf{A}}\mathbf{W} = \mathbf{W}\boldsymbol{\Lambda}$
*	$\boldsymbol{\Phi} = \mathbf{X}'\mathbf{V}\boldsymbol{\Sigma}^{-1}\mathbf{W}$



Eigenvalues: growth/decay, oscillations

DMD modes: spatial correlations between measurements



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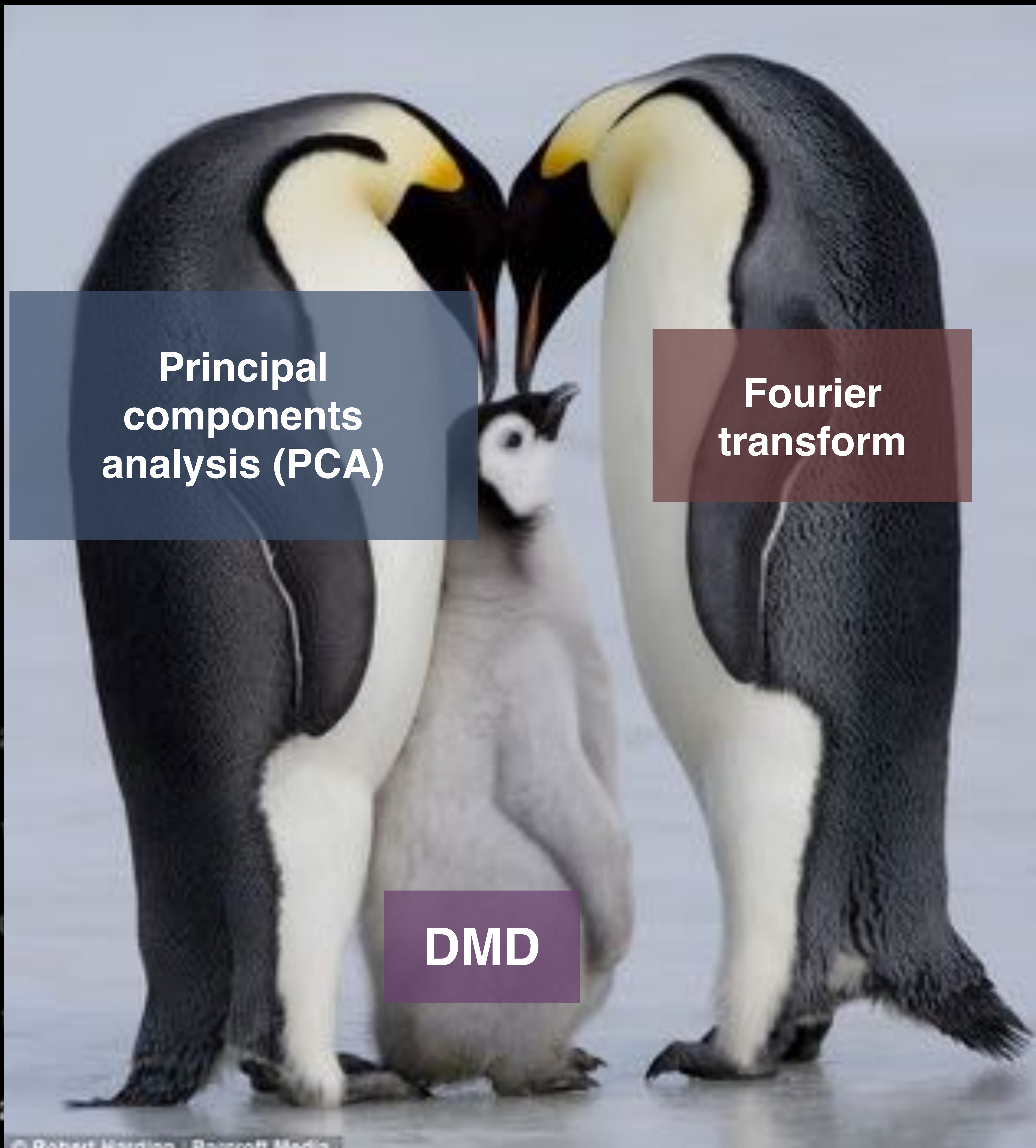
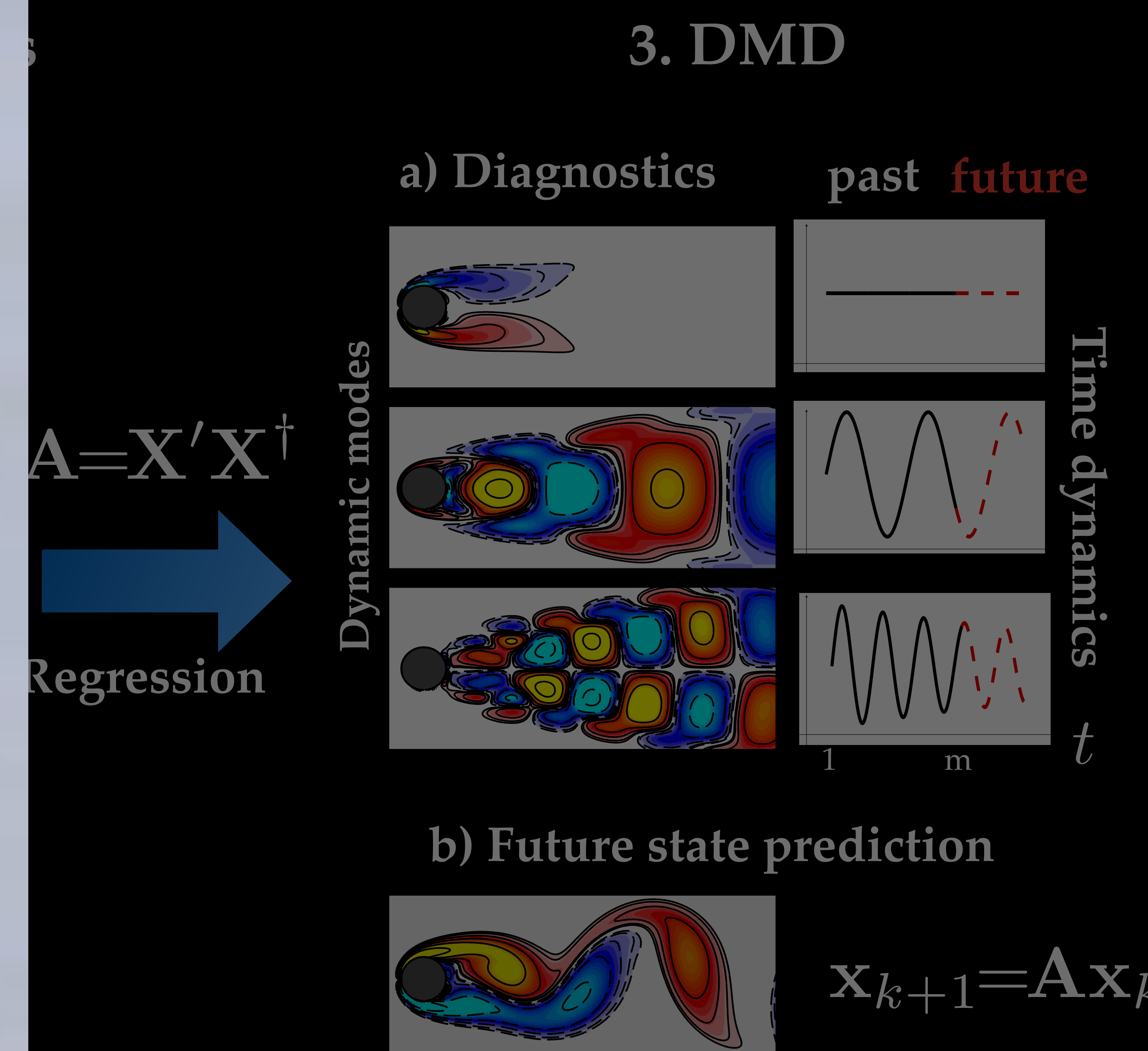
Dynamic Mode Decomposition (DMD)





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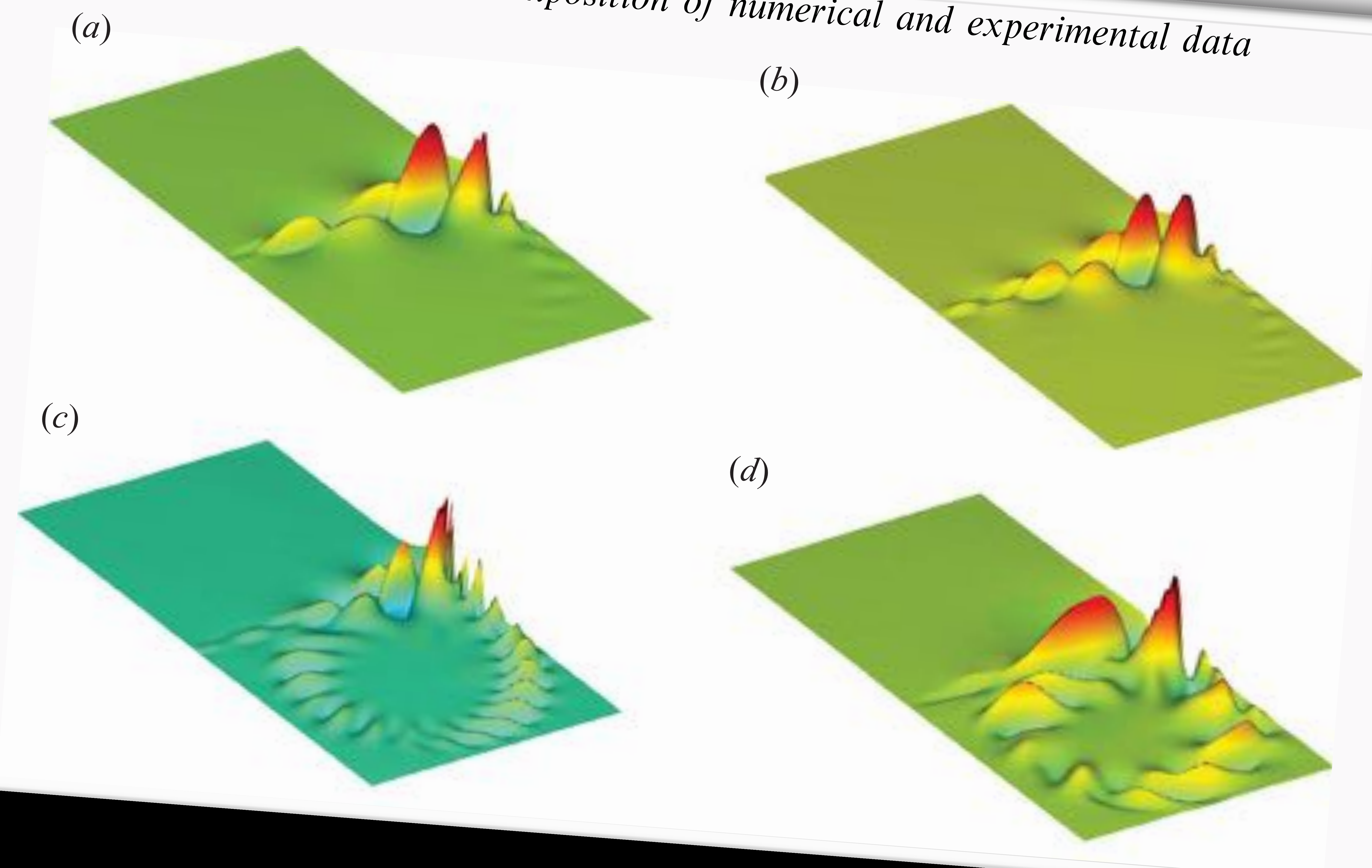
Kutz, Brunton, Brunton, Proctor, *SIAM* 2016.

J. Fluid Mech. (2010), *vol.* 656, *pp.* 5–28.
doi:10.1017/S0022112010001217

Dynamic mode decomposition of numerical and experimental data

PETER J. SCHMID†
91128 Palaiseau, France

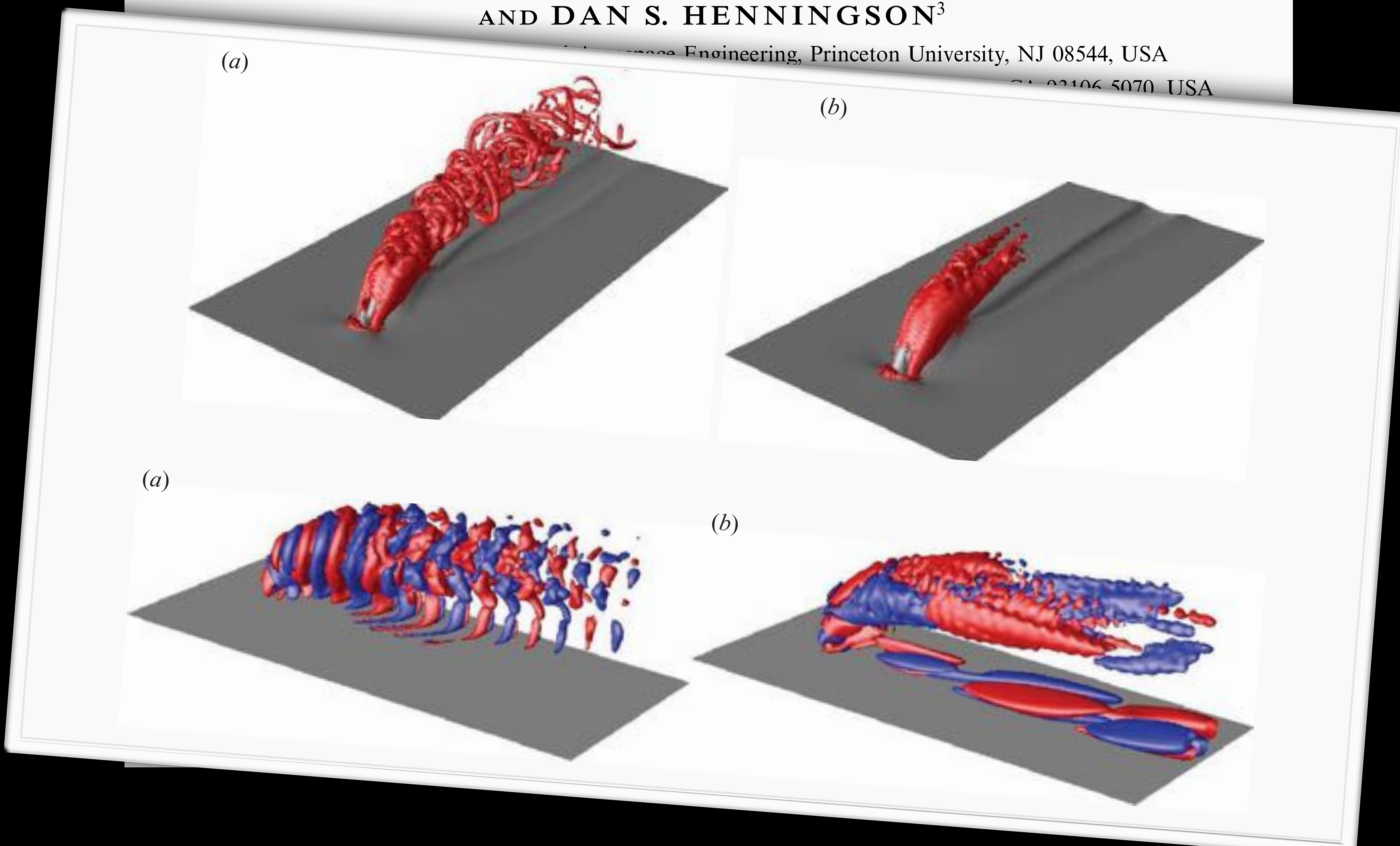


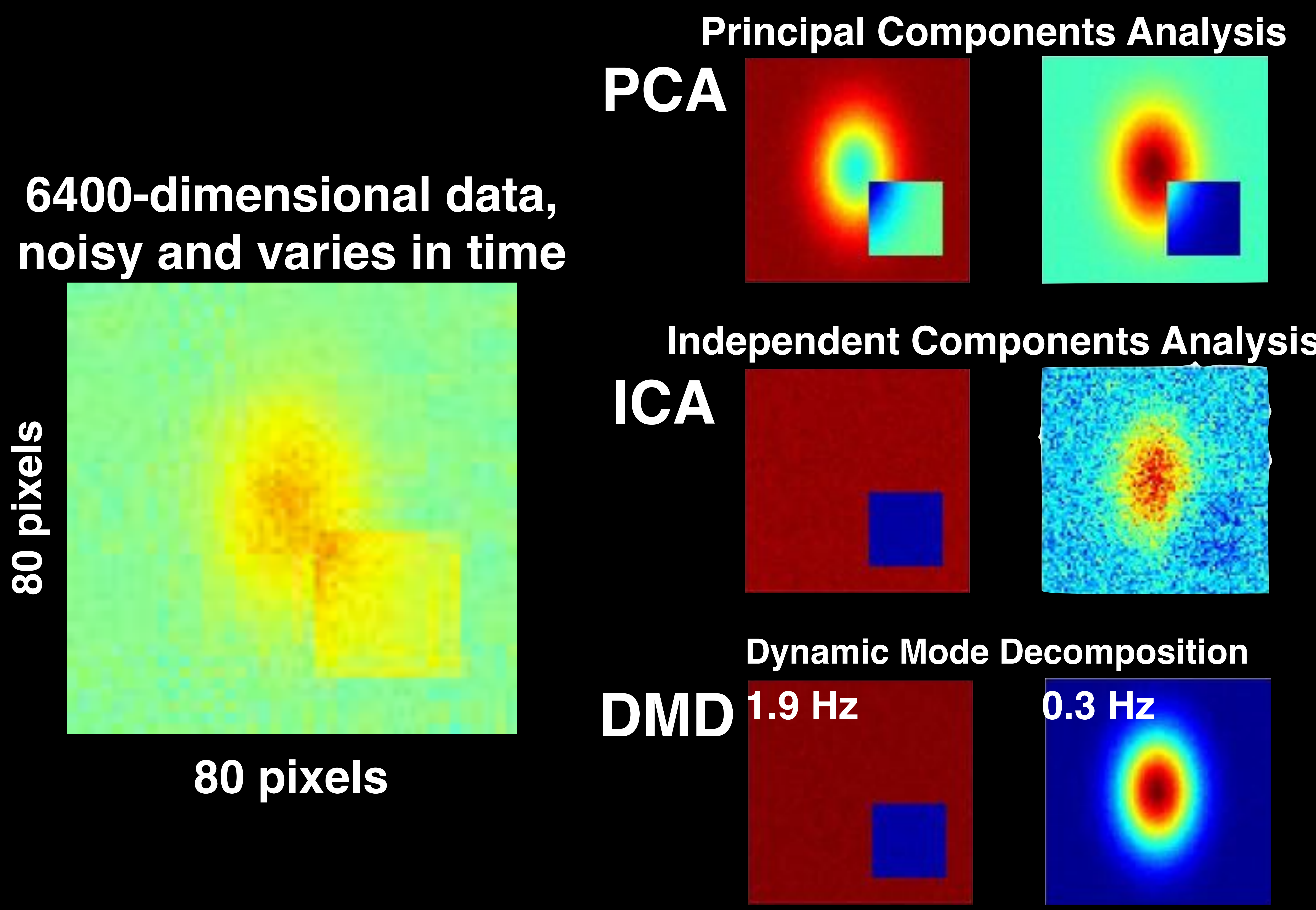
J. Fluid Mech., page 1 of 13 © Cambridge University Press 2009 1
doi:10.1017/S0022112009992059

