Minimal sets in $\mathbb{R}^3$, and J. Taylor's regularity theorem

Guy David

1. MS-minimal sets

First, what do we mean by minimal sets?

Let $E \subset \mathbb{R}^3$ be closed, with locally finite Hausdorff measure H^2 of dimension 2 (think about surface measure). The most natural notion of minimality is that

$$H^2(E \setminus F) \leq H^2(F \setminus E) \tag{1}$$

for every competitor F for E.

There are different notions of competitors. Here is the notion that is relevant for the Mumford-Shah functional. The closed set F is called a MS-competitor for E if there is a ball B such that

$$F \setminus B = E \setminus B$$

and

F separates y from z whenever $y, z \in \mathbb{R}^3 \setminus (B \cup E)$ are separated by E.

We call MS-minimal a closed set $E \subset \mathbb{R}^3$ such that (1) holds for every MS-competitor F for E.

The following question is the first step when one tries to classify the global Mumford-Shah minimizers in $\mathbb{R}^3$:

$$(Q1) \qquad \text{what are the MS-minimal sets in } \mathbb{R}^3?$$

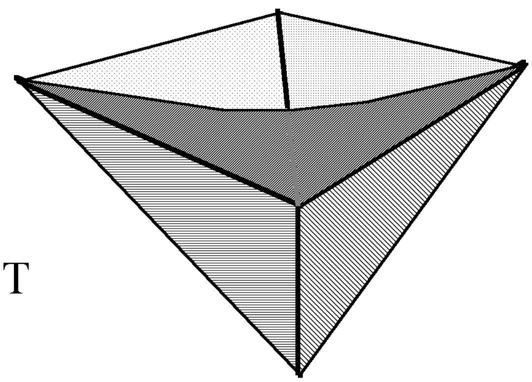
Indeed MS-minimal sets correspond to Mumford-Shah minimizers (u, K) such that u is locally constant.

The answer is a rather simple consequence of J. Taylor's theorems, but surprisingly I did not find it anywhere: modulo a set of vanishing H^2-measure,

$$(A) \qquad E \text{ is empty, a plane, a } Y\text{, or a } T.$$

Here a Y is a union of three half planes, with a common boundary (a line L), and that make $120°$ angles along L.

And a T is the image by a translation of the cone (centered at 0) over the union of the 6 edges of a regular tetraedron centered at 0. Thus T is composed of 6 faces that are angular sectors of angle $\alpha \sim 109°$ that meet along 4 half lines starting from 0, and T separates $\mathbb{R}^3$ into 6 "equal" regions.



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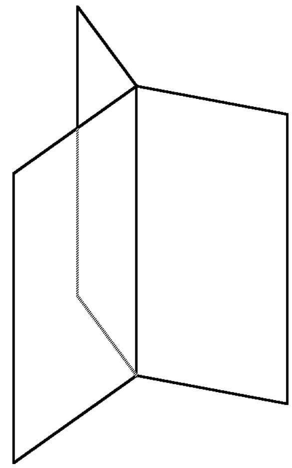

Y

2. Almgren-minimal sets

Here is another notion of competitors, that leads to a description of soap films. An Almgren competitor for E is a closed set of the form $F = \varphi(E)$, where $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$ is a Lipschitz mapping such that

$$W_\varphi = \{x \in \mathbb{R}^3 \, ; \, \varphi(x) \neq x\}$$

is bounded. Notice that φ does not need to be injective.

We say that the closed set $E \subset \mathbb{R}^3$ is Almgren-minimal when (1) holds for every Almgren competitor for E.

Think about soap films. Examples are planes, Y, T.

Minimal sets are not the same as "minimal surfaces"; the difference is the same as for "size" vs. "mass" for curents.

Supports of size-minimizing curents are (locally) Almgren-minimal sets (I think).

