Multilinear Oscillatory Integrals

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Three parts:
(1) Joint work with Li, Tao and Thiele
(2) Related to work of Bennett, Carbery, and Tao
(3) Joint work with J. Holmer

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$$T_\lambda(f_1, \cdots, f_n) = \int_{\mathbb{R}^d} e^{i\lambda P(y)} \prod_{j=1}^{n} f_j \circ \ell_j(y)\, \eta(y)\, dy$$

where

• P is a real-valued polynomial,

• $\lambda \in \mathbb{R}$ is a large parameter,

• η is a smooth compactly supported cutoff function,

• $\ell_j : \mathbb{R}^d \mapsto \mathbb{R}^{d_j}$ are surjective linear transformations.

Question. Is

$$|T_\lambda(\{f_j\})| \le C|\lambda|^{-\delta} \prod_j \|f_j\|_{L^\infty}$$

uniformly for all functions f_j as $|\lambda| \to \infty$?

This part of talk is joint with Li, Tao, Thiele.

We started with question posed by Lacey: Does bilinear Hilbert transform with oscillatory factor

$$\int_{\mathbb{R}} e^{it^k} f_1(x+t) f_2(x-t)\, t^{-1} dt$$

have same L^p mapping properties as the bilinear Hilbert transform without the oscillatory factor? We showed that it does, and the main step was to prove that

$$\Big| \iint e^{i\lambda t^k} f_1(x+t) f_2(x-t) f_3(x)\, \eta(x,t)\, dt\, dx \Big| \lesssim |\lambda|^{-\delta} \prod_i \|f_i\|_{L^2}.$$

We then realized that the nonsingular problem was the real issue.

Most fundamental example

$$\Big|\iint_{\mathbb{R}^d\times\mathbb{R}^d} e^{-i\lambda x\cdot y} f(x)g(y)\eta(x,y)\,dx\,dy\Big|$$
$$\le C|\lambda|^{-d/2}\|f\|_2\|g\|_2.$$

This inequality, after rescaling, implies the L^2 boundedness of the Fourier transform.

Note that here every point $x \in \mathbb{R}^d$ interacts with every point $y \in \mathbb{R}^d$. Our work is concerned with generalizations where the integral is not over $\prod_j \mathbb{R}^{d_j}$, but rather over a d-dimensional linear subspace of $\prod_j \mathbb{R}^{d_j}$. Thus most n-tuples of points $(x_1, \cdots, x_n) \in \prod_j \mathbb{R}^{d_j}$ *do not interact*.

In the linear/bilinear case $n = 2$ this problem has been studied intensively, in particular by Stein and by Phong-Stein but also by many others. For

$$\iint_{\mathbb{R}^{d+d}} e^{i\lambda P(x,y)} f(x) g(y) \eta(x,y)\, dx\, dy$$

a power decay bound holds if and only if there exist $\alpha \neq 0$ and $\beta \neq 0$ for which

$$\partial_x^\alpha \partial_y^\beta P(x,y) \text{ does not vanish identically;}$$

that is, if and only if P is not of the form $p(x) + q(y)$.

• In the truly multilinear case quite little is known. The focus here is on the basic question of whether there is any decay at all.

• From linear experience we expect the case of polynomial phase P to be fundamental, and then if there is any decay at all it seems reasonable to hope for power decay.

• We're putting the strongest norm on the functions f_j, and aren't trying to quantify δ. If there's any decay for L^∞, then there is also for any $(p_1, \cdots, p_n)$ for which the integral converges absolutely, except for endpoints.

• If some smoothness condition is imposed on the f_j it's a completely different problem.

• It is equivalent to ask whether
$\int e^{i\lambda(P - \sum_j h_j \circ \ell_j)} \eta \, dy = O(|\lambda|^{-\delta})$
uniformly for all measurable real-valued functions h_j.

Obvious necessary condition. If

$$\boxed{P = \sum_j h_j \circ \ell_j}$$

for some functions h_j then there's no decay (take $f_j = e^{-i\lambda h_j}$) to cancel out all the apparent oscillation.

Definition. P is nondegenerate relative to $\{\ell_j\}$ if P can not be represented as $\sum_j q_j \circ \ell_j$ for any functions q_j.

Question. Does power decay always hold for nondegenerate polynomial phase functions P?

This remains open, even for quadratic polynomials in three variables.

Suppose P homogeneous, to simplify statements.