Spectral analysis of nonlinear flows

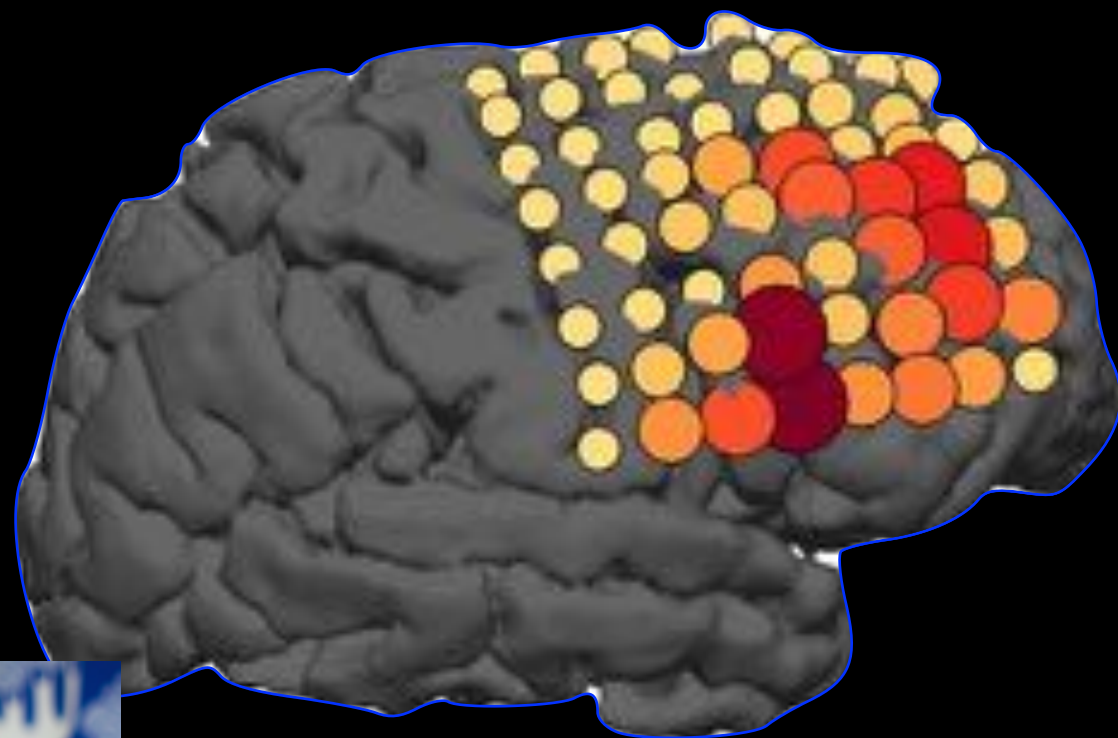
CLARENCE W. ROWLEY[1]†, IGOR MEZIĆ[2],
SHERVIN BAGHERI[3], PHILIPP SCHLATTER[3]
AND DAN S. HENNINGSON[3]

... Aerospace Engineering, Princeton University, NJ 08544, USA
... CA 93106-5070, USA




6400-dimensional data,
noisy and varies in time
80 pixels
80 pixels
Principal Components Analysis
PCA
Independent Components Analysis
ICA
Dynamic Mode Decomposition
DMD
1.9 Hz
0.3 Hz

DMD/Koopman: Highly applicable



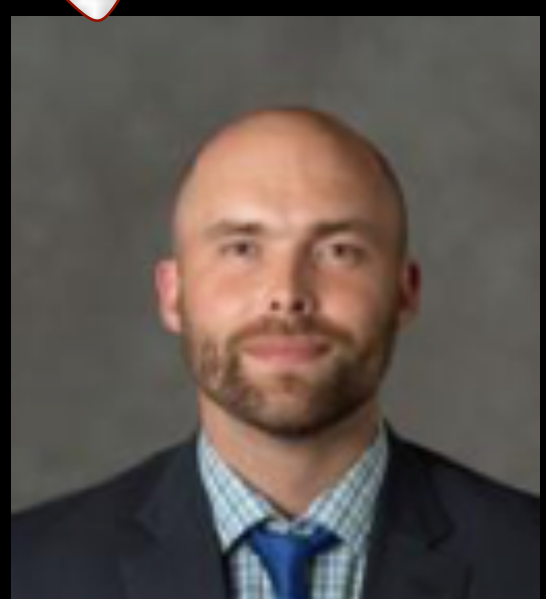
Bing Brunton

Neuroscience

Magnitude

Phase

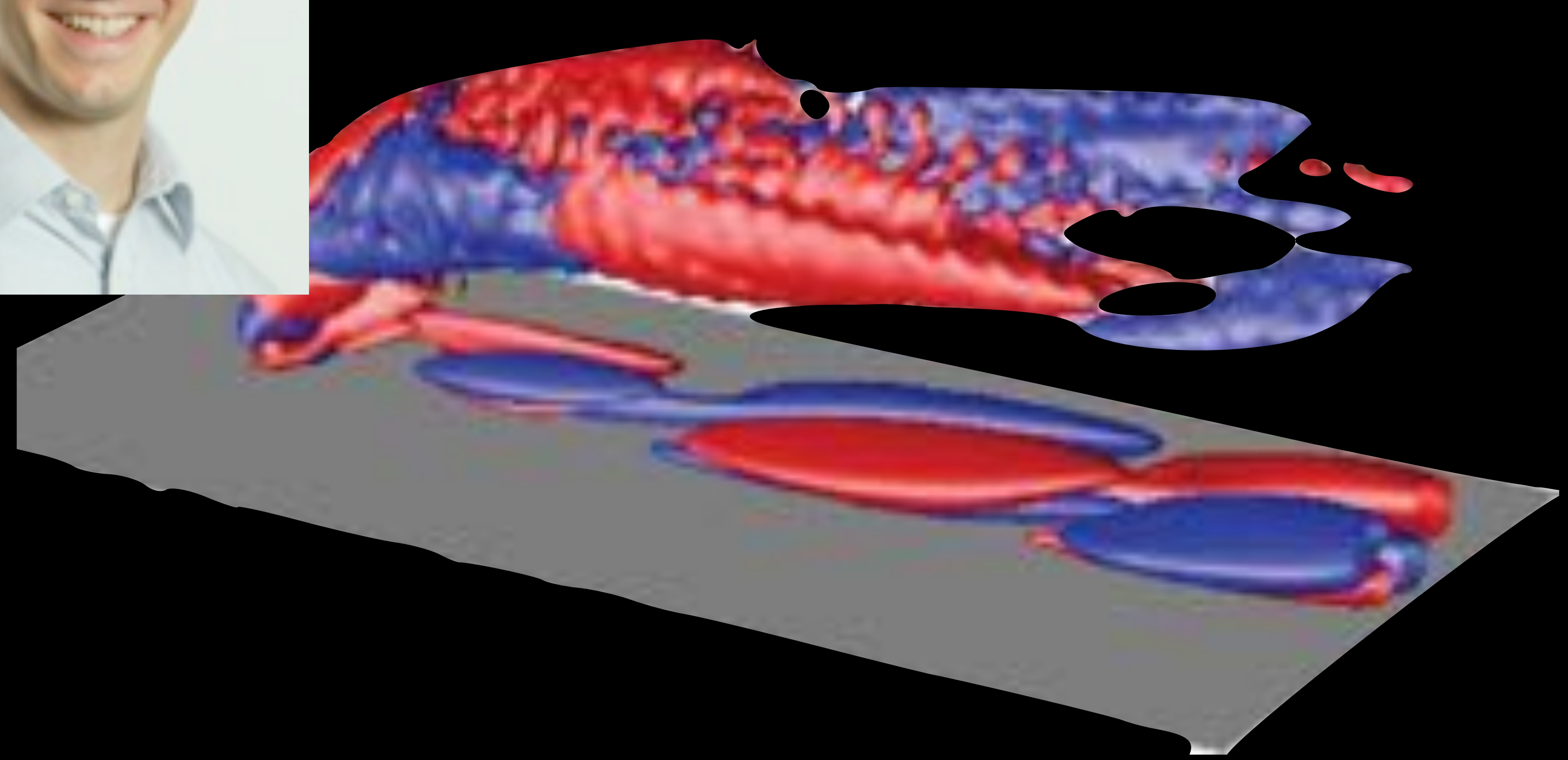
Disease and Epidemiology



Josh Proctor

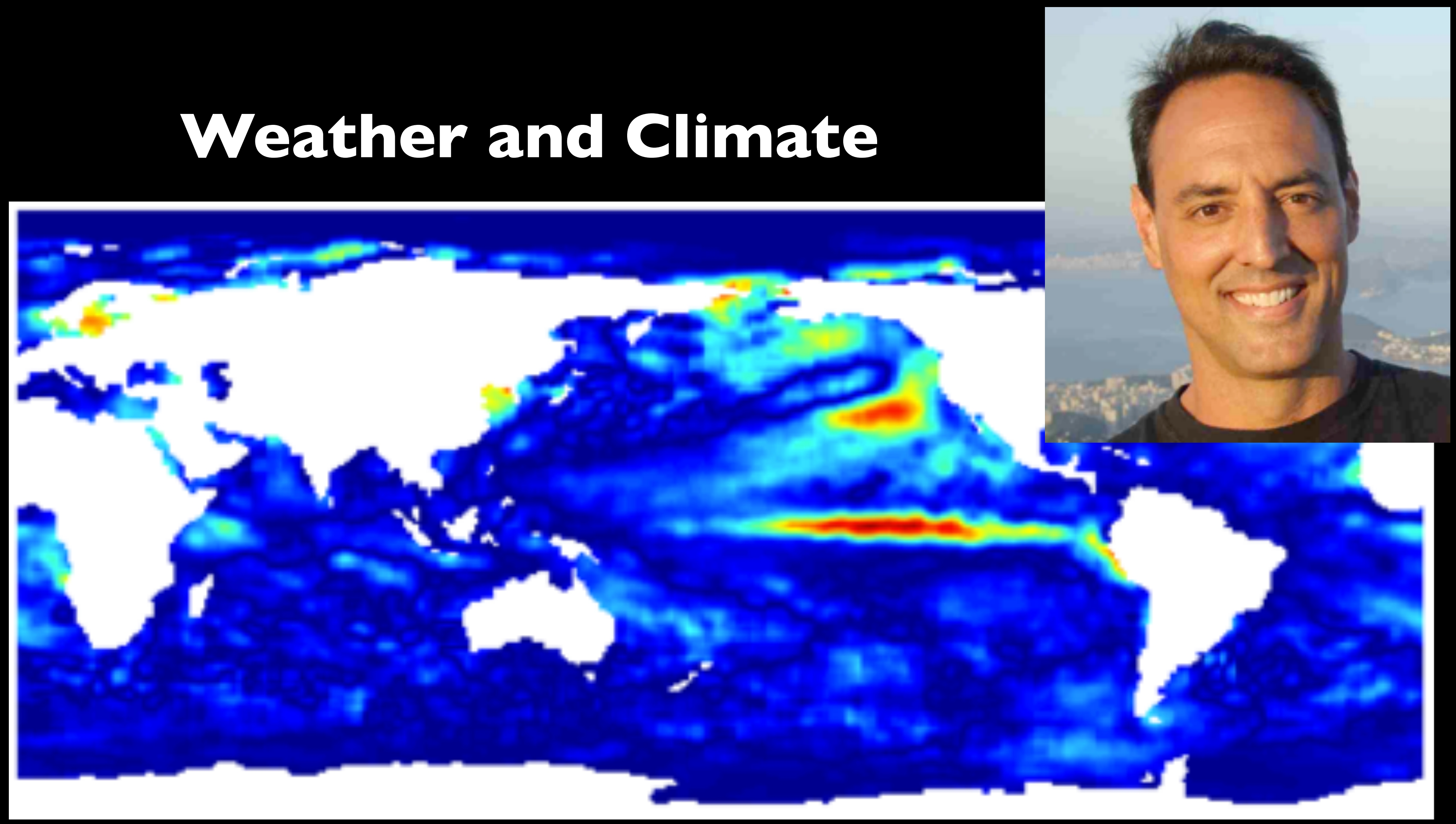
Clancy Rowley

Fluid Dynamics

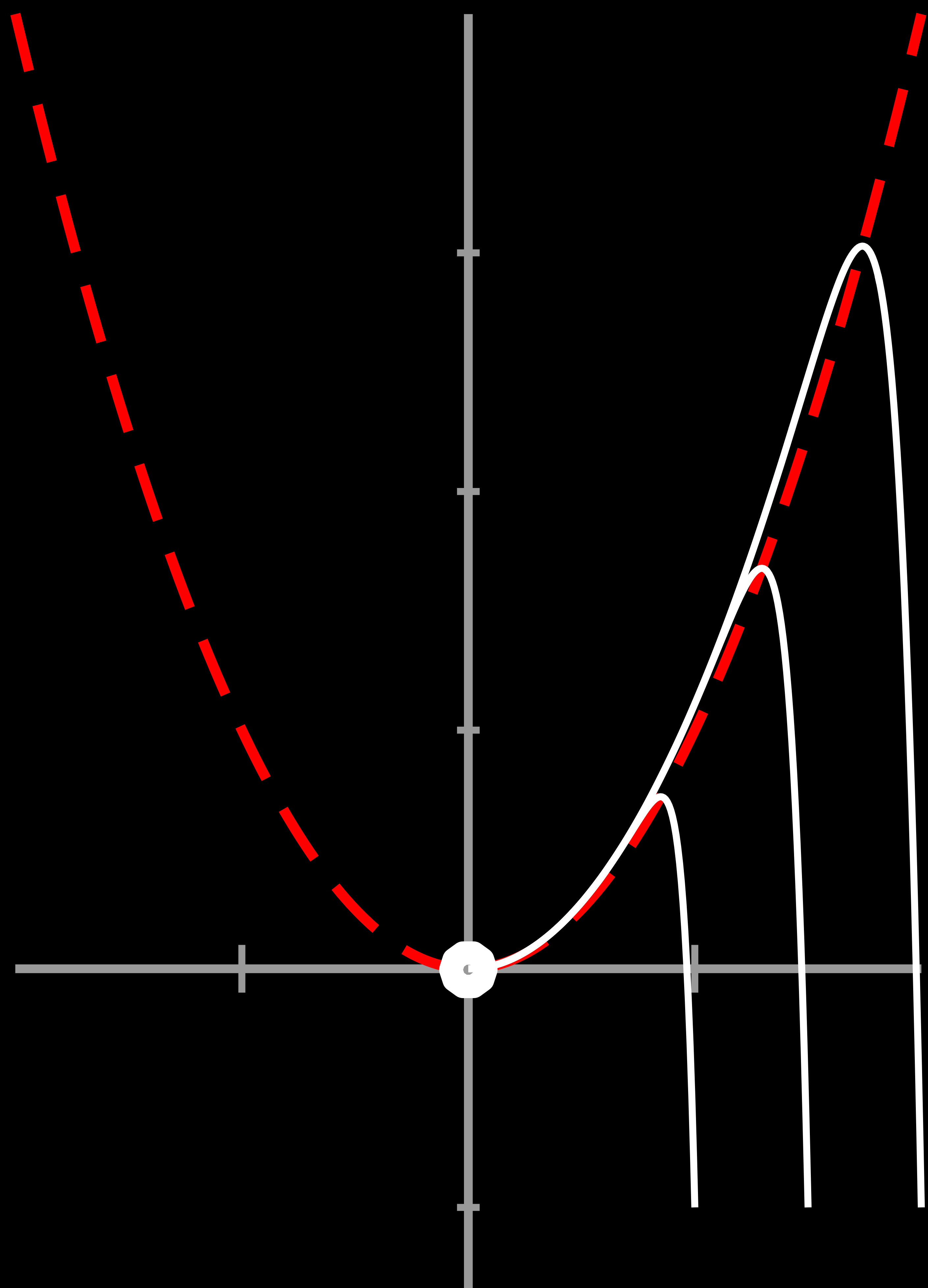


Nathan Kutz

Weather and Climate



Discovering Coordinate Systems



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Bernard Koopman

BERNARD KOOPMAN

HAMILTONIAN SYSTEMS AND TRANSFORMATIONS IN HILBERT SPACE

By B. O. Koopman

Department of Mathematics, Columbia University

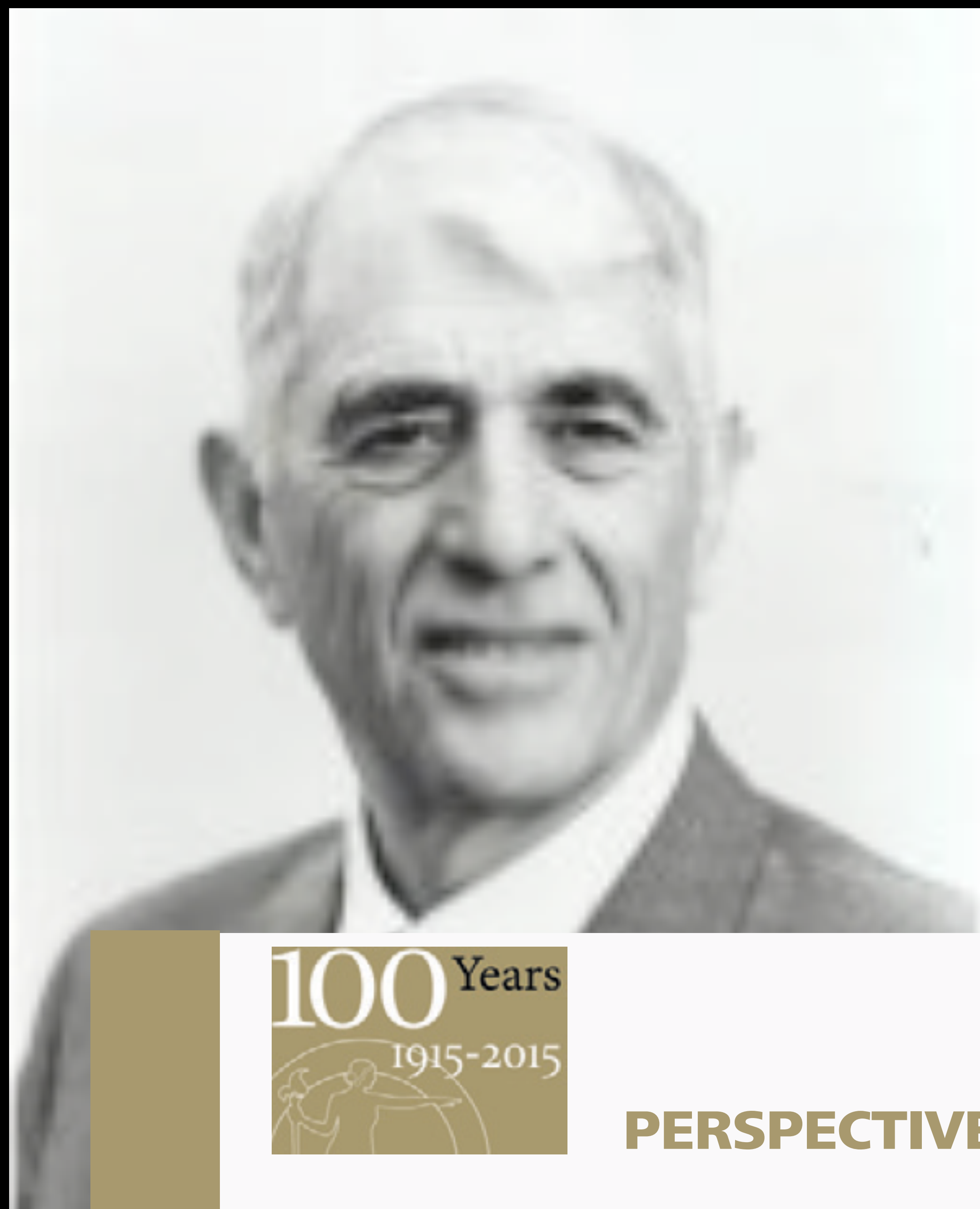
Communicated March 23, 1931

In recent years the theory of Hilbert space and its linear transformations has come into prominence.[1] It has been recognized to an increasing extent that many of the most important departments of mathematical physics can be subsumed under this theory. In classical physics, for example in those phenomena which are governed by linear conditions—linear differential or integral equations and the like, in those relating to harmonic analysis, and in many phenomena due to the operation of the laws of chance, the essential rôle is played by certain linear transformations in Hilbert space. And the importance of the theory in quantum mechanics is known to all. It is the object of this note to outline certain investigations of our own in which the domain of this theory has been extended in such a way as to include classical Hamiltonian mechanics, or, more generally, systems defining a steady n-dimensional flow of a fluid of positive density.

