A Characterization of Subsets of Rectifiable Curves in Hilbert Space

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Motivation

- Want to discuss the geometry of sets of points in Hilbert space.
- Results about sets lying in $\mathbb{R}^d$ usually have constants that depend **exponentially** on d.
- This is called 'the curse of dimensionality'

Outline

- Introduction
  - dimension free estimates in harmonic analysis
  - traveling salesmen theorems. Jones and Okikiolu
  - dictionary
  - main result - thesis work
- Our proof of thm 1
  - 3 types of balls
  - type 2 balls - more details. two subtypes
  - type 1 (3) balls - more details

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Dimension free estimates - a sample

Theorem: ([SS83])
Ball maximal function is L^p bounded.
Suppose $f : \mathbb{R}^d \to \mathbb{C}$. Let

$$M_d(f)(x) = \sup_{r>0} \frac{1}{\text{Volume}(\text{Ball}(0,r))} \int_{\text{Ball}(0,r)} |f(x-y)| dy.$$

Then

$$\|M_d(f)\|_p \leq C_p \|f\|_p$$

for $1 < p \leq \infty$. The constant C_p is **independent** of d.

Dimension free estimates - a sample

Theorem: ([Ste83])
Size of Riesz vector is L^p bounded.
If $f : \mathbb{R}^d \to \mathbb{C}$ and $R_j(f)$ is the j^{th} Riesz transform, then

$$\|(\sum_{j=1}^{d} |R_j(f)|^2)^{\frac{1}{2}}\|_p \leq C_p \|f\|_p$$

for $1 < p < \infty$. The constant C_p is **independent** of d.
Reminder: j^{th} Riesz transform is defined by

$$(\widehat{R_j(f)})(\xi) = i\frac{\xi_j}{\|\xi\|}\widehat{f}(\xi)$$

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GMT

Before quantitative rectifiabilty:
A set K is called 1-rectifiable iff

$$K \subset \bigcup_{i \in \mathbb{N}} \Gamma_i$$

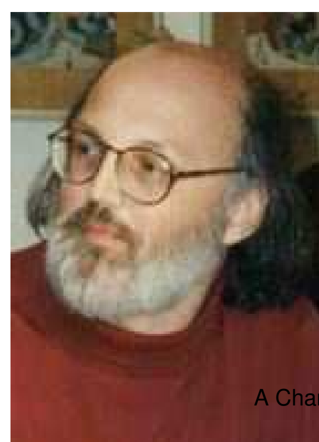
except for a set of 1-dim Hausdorff measure 0, where Γ_i is the image of a Lipschitz function (= a curve).

- Note that there is no mention of the length of a curve or how many of them there are!
- When is a set K contained inside a single curve of finite length?
- How long is the shortest curve?

Quantitative Rectifiability

This was answered in the setting of $\mathbb{R}^d$ by Peter Jones and Kate Okikiolu.

- Intuitive Picture:
  - A connected set of finite length is ‘flat’ on most scales and in most locations.
  - This can be used to characterize subsets of finite length connected sets.
  - One can give a quantitative version of this using multiresolutional analysis.

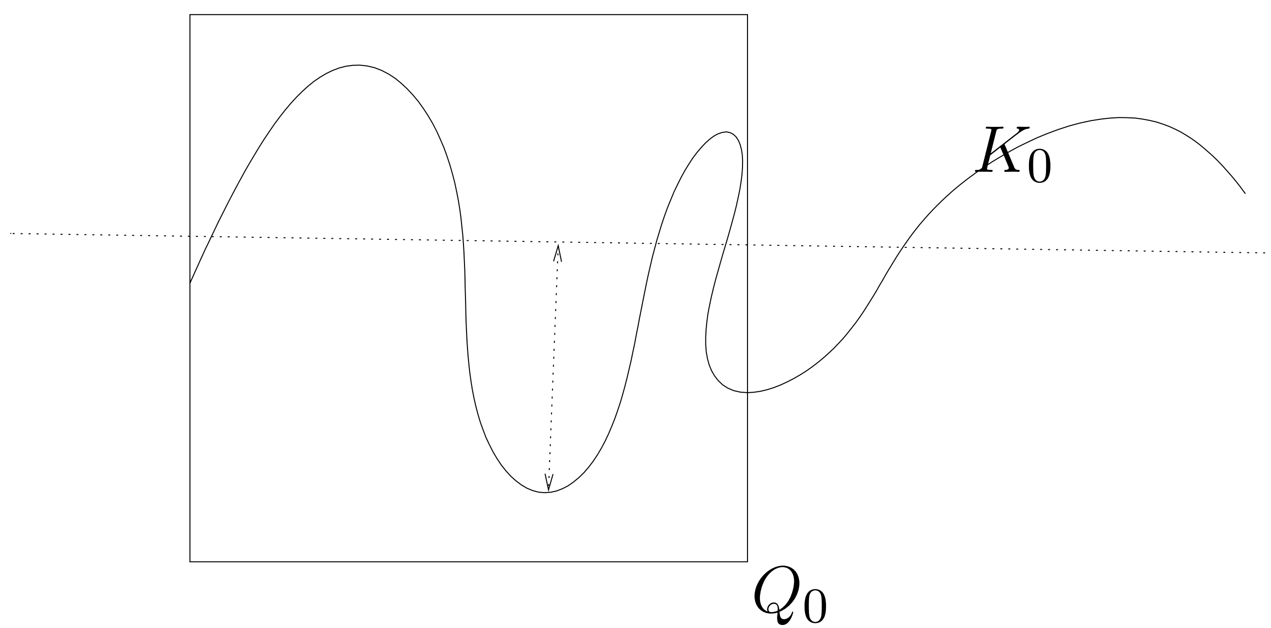


Quantitative Rectifiability

Definition: (Jones β number)

$$\beta_{K_0}(Q_0) = \frac{1}{\text{diam}(Q_0)} \inf_{L \text{ line}} \sup_{x \in K_0 \cap Q_0} \text{dist}(x, L)$$

$$= \frac{\text{radius of the thinest tube containing } K_0 \cap Q_0}{\text{diam}(Q_0)}.$$



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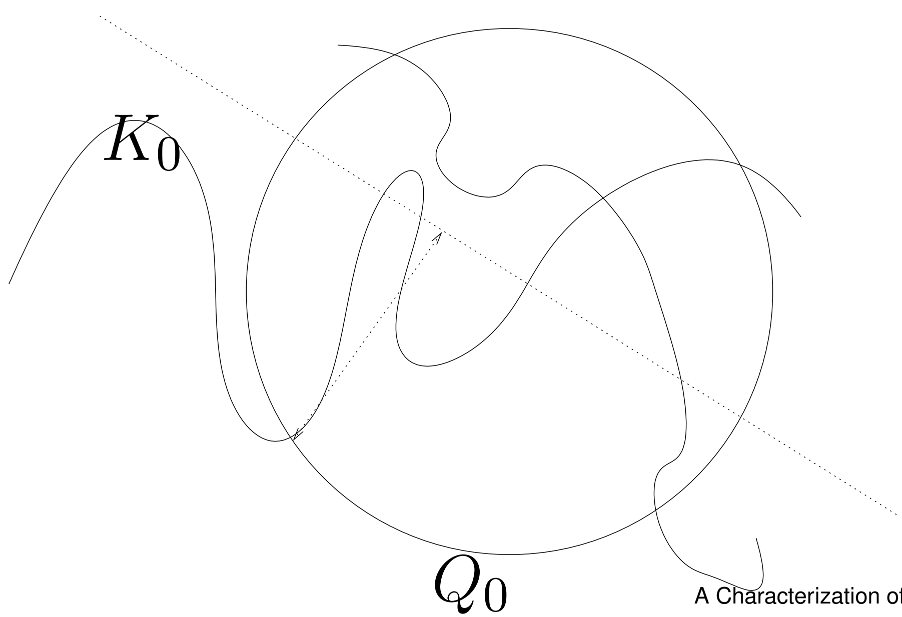

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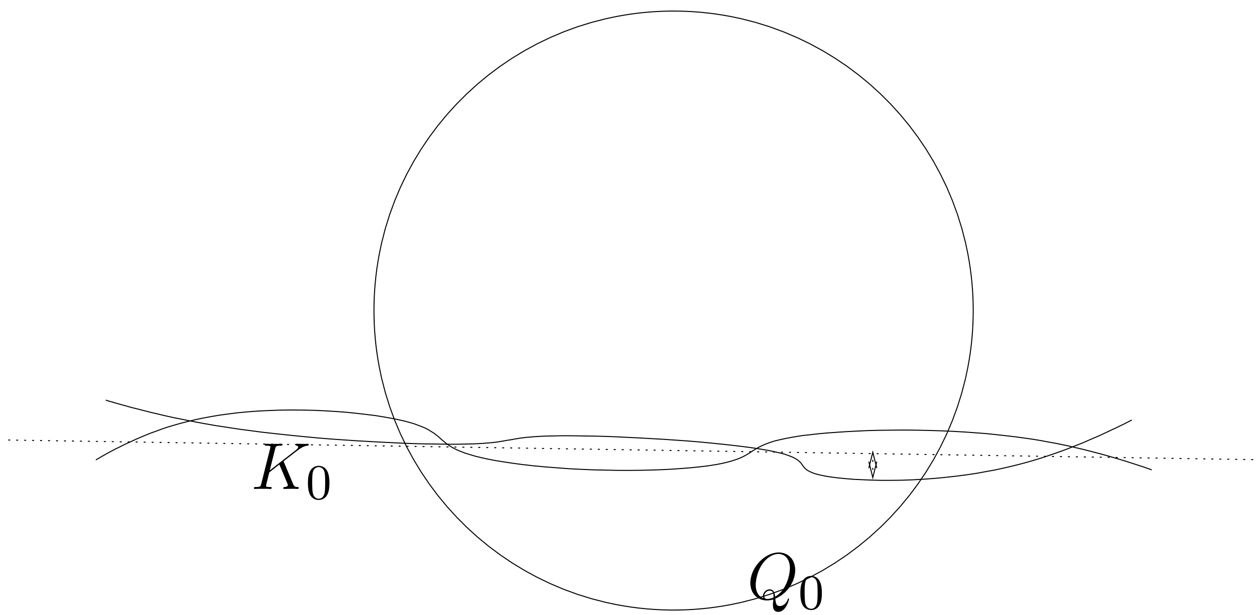

