Hilbert Transform on $C^{1+\epsilon}$ Families of Lines

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Outline

Besicovitch Set

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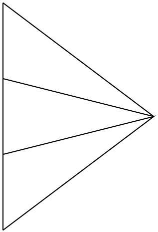
- This set necessarily has Hausdorff dimension two, a fundamental fact in this subject.
- One constructs highly eccentric rectangles which have small union, but the translates along their long direction, their "reach," are essentially disjoint.
- We briefly outline the construction of this set.

Besicovitch Set Construction



The triangle contains unit length line segments in a full angle of directions.

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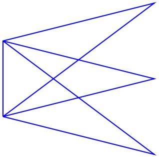
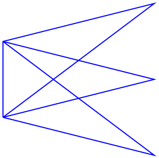


The thirds of the triangle are moved so that they share a common base.

Besicovitch Set Construction

Reflect the triangles about their vertexes.



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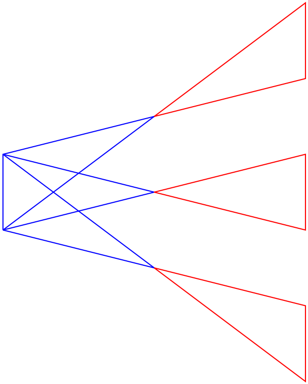

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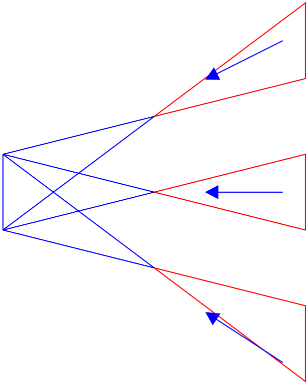

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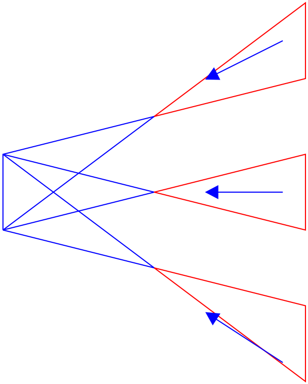

The red triangles are essentially disjoint, and are called the "reach" of the set. A vector field, which points into the set, defined in the "reach" can be Hölder continuous, of any index strictly less than one. Conversely, if v is Lipschitz, then the "Besicovitch set" is has can't have zero Lebesgue measure.

Zygmund Conjecture

If v is Lipschitz, then for all $f \in L^2(\mathbb{R}^2)$,

$$f(x) = \lim_{\epsilon \downarrow 0} (2\epsilon)^{-1} \int_{-\epsilon}^{\epsilon} f(x - yv(x))\, dy \qquad \text{a.e.}(x)$$

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Both conjectures are open. They represent very subtle statements about the nature of the Besicovitch set.

Main Results of Xiaochun Li and L.

Theorem (L. & Li)

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Key Proposition (Scale Invariant Formulation)

If v is Lipschitz, then

$$\|H_v \lambda_k\|_2 \lesssim 1 + \log 2^{-k}\|v\|_{Lip}.$$

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- There is a rich and beautiful theory of Radon Transforms, as developed by Christ, Nagel, Stein, and Wainger.
- The main point is that these results are true in absence of (a) geometric conditions on v (b) minimal smoothness conditions.
- Genuinely two dimensional time frequency analysis.

Key Proposition Implies Main Theorem

With $1+\epsilon$ smoothness, one can show this:

$$\|H_v \lambda_k * f - \lambda_{k'} * (H_v \lambda_{k'} * f)\|_2 \lesssim 2^{-\epsilon'|k-k'|}\|f\|_2$$

And this proves the Main Theorem from the Key Proposition.

Important Obstacle in Extensions

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This orthogonality takes some care to formalize correctly.

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Assuming our Main Theorem, we can show that for smooth $N(x)$, the operator above is bounded, with norm independent of $\|N\|_{C^2}$.

