A generalization of Reifenberg's theorem in $\mathbb{R}^3$

G. David, T. DePauw & T. Toro

1. Reifenberg's work

1960 - Plateau problem for m-dimensional surfaces in $\mathbb{R}^k$.

Question: Is the infimum of the area of surfaces with a given boundary attained? (What does *surface* mean?)

Let $\Gamma \subset \mathbb{R}^k$ be a given set homeomorphic to $\mathbb{S}^{m-1}$

$$\mathcal{F} = \Big\{\Sigma \subset \mathbb{R}^k : \partial\Sigma = \Gamma, \Sigma \text{ is proper} \\ \Sigma \text{ is not a retraction of } \Gamma\Big\}$$

Theorem. *There exists $\Sigma_0 \in \mathcal{F}$ such that*

$$\mathcal{H}^m(\Sigma_0) = \inf_{\Sigma \in \mathcal{F}} \mathcal{H}^m(\Sigma).$$

Moreover for $\mathcal{H}^m$ a.e $x \in \Sigma_0$ there exist a neighborhood of x in Σ_0 which is a topological disk.

Key ideas:

1.Define

$$\theta(x,r) = \frac{\mathcal{H}^m(\Sigma \cap B(x,r))}{\omega_m r^m}.$$

2. Monotonicity : If Σ is a minimizer then for each x, $\theta(x,r)$ is an increasing function of r, and for a.e. x,

$$\theta(x) = \lim_{r \to 0} \theta(x,r) \geq 1$$

3. Given $\delta > 0$ there exists $\epsilon > 0$ such that if $1 \leq \theta(x_0, 2r_0) < 1 + \epsilon$ then $\Sigma \cap B(x,r)$ is within δr of an m-plane through x whenever $B(x,r) \subset B(x_0, r_0)$.

4. Topological disk property.

1964 - The Epiperimetric inequality yields a rate of decay of $\theta(x,r)$ toward 1, which implies that for

a.e $x \in \Sigma$ there exists $r_0 > 0$ such that for $r < r_0$,

$\Sigma \cap B(x,r)$ is within $r^{1+\alpha}$ of an m-plane through x

Locally Σ is a $C^{1,\alpha}$ submanifold, thus real analytic.

For a minimizer if $\theta(x,r)$ is close to the density of the tangent cone then the set is close to the tangent cone in Hausdorff distance.

Applications of Reifenberg's ideas:

1. Hardt - Lin show that the singular set of an energy minimizing harmonic map from B^4 into $\mathbb{S}^2$ is (locally) a finite set and a finite union of Hölder continuous curves.

2. Criteria for existence on biLipschitz parameterizations for subsets of Euclidean space

3. Construction of snowballs.

2. Taylor's work

1976 - Classification of the tangent cones in $\mathbb{R}^3$ for Almgren (locally) almost-minimal sets ("size minimizers").

E is (locally) (α, δ) almost-minimal if

$$\mathcal{H}^2(E \cap W_\varphi) \leq (1 + Cr^\alpha)\mathcal{H}^2(\varphi(E \cap W_\varphi)),$$

whenever $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$ is a Lipschitz mapping such that

$$\text{diam}\,(\mathrm{W}_\varphi \cup \varphi(\mathrm{W}_\varphi)) = \mathrm{r} < \delta,$$

where

$$W_\varphi = \{x \in \mathbb{R}^3 : \varphi(x) \neq x\}.$$

E is almost-minimal if $\delta = \infty$.

Classification of Minimal cones in $\mathbb{R}^3$:
(Modulo rotation and translation)

• A 2-plane: P through the origin

• $Y = Y_0 \times \mathbb{R}$ where Y_0 is the union of 3 half-lines in a plane, intersecting at the origin, and making equal angles of $3\pi/2$. The spine of Y is the line of intersection of the 3 half-planes that compose it.

• T the cone over the 1-skeleton of the tetrahedron (spine), i.e a set composed of 6 angular sectors bounded by 4 half lines that start at the origin and make maximal equal angles.

Theorem.[**Taylor**] *An* (α, δ) *almost-minimal set is (locally) a* $C^{1,\beta}$ *image of either* $P \cap B(0,1)$*,* $Y \cap B(0,1)$ *or* $T \cap B(0,1)$*.*

Tools:

1. Monotonicity formula

2. For a.e $x \in E$,

$$\theta(x) \in \left\{1, 3/2, \frac{3}{\pi}\cos^{-1}(-1/3)\right\}.$$

Classification of the possible tangent cones.

3. The Epiperimetric inequality establishes the rate at which $\theta(x,r)$ converges to $\theta(x)$.

If $\theta(x,r)$ is close to the density of the tangent cone then the set is close to the tangent cone in Hausdorff distance.