Bernard Koopman

John von Neumann

George Birkhoff



B

100 Years 1915-2015

PERSPECTIVE

CrossMark click for updates

Ergodic theorem, ergodic theory, and statistical mechanics

Calvin C. Moore[1]

Department of Mathematics, University of California, Berkeley, CA 94720

Edited by Kenneth A. Ribet, University of California, Berkeley, CA, and approved January 9, 2015 (received for review November 13, 2014)

This perspective highlights the mean ergodic theorem established by John von Neumann and the pointwise ergodic theorem established by George Birkhoff, proofs of which were published nearly simultaneously in PNAS in 1931 and 1932. These theorems were of great significance both in mathematics and in statistical mechanics. In statistical mechanics they provided a key insight into a 60-y-old fundamental problem of the subject—namely, the rationale for the hypothesis that time averages can be set equal to phase averages. The evolution of this problem is traced from the origins of statistical mechanics and Boltzman's ergodic hypothesis to the Ehrenfests' quasi-ergodic hypothesis, and then to the ergodic theorems. We discuss communications between von Neumann and Birkhoff in the Fall of 1931 leading up to the publication of these papers and related issues of priority. These ergodic theorems initiated a new field of mathematical-research called ergodic theory that has thrived ever since, and we discuss some of recent developments in ergodic theory that are relevant for statistical mechanics.

George D. Birkhoff (1) and John von Neumann (2) published separate and virtually simultaneous path-breaking papers

(a concept to be defined below). First of all, these two papers provided a key insight into a 60-y-old fundamental problem of statistical

container. The molecules are in motion, colliding with each other and with the hard walls of the container. The molecules can be

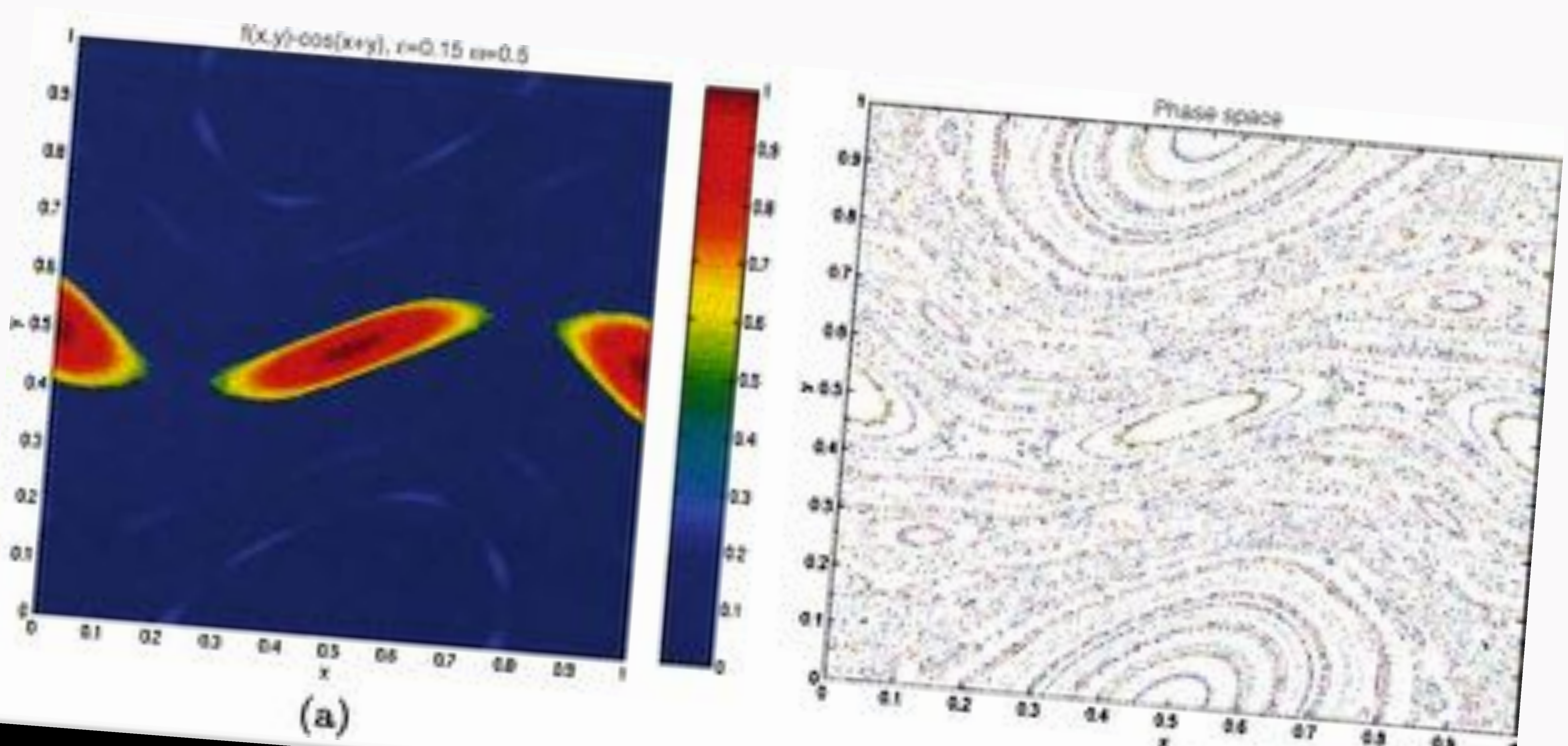
Nonlinear Dynamics (2005) 41: 309–325

Spectral Properties of Dynamical Systems, Model Reduction and Decompositions

IGOR MEZIĆ
Department of Mechanical and Environmental Engineering and Department of Mathematics, University of California, Sant… CA 93105-5070, U.S.A. (e-mail: mezic@engineering.ucsb.edu; fax: +1-805-893-8651)

(Re…

$$v_x^n(m) = U_s^n v_x(m) = v^*(x) + \sum_{j=1}^{k} \lambda_j^n f_j(m) s_j(x) + \int_0^1 \exp(i2\pi\alpha)\, dE(\alpha) v(x, m),$$



(a) (b)

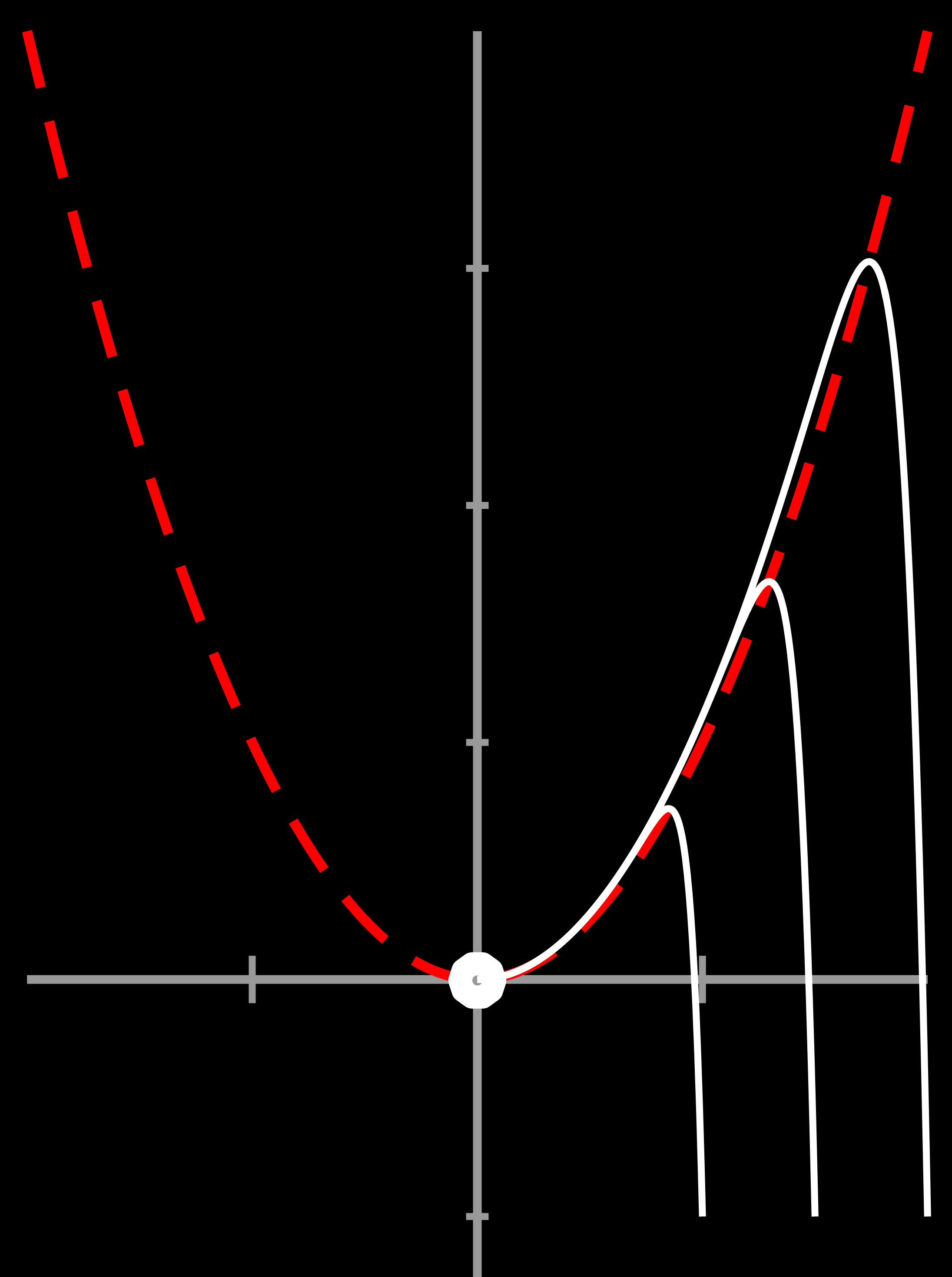
Example: Koopman Linear Embedding

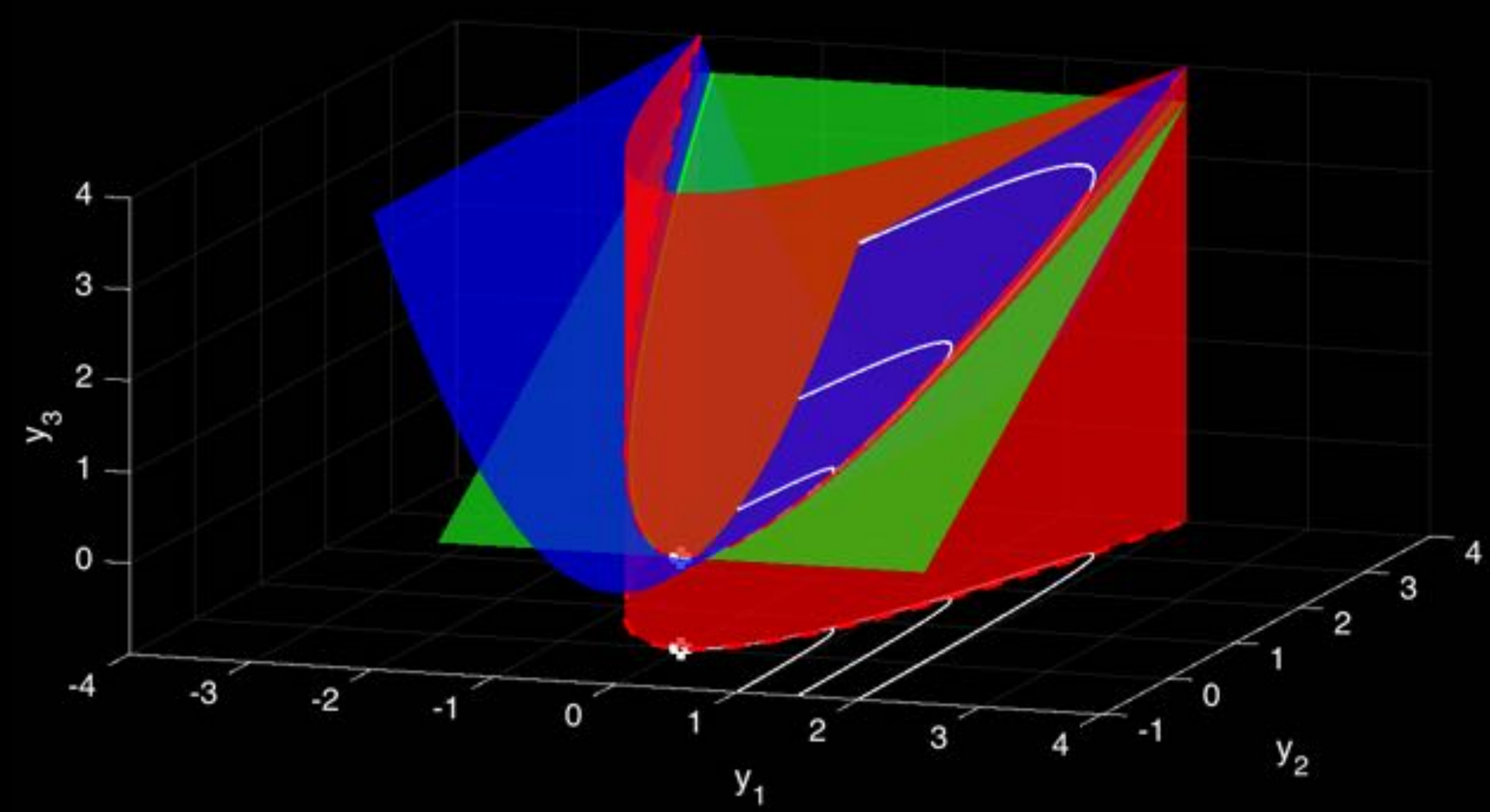
Nonlinear dynamics:

$$\dot{x}_1 = \mu x_1$$
$$\dot{x}_2 = \lambda(x_2 - x_1^2)$$

Koopman linear system:

$$\frac{d}{dt}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} \mu & 0 & 0 \\ 0 & \lambda & -\lambda \\ 0 & 0 & 2\mu \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \quad \text{for} \quad \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_1^2 \end{bmatrix}$$