Construction of v

Calculate the symbols of both operators where $\sigma(\xi) = \int_{-1}^{1} e^{ix\xi} dy/y$.

$$H_v f(x) = \int_{\mathbb{R}^2} \sigma(\xi \cdot v(x)) \widehat{f}(\xi) e^{ix\cdot\xi} \, d\xi$$

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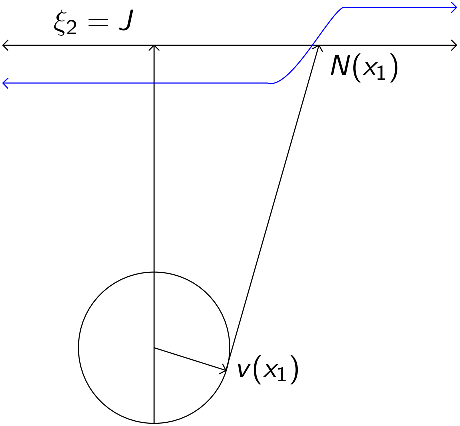
Then you choose $v(x_1, x_2) \simeq (1, N(x_1)/J)$, so that

$$\xi \cdot v(x) = x_1\xi_1 - N(x_1) \qquad \text{on the line } \xi_2 = J.$$

The Picture for v



- The Blue line is the function $\sigma(\xi_1 - N(x_1))$.

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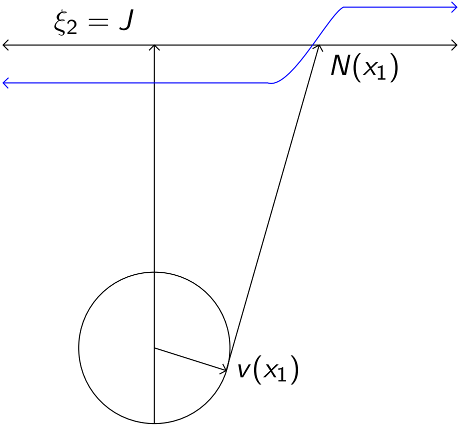

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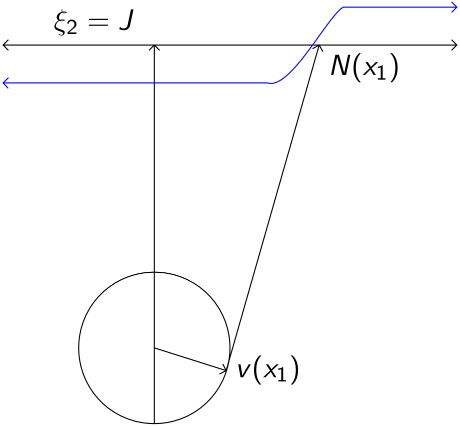

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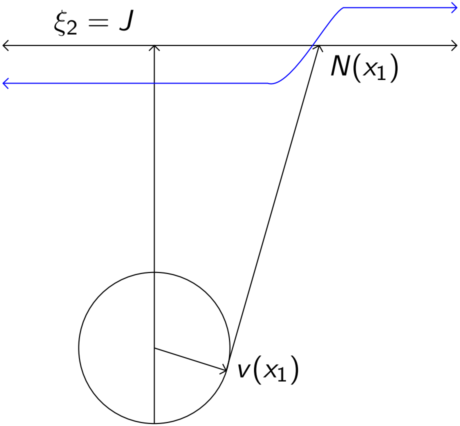

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- Then $v(x_1) \simeq (1, -N(x_1)/J)$, so take $J >> \|v\|_{C^2}$
- The symbol of H_v and C_N agree on the line $\xi_2 = J$.
- Note that small oscillations give oscillations in frequency, that increase as frequency increases.

The Proof of Main Theorem

3. Lemma Related To Carleson's Theorem
   - The Weak L^2 estimate is Sharp

4. Annular Tiles
   - The Functions associated to a Tile

Lemma Related to Carleson's Theorem

Using the methods of Carleson's Theorem, as proved in [7], one can show that

Lemma for Measurable Vector Fields

If v is measurable, one has

$$\|H_v \lambda_0\|_{2\to 2,\infty} < \infty,$$
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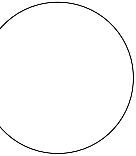
- The L^2 to weak L^2 estimate is optimal.
- And these estimate are critical to an interpolation argument that we use to prove the Key Proposition.