It is fairly easy to check that Almgren competitors are MS-competitors (because φ preserves the separation of faraway points), so MS-minimal sets are Almgren-minimal. Probably the question

$(Q2)$ what are the Almgren-minimal sets in $\mathbb{R}^3$?

has the same answer (A) (again modulo sets of zero H^2-measure) as (Q1), but I don't know a proof yet (See later).

J. Taylor proved that the Almgren-minimal sets that are cones are as in (A).

3. Almgren almost-minimal sets

The notion of Almgren-minimal set can be localized and also weakened. First, in Almgren's original definition, (1) was replaced with

$$H^2(E \cap W_\varphi) \leq H^2(\varphi(E \cap W_\varphi)) \tag{1'}$$

when φ and W_φ are as above. It looks slightly different (because $\varphi(E \cap W_\varphi)$ may meet $E \setminus W_\varphi$), but it is fairly easy to see that the two give the same classes of minimal sets.

Let us define almost-minimal sets too. Let a continuous, non-decreasing function $\varepsilon : [0, +\infty) \to [0, +\infty)$ be given, with $\varepsilon(0) = 0$. We'll say that the closed set E is almost-minimal, with gauge function ε, if

$$H^2(E \cap W_\varphi) \leq (1 + \varepsilon(r)) H^2(\varphi(E \cap W_\varphi)) \tag{2}$$

whenever $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$ is a Lipschitz mapping such that the diameter of $W_\varphi \cup \varphi(W_\varphi)$ is at most r. This is a special case of Almgren's notion of restricted set.

Localization is easy: we only consider competitors for which $W_\varphi \cap \varphi(W_\varphi)$ is compactly contained in a domain, or/and has small enough diameter.

We shall often assume that $\varepsilon(r)$ tends to 0 fast enough with r (any positive power would do).

4. J. Taylor's regularity result (partially)

The following local description of any Almgren-minimal set, or even locally almost-minimal set E with small enough gauge function, follows from J. Tayor's results (and modulo sets of H^2-measure zero).

For every point of E, there is a radius $r > 0$, a minimal cone Z (that is, a plane, a Y, or a T) that contains 0, and a bi-Hölder mapping $f : B(0,r) \to \mathbb{R}^3$, with $f(0) = x$,

$$C^{-1}|y-z|^{101/100} \leq |f(y)-f(z)| \leq C|y-z|^{99/100},$$

$$B(x,r) \subset f(B(0,r)) \subset B(x,2r),$$

$$E \cap B(x,r) \subset f(Z \cap B(0,r)) \subset E \cap B(x,2r).$$

[More is true; see [T-76].]

Plan for the rest of the lecture: try to explain why this is true (without currents or varifolds), and the conection to (Q1) and (Q2). Our tools will be:

- The (almost) monotonicity of density for (almost) minimal sets;

- A result on limits of almost-minimal sets (and the lowersemicontinuity of H^2 when we restrict to such sets);

- a Reifenberg-type theorem (obtained with Thierry De Pauw and Tatiana Toro; see Tatiana's lecture) that gives the description above when E is ε-close to minimal cones at all scales and locations.

5. Monotonicity of density (A key ingredient)

To simplify, assume that E is Almgren-minimal and "reduced" (equal to the closed support of $H^2_{|E}$), to get rid of sets of zero H^2 measure. Then

$$\theta(x,r) = \frac{1}{\pi r^2} H^2(E \cap B(x,r)) \tag{3}$$

is a nondecreasing function of r.

Idea of proof (with many simplifications!) Suppose $x = 0$. We want to compare $E \cap B(x,r)$ with the cone over $E \cap \partial B(x,r)$.

This is not exactly a competitor, but almost: pick η very small and try $F = \varphi(E)$, where φ is radial (i.e., $\varphi(\rho\xi) = h(\rho)\xi$ when $\rho \geq 0$ and $|\xi| = 1$), and with $h(t) = t$ for $t \geq r$, $h(0) = 0$, $h(1-\eta) = \eta$, and h is picewise linear on $[0, r-\eta]$ and $[r-\eta, r]$. Then let η tend to 0. Comparizon yields

$$H^2(E \cap B(x,r)) \leq \frac{r}{2} H^1(E \cap \partial B(x,r)). \tag{4}$$

But if we set $\Psi(r) = H^2(E \cap B(x,r))$ (a nondecreasing function) then $\Psi'(r) \geq H^1(E \cap \partial B(x,r))$. Then (3) follows from (4).