Lemma. The following are equivalent:

• $P \neq \sum_j q_j \circ \ell_j$ for $\boxed{\text{polynomials}}$ q_j of degrees $\leq$ degree(P).

• $P \neq \sum_j h_j \circ \ell_j$ for any distributions h_j.

• There exists a constant-coefficient homogeneous linear partial differential operator $\mathcal{L}$ satisfying

$$\mathcal{L}(f_j \circ \ell_j) \equiv 0 \text{ for all functions } f_j, \text{ for all } j,$$
$$\mathcal{L}(P) \neq 0.$$

Warning. Nondegeneracy of P relative to $\{\ell_j : 1 \leq j \leq n\}$ imposes no bound whatsoever on n in terms of the degree of P and the ambient dimension d.

Two formally stronger notions of nondegeneracy emerge:

Definition. P is *simply nondegenerate* if there exists $\mathcal{L}$ of the form

$$\mathcal{L} = \prod_j (v_j \cdot \nabla)$$

which kills all functions $f_j \circ \ell_j$, yet $\mathcal{L}(P)$ does not vanish identically.

Definition. P is *discretely nondegenerate* if there exist a finite set S and coefficients c_s such that

$$\sum_{s \in S} c_s f_j \circ \ell_j(s) = 0 \text{ for all functions } f_j$$
$$\sum_{s \in S} c_s P(s) \neq 0.$$

Obviously both simple nondegeneracy and discrete nondegeneracy imply nondegeneracy.

Example: *Simple nondegeneracy is a strictly stronger notion than nondegeneracy.*

• Start with any homogeneous constant-coefficient linear PDO $\mathcal{L}$ such that the zero variety of its symbol σ is not a finite union of subspaces.

• Choose any homogeneous P of the same degree so that $\mathcal{L}(P) \neq 0$.

• Choose any finite set of distinct vectors v_j such that $\sigma(v_j) = 0$.

• Define $\ell_j(y) = \langle y, v_j \rangle$, mapping $\mathbb{R}^d$ to $\mathbb{R}^1$.

• If enough v_j are chosen then P won't be simply nondegenerate.

Theorem. If P is simply nondegenerate then it satisfies a power decay bound.

Recall notation: $\ell_j : \mathbb{R}^d \mapsto \mathbb{R}^{d_j}$.
Proposition. When each $d_j = d - 1$, simple nondegeneracy is equivalent to nondegeneracy.

Corollary. Nondegeneracy is equivalent to the power decay property in the codimension one case $d_j = d - 1$.

Theorem. If each $d_j = 1$ and if the number of functions n satisfies

$$n < 2d$$

then any nondegenerate polynomial P satisfies a power decay bound (under an auxiliary general position hypothesis on $\{\ell_j\}$).

The proof for simply nondegenerate P is based on a TT^*-type argument, related to Weyl's analysis of equidistribution for sequences $(n^2\gamma)_{\mathrm{mod} 1}$, and to more recent work of Carbery, Christ, and Wright on higher-dimensional analogues of van der Corput's lemma.

The proof of the theorem for $n < 2d$ is in part an L^2 Fourier-based analysis, relying on the simply nondegenerate case. A key idea is related to the work of Roth and Gowers on Szemerédi's theorem: one distinguishes the case where there exist a polynomial q and coefficient c such that

$$\|f_n - ce^{iq}\|_{L^2} \leq (1 - |\lambda|^{-\rho})\|f_n\|_{L^2}$$

for a suitably chosen exponent ρ.

$T_\lambda(f_1, \cdots, f_{n-1}, ce^{iq})$ is handled by induction on n, while the contribution of $f_n - ce^{iq}$ is OK because the norm has decreased.

The case where none of the f_j can be approximated in even this very weak sense by pure exponentials is (roughly) split into a simply nondegenerate case and a case where P is nearly zero; the latter becomes very simple when rewritten terms of Fourier transforms when all f_j are badly approximable.

Sublevel set bounds. Fix a bounded region B. Define $E_\varepsilon = E_\varepsilon(P, f_1, \cdots, f_n)$ by

$$E_\varepsilon = |\{x \in B : |P(x) - \sum_j f_j(\ell_j(x))| < \varepsilon\}.$$

If a power decay bound holds for the multilinear oscillatory integral operator with phase P then

$$|E_\varepsilon| \lesssim \varepsilon^\delta$$

uniformly for all measurable functions f_j.