Quantitative Rectifiability

Jones ([Jon90]):

Theorem 1: For any connected $\Gamma \subset \mathbb{C}$

$$\sum_{Q \in \text{dyadic grid}} \beta_\Gamma^2(3Q)\operatorname{diam}(Q) \lesssim \ell(\Gamma)$$

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Theorem 2: For any set $K \subset \mathbb{R}^d$, there exists $\Gamma_0 \supset K$, Γ_0 connected, such that

$$\ell(\Gamma_0) \lesssim \sum_{Q \in \text{dyadic grid}} \beta_K^2(3Q)\mathrm{diam}(Q) + \mathrm{diam}(K)$$

(and in particular $K \subset \mathbb{C}$).

Quantitative Rectifiability

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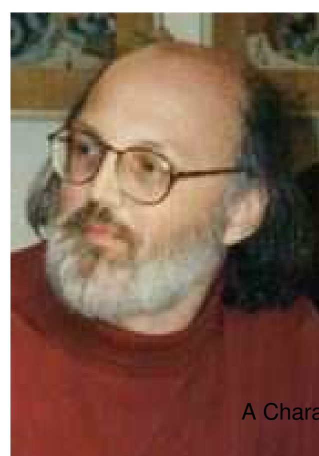


Quantitative Rectifiability

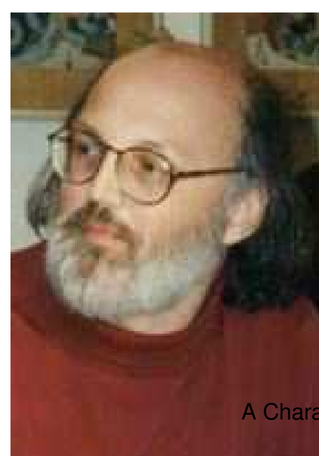
Jones ([Jon90]): **+ Okikiolu ([Oki92])**

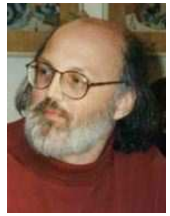
Theorem 1: For any connected $\Gamma \subset \mathbb{C}$ **or** $\Gamma \subset \mathbb{R}^d$

$$\sum_{Q \in \text{dyadic grid}} \beta_\Gamma^2(3Q)\operatorname{diam}(Q) \lesssim \ell(\Gamma)$$

Theorem 2: For any set $K \subset \mathbb{R}^d$, there exists $\Gamma_0 \supset K$, Γ_0 connected, such that

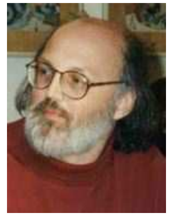
$$\ell(\Gamma_0) \lesssim \sum_{Q \in \text{dyadic grid}} \beta_K^2(3Q)\operatorname{diam}(Q) + \operatorname{diam}(K)$$



Corollary:
For any connected set $\Gamma \subset \mathbb{R}^d$

$$\operatorname{diam}(\Gamma) + \sum_{Q \in \text{dyadic grid}} \beta_\Gamma^2(3Q)\operatorname{diam}(Q) \sim \ell(\Gamma)$$

More generally:
For any set $K \subset \mathbb{R}^d$

$$\mathrm{diam}(K) + \sum_{Q \in \text{dyadic grid}} \beta_K^2(3Q)\mathrm{diam}(Q) \sim \ell(\Gamma_{MST})$$

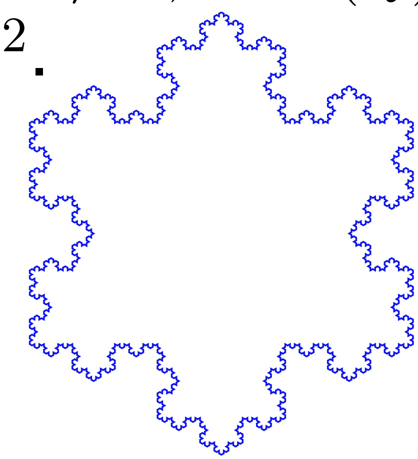
where Γ_{MST} is the shortest curve containing K.

Proof of corollary:

$$
\begin{aligned}
\ell(\Gamma_{MST}) & \\
&\leq \ell(\Gamma_0) \\
&\lesssim \operatorname{diam}(K) + \sum_{Q \in \text{dyadic grid}} \beta_K^2(3Q)\operatorname{diam}(Q) \\
&\leq \operatorname{diam}(K) + \sum_{Q \in \text{dyadic grid}} \beta_{\Gamma_{MST}}^2(3Q)\operatorname{diam}(Q) \\
&\lesssim \ell(\Gamma_{MST})
\end{aligned}
$$

- Allowed people to prove results about one dimensional sets in $\mathbb{C}$ or $\mathbb{R}^d$. (e.g. [BJ90])
- Another example:([BJ97])

  Γ connected, with
  $\beta_\Gamma(3Q) \geq \epsilon, \forall Q \cap \Gamma \neq \emptyset, \mathrm{diam}(Q) \leq \mathrm{diam}(\Gamma)$ then
  $dim_H(\Gamma) \geq 1 + c\epsilon^2$.



- Set the basis for 'The Theory of Quantitative Rectifiability' (developed by David, Semmes, Pajot , Verdera, Lerman and others)

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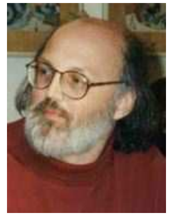

Dictionary discovered by Peter Jones.

wavelets	Jones' β numbers
$\{a_{j,k}\}$ for function f	$\{\beta(Q)\}$ for set K
analysis and synthesis of the function f	analysis and synthesis of curve $\Gamma \supseteq K$
$\|f\|^2 = \sum \|a_{j,k}\|^2$	$l(\Gamma) \sim \sum \beta(Q)^2 \cdot \mathrm{diam}(Q) + \mathrm{diam}(\Gamma)$
Wavelet square function $W_\psi(x)^2$	Jones' function $J(x)$

Slide by Gilad Lerman.

(define $J(x)$ on board)

 

Reminder

Theorem 1: For any connected $\Gamma \subset \mathbb{R}^d$

$$\sum_{Q \in \text{dyadic grid}} \beta_\Gamma^2(3Q)\operatorname{diam}(Q) \lesssim \ell(\Gamma)$$

Theorem 2: For any set $K \subset \mathbb{R}^d$, there exists $\Gamma_0 \supset K$, Γ_0 connected, such that

$$\ell(\Gamma_0) \lesssim \sum_{Q \in \text{dyadic grid}} \beta_K^2(3Q)\operatorname{diam}(Q) + \operatorname{diam}(K)$$

Issue to fix

Constants that make inequalities true are **exponential in d**.

- **Want:** Hilbert Space version ($\implies$ dim free) of above (would allow Quantitative Rectifiability in Hilbert space).
- Some results for Ahlfors-David curves have proofs that work (either as is or with small variations) in Hilbert space! Examples appear in [Dav91, DS93].

 

Different formulation

Take $\{X_n\}_n$ a sequence of nested nets.

$X_n \subset K$ a 2^{-n} net, $X_n \subset X_{n+1}$

$$\mathcal{G}^K = \{B(x, A2^{-n}) : x \in X_n; n \in \mathbb{Z}\}$$

Theorem 1: For any connected $\Gamma \subset \mathbb{R}^d, \Gamma \supset K$

$$\sum_{Q \in \mathcal{G}^K} \beta_\Gamma^2(Q)\mathrm{diam}(Q) \lesssim \ell(\Gamma)$$

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constants that make inequalities true are **exponential in d**

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My Thesis Work

Take $\{X_n\}_n$ a sequence of nested nets.
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With constants **independent of dimension!**

We actually show the theorems for Γ or K in **Hilbert space**.

My Thesis Work

Corollary:
For any set $K \subset$ Hilbert Space

$$\mathrm{diam}(K) + \sum_{Q \in \mathcal{G}^K} \beta_K^2(Q)\mathrm{diam}(Q) \sim \ell(\Gamma_{MST})$$

where Γ_{MST} is the shortest curve containing K.