3. Our results

For A, B closed sets in $\mathbb{R}^3$ intersecting $B(x, r)$ define

$$D_{x,r}(A, B) = \frac{1}{r} \sup_{z \in B \cap B(x,r)} \{\text{dist}\,(z, A)\} + \frac{1}{r} \sup_{z \in A \cap B(x,r)} \{\text{dist}\,(z, B)\}$$

Let $E \subset \mathbb{R}^3$ be a closed set. E is ϵ-*minimal* if for each $r \leq 2$

$$\inf_{x \in Z} D_{x,r}(E, Z) \leq \epsilon$$

where the infimum is taken over all sets Z of type P, Y, and T containing x

i.e. there exists a minimal cone $Z(x, r)$ such that $x \in Z(x, r)$ and satisfying

$$D_{x,r}(E, Z(x, r)) \leq \epsilon.$$

Theorem. *If E is ϵ-minimal there exists $A = A(\epsilon)$ so that for each $x \in E$ there exists a map $f : B(0,1) \to \mathbb{R}^3$ with $f(0) = x$ satisfying*

$$A^{-1}|y-z|^{1+C\epsilon} \leq |f(y)-f(z)| \leq A|y-z|^{1-C\epsilon}$$

$$B(x,r) \subset f(B(0,r)) \subset B(x,2r)$$

$$E \cap B(x,r) \subset f(Z \cap B(0,r)) \subset E \cap B(x,2r),$$

where Z is either P, Y or T.

4. Preliminaries

Let E be ϵ minimal and $\epsilon < 10^{-4}$.

$$a(x,r) = \inf\left\{D_{x,r}(E,P); x \in P\right\},$$

$$b(x,r) = \inf\left\{D_{x,r}(E,Y); x \text{ in the spine of } Y\right\},$$

and

$$c(x,r) = \inf\left\{D_{x,r}(E,T); T \text{ centered at } x\right\}.$$

Define

$$E_P = \{x \in E : a(x,r) \le 2\epsilon\}$$

$$E_Y = \{x \in E : b(x,r) \le 500\epsilon\}$$

$$E_T = \{x \in E : c(x,r) \le 80\epsilon\}$$

Claim 1:

$$E = E_P \cup E_Y \cup E_T,$$

E_T is a discrete set and $E_Y \cup E_T$ is a closed set.

Proof:

- Minimal cones of different types are far away from each other in Hausdorff distance.

- If $a(x, 2r) \leq 10^{-4}$ then $E \cap B(x,r) \subset E_P$ (i.e. E_P is Reifenberg flat).

- For each $x \in E$, $E_T \cap B(x, 3/2)$ contains at most one point.

- If $b(x, 2r) \leq 10\epsilon$ then $B(x,r) \cap E_T = \emptyset$ and $D_{x,r}(L, E_Y) \leq 750\epsilon$, where L is the spine of $Y = Z(x, 2r)$.

- If $x \notin E_P$ then x is very close to the spine of $Z(x,r)$.

Claim 2: E_Y is Reifenberg flat of dimension 1. Moreover near E_T, E_Y is close to the spine of a tetrahedron.

Assume that $0 \in E$. Two distinct cases:

- $Z(0,2) = Y$. $(*)$

- $Z(0,2) = T$.

5. Idea of the proof

The parameterization f is constructed by successive deformations of the set $Z(0,2)$ near $E \cap Z(0,2)$.

- Hierarchical construction of f: E_Y, E_P, $B(0,2)$.

- Technical ingredient: good choice of appropriate partitions of unity.

Step 1:

1. Cover $E_Y \cap B(0,3/2)$ by balls of radius 2^{-n-20} with centers in E_Y at distance at least 2^{-n-20} from each other. Let V_Y be the union of these balls.

2. Cover $E_P \cap B(0,3/2) \backslash V_Y$ by balls of radius 2^{-n-30} with centers in E_P at distance at least 2^{-n-30} from each other. Let V_P be the union of these balls. (Note that E_Y is *far* from V_P.)

3. Build partitions of unity associated to these coverings.

Step 2:

Let L be the spine of $Y = Z(0,2)$. Since E_Y is Reifenberg flat closed set of dimension 1, Reifenberg's work yields a parameterization

$$f^* : \Gamma = L \cap B(0,3/2) \to E_Y.$$

f^* is the limit of mappings f_n^* defined on Γ.

For $f_{n+1}^* = g_n^* \circ f_n^*$ with $f_0^* = id$, where g_n^* is defined near $f_n^*(\Gamma) = \Gamma_n$ as a weighed sum of suitable deformations which push points closer to E_Y.

To show that f^* is biHölder note that

$$(1-C\epsilon)|y-z| \leq |g_n^*(y)-g_n^*(z)| \leq (1+C\epsilon)|y-z|$$

for $y, z \in \Gamma_n$ and $|y-z| \leq 2^{-n}$.

Step 3:

Build f as a limit of f_n's defined on $Y \cap B(0,1)$ by $f_{n+1} = g_n \circ f_n$ with $f_0 = id$, where g_n is defined near $f_n(Y \cap B(0,1))$ as a weighed sum of suitable deformations which push points closer E. Moreover $g_n = g_n^*$ on Γ_n.

To show that f^* is biHölder we show that

$$(1-C\epsilon)|y-z| \leq |g_n^*(y)-g_n^*(z)| \leq (1+C\epsilon)|y-z|$$

for $y, z \in f_n(Y \cap B(0,1))$ and $|y-z| \leq 2^{-n}$.