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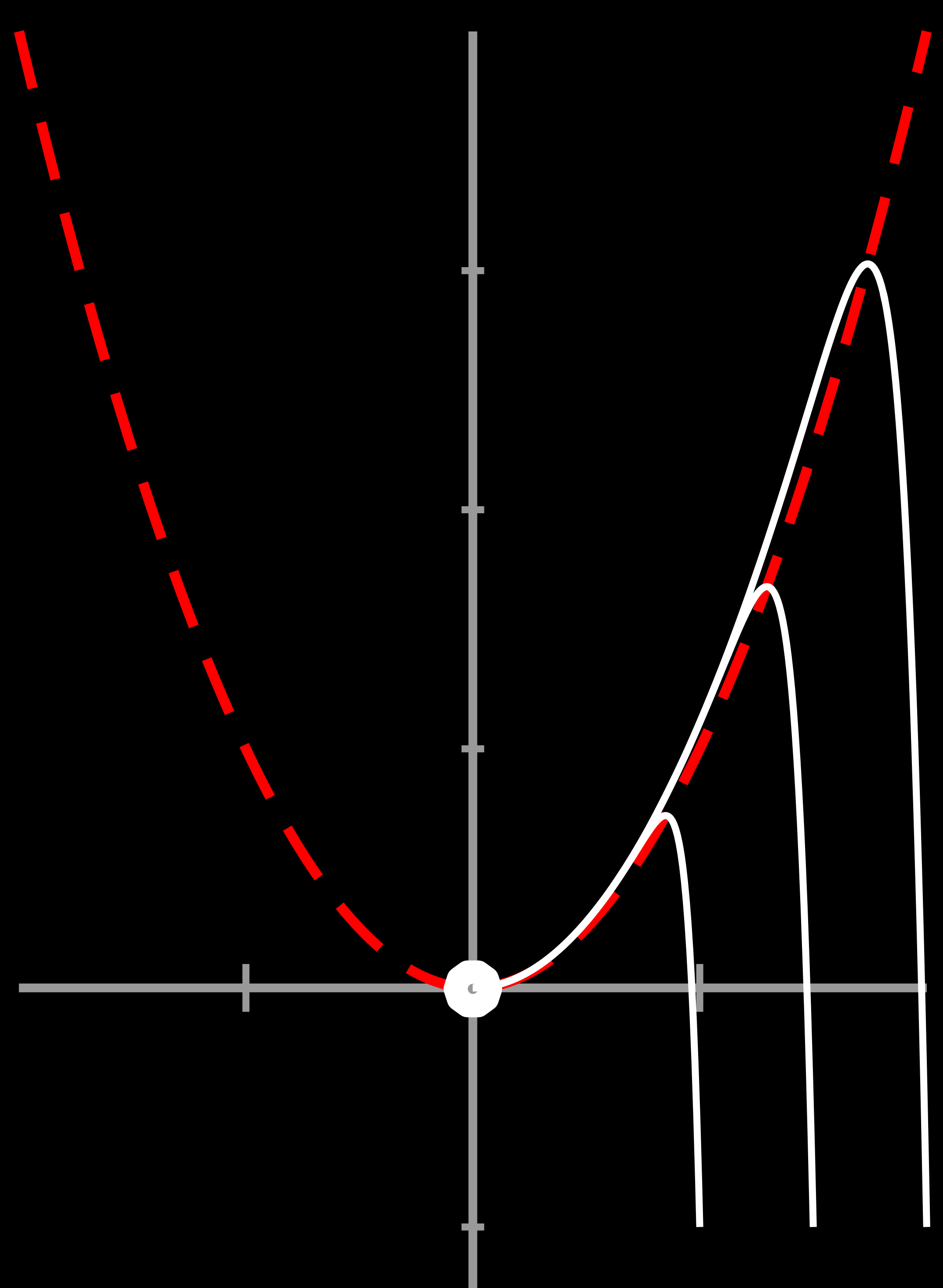
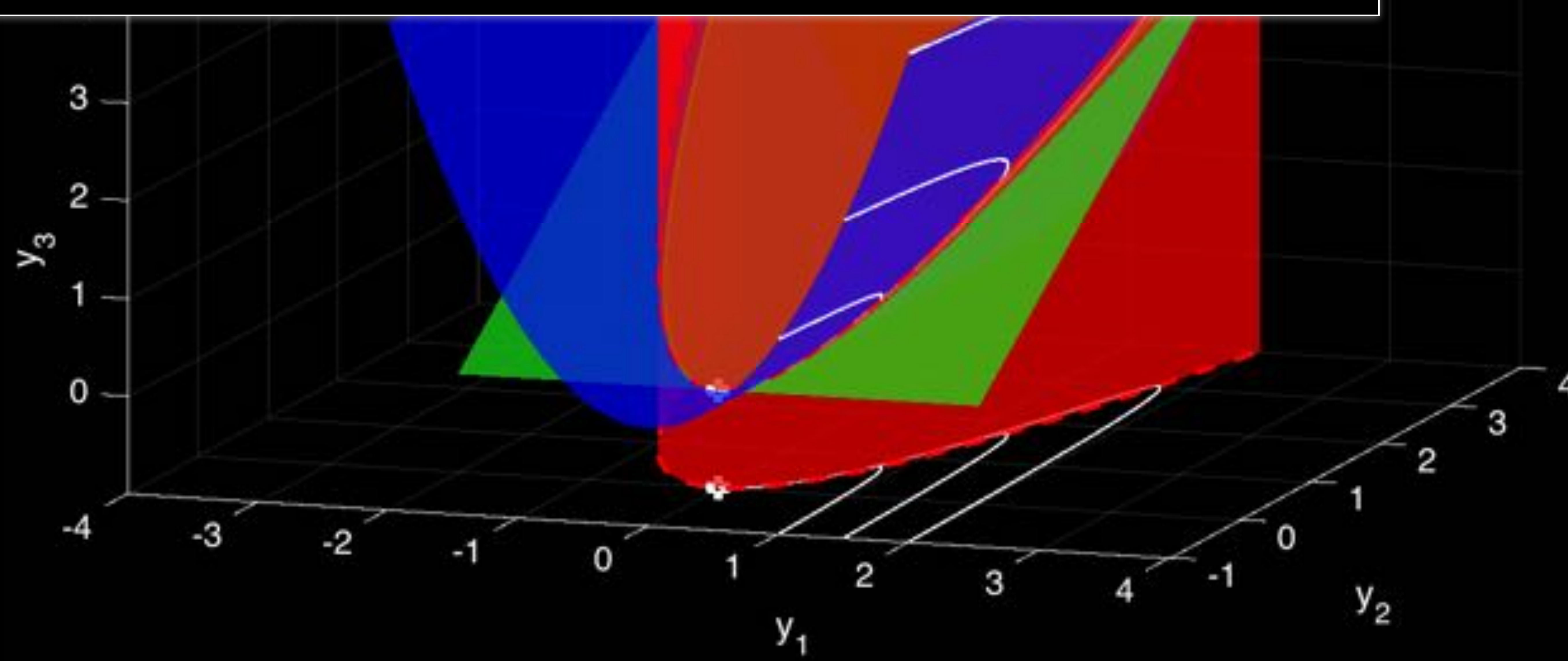
Eigen-observables: $\varphi_\alpha(\mathbf{x}) = \boldsymbol{\xi}_\alpha \mathbf{y}(\mathbf{x}), \quad \text{where} \quad \boldsymbol{\xi}_\alpha \mathbf{K} = \alpha \boldsymbol{\xi}_\alpha.$



$$\varphi_\mu = x_1, \quad \text{and} \quad \varphi_\lambda = x_2 - b x_1^2 \quad \text{with} \quad b = \frac{\lambda}{\lambda - 2\mu}.$$

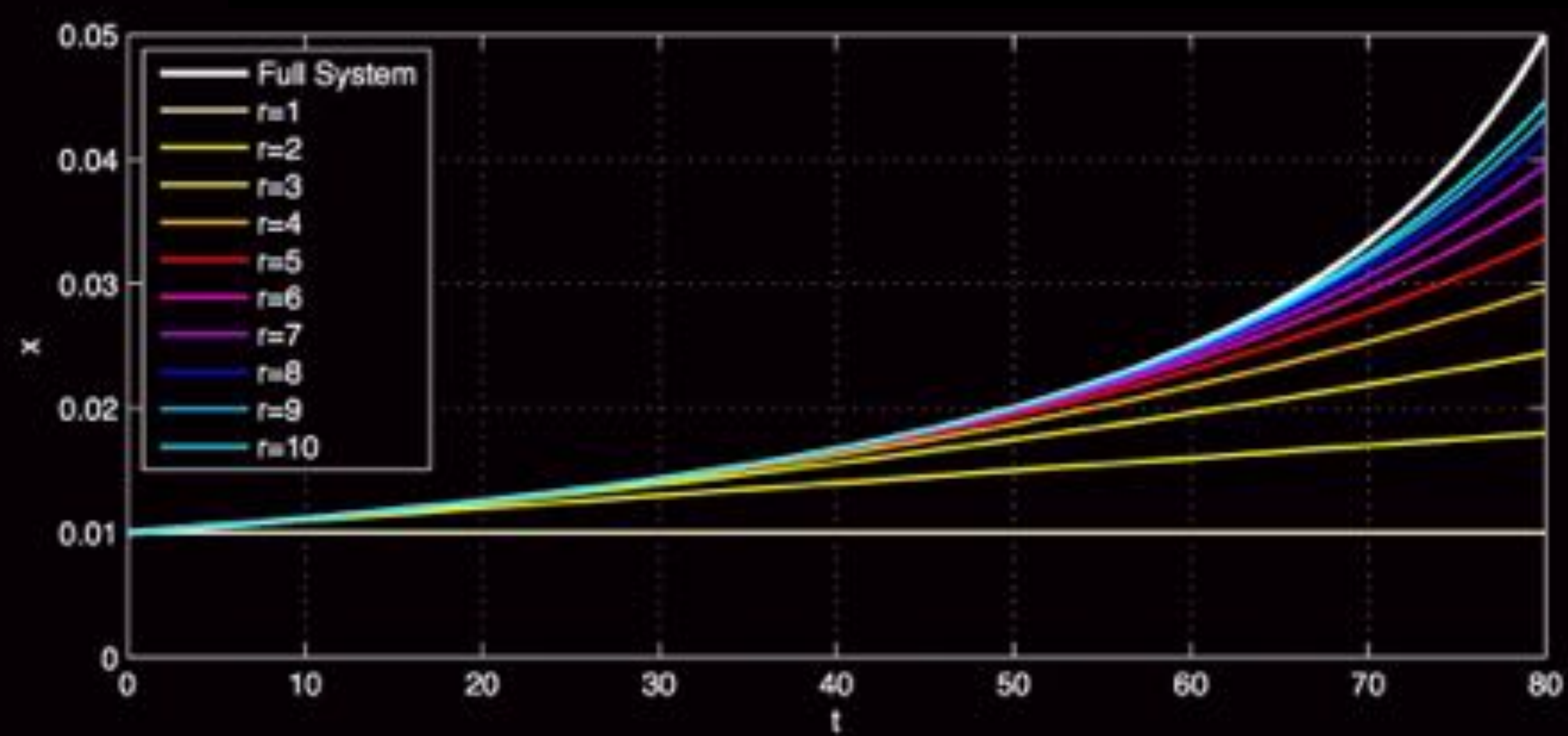


Koopman Operator Optimal Control

$$\frac{d}{dt}x = x^2 \quad \Rightarrow \quad \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x \\ x^2 \end{bmatrix}$$

$$\Rightarrow \quad \frac{d}{dt}y_2 = 2x\dot{x} = 2x^3$$

$$\Rightarrow \quad \frac{d}{dt}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 2 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 3 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 4 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ \vdots \end{bmatrix} \quad \text{where} \quad \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ \vdots \end{bmatrix} = \begin{bmatrix} x \\ x^2 \\ x^3 \\ x^4 \\ x^5 \\ \vdots \end{bmatrix}.$$



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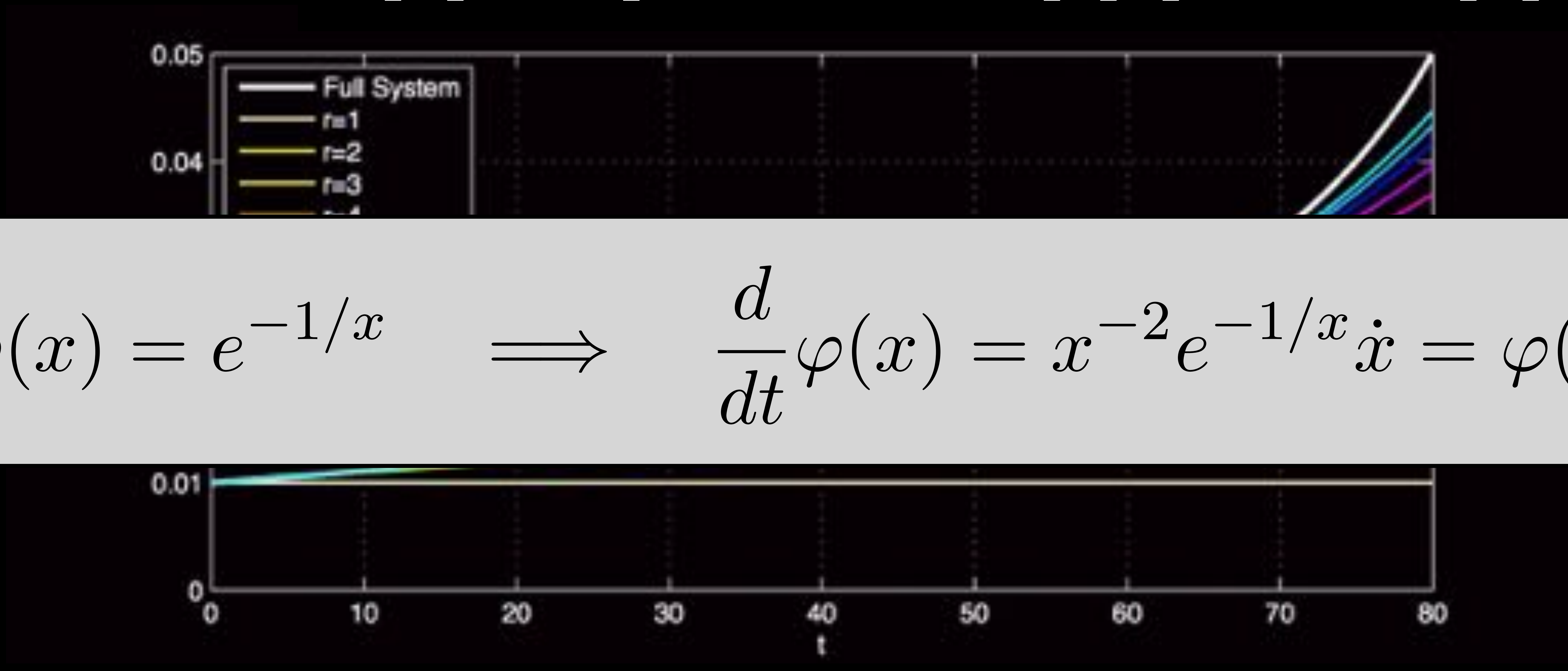


$$\varphi(x) = e^{-1/x} \quad \Longrightarrow \quad \frac{d}{dt}\varphi(x) = x^{-2}e^{-1/x}\dot{x} = \varphi(x).$$

Dynamical Systems: Koopman and Operators

Dynamics

$$\frac{d}{dt}\mathbf{x} = \mathbf{f}(\mathbf{x}), \quad \Rightarrow \quad \mathbf{F}_t(\mathbf{x}(t_0)) = \mathbf{x}(t_0+t) = \mathbf{x}(t_0) + \int_{t_0}^{t_0+t} \mathbf{f}(\mathbf{x}(\tau))\, d\tau.$$

$$\Rightarrow \quad \mathbf{x}_{k+1} = \mathbf{F}_t(\mathbf{x}_k),$$ Discrete-time update

Koopman operator

$$\mathcal{K}_t g = g \circ \mathbf{F}_t \quad \Rightarrow \quad \mathcal{K}_t g(\mathbf{x}_k) = g(\mathbf{F}_t(\mathbf{x}_k)) = g(\mathbf{x}_{k+1}).$$

$$\Rightarrow \quad g(\mathbf{x}_{k+1}) = \mathcal{K}_t g(\mathbf{x}_k).$$ Discrete-time update

Koopman invariant subspace:

$$g = \sum_{k=1}^{\infty} \alpha_k y_k.$$

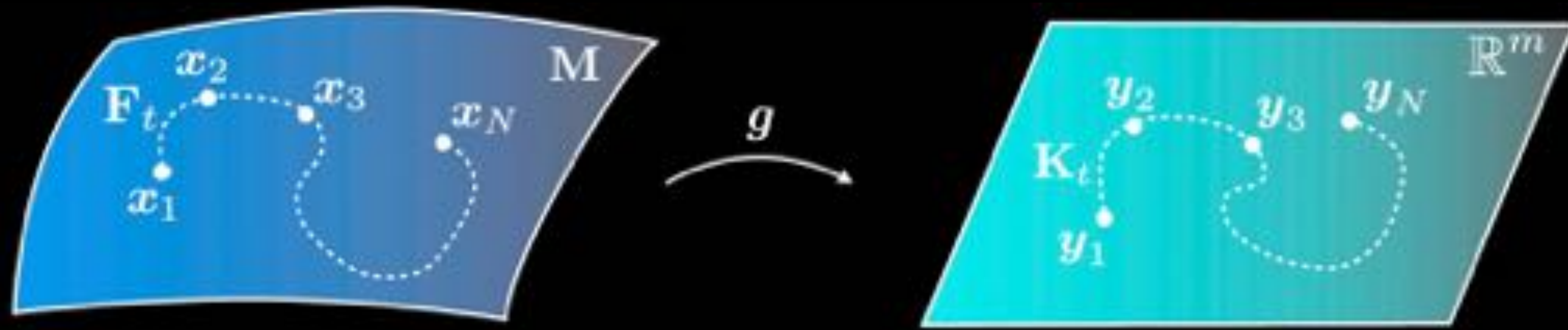
$$g = \alpha_1 y_{s_1} + \alpha_2 y_{s_2} + \cdots + \alpha_m y_{s_m},$$

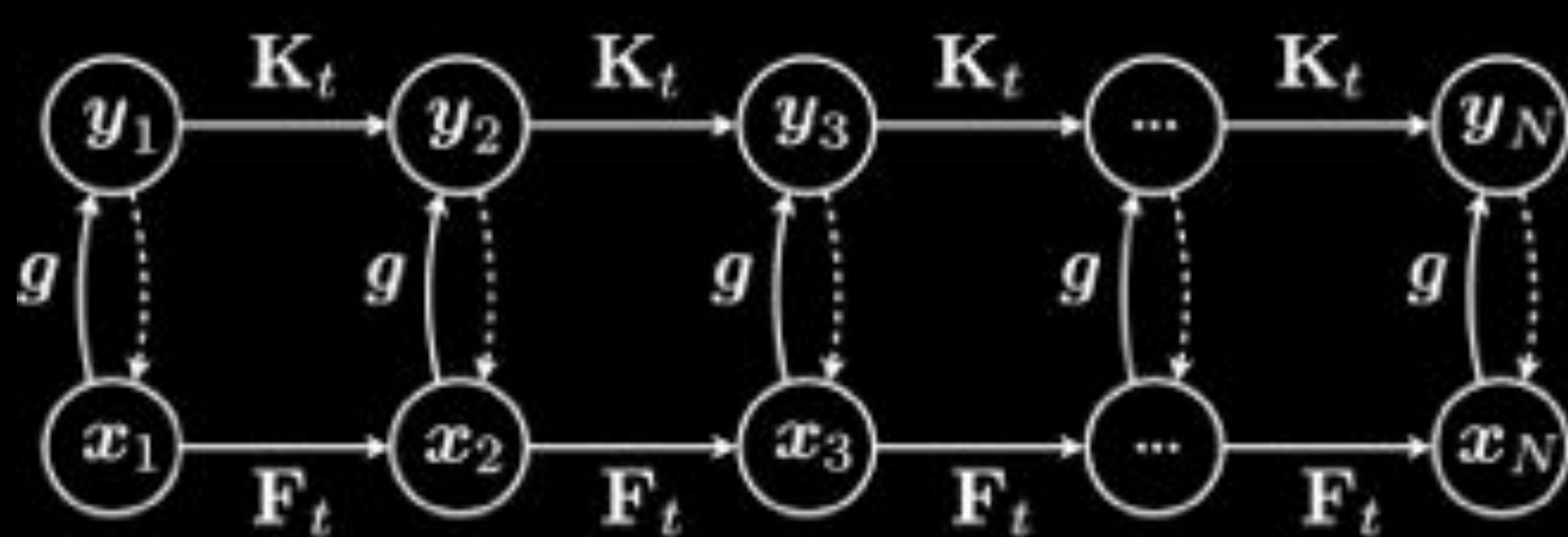
$$\mathcal{K} g = \beta_1 y_{s_1} + \beta_2 y_{s_2} + \cdots + \beta_m y_{s_m}.$$

Koopman operator $\mathcal{K}$ is infinite dimensional and *linear*

Koopman, *PNAS* 1931.
Mezic, *Nonlinear Dynamics* 2005.
Mezic, *ARFM* 2013.
Williams, Kevrekidis, Rowley, *JNS* 2015.