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Pf of thm 1; (main ideas)

A Few Notes Before We Start:

- We must deal with balls, not dyadic cubes. For each scale they are highly overlapping.
- It suffices to prove the theorem for $\Gamma \subset \mathbb{R}^d$ as long as the constants we get are independent of d.
- Given a compact connected set Γ, we fix a parametrization γ. The parametrization arclength is $\sim$ to the one dimensional Hausdorff length.
- Okikiolu's outline

Pf of thm 1; (main ideas)

$$\begin{aligned} \Lambda(Q) &:= \{\tau \subset \Gamma \cap Q : \partial\tau \subset \partial Q; \tau \text{ is a subarc of } \gamma\} \\ &= \gamma(\text{connected components of } \gamma^{-1}(Q)). \end{aligned}$$

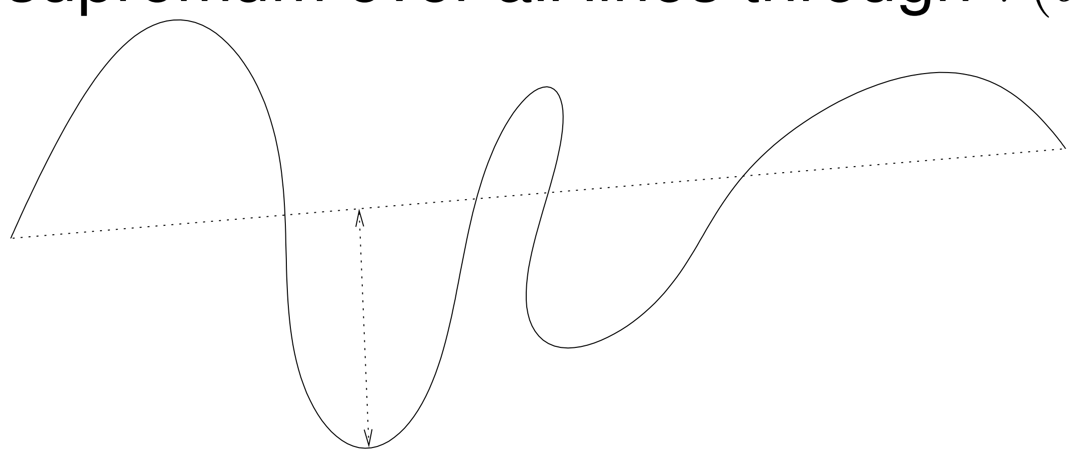
(draw on board)

Pf of thm 1; (main ideas)

We will call $\tau : [a,b] \longrightarrow \mathbb{R}^d \in \Lambda(Q)$ *almost straight* iff

$$\beta(\tau) := \sup_{t \in [a,b]} \frac{\text{dist}(\tau(t), < [\tau(a), \tau(b)] >)}{\text{diam}(Q)} < \epsilon_2 \beta(Q)$$

where $\tau \in \Lambda(Q)$ and $< [x,y] >$ is the line containing the segment $[x,y]$ (this is how we define the Jones β number of an arc). If $< [\tau(a), \tau(b)] >$ is not well defined (i.e. $\tau(a) = \tau(b)$) take the supremum over all lines through $\tau(b)$.



Pf of thm 1; (main ideas)

$$S_Q := \{\tau \in \Lambda(Q) : \beta(\tau) < \epsilon_2 \beta(Q)\}$$

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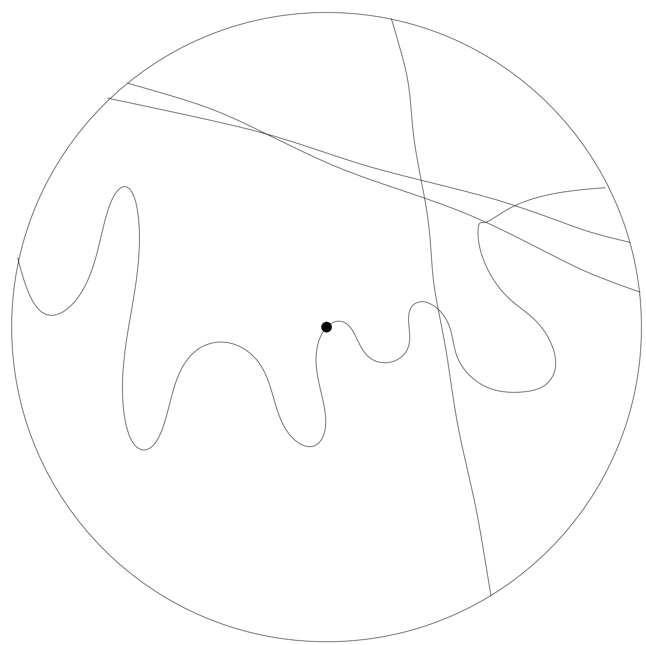

Example of Q

Pf of thm 1; (main ideas)

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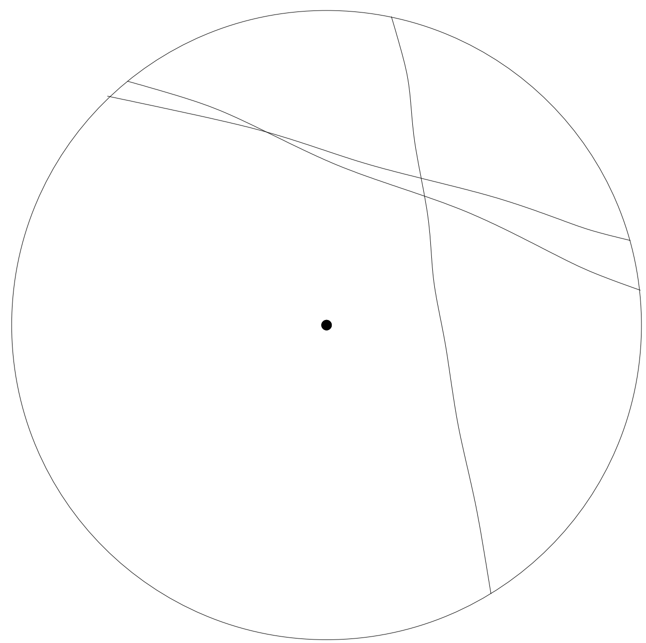

Example of S_Q

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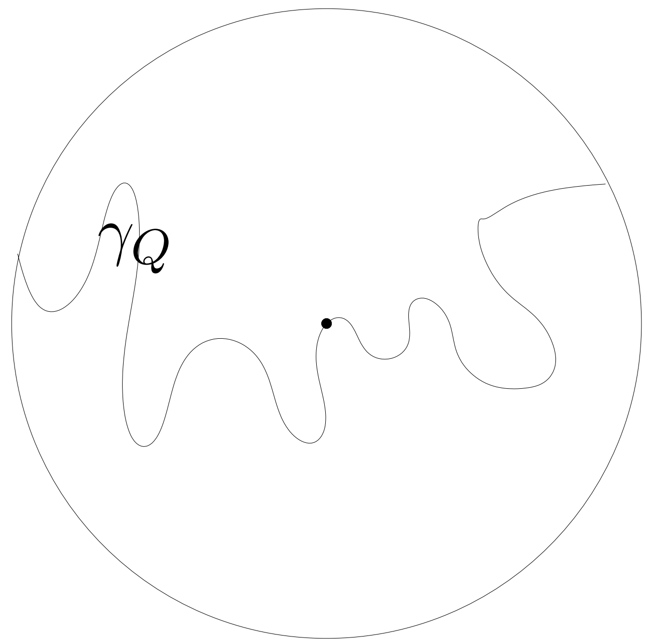
Pf of thm 1; (main ideas)

Fix $\gamma_Q \in \Lambda_Q$ an arc containing the center of Q.

Type 1: $\gamma_Q \notin S_Q$

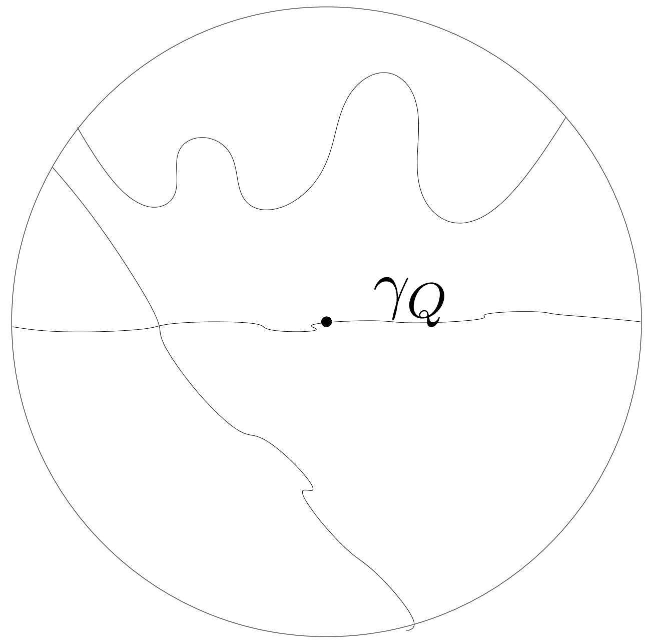


The ideas are a combination of ones that appear in [Dav91] and of Okikiolu's use of l^2 techniques.