In fact, not only $\Psi'(r) \geq H^1(E \cap \partial B(x,r))$, but also

$$\Psi'(r) \geq \int_{E\cap\partial B(x,r)} \frac{dH^1(y)}{\cos(\alpha(y))}\ , \tag{5}$$

where $\alpha(y)$ is the angle of the radius $[0,y]$ with the tangent plane to E at y.

Because of (5) we even get that if $\theta(x,\cdot)$ is constant on an interval (a,b), then $\alpha(y) = 0$ almost-everywhere on the annulus $E \cap \{a < |y-x| < b\}$ and (with some work) that E coincides with a cone on $\{a < |y-x| < b\}$.

Then the cone is a minimal set too, and "one" can check that it is a plane, a Y, or a T.

So minimal sets for which $\theta(x,\cdot)$ is constant are under control.

For almost-minimal sets, we still get some almost-monotonicity.

In particular, if $\displaystyle\int_0^1 \varepsilon(r)\frac{dr}{r} < +\infty$, then $h(r)\theta(x,r)$ is a nondecreasing function of $r \in (0,1)$ for some positive function $h(r)$ such that $\lim_{r\to 0} h(r) = 1$.

6. Limits

We'll need to know that if the sequence $\{E_k\}$ of reduced Almgren-minimal sets converges (locally in $\mathbb{R}^3$ in Hausdorff distance) to the closed set E, then:

$$H^2(E \cap V) \leq \liminf_{k \to +\infty} H^2(E_k \cap V) \quad \text{for } V \subset \mathbb{R}^3 \text{ open};$$

$$H^2(E \cap H) \geq \limsup_{k \to +\infty} H^2(E_k \cap H) \quad \text{for } H \subset \mathbb{R}^3 \text{ closed};$$

$$E \quad \text{is Almgren-minimal too.}$$

Thus we also get some convergence of densities.

This stays true in different dimensions, for Almgren almost-minimal sets with a given gauge function, and even Almgren restricted sets with given parameters. See [D-03].

The main point is the first one, obtained because restricted sets satisfy the "concentration property" of Dal Maso, Morel, and Solimini [DMS-92].

Notice that from every sequence of reduced Almgren-minimal sets, it is easy to extract convergent subsequences.

7. Blow up, blow in

Return to a fixed (nonempty) minimal set $E \subset \mathbb{R}^3$.

By monotonicity, $\theta_\infty = \lim_{r\to+\infty} \theta(x,r)$ exists for every x. It does not depend on x, and is finite.

Similarly, $\theta(x) = \lim_{r\to 0} \theta(x,r)$ exists for every $x \in \mathbb{R}^3$, and $\theta(x) \leq \theta_\infty$.

A Blow-in limit of E is any limit E_∞ of some sequence $\{r_k^{-1}E\}$, with $\lim_{k\to+\infty} r_k = +\infty$. By the discussion above, E_∞ is a minimal set, and the analogue for E_∞ of $\theta(0,\cdot)$ is constant equal to θ_∞.

Then E_∞ it is one of the three minimal cones above, and θ_∞ can only take 3 values, namely, $\theta_\infty = 1$, or $3/2$, or $3\alpha/\pi \sim 1.82$.

A Blow-up limit of E at $x \in E$ is any limit E_0 of a sequence $\{r_k^{-1}(E-x)\}$, with $\lim_{k\to+\infty} r_k = 0$.

Again E_0 is a minimal set, and the analogue of $\theta(0,\cdot)$ for E_0 is constant and equal to $\theta(x)$. So E_0 it is one of the three minimal cones above, and $\theta(x)$ can only take the 3 values above.