Weak L^2 estimate Is Sharp



Consider the radial vector field v, and a smooth bump function φ.

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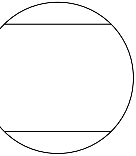

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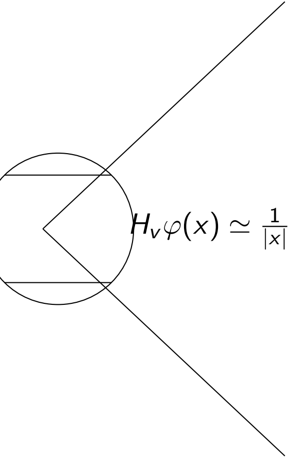

Consider the radial vector field v, and a smooth bump function φ. Modulate φ so that it is supported in frequency in the annulus $1 \leq |\xi| \leq 2$. Calculate H_v at any point inside the lines, and there will be no cancellation.

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- The proof breaks down completely below L^2, as it requires essentially the boundedness of the Kakeya maximal function.
- A key innovation is to replace the Kakeya maximal function by a variant associated to the vector field v.

Annular Tiles

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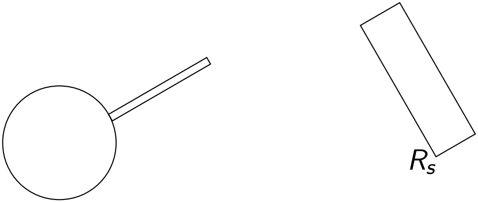

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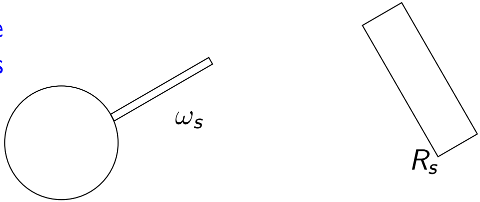

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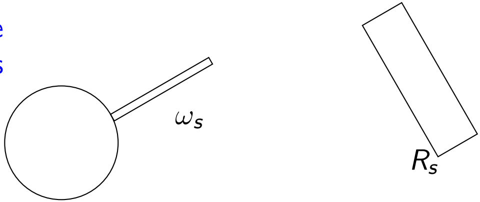

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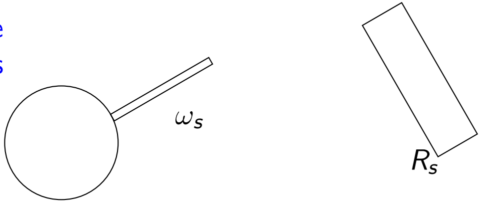
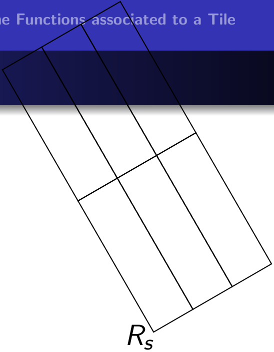
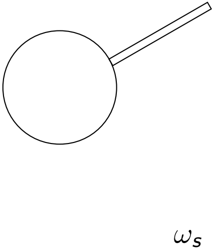


More Pictures of Tiles





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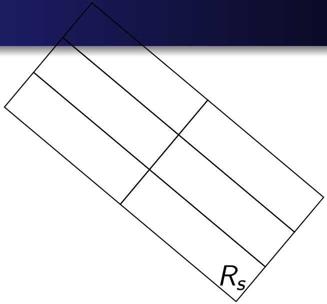