Pf of thm 1; (main ideas)

Pf of thm 1; (main ideas)

Type 2: $\gamma_Q \in S_Q$ and $\epsilon_1 \beta(Q) \leq \beta_{S_Q}(Q)$



Pf of thm 1; (main ideas)

Type 2: $\gamma_Q \in S_Q$ and $\epsilon_1 \beta(Q) \leq \beta_{S_Q}(Q)$
Geometric ideas. For each ball Q a density w_Q is chosen, such that

$$\text{supp}(w_Q) \subset \Gamma \cap Q$$

$$\beta(Q)\mathrm{diam}(Q) \lesssim \int_Q w_Q$$

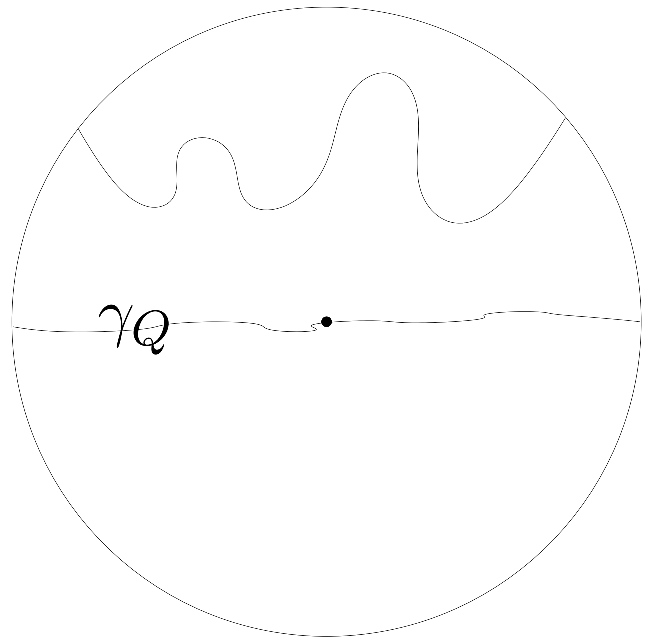
$$\sum_{Q \ni x} w_Q(x) \lesssim 1 \text{ for a.e. } x$$

(small lie)

Also uses some 'type 1' techniques (to get rid of lie).

Pf of thm 1; (main ideas)

Type 3: $\gamma_Q \in S_Q$ and $\epsilon_1 \beta(Q) \geq \beta_{S_Q}(Q)$



Similar ideas to 'type1' balls. A little more technical.

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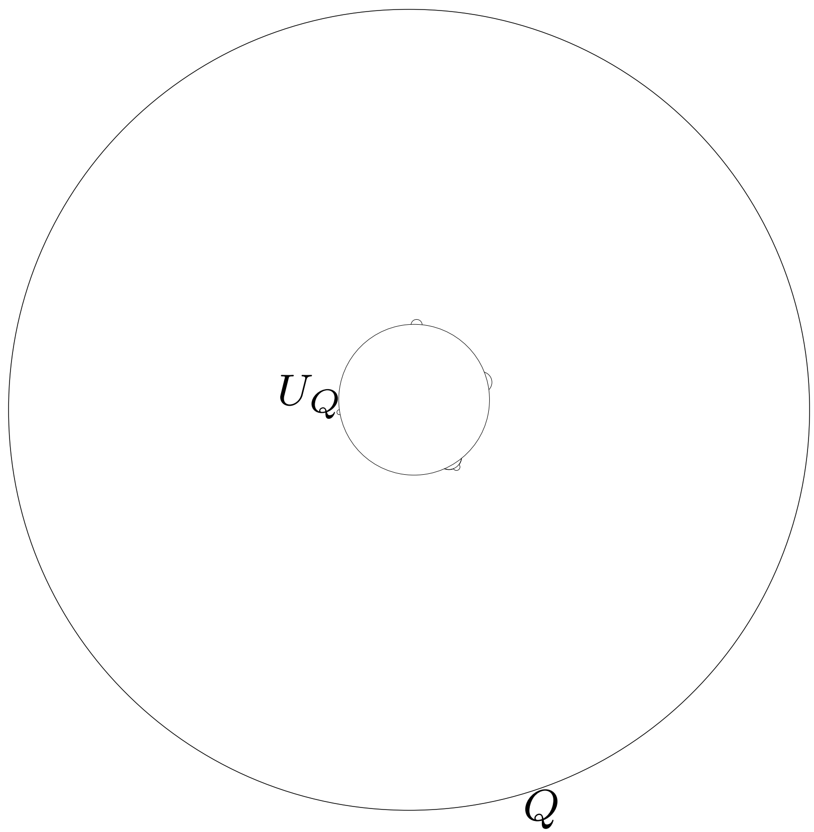
Type 2 Balls

For type 2 ball Q we have a core U_Q.
The cores are divided into families, such that in each family have nice nesting properties.

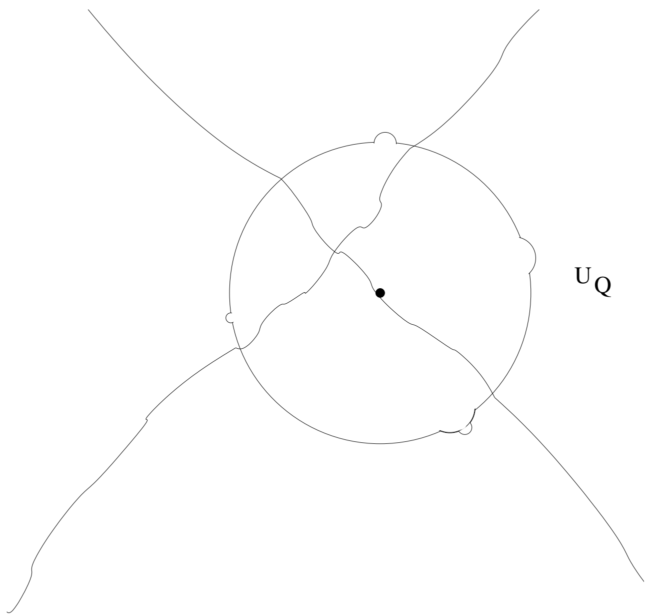


Type 2 Balls

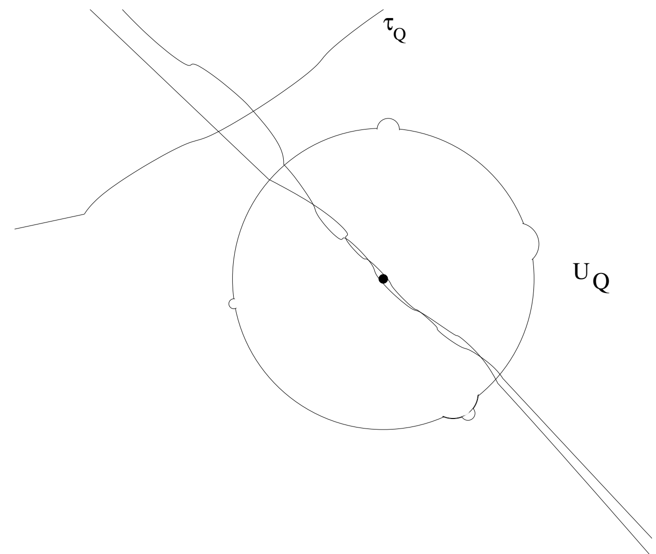
Consider one family. Have two subtypes of balls:

1^{st} subtype: $C\beta_{S_Q}(U_Q) > \beta_{S_Q}(Q)$

2^{nd} subtype: the rest



subtype 1



subtype 2

Type 2 Balls

Focus on $\Delta \subset 1^{st}$ subtype such that $\beta(U_Q) > \frac{1}{2}, \forall Q \in \Delta$

Assume $\Gamma \cap Q =$ union of straight lines $\forall Q \in \Delta$.

Lemma: Suppose $\Delta \subset$ type 2 balls such that $\beta(U_Q) > \mathbf{\frac{1}{2}}, \forall Q \in \Delta$. Then

$$\sum_{Q \in \Delta} \beta(Q)\mathrm{diam}(Q) \lesssim \ell(\Gamma).$$

idea : construct weights w_Q supported on $U_Q \cap \Gamma$ such that

$$\beta(Q)\mathrm{diam}(Q) \lesssim \int_Q w_Q$$

$$\sum_{Q \ni x} w_Q(x) \lesssim 1 \text{ for a.e. } x$$

Type 2 Balls

(draw on board)

$$U_Q = (\bigcup_i U_{Q^i}) \cup R_Q$$

U_{Q^i} maximal in U_Q, such that $Q^i \in \Delta$
$R_Q = U_Q \setminus \bigcup_i U_{Q^i}$.
(how would we change this if we had $\frac{1}{32} \leq \beta(U_Q) < \frac{1}{16}$??)

Type 2 Balls - Constructing w_Q

Set
$I_{Q'} :=$ large connected component of $\gamma_{Q'} \cap U_{Q'}$.
Set

$$\int_{U_Q} w_Q = \ell(I_Q).$$

We use $U_{Q'} = (\bigcup_i U_{Q'i}) \cup R_{Q'}$ to construct w_Q as a martingale.

Type 2 Balls - Constructing w_Q

We divide the mass non-evenly :

$$\int_{R_Q} w_Q = \frac{\int_{U_Q} w_Q}{s} \ell(R_Q) \text{ and } \int_{U_{Q^i}} w_Q = \frac{\int_{U_Q} w_Q}{s} \ell(I_{Q^i})$$

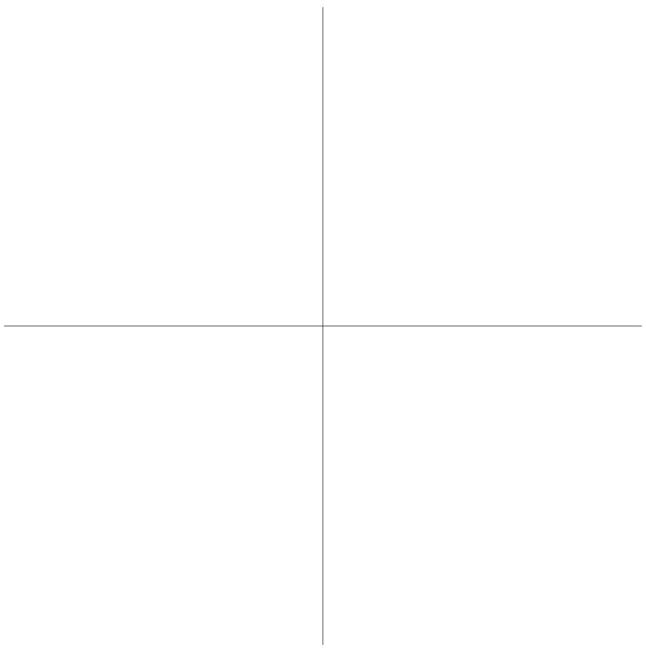
where $s = \ell(R_Q) + \sum_i \ell(I_{Q^i})$.