Question(s). If we can't establish power decay for oscillatory integrals, can we at least prove the consequence $|E_\varepsilon| = O(\varepsilon^\delta)$ uniformly?

If not, can we at least prove that $|E_\varepsilon| \to 0$ as $\varepsilon \to 0$, uniformly for all f_j?

Observations.

(1) If $\{\ell_j\}$ are rationally commensurate, and if $|E_\varepsilon| \leq \Theta(\varepsilon)$ where $\Theta \to 0$ as $\varepsilon \to 0$ then P must be discretely nondegenerate.

(2) If P is discretely nondegenerate then

$$|E_\varepsilon(P, f_1, \cdots, f_n)| \leq \Theta(\varepsilon)$$

where $\Theta \to 0$ as $\varepsilon \to 0$.

(3) If $\{\ell_j\}$ are rationally commensurate, if all $d_j = 1$, and if a polynomial P is nondegenerate, then P is discretely nondegenerate.

Combining (2) and (3): If all $d_j = 1$ and $\{\ell_j\}$ are rationally commensurate then for any nondegenerate polynomial phase P, a weak sublevel bound $|E_\varepsilon| \leq \Theta(\varepsilon)$ holds. (But no effective bound on rate at which $\Theta \to 0$.)

Ideas of proofs for sublevel set observations.

(2) (Assume that P homogeneous, and any lower degree polynomial is degenerate, for simplicity.) Set $Q = P - \sum_j f_j \circ \ell_j$. Then

$$\sum_{s \in S} c_s Q(x + rs) \equiv cr^D$$

for all x, r, where $c \neq 0$ and $D = \text{degree}(P)$.

Unless $r^D \lesssim \varepsilon$ this yields a contradiction whenever $x + rs \in E_\varepsilon$ for each $s \in S$. Thus certain translations and dilations of S cannot lie in E_ε.

By a generalization of Szemerédi's theorem due to Furstenberg and Katznelson, this forces $|E_\varepsilon| \le \Theta(\varepsilon)$.

(3) Consider a large finite "sublattice" L of $\mathbb{Z}^d$. If P is not discretely nondegenerate then $P = \sum_j f_j \circ \ell_j$ on L.

By working with finite difference operators, in same spirit as differential operators were used to discuss nondegeneracy, show that each f_j must agree on $\mathbb{Z}^1$ with a polynomial whose degree is bounded by a quantity independent of the size of L.

If L is sufficiently large then $P - \sum_j f_j \circ \ell_j$ vanishes on too large a set, relative to its degree.

Even more elementary questions.

For what exponents $p_j \in [1, \infty]$ does the integral

$$\int_{\mathbb{R}^d} \prod_{j=1}^{n} f_j \circ \ell_j(y)\, \eta(y)\, dy$$

converge absolutely for all $f_j \in L^{p_j}(\mathbb{R}^{d_j})$? And what about the global version

$$\int_{\mathbb{R}^d} \prod_{j=1}^{n} f_j \circ \ell_j(y)\, dy?$$

This enters into discussion of the multilinear oscillatory integral operators in three distinct ways:

(A) If power decay holds at all, then it holds for all exponents for which the integral is guaranteed to converge absolutely, except endpoints.

(B) It's reasonable to guess that bounds in terms of $\prod_j \|f_j\|_{L^2}$ might play a fundamental role in the theory, but such a bound is conceivable only if absolute convergence holds for all $f_j \in L^2$.

(C) is to be explained later in the talk.

Example. Let $d_j = d - 1$ for $1 \leq j \leq n = d$. Identity the j-th target space $\mathbb{R}^{d_j}$ with $\{(x_1, x_2, \cdots, x_d) : x_j = 0\}$ and let π_j be the orthogonal projection of $\mathbb{R}^d$ onto this subspace.

Loomis and Whitney proved

$$\Big| \int_{\mathbb{R}^d} \prod_{j=1}^{d} f_j \circ \pi_j \Big| \leq C \prod_j \|f_j\|_{L^{p_j}(\mathbb{R}^{d-1})}$$

when all $p_j = d - 1$.

There is no such inequality for any other d-tuple of exponents $(p_1, \cdots, p_d)$.