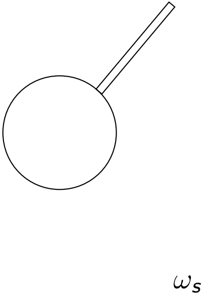

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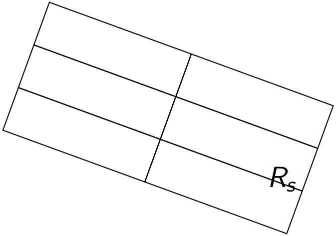

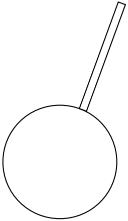

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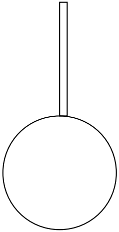

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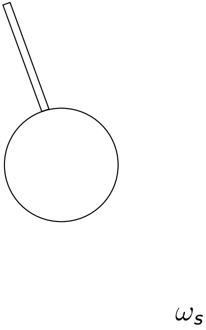

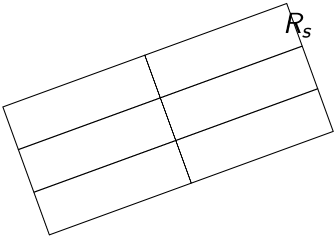

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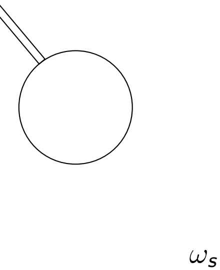

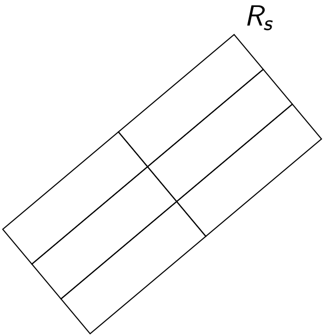

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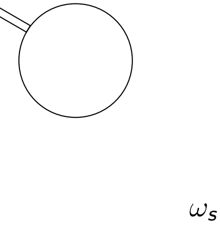
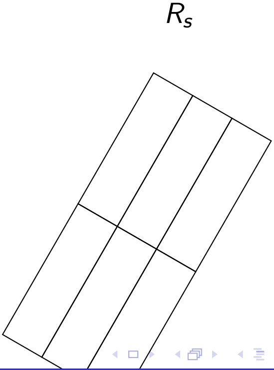
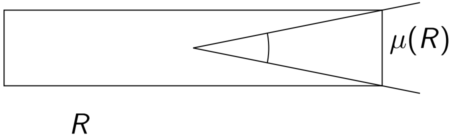




The Uncertainty Intervals



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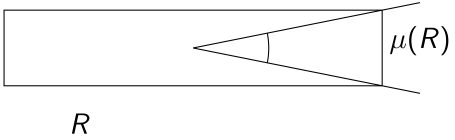

Interval of uncertainty associated with rectangle R is a sub arc of the unit circle. Its center is the long direction of the rectangle. Its length is the width of R divided by length of R. Basis changes are permitted, up to a tolerance level dictated by the angle of uncertainty.

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$\operatorname{scl}(s) =$ the short side of ω_s. ψ is a Schwartz function on $\mathbb{R}$, with Fourier support in a small neighborhood of one.

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$\operatorname{scl}(s) =$ the short side of ω_s. ψ is a Schwartz function on $\mathbb{R}$, with Fourier support in a small neighborhood of one. Let $\mathcal{AT}$ be all annular tiles with ω_s contained in $1 \leq |\xi| \leq 2$.

The Main Lemma for Sums Over Tiles

For a measurable vector field, we have the estimate

$$\Big\| \sum_{s \in \mathcal{AT}} \langle f, \varphi_s \rangle \phi_s \Big\|_{2,\infty} \lesssim \|f\|_2$$