We reiterate: $U_{Q'} = (\cup U_{Q'^j}) \cup R_{Q'}$ then

$$\int_{R_{Q'}} w_Q = \frac{\int_{U_{Q'}} w_Q}{s'} \ell(R_{Q'}) \text{ and } \int_{U_{Q'^j}} w_Q = \frac{\int_{U_{Q'}} w_Q}{s'} \ell(I_{Q'^j})$$

where $s' = \ell(R_{Q'}) + \sum_j \ell(I_{Q'^j})$.

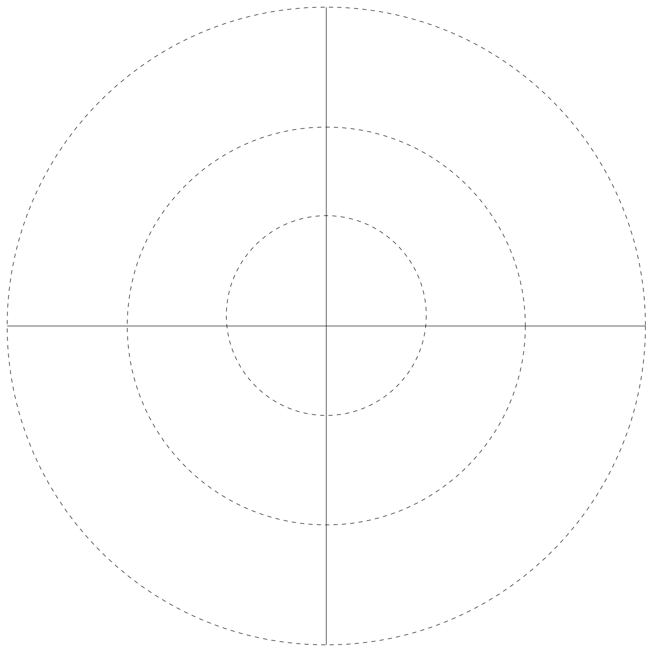
Type 2 Balls - Constructing w_Q



2 crossing segments

Figure 1: An (unnatural) example of weight distribution arising form the martingale.

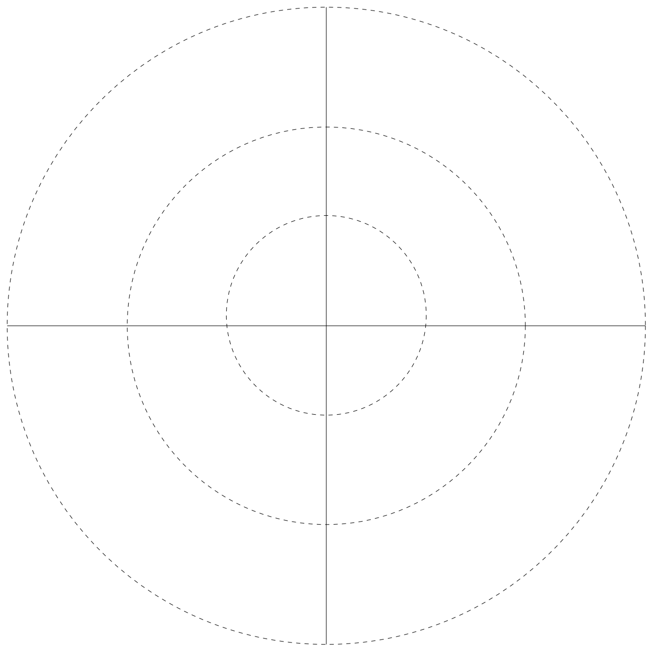
Type 2 Balls - Constructing w_Q



2 crossing segments with $U_{Q_1} \supset U_{Q_2} \supset U_{Q_3}$.
We have $M = 1$.
We assume these are the only U_Q's.

Figure 1: An (unnatural) example of weight distribution arising form the martingale.

Type 2 Balls - Constructing w_Q



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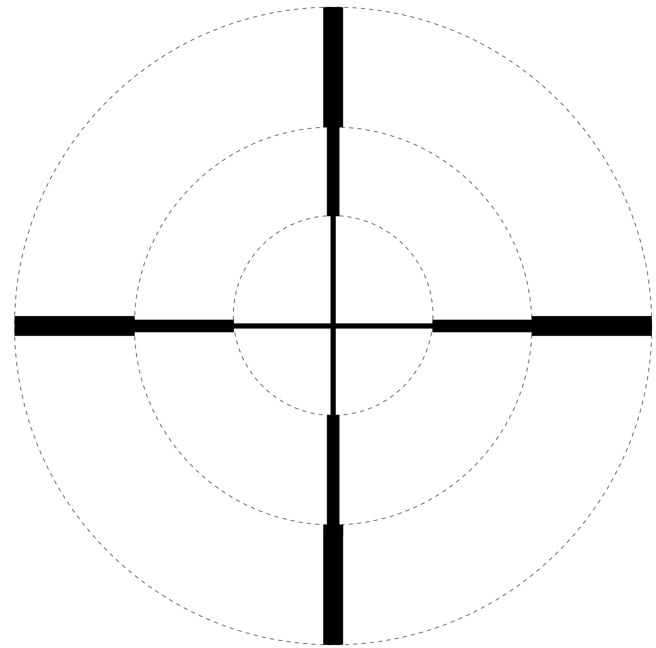
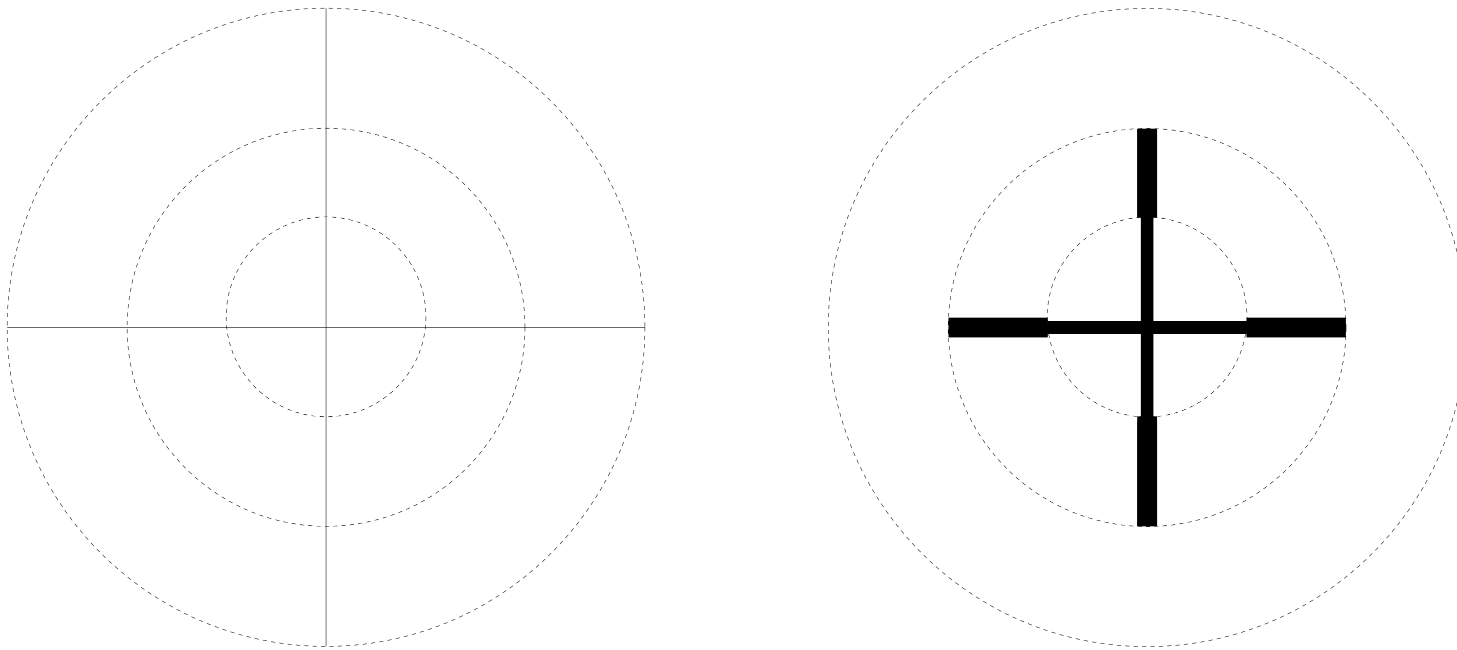


w_{Q_1}

Figure 1: An (unnatural) example of weight distribution arising form the martingale.