For the global version, Bennett, Carbery, and Tao proved

Theorem. Let $\ell_j : \mathbb{R}^d \mapsto \mathbb{R}^{d_j}$ be surjective linear transformations. Then

$$\int_{\mathbb{R}^d} \prod_j f_j \circ \ell_j \, dy \le C \prod_j \|f_j\|_{L^{p_j}}$$

if and only if

$$\sum_j p_j^{-1} d_j = d$$

and

$$\sum_j p_j^{-1} \operatorname{dim}(\ell_j(V)) \ge \operatorname{dim}(V)$$

for every subspace $V \subset \mathbb{R}^d$.

Warning. Rearrangement does not work for this type of inequality, unless each $d_j = 1$.

Those authors have a clever, and to me surprising, proof (please see Carbery's lecture on Wednesday). There is an alternative proof which also establishes the following closely related variants:

Theorem. A necessary and sufficient condition for the inequality

$$\int_{|y|\leq 1} \prod_j f_j \circ \ell_j(y)\, dy \leq C \prod_j \|f\|_{L^{p_j}}$$

is that for every subspace V of $\mathbb{R}^d$,

$$d - \text{dim}(V) \geq \sum_j p_j^{-1}(d_j - \text{dim}(\ell_j(V))).$$

Next theorem unifies these local and global versions:

Theorem.

$$\int_{\mathbb{R}^d \cap \{y: |\ell_0(y)| \leq 1\}} \prod_{j=1}^{n} f_j \circ \ell_j(y)\, dy \leq C \prod_{j=1}^{n} \|f_j\|_{L^{p_j}}$$

for all nonnegative measurable f_j if and only if every subspace $V \subset \mathbb{R}^d$ satisfies

$$d - \text{dim}(V) \geq \sum_j p_j^{-1}\Big(d_j - \text{dim}(\ell_j(V))\Big)$$

and furthermore

$$\sum_j p_j^{-1}\, \text{dim}(\ell_j(V)) \geq \text{dim}(V) \quad \text{if } V \subset \text{kernel}(\ell_0)$$

Theorem. Let G and $\{G_j : j \le i \le N\}$ be finitely generated Abelian groups. Let $\varphi_j : G \mapsto G_j$ be homomorphisms whose ranges are subgroups of finite indices. Then

$$\sum_{y \in G} \prod_{j=1}^{N} f_j \circ \varphi_j(y) \le C \prod_j \|f_j\|_{\ell^{p_j}(G_j)}$$

for all nonnegative f_j if and only if

$$\sum_j p_j^{-1} \operatorname{rank}(\varphi_j(H)) \ge \operatorname{rank}(H)$$

for every subgroup H of G.

There's also a version for amalgamated spaces $\ell^p(L^\infty)(\mathbb{R}^{d_j})$, with functions that are locally bounded and globally in L^p.

Idea of proofs:
(1) If all indices p_j are 1 or ∞ then result is straightforward. Try to reduce to this by interpolation; perturb exponents.

Get stuck if equality holds in the hypothesized inequalities for some nonzero proper subspace $W \subset \mathbb{R}^d$.

(2) Then $\int_W \prod_j f_j \circ \ell_j$ is a lower-dimensional instance of the same problem. The hypotheses are inherited by W and $\ell_j|_W$ (mapping W to $\ell_j(W)$, not to $\mathbb{R}^{d_j}$).

(2a) Foliate $\mathbb{R}^d$ by translates of W. Apply induction hypothesis on each copy of W.

(2b) What's left turns out to be another instance of same problem of form $\int_{W^\perp} \prod_j F_j \circ L_j$, with function $F_j(\cdot)$ equal to L^{p_j} norm of f_j over a translate of $\ell_j(W)$. Again the hypothesized inequalities turn out to be inherited (not obvious!). Apply induction hypothesis again to conclude.

Lament: The results I stated earlier on multilinear oscillatory integral operators fail to cover a well-known and understood example, and the techniques don't yield optimal decay exponents δ.

Example: Twisted convolution.

$$\Big| \iint_{\mathbb{C}^n \times \mathbb{C}^n} e^{i\lambda \, \mathrm{Im}(z \cdot \bar{w})} f_1(z) f_2(w) f_3(z-w) \, dz \, dw \Big| \lesssim |\lambda|^{-n/2} \prod_j \|f_j\|_2.$$

This problem is self-dual in sense that if we rewrite it as a trilinear expression in the three Fourier transforms $\widehat{f_j}$, we obtain precisely the same expression except for changes in various constants.