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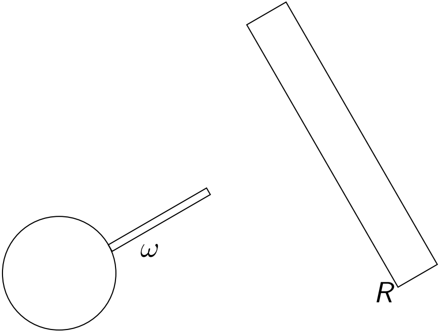
We also have the L^p inequality for $p > 2$. If in addition $\|v\|_{Lip} < \infty$, then we have the L^2 inequality

$$\Big\| \sum_{\substack{s \in \mathcal{AT} \\ \mathrm{scl}(s) > 100\|v\|_{Lip}}} \langle f, \varphi_s \rangle \phi_s \Big\|_2 \lesssim \|f\|_2$$

The 1–Trees



A 1–tree has frequency intervals that progress, counterclockwise, around the circle, at a regular rate.

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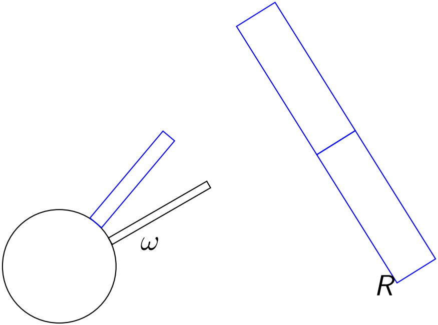

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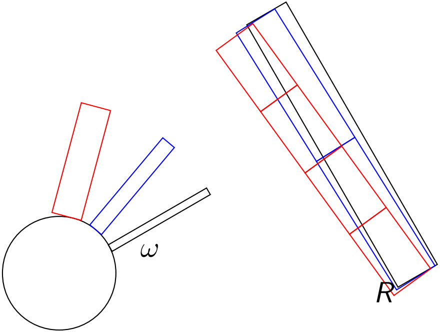


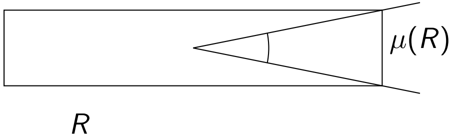
A 1–tree has frequency intervals that progress, counterclockwise, around the circle, at a regular rate.

The spatial intervals are all contained in R, up to an angle of uncertainty.

This looks like a singular integral computed in the direction v.

Outline

The Uncertainty Intervals



Interval of uncertainty associated with rectangle R is a sub arc of the unit circle. Its center is the long direction of the rectangle. Its length is the width of R divided by length of R.

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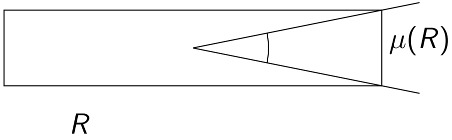

Interval of uncertainty associated with rectangle R is a sub arc of the unit circle. Its center is the long direction of the rectangle. Its length is the width of R divided by length of R. Basis changes are permitted, up to a tolerance level dictated by the angle of uncertainty.

Density for Tiles

$$\mathrm{dense}(R) = \frac{|R \cap v^{-1}(\mu(R))|}{|R|}$$

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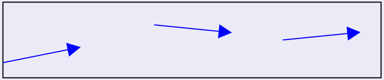
A rectangle and three vectors in the angle of uncertainty



Definition of the Maximal Function

Let $\mathcal{R}$ be a collection of rectangles with $\text{dense}(R) > \delta$ for all $R \in \mathcal{R}$. and they **all have the same width,** and lengths at most $\|v\|_{Lip}/100$. Let

$$\mathrm{M}_{\mathcal{R}} f = \sup_{R \in \mathcal{R}} \frac{\mathbf{1}_R}{|R|} \int_R f(y)\, dy$$

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- Here, it is essential that the estimate be *independent of eccentricity.*

Maximal Function Lemma

For all $0 < \delta < 1$, the maximal function satisfies

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- And a norm estimate of δ^{-N} for **any** $N < \infty$.
- The method of proof is a careful analysis, in the style of Fefferman and Cordoba.

First Lemma on Lipschitz vector fields

Lemma 1: Suppose there are $R_1, \ldots, R_n \in \mathcal{R}$, all containing a common point. Then, $n < \delta^{-1}$.