Dynamical Systems: Koopman and Operators





$\mathbf{F}_t : x_k \mapsto x_{k+1}$

$g : x_k \mapsto y_k$

$\mathbf{K}_t : y_k \mapsto y_{k+1}$

Koopman invariant subspace:

$$g = \sum_{k=1}^{\infty} \alpha_k y_k.$$

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Koopman, *PNAS* 1931.
Mezic, *Nonlinear Dynamics* 2005.
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Williams, Kevrekidis, Rowley, *JNS* 2015.

Koopman Eigenfunctions Define Invariant Subspaces

Linear dynamics in eigenfunction coordinates

$$\frac{d}{dt}\varphi(\mathbf{x}) = \lambda\varphi(\mathbf{x})$$

Nonlinear dynamics in original

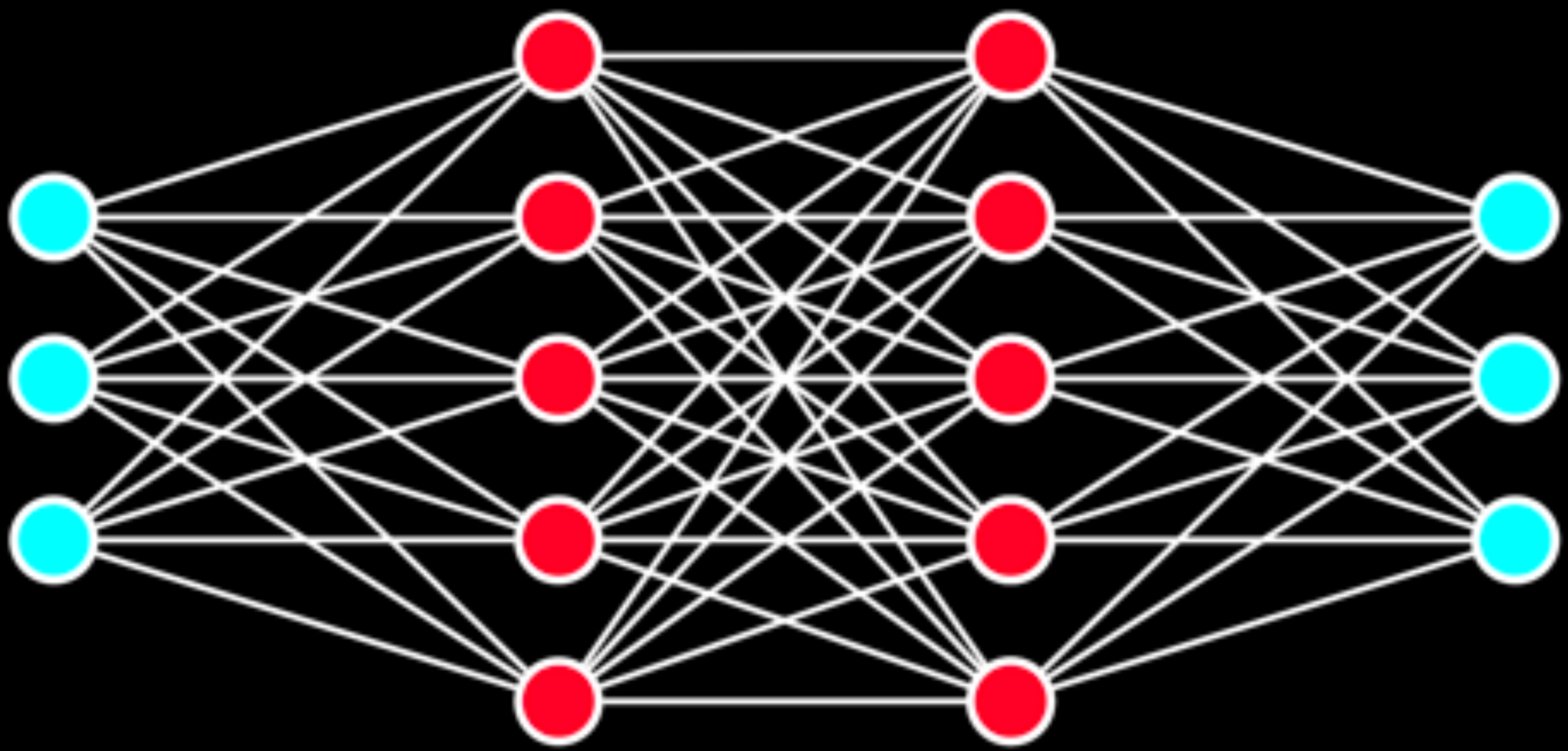
$$\frac{d}{dt}\mathbf{x} = \mathbf{f}(\mathbf{x})$$

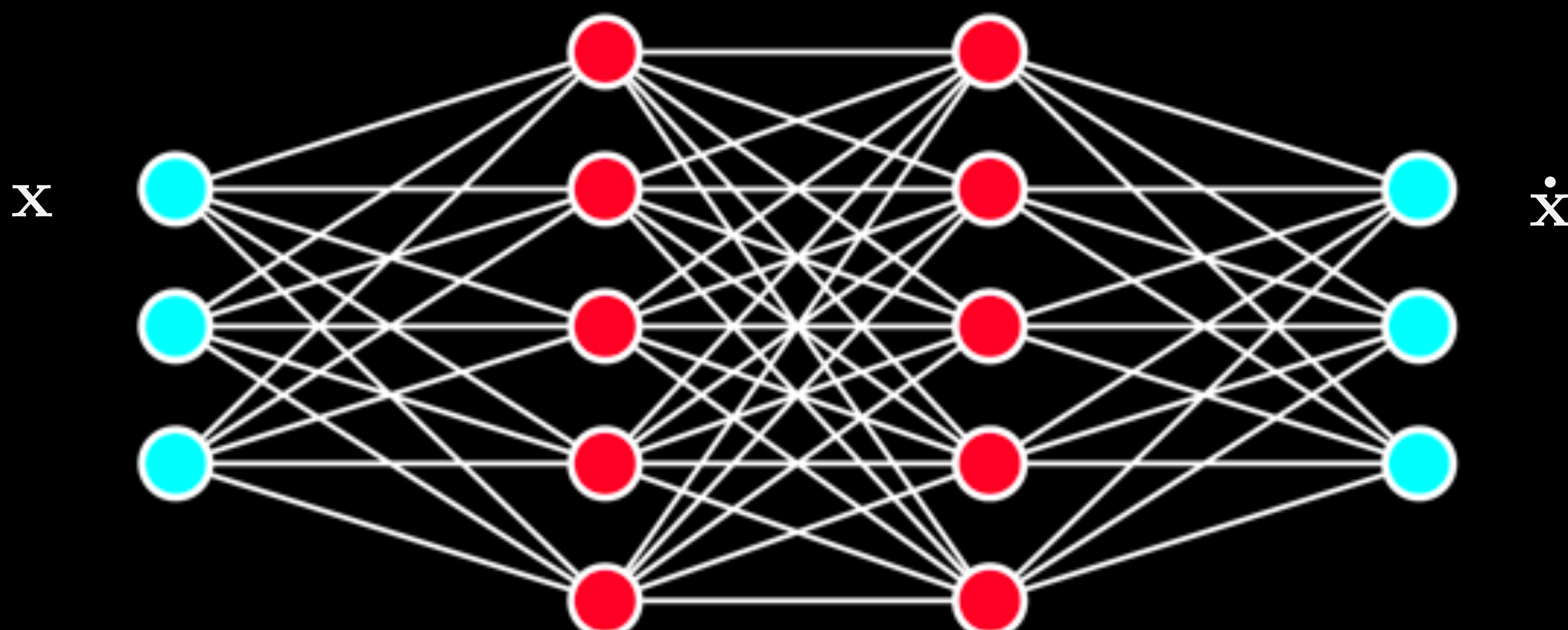
$$\frac{d}{dt}\varphi(\mathbf{x}) = \nabla\varphi(\mathbf{x})\cdot\mathbf{f}(\mathbf{x}) \implies \boxed{\nabla\varphi(\mathbf{x})\cdot\mathbf{f}(\mathbf{x}) = \lambda\varphi(\mathbf{x})}$$

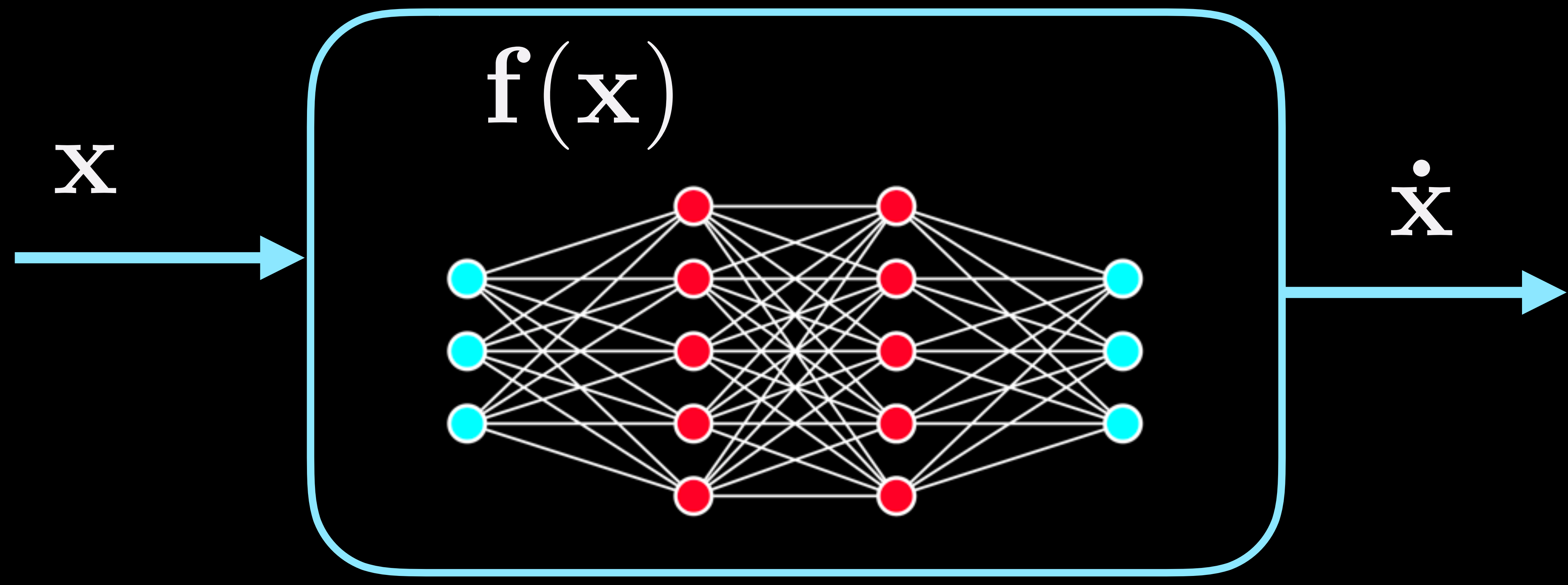
PDE for Koopman Eigenfunctions!

$$\frac{d}{dt}x = x^2 \quad \rightarrow \quad \varphi(x) = e^{-1/x} \implies \frac{d}{dt}\varphi(x) = x^{-2}e^{-1/x}\dot{x} = \varphi(x).$$

Neural Networks for Dynamics

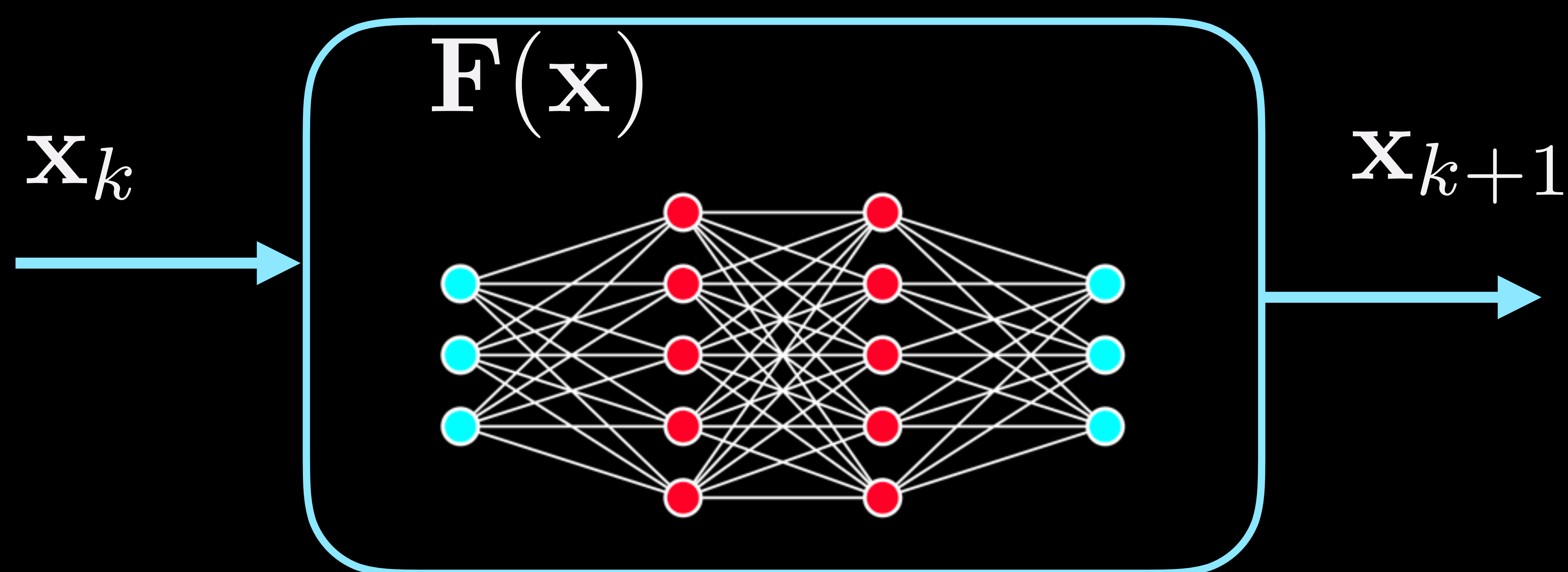


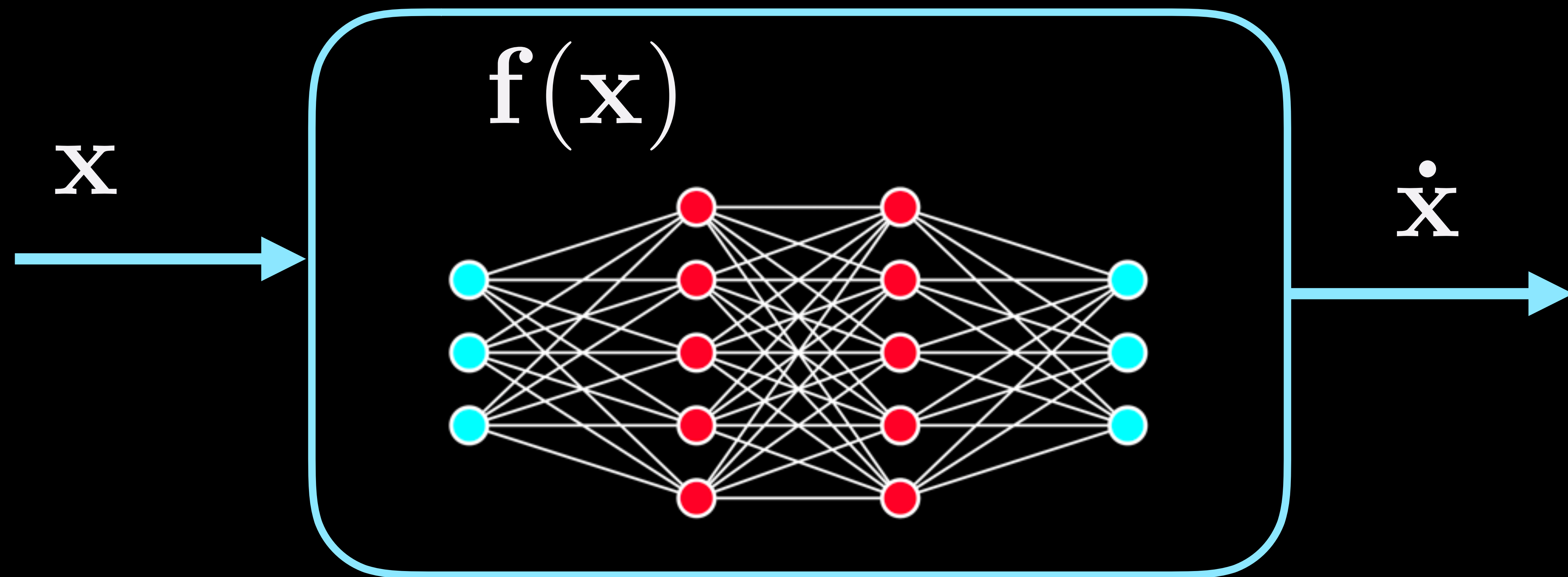

$\mathbf{x}$
$\dot{\mathbf{x}}$



Integrate:

$$\mathbf{x}_{k+1} = \mathbf{x}_k + \Delta t \mathbf{f}(\mathbf{x})$$

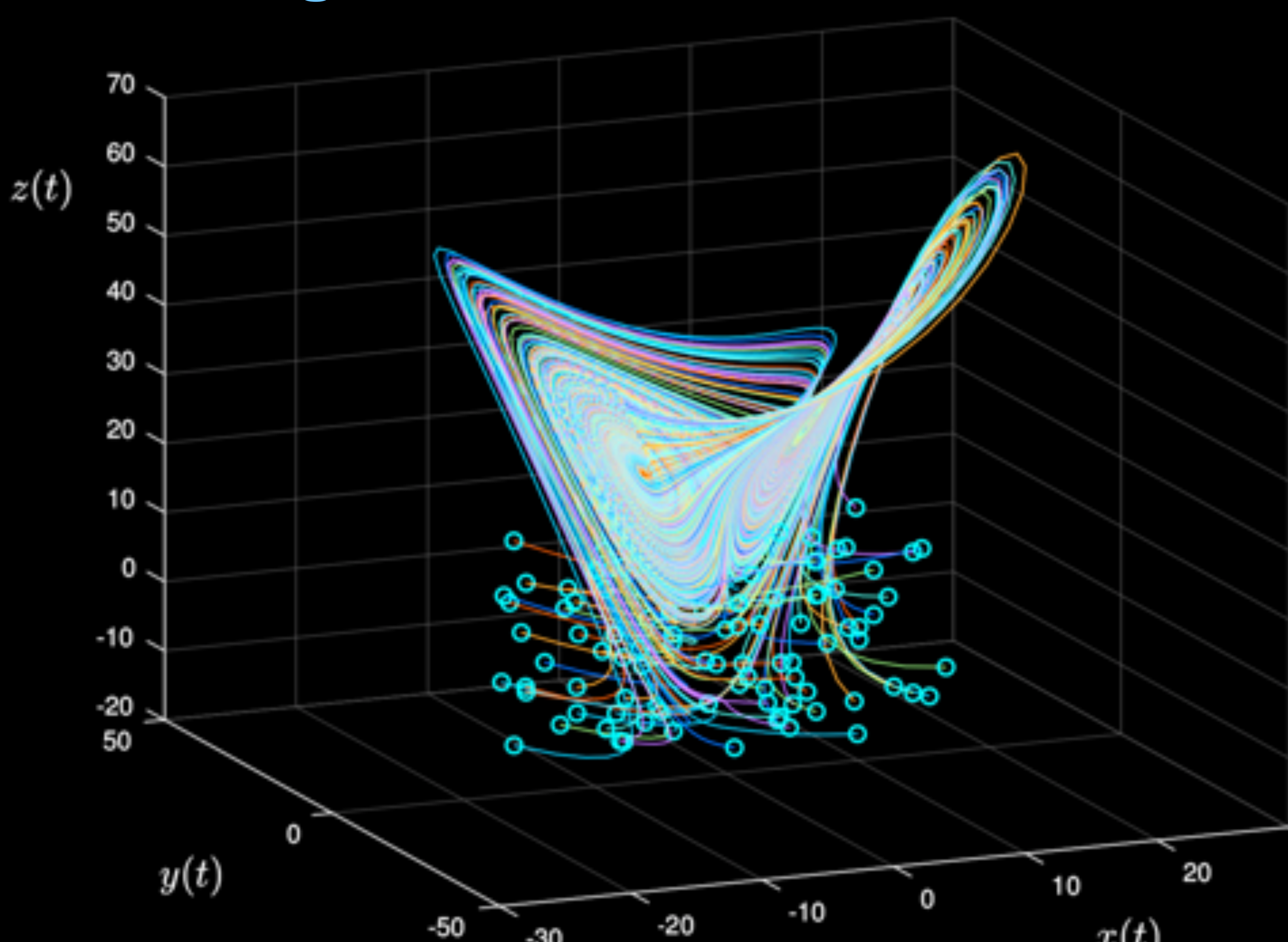




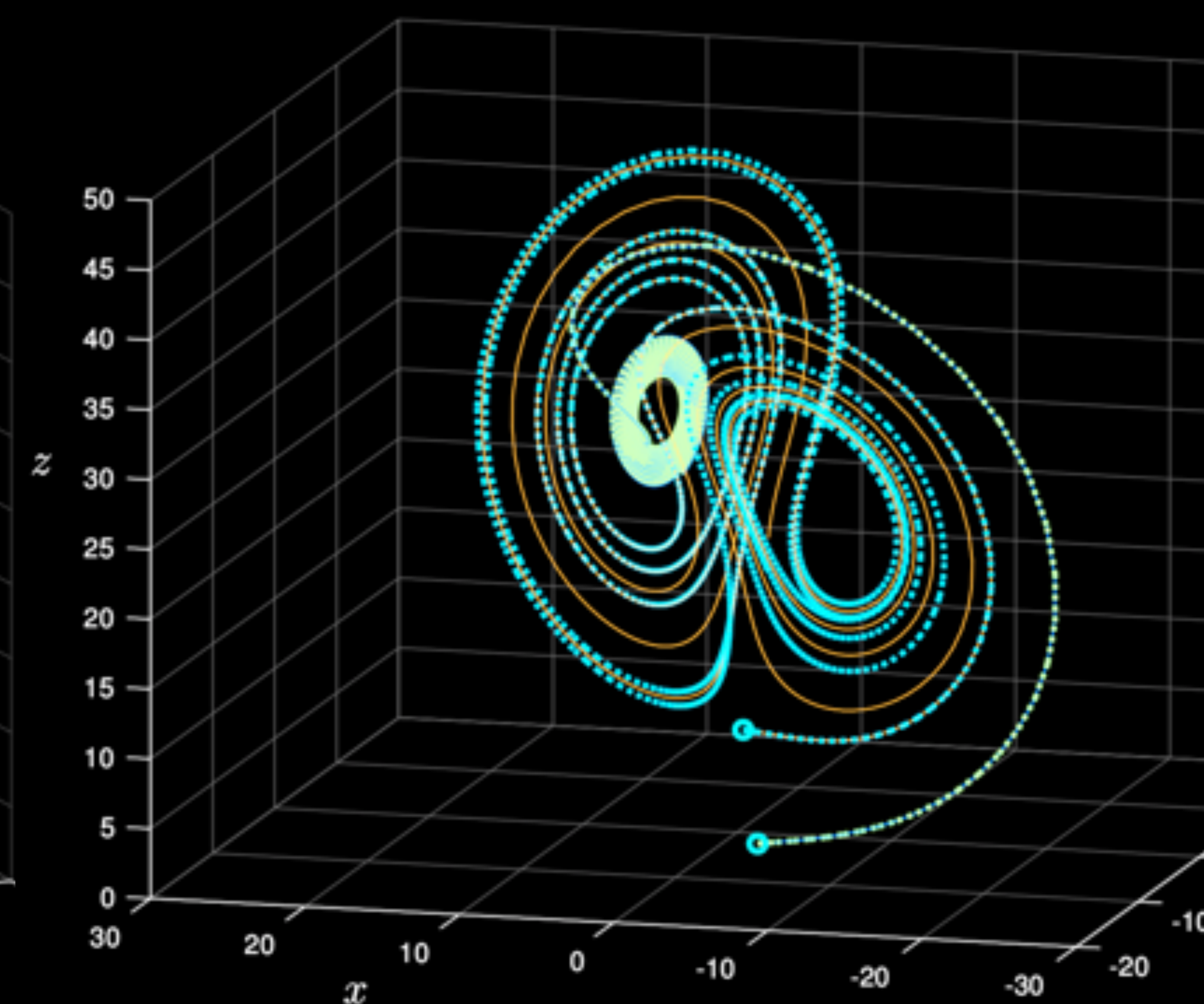
Integrate:

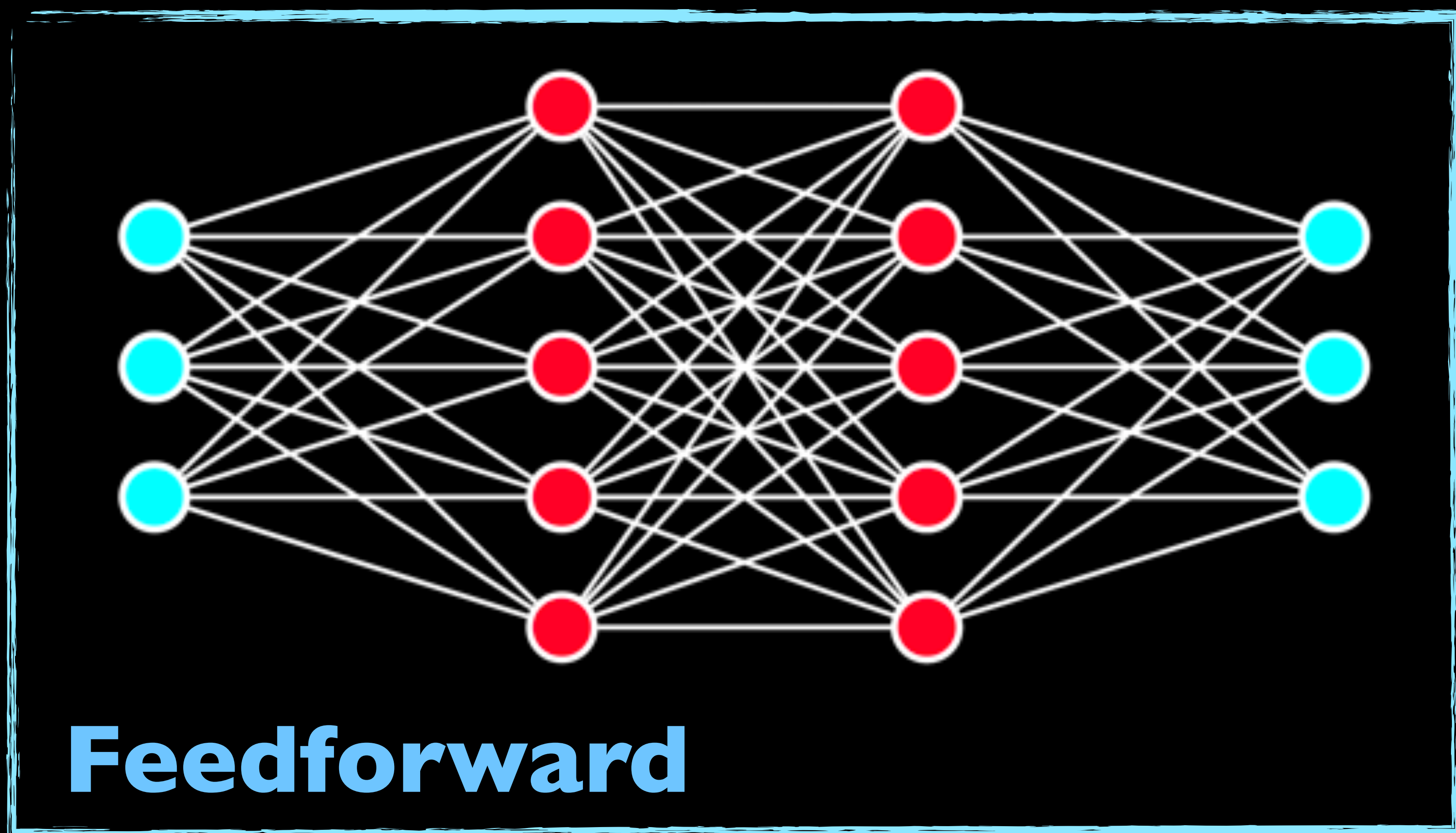
$$\mathbf{x}_{k+1} = \mathbf{x}_k + \Delta t \mathbf{f}(\mathbf{x})$$