Type 2 Balls - Constructing w_Q

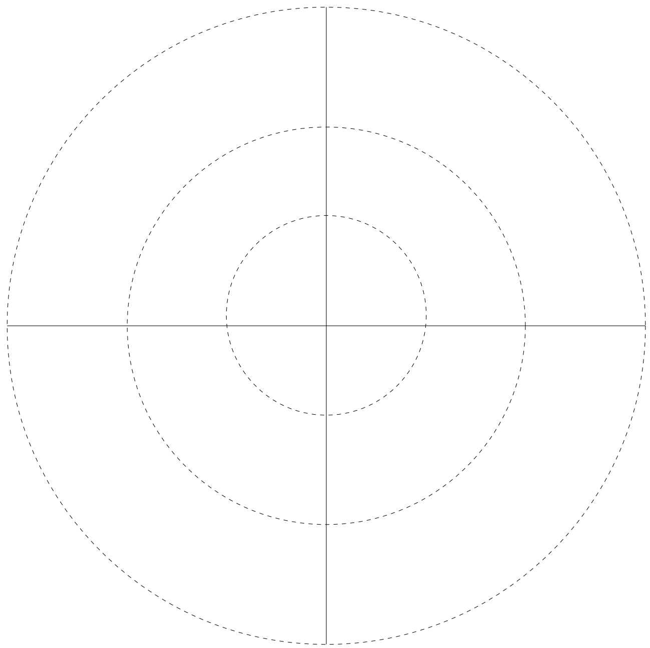


2 crossing segments with $U_{Q_1} \supset U_{Q_2} \supset U_{Q_3}$.
We have $M = 1$.
We assume these are the only U_Q's.

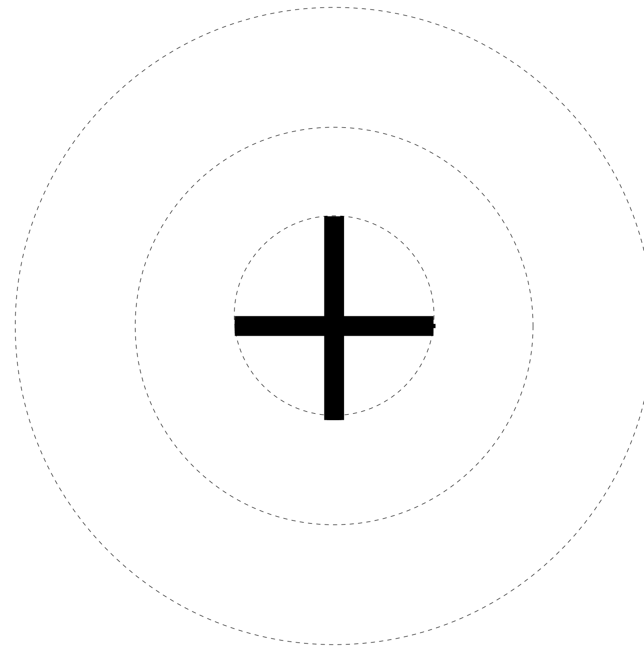
w_{Q_2}

Figure 1: An (unnatural) example of weight distribution arising form the martingale.

Type 2 Balls - Constructing w_Q



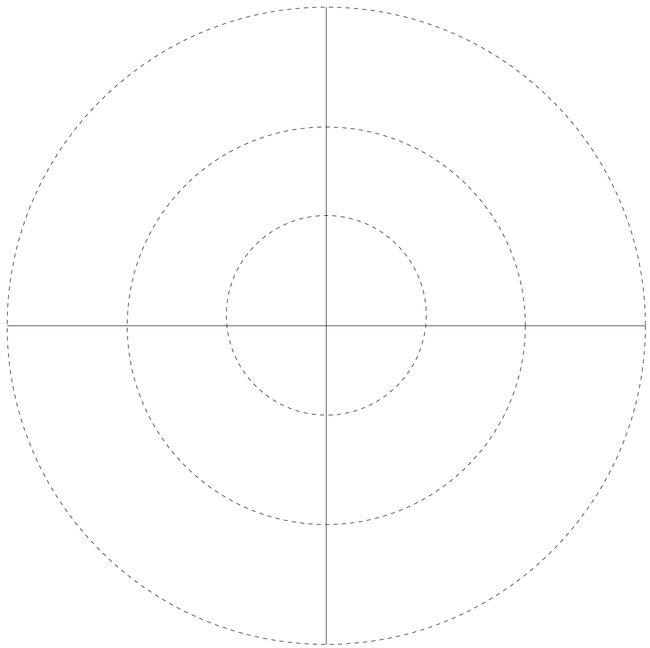
2 crossing segments with $U_{Q_1} \supset U_{Q_2} \supset U_{Q_3}$.
We have $M = 1$.
We assume these are the only U_Q's.



w_{Q_3}

Figure 1: An (unnatural) example of weight distribution arising form the martingale.

Type 2 Balls - Constructing w_Q



2 crossing segments with $U_{Q_1} \supset U_{Q_2} \supset U_{Q_3}$.
We have $M = 1$.
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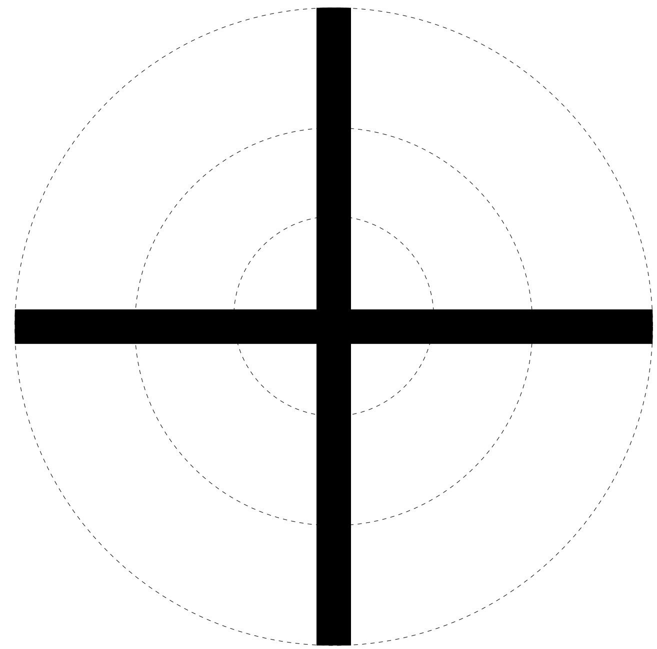


$$w_{Q_1} + w_{Q_2} + w_{Q_3}.$$

Figure 1: An (unnatural) example of weight distribution arising form the martingale.

Type 2 Balls - Constructing w_Q

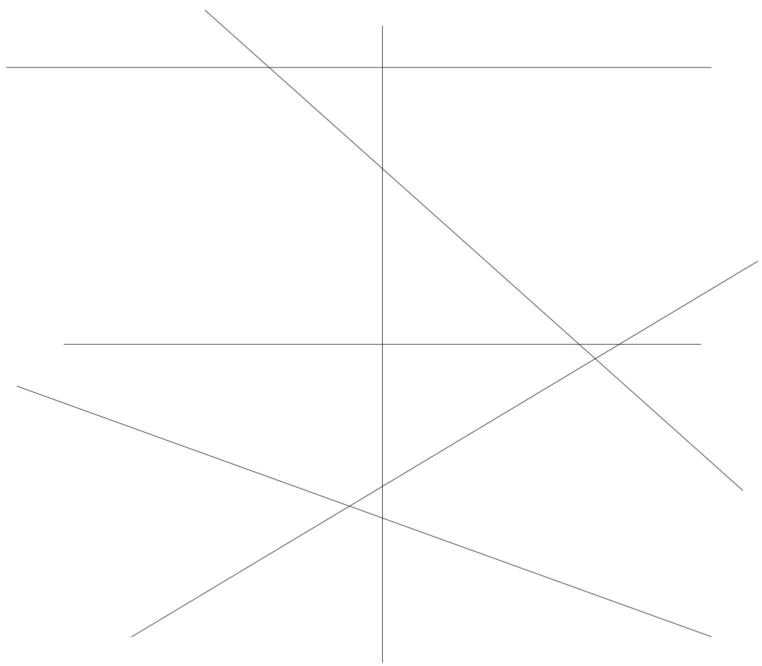


Figure 2: A more complicated (unnatural) example of weight distribution arising form the martingale.

Type 2 Balls - Constructing w_Q

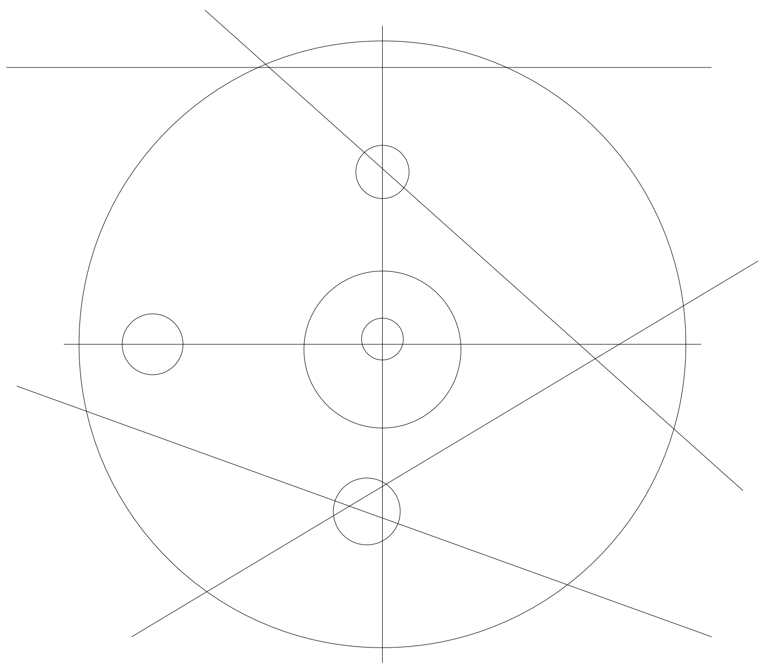


Figure 2: A more complicated (unnatural) example of weight distribution arising form the martingale.