This last part of the talk is a preliminary report on joint work with Justin Holmer. We've analyzed the inequality

$$\Big| \int_{\mathbb{R}^d} e^{i\lambda Q(y)} \prod_{j=1}^{n} f_j \circ \ell_j(y) \eta(y)\, dy \Big| \le C|\lambda|^{-\delta_0} \prod_j \|f_j\|_{L^2}$$

where Q is a homogeneous quadratic polynomial, all $d_j = D$, all norms on the right-hand side are L^2 norms, and

$$\delta_0 = \frac{d}{2} - \frac{nD}{4}$$

is the largest exponent for which such an estimate isn't ruled out by scaling considerations. Thus we're trying to characterize the maximally nondegenerate phase functions.

We've established a sufficient condition which we believe is also necessary. Unfortunately, we don't yet have a palatable formulation of our sufficient condition, so I'll explain the method of proof without formulating the result.

FBI transform. Define

$$\mathcal{F}(f)(x,\xi) = \langle f, \varphi_{(x,\xi)} \rangle$$

where

$$\varphi_{(x,\xi)}(y) = e^{iy\cdot\xi} e^{-|x-y|^2/2}.$$

One has

$$\|f\|_{L^2(\mathbb{R}^d)} = c_d \|\mathcal{F}(f)\|_{L^2(T^*\mathbb{R}^d)}$$

and

$$f(y) = c_d \int_{T^*(\mathbb{R}^d)} \varphi_{(x,\xi)}(y) \mathcal{F}(f)(x,\xi)\, dx\, d\xi.$$

Proving the desired multilinear L^2 bound is equivalent to proving a global inequality without any large parameter:

$$\Big| \int_{\mathbb{R}^d} e^{iQ} \prod_j f_j \circ \ell_j \Big| \le C \prod_j \|f_j\|_{L^2}.$$

Here there is a preferred unit scale. With respect to the FBI transform there is no longer any self-duality.

Expressing each f_j in terms of $\mathcal{F}(f_j)$ yields:

$$\int_{\oplus_j T^*(\mathbb{R}^D)} a(x,\xi) \prod_j \mathcal{F}(f_j)(x_j,\xi_j)\, dx\, d\xi$$

where $(x,\xi) = (x_1,\xi_1,\cdots,x_n,\xi_n) \in (\mathbb{R}^{2D})^n$ and

$$|a(x,\xi)| \lesssim e^{-c\,\text{distance}((x,\xi),\Sigma)^2}$$

where the linear subspace Σ equals the set of all (x,ξ) for which there exists $y \in \mathbb{R}^d$, necessarily unique, such that

$$\ell_j(y) = x_j \text{ for all } j$$
$$\nabla Q(y) + \sum_j \ell_j^*(\xi_j) = 0.$$

Moreover a exhibits no useful cancellation or decay on Σ. Thus this expression is essentially

$$\int_\Sigma \prod_j \mathcal{F}(f_j)(x_j,\xi_j)\, d\sigma$$

where σ is Lebesgue measure on Σ. This is a nonoscillatory multilinear integral operator of precisely the type discussed in the middle portion of this talk.

Now a crucial observation is that (under certain hypotheses of general position on $\{\ell_j\}$) the $\boxed{\text{dimension of } \Sigma \text{ is always half}}$ of the dimension of the ambient space $\oplus_j T^*(\mathbb{R}^{d_j})$. Thus scaling considerations are consistent with a bound

$$\Big|\int_\Sigma \prod_j F_j(x_j,\xi_j)\,d\sigma\Big| \le C\prod_j \|F_j\|_{L^2(T^*(\mathbb{R}^{d_j})},$$

and we have $F_j = \mathcal{F}(f_j) \in L^2$ if $f_j \in L^2$ by the Plancherel identity for the FBI transform.

Our preliminary theorem says that the original multilinear oscillatory integral operator satisfies the strongest possible L^2 decay estimate provided that Σ (that is, Σ together with the collection of mappings $\pi_j|_\Sigma$ where $\pi_j : \oplus_i T^*(\mathbb{R}^{d_i}) \mapsto T^*(\mathbb{R}^{d_j})$ is the canonical projection) satisfies the hypothesis of the theorem of Bennett, Carbery, and Tao with all exponents $p_j = 2$.

Special cases include the inequality for twisted convolution, and Plancherel's inequality itself.