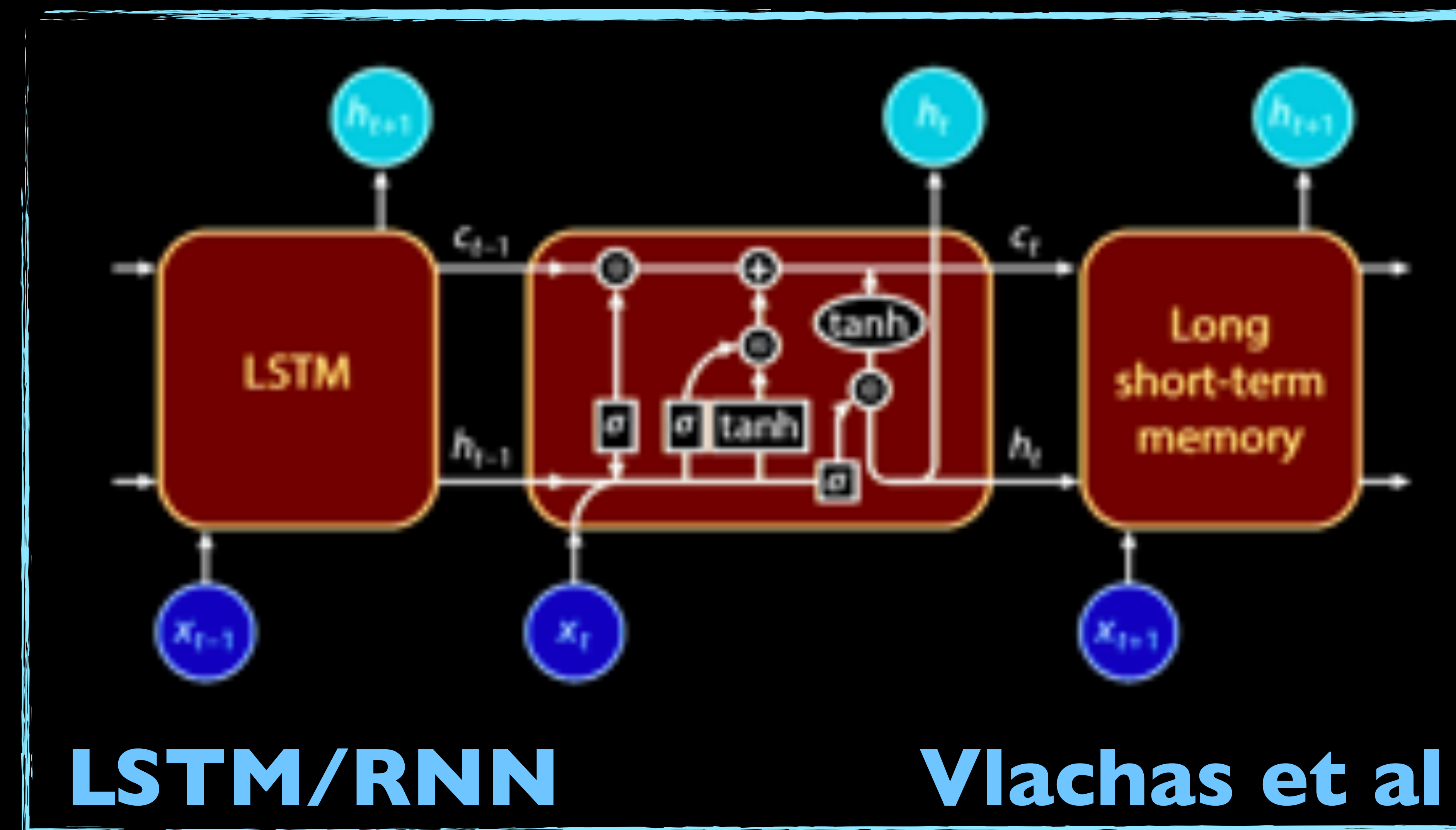
Training

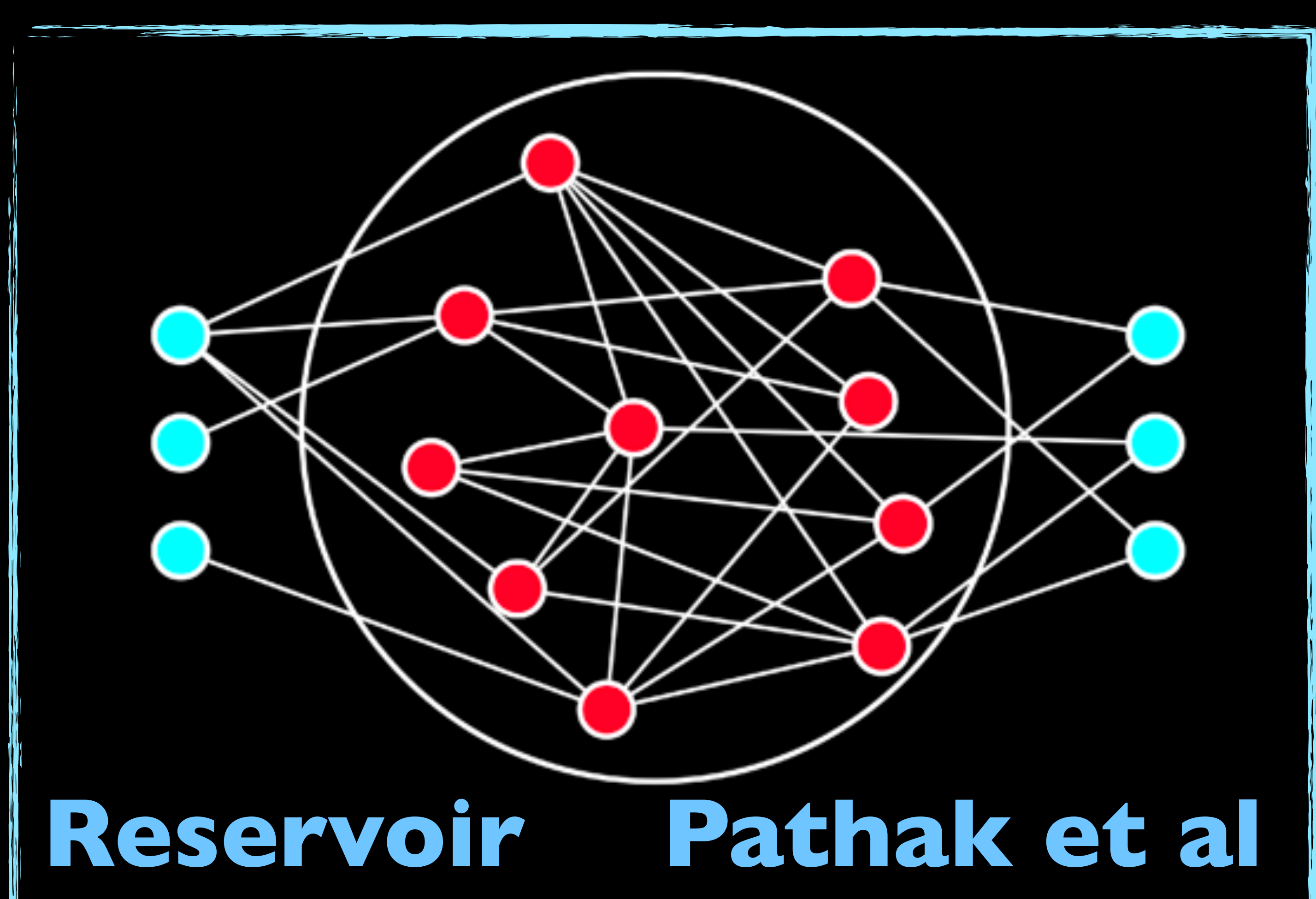


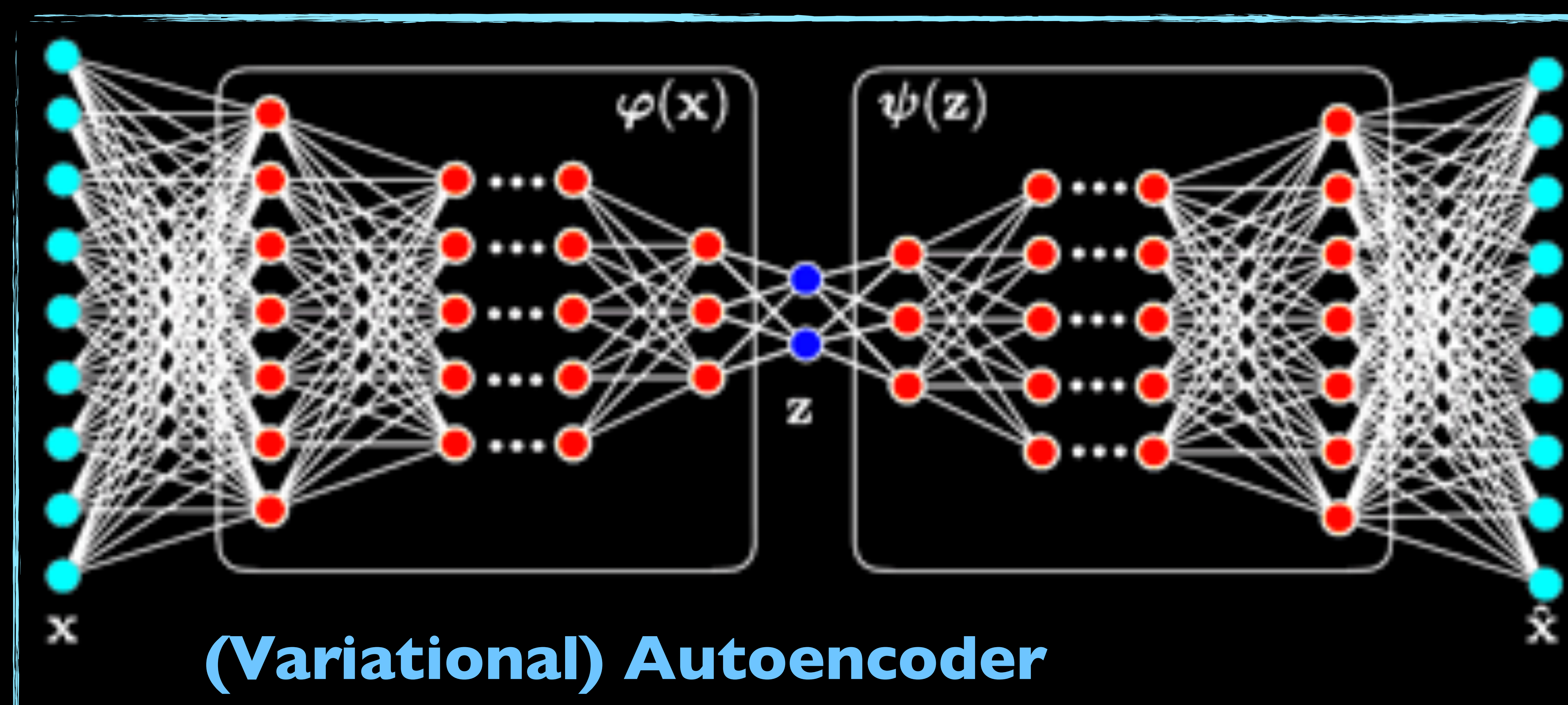
Test

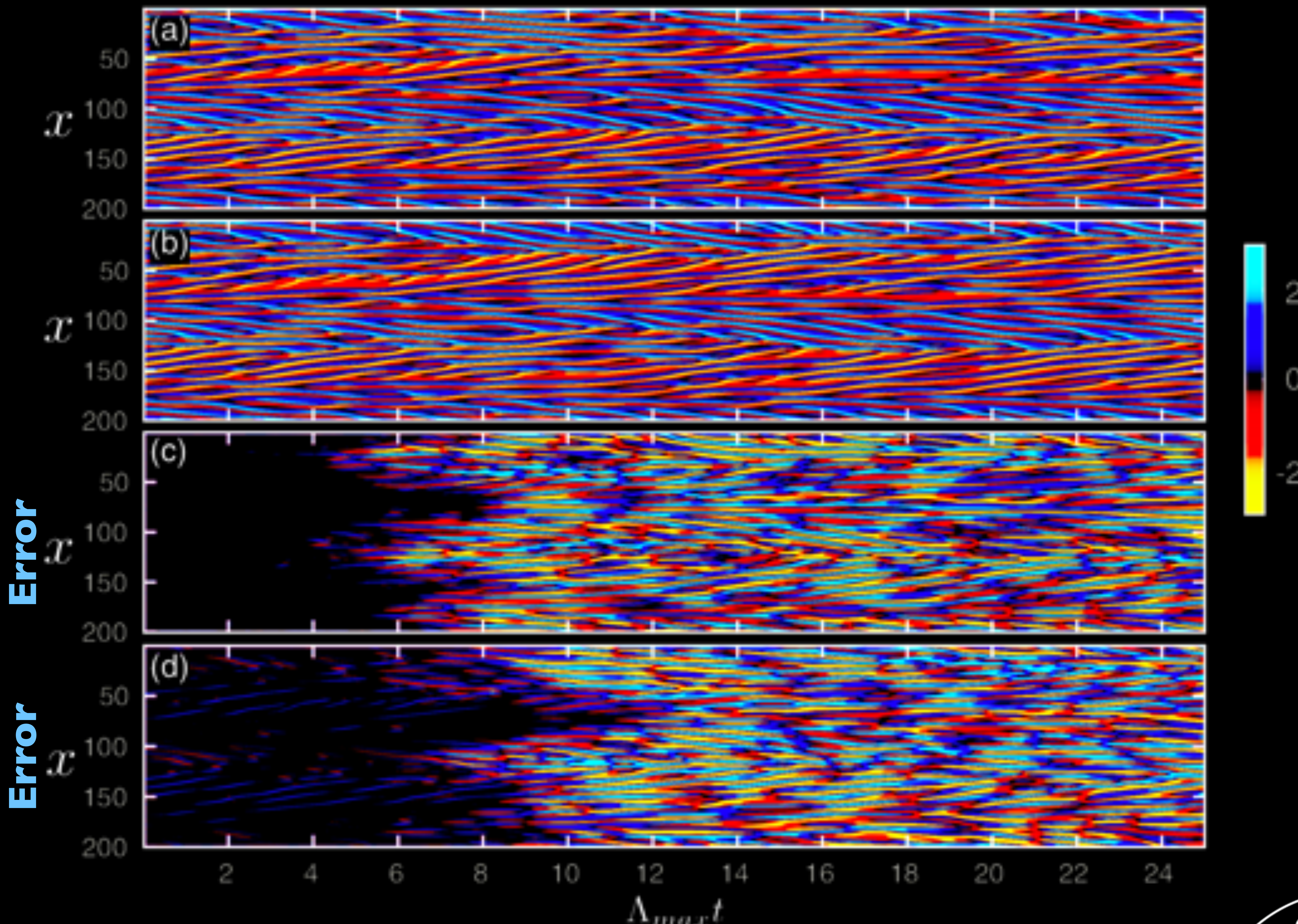


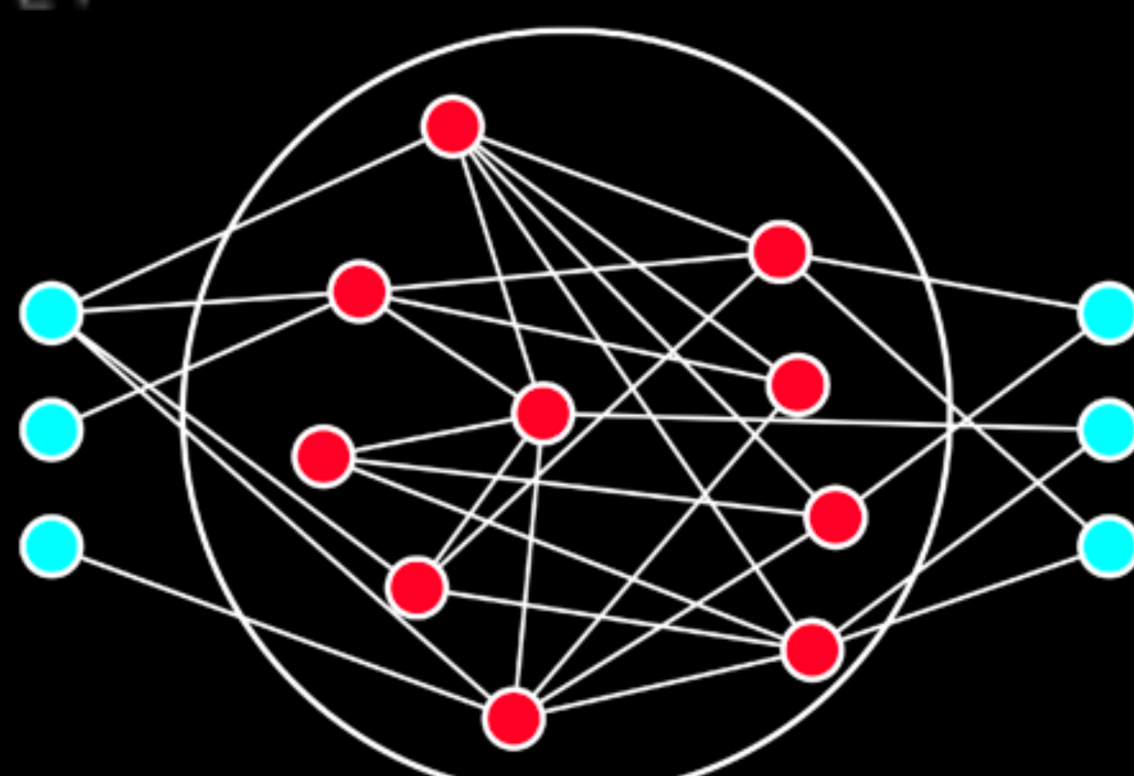

Feedforward


LSTM
tanh
Long short-term memory
LSTM/RNN
Vlachas et al


Reservoir
Pathak et al


$\varphi(\mathbf{x})$
$\psi(\mathbf{z})$
z
x
$\hat{\mathbf{x}}$
(Variational) Autoencoder





Pathak et al, PRL 2018

HISTORY

Ptolemaic System

Ptolemy


CŒLIFER ATLAS
Hic canet errantē Lunam, Solisq; labores
Arcturūq;, pluuiasq; hyad. gēmosq; triōes

Armillary Sphere

Armillary Sphere

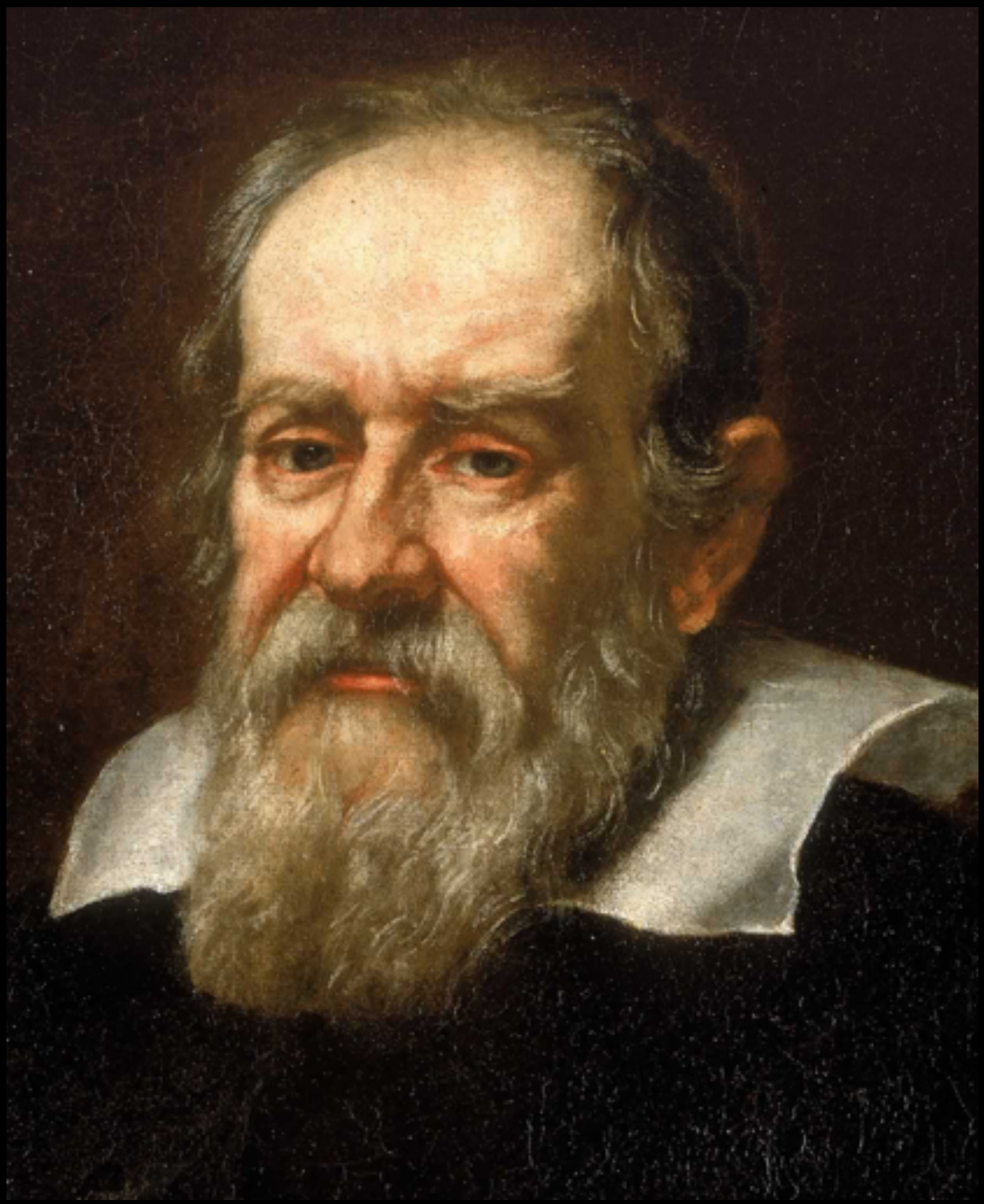
Orrery

3blue1brown

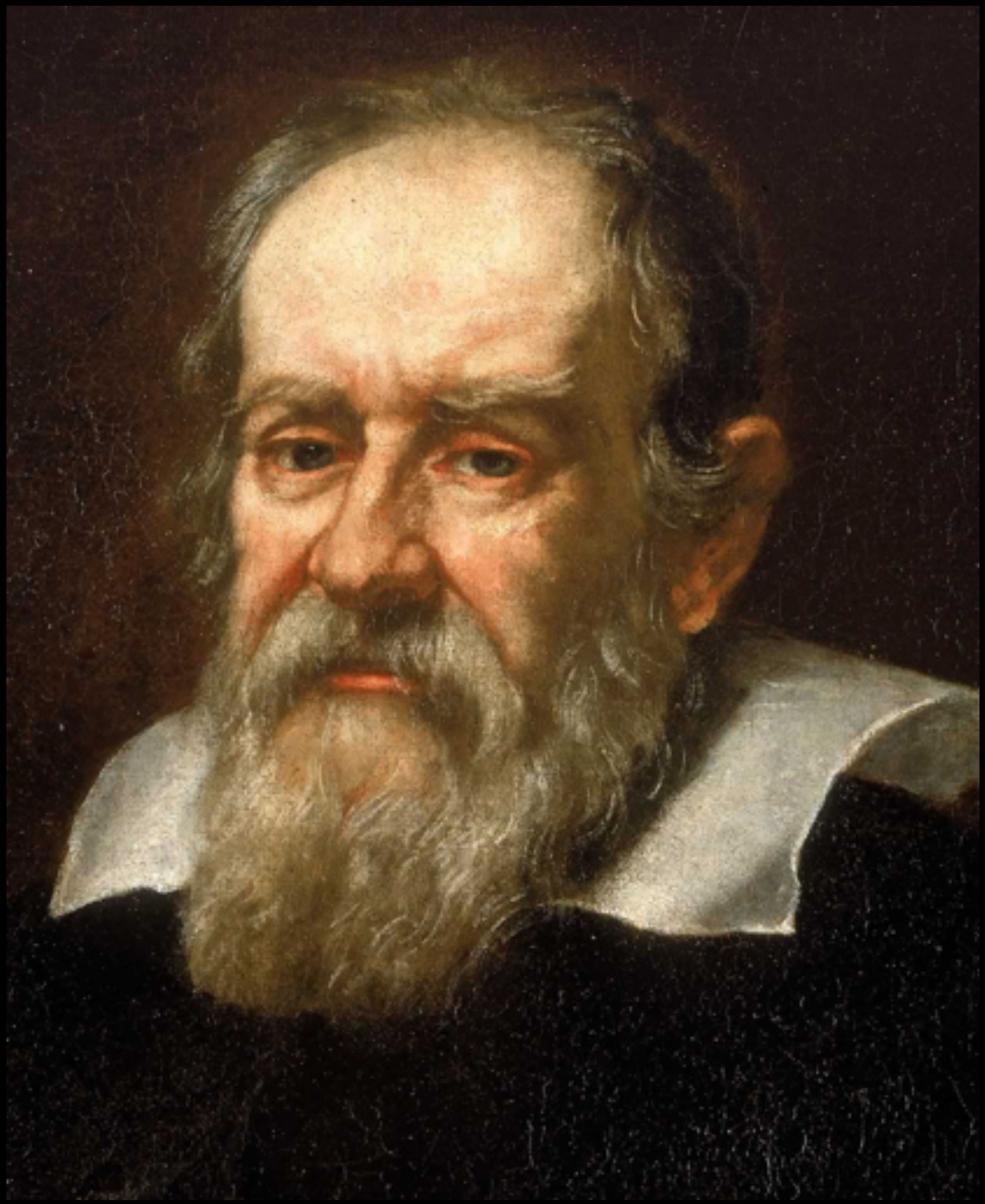
https://www.youtube.com/watch?v=r6sGWTCMz2k&feature=youtu.be