Type 2 Balls - Constructing w_Q

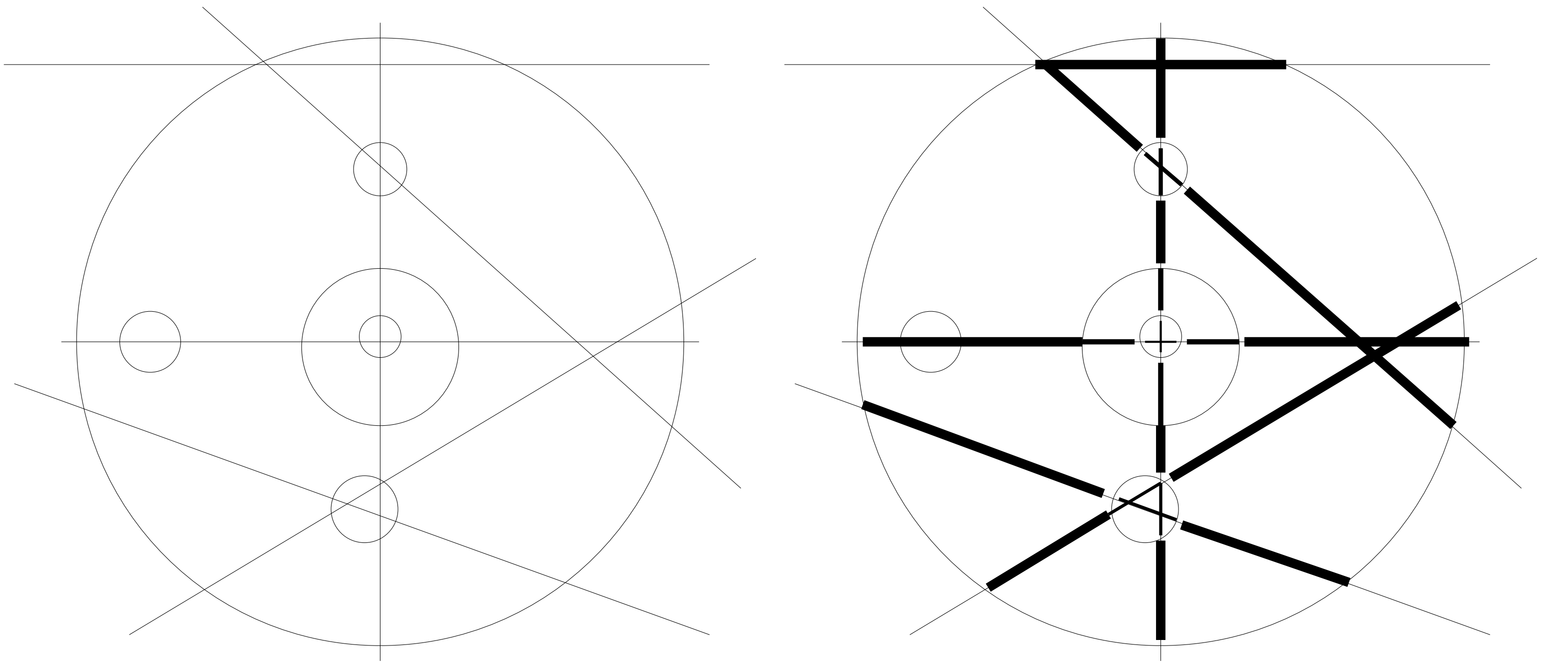


Figure 2: A more complicated (unnatural) example of weight distribution arising form the martingale.

Type 2 Balls - Constructing w_Q

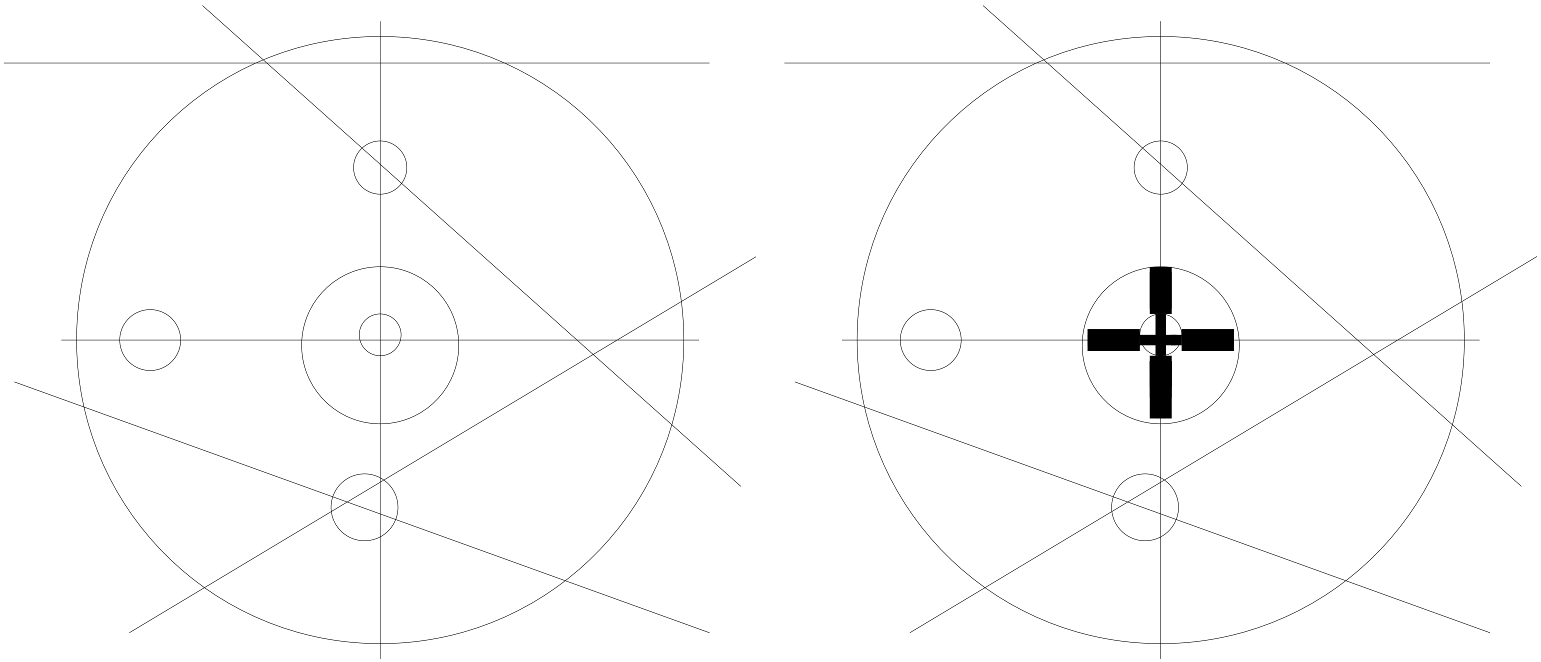


Figure 2: A more complicated (unnatural) example of weight distribution arising form the martingale.

Type 2 Balls

Focus on $\Delta \subset 1^{st}$ subtype such that
$2^{-M} \leq \beta(U_Q) < 2^{-M+1}, \forall Q \in \Delta$

Lemma: Suppose $\Delta \subset$ type 2 balls such that $2^{-M} \leq \beta(U_Q) < 2^{-M+1}, \forall Q \in \Delta$. Then

$$\sum_{Q \in \Delta} \beta(Q) \mathrm{diam}(Q) \lesssim \mathbf{M2^{-M}} \ell(\Gamma).$$