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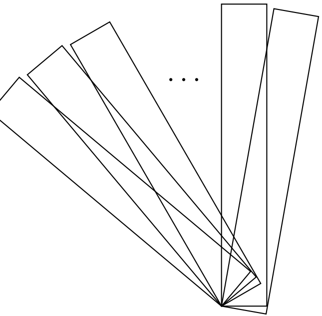

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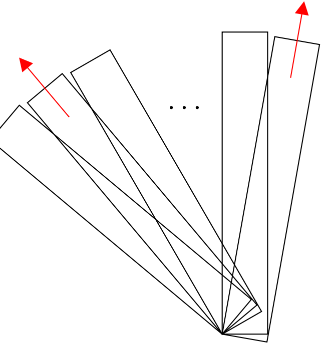

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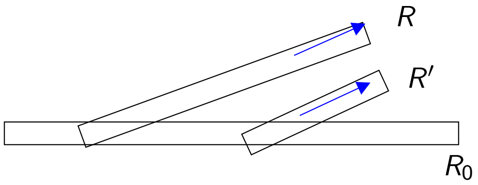
Suppose not. Then we can find two rectangles, and points in the rectangles, where the vector field points in the long direction of the rectangle.

At these two points, vector field is nearly radial. But these two points are very close together. Less than angle $\times$ length. That is a contradiction.

Second Lemma on Lipschitz



Lemma 2

Consider three rectangle, R_0, R, and R' as pictured. Key assumption is that there is a point $x \in R$ with $v(x) \in \mu(R)$, and $x' \in \mu(R')$, which share the same projection onto R_0. Then the uncertainty intervals $\mu(R)$ and $\mu(R')$ are very close.

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Standard Arguments then prove the Maximal Function Estimate.

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Let M_{100} be a maximal function computed in 100 uniformly distributed directions of the plane. This operator maps $L^1(\mathbb{R}^2)$ to weak L^1.

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Update, $\mathcal{R}' := \mathcal{R}' \cup \{R'\}$. Remove from $\mathcal{R}^{\text{stock}}$ any rectangle R such that

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Focusing on the Principal Inequality

The main point to prove is

$$\Big\| \sum_{\substack{R\in\mathcal{R}' \\ \mathrm{lngth}(R)<\mathrm{lngth}(R_0)}} \mathbf{1}_{R\cap R_0} \Big\|_n \lesssim \delta^{-2} |R_0|^{1/n} \qquad R_0 \in \mathcal{R}'.$$

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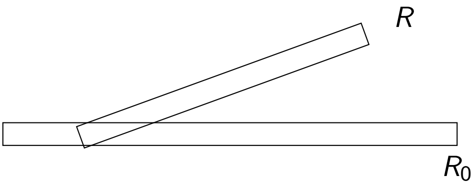
We will need to select a distinguished subset of $\mathcal{R}_0$.

Selecting a Distinguished Subset of $\mathcal{R}_0$, Part 1



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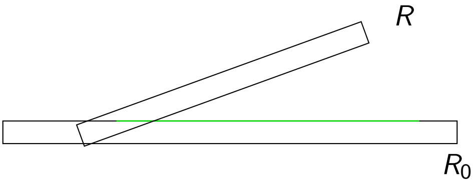

$I(R)$ is the projection of all of R onto the top side of R_0. I_0 is the top side of R_0.

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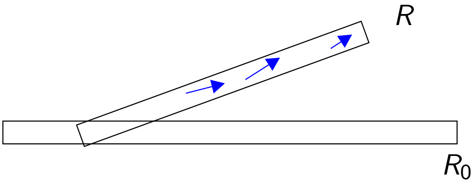

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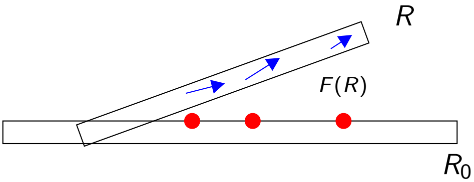

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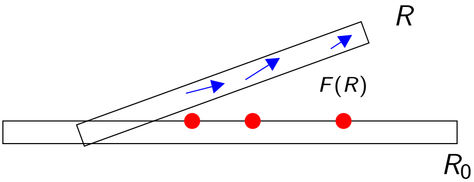

The rectangle R has density at least δ. Project those $x \in R$ with $v(x)$ in the angle of uncertainty of R, onto the top side of R_0. Call that set $F(R)$. This set is at least as big as $\delta \text{lngth}(R)$.