Aristotle and Galileo

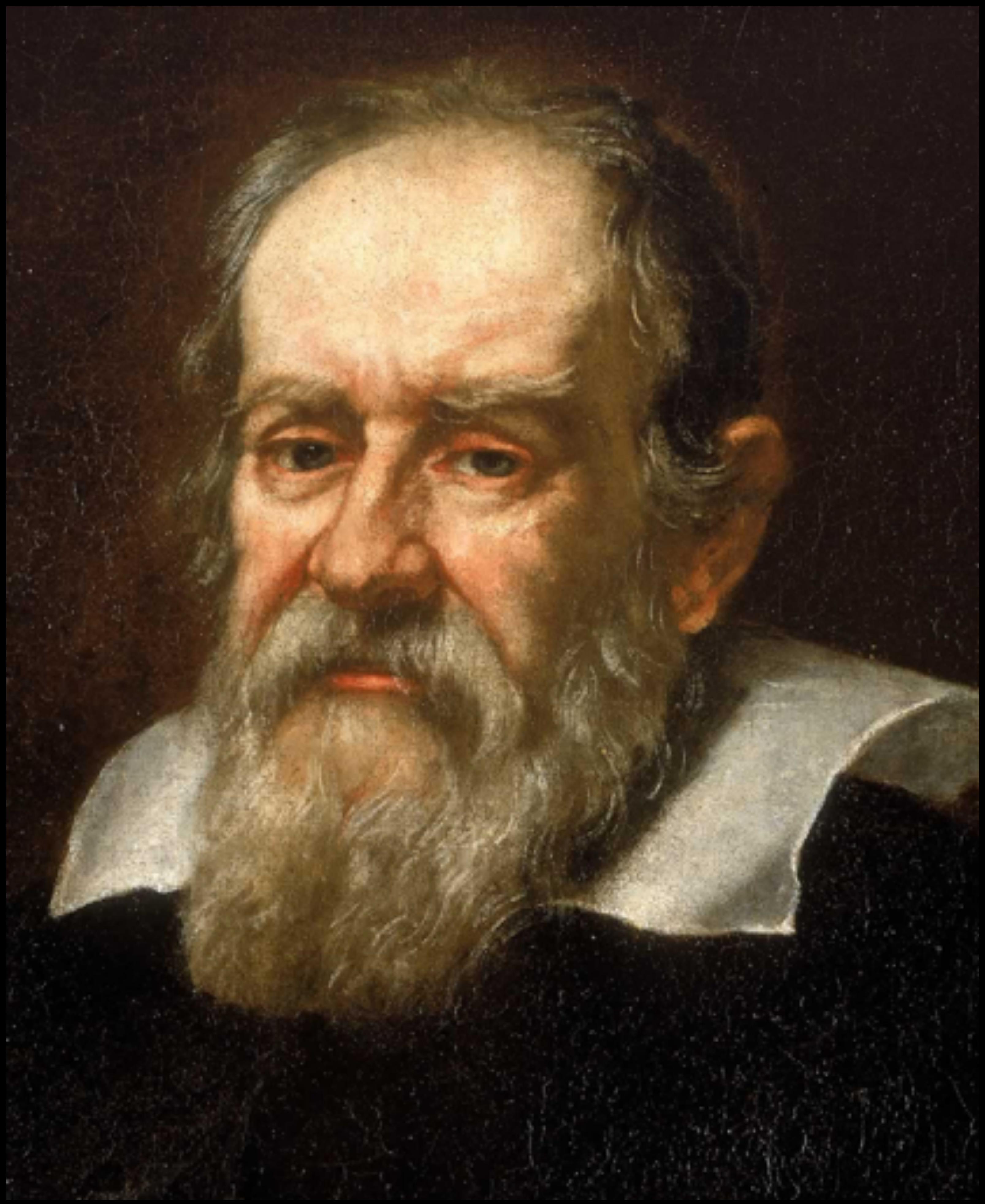
Galileo



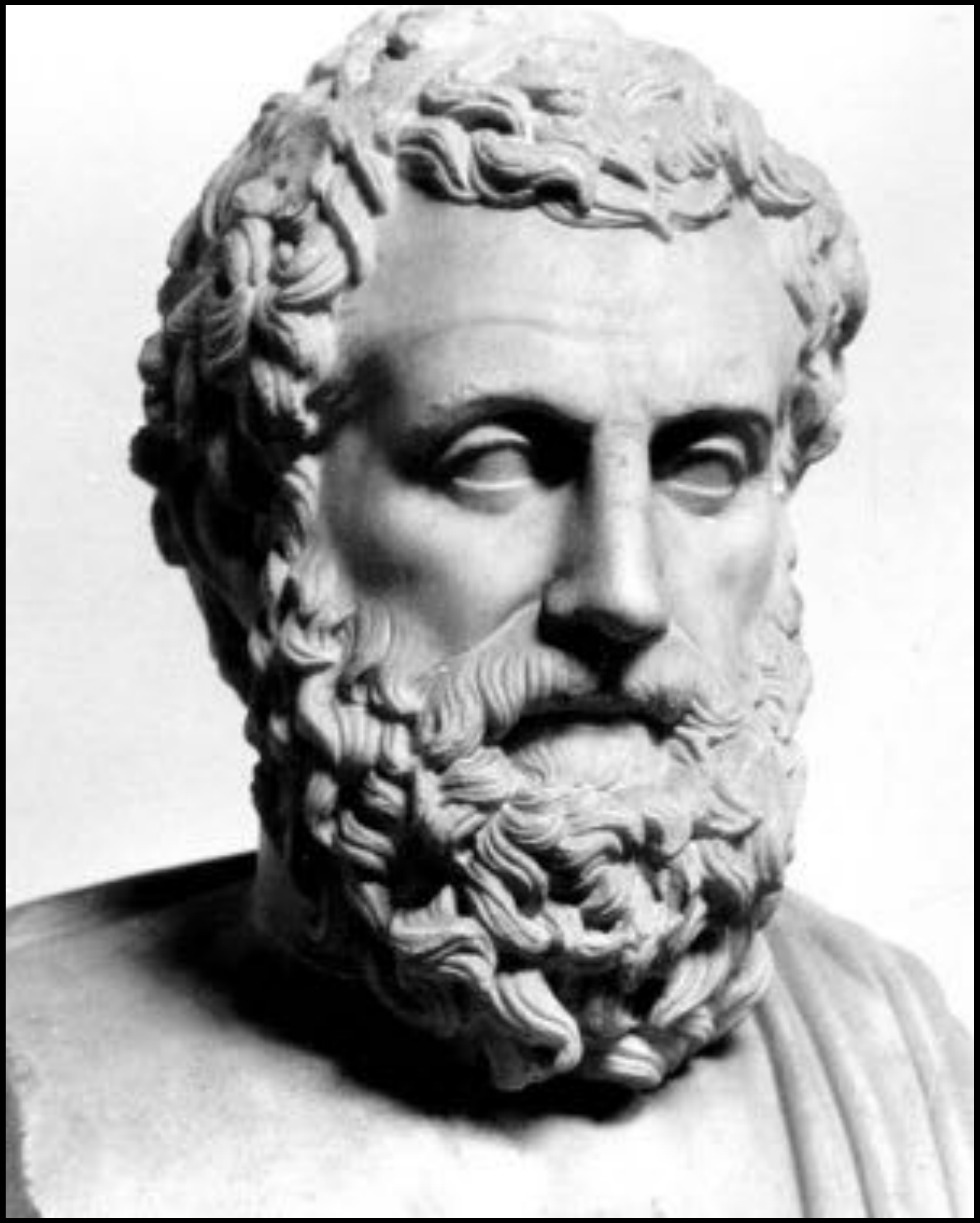
Galileo



Galileo

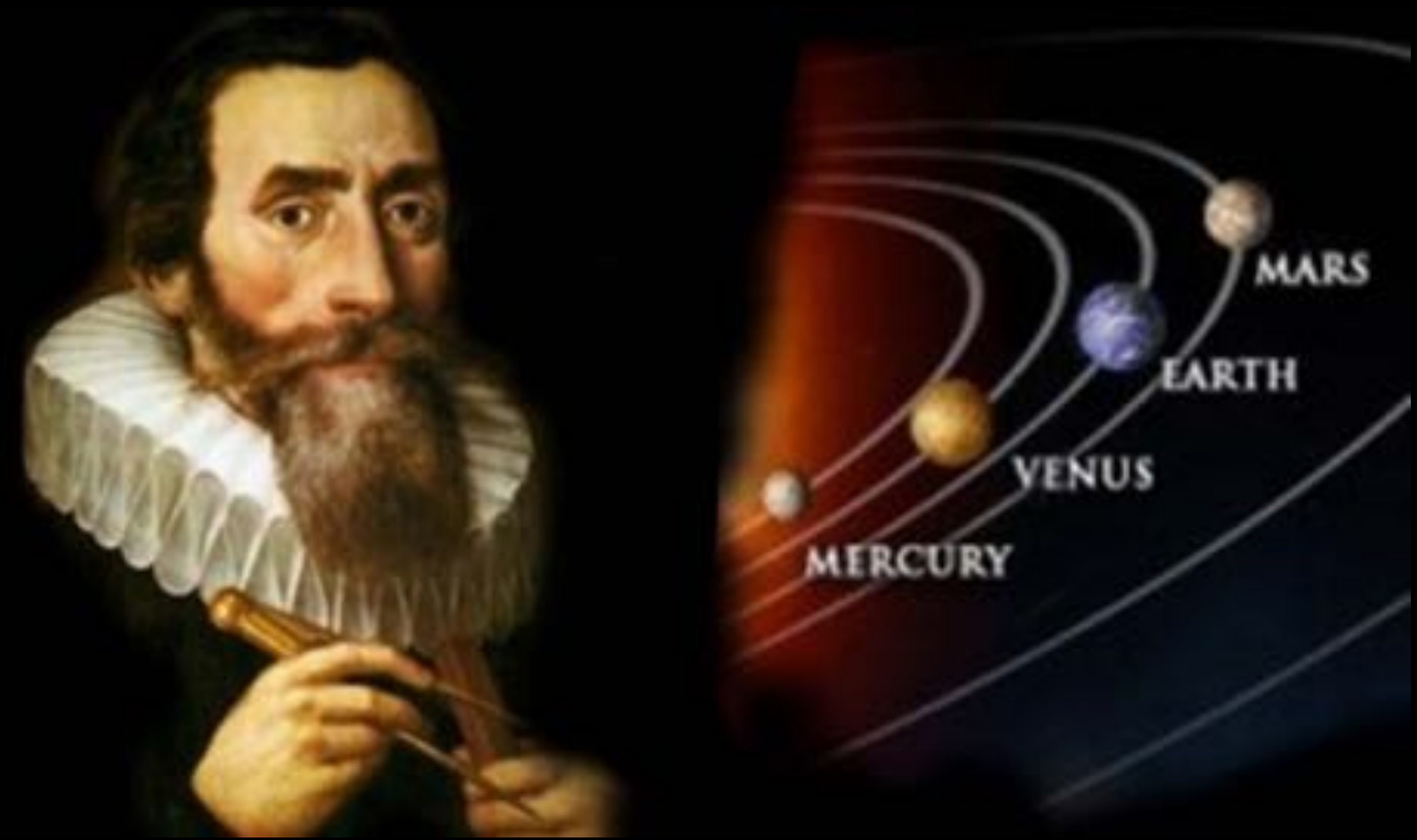


Aristotle

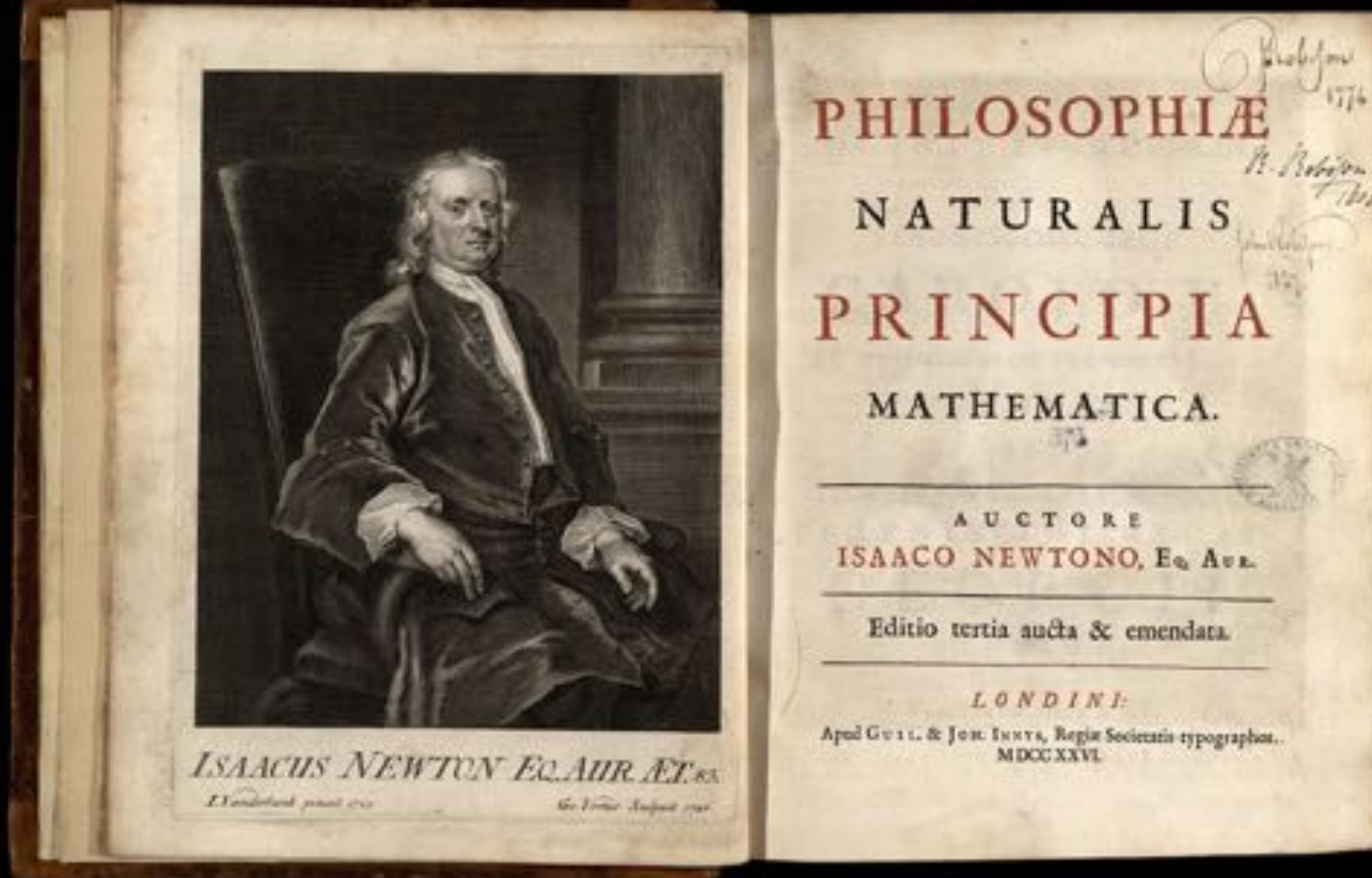


Kepler and Newton

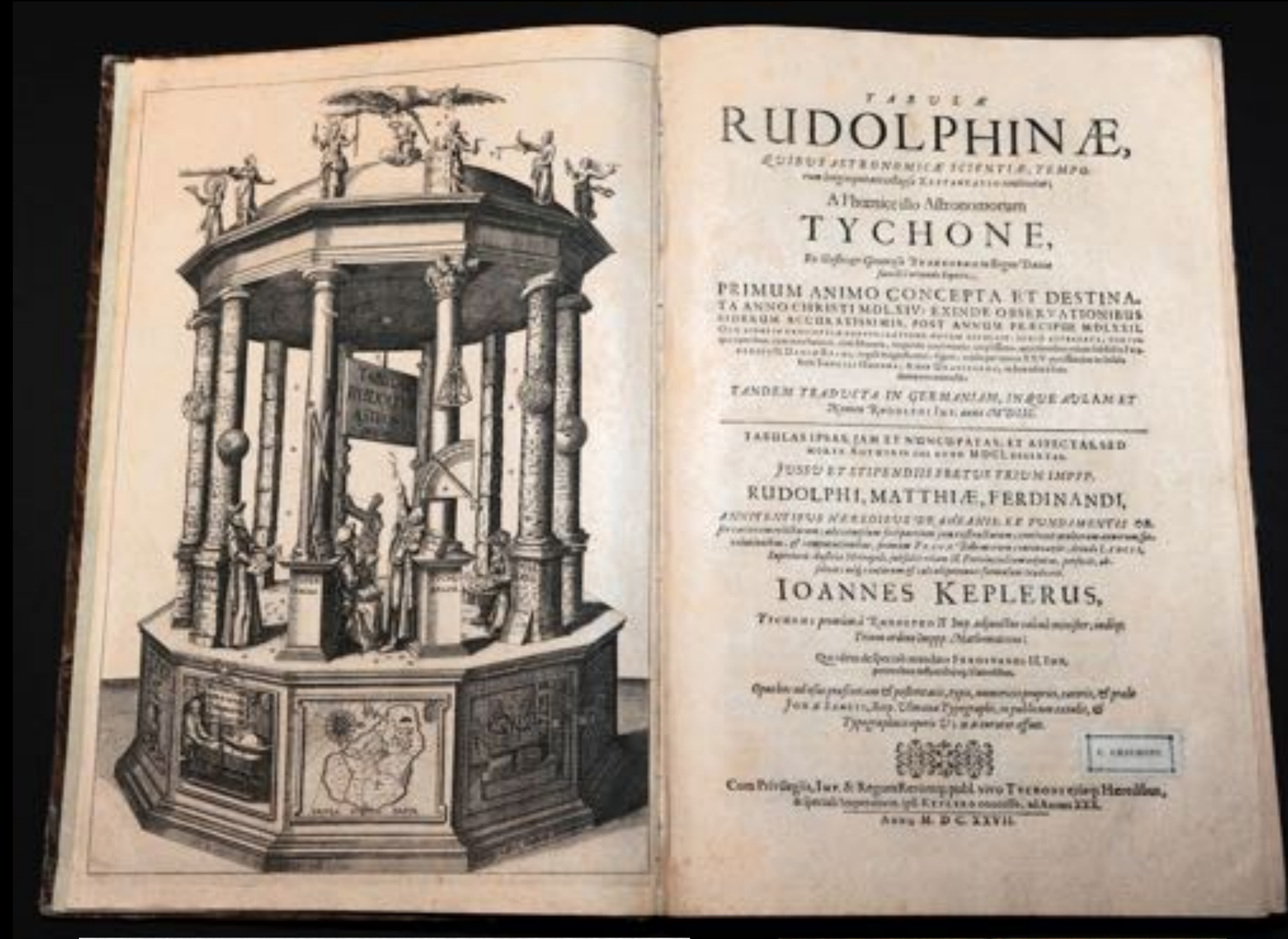
Kepler



Newton


ISAACUS NEWTON EQ. AUR. ÆT. 83
PHILOSOPHIÆ
NATURALIS
PRINCIPIA
MATHEMATICA.
AUCTORE
ISAACO NEWTONO, Eq. Aur.
Editio tertia aucta & emendata.
LONDINI:
Apud Guil. & Joh. Innys, Regiæ Societatis typographos.
MDCCXXVI.

Brahe


TABULÆ
RUDOLPHINÆ,
TYCHONE,
PRIMUM ANIMO CONCEPTA ET DESTINA-
RUDOLPHI, MATTHIÆ, FERDINANDI,
IOANNES KEPLERUS,