idea : weights w_Q supported on $U_Q \cap \Gamma$ such that

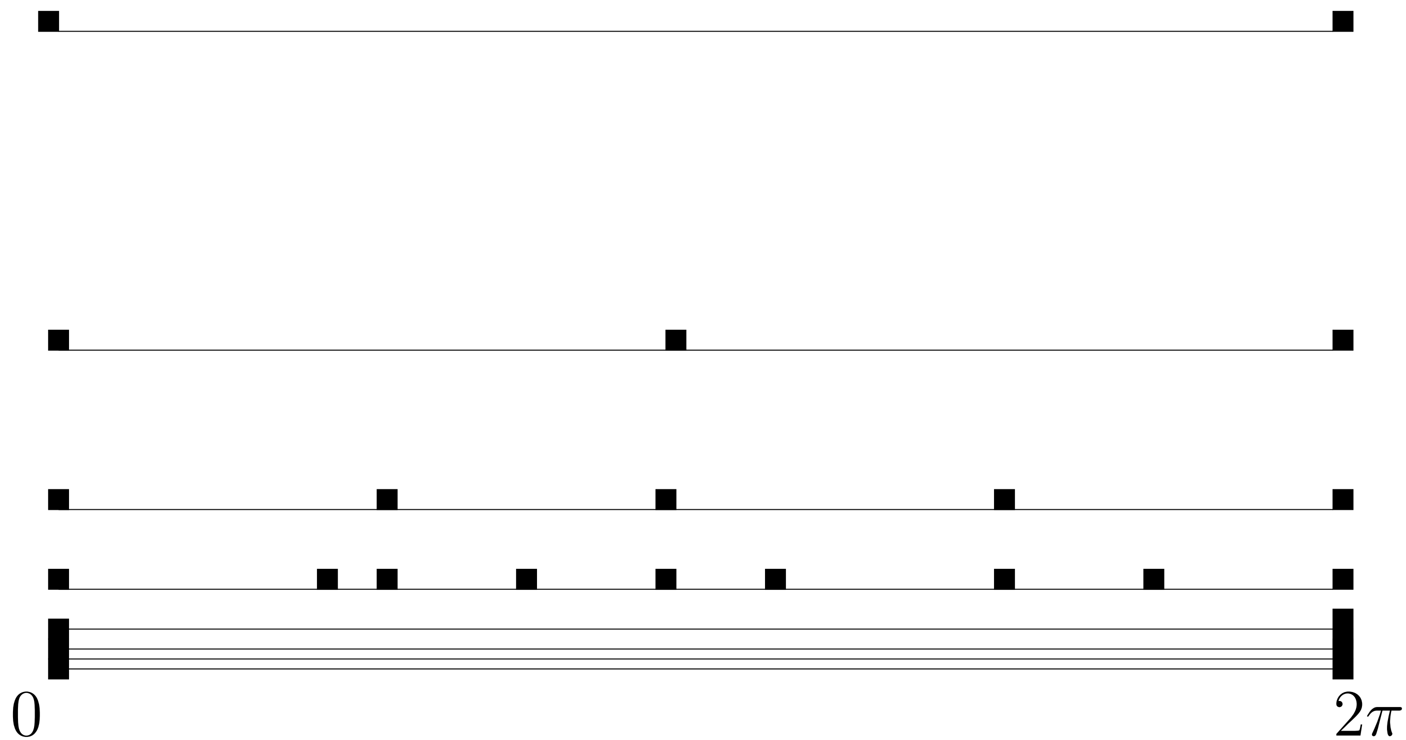
$$\beta(Q) \mathrm{diam}(Q) \lesssim \int_Q w_Q$$

$$\sum_{Q \ni x} w_Q(x) \lesssim \mathbf{M2^{-M}} \text{ for a.e. } x$$

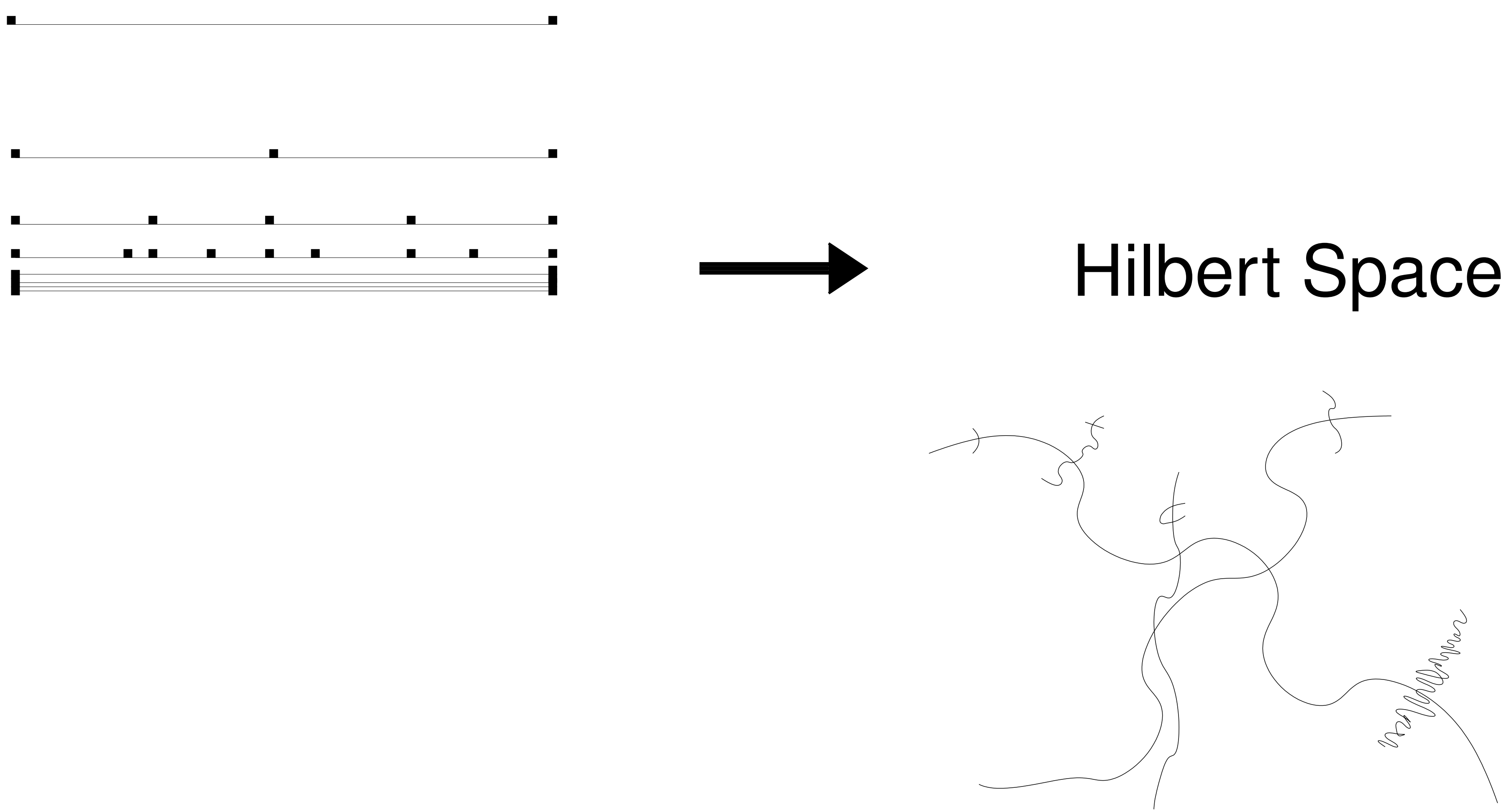
Outline

- Introduction
  - dimension free estimates in harmonic analysis
  - traveling salesmen theorems. Jones and Okikiolu
  - dictionary
  - main result - thesis work
- Our proof of thm 1
  - 3 types of balls
  - type 2 balls - more details. two subtypes
  - **type 1 (3) balls - more details**

Type 1 (and 3) Balls



Type 1 (and 3) Balls



Type 1 (and 3) Balls

Lemma: Suppose we are given a family of arcs $\mathcal{F} = \bigcup_{i=0}^{\infty} \mathcal{F}_i$ with the following properties:

(1) $\tau' \in \mathcal{F}_{n+1} \implies \exists! \tau \in F_n$ such that $\tau' \subset \tau$
(2) $\tau \in \mathcal{F}_n \implies 2^{-nJ} \leq \mathrm{diam}(\tau) \leq A 2^{-nJ+2}$
(3) $\tau, \tau' \in \mathcal{F}_n \implies \sharp(\tau \cap \tau') \in 0, 1, 2$ (the intersection is an empty set, a single point, or two points)
(4) $\bigcup_{\mathcal{F}_0} \tau = \bigcup_{\mathcal{F}_n} \tau \; \forall n$

(we will call such a family a filtration). Then we have:

$$\sum_{\tau \in \mathcal{F}} \beta(\tau)^2 \mathrm{diam}(\tau) \lesssim \ell(\bigcup_{\mathcal{F}_0} \tau)$$

proof copied almost word for word from [Oki92]

Type 1 (and 3) Balls

- How do we build a filtration relevant to our connected set?
- How do we relate $\beta(Q)$ to $\beta(\tau)$ for some τ in our filtration?

Type 1 (and 3) Balls

Lemma: Given a family of arcs $\mathcal{F}^0 = \bigcup_{i=0}^{\infty} \mathcal{F}_i^0$ with the following properties:

(1) $\tau \in \mathcal{F}_n^0 \implies \sharp\{\tau' \in \mathcal{F}_n^0 : \tau' \cap \tau \neq \emptyset\} \leq C$

(2) $\tau \in \mathcal{F}_n^0 \implies 2^{-n} \leq \mathrm{diam}(\tau) \leq A2^{-n+1}$.

Then we have $2CJ$ filtrations. Further more:
$\forall \tau' \in \mathcal{F}_n^0 \exists \tau \in \mathcal{F}_{nJ}$ for one of the filtrations, such that $\tau' \subset \tau$ and $\mathrm{diam}(\tau) < 2\mathrm{diam}(\tau')$ (and hence $\beta(\tau) \geq \frac{1}{4}\beta(\tau')$).
This mapping can be made to be injective.
J is a constant we fix; $J \geq 10$ will be more then enough.

- How do we get such a family?

Type 1 (and 3) Balls

Type 1:

$\{\gamma_Q\}_{Q \text{ type } 1} \rightarrow \{\tau_Q\}_{Q \text{ type } 1} \rightarrow$

lemma 2 $\rightarrow$ lemma 1

where $\tau_Q \subset \gamma_Q$ chosen to have some properties.

Type 3:

More technical

Metric spaces

Metric spaces

- Use Menger curvature to define β (As shown on tuesday).
- We need more axioms. Didn't have enough time to get a decent list that I have enough confidence with to present.
- In particular, axioms to give sensible $\beta(\tau)$
- Axioms to assure that if we have two 'almost straight' arcs then one has a large subarc that is far away(relative to their joint β) from the other.

Conclusion

- Have a characterization of subsets of rectifiable curves in Hilbert space.
- Have constructed weights that contain information about a large collection of straight lines that lie in Euclidean space.
- ideas *should* cary over to metric spaces with Menger curvature satisfying some axioms

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