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This is essentially immediate from the disjointness of the sets $F(R)$ for $R \in \mathcal{R}_1$.

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This is essentially a *BMO* estimate, so it implies the higher moments condition we need.

The Essential Geometric Observation

Suppose that there is an interval $I \subset I_0$ and a choice of $R_1 \in \mathcal{R}_1$ such that

$$\sum_{\substack{R \in \mathcal{R}(R_1) \\ \text{lngth}(R) \geq 4|I|}} |R \cap I \times J| \geq 10^3 \delta^{-1} |I \times J|.$$

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Then, for either $\varepsilon = +1$ or $\varepsilon = -1$, there can be no $R' \in \mathcal{R}(R_1)$ with $2\text{lngth}(R') < |I|$ and R' intersects $\frac{1}{2}(I + \varepsilon|I|) \times J$.

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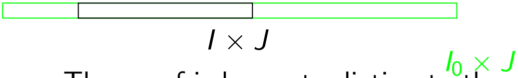
The Proof of the Essential Geometric Observation



The proof is by contradiction to the construction of $\mathcal{R}'$, and in particular, the use of the maximal function in 100 different directions.

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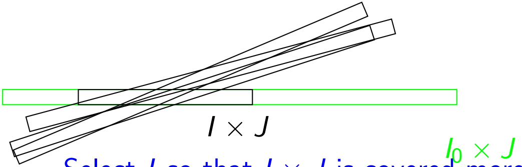

Select I so that $I \times J$ is covered more and $10^3\delta^{-1}$ times by longer intervals.

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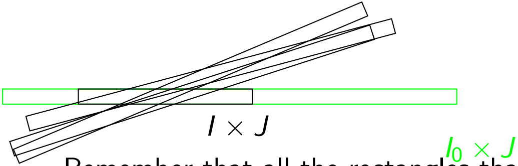

Remember that all the rectangles that cover $I \times J$ have to have angles of uncertainty that are very close.

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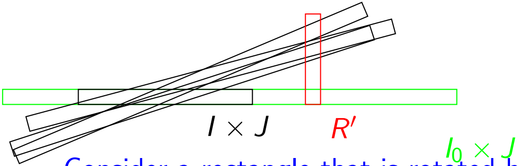

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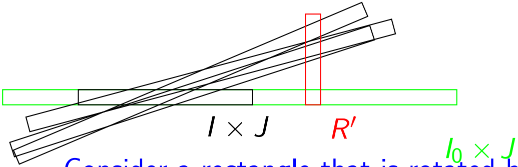

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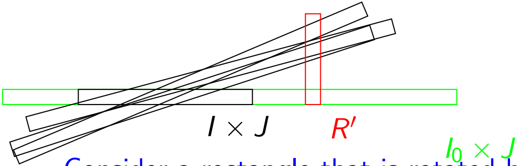

Consider a rectangle that is rotated by 90^o, has width the same as all other rectangles, and is translated by about $|I|$. And height approximately angle $\times\ |I|$.

This rectangle is contained in the removed set. A contradiction. This proves the Essential Geometric Observation.

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[1] J. Bourgain, *A remark on the maximal function associated to an analytic vector field*, Analysis at Urbana, Vol. I (Urbana, IL, 1986–1987), London Math. Soc. Lecture Note Ser., vol. 137, Cambridge Univ. Press, Cambridge, 1989, pp. 111–132. MR 90h:42028

[2] Lennart Carleson, *On convergence and growth of partial sumas of Fourier series*, Acta Math. **116** (1966), 135–157. MR 33 #7774

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