If $\theta(x) = \theta_\infty$ for some x, then E is a cone.

We are left with the case when $\theta(x) < \theta_\infty$ for all x.

8. Local regularity near a point of type P

(that is, a point $x \in E$ such that $\theta(x) = 1$).

So let E be a (reduced) minimal set in $\mathbb{R}^3$ (almost-minimal with small $\varepsilon(r)$ would work too).

Useful fact: if $\theta(x,r)$ varies very little between r_1 and $r_2 >> r_1$, then inside $B(x, \sqrt{r_1 r_2})$, E is close to a minimal cone centered at x, both in Hausdorff distance and in density.
Proof by compactness.
Notice that the type of cone is determined by density.

Assume that $x \in E$ and $\theta(x,r) \leq 1 + \varepsilon$, with ε very small.
By monotonicity, $1 \leq \theta(x,\rho) \leq 1 + \varepsilon$ for $\rho \leq r$.
By the useful fact, E is quite close to a plane in $E \cap B(x, r/C)$.
Then $\theta(y,\rho) \leq 1 + \varepsilon'$ for $y \in B(x, r/2C)$ and $t = r/2C$.
By monotonicity again, this stays true for $t < r/2C$.
By the useful fact again, E is close to planes in all balls B such that $CB \subset B(x,r)$.
By the standard Reifenberg theorem, E is biHölder-equivalent to a plane near x.

Notice that our assumption $\theta(x,r) \leq 1+\varepsilon$ holds when E is close enough (in Hausdorff distance) to a plane in $B(x, Cr)$ (if not, take a subsequence that converges and use the convergence of densities).

Also $\theta(x,r) \leq 1 + \varepsilon$ for r small for every x of type P.
And notice that x is of type P as soon as $\theta(x,r) < 3/2$ for some r.

9. Points of type Y (that is, such that $\theta(x) = 3/2$).

First, existence: we claim that if E is very close to a Y in $B(x,r)$ then there is a point of type Y near x.

[Close in Hausdorff distance, hence also in density.]

Suppose not. All points near x are of type P (there is not enough density for a point of type T), and so E is locally biHölder-equivalent to a plane.

Call x_1, x_2, x_3 the three points of $B(x,r/4)$ that lie at maximal distance from Y (assume Y goes through x). We claim that E separates x_i from x_j, $j \neq i$, in $B(x,r/2)$.

First, we construct a path $\gamma_{i,j}$ from x_i to x_j, that meets E exactly once. A first attempt is $[x_i, x_j]$, which we modify near the face of Y, using Section 8 and the local description of E far from the spine of Y.

The claim is now proved by deformation. Suppose γ is a path from x_i to x_j, in $B(x,r/2) \setminus E$. We deform γ into $\gamma_{i,j}$ correctly, and along the deformation the number of intersections with E always changes by ± 2. This is impossible. So E separates x_i from x_j in $B(x,r/2)$.

Recall that E is locally biHölder-equivalent to a plane. Near every point $y \in E \cap B(x, r/3)$, there are two components of $B(x, r/2) \setminus E$ near y. And the (unordered) pair of components is locally constant near y.

At the point $z_{i,j}$ of $\gamma_{i,j} \cap E$, the two components are distinct, and the three choices of $\{i, j\}$ give three different pairs (by the claim).

Also, the $z_{i,j}$ lie in the same component of E (by separation; essentially look in the plane that contains the three $z_{i,j}$).

This is impossible, so there is a point of type Y in $B(x, r)$.

By the same argument, all the points of the spine of Y are as close as we want to E_Y, the set of points of type Y.

Since E is close to Y also in density, the points of E that are far from L have density $\theta(y) < 3/2$, so they are of type P.

Recall that there is no point of type T.

Altogehter, E_Y is close to a line near x.

10. Regularity of E near points of type Y

If $x \in E_Y$, the argument above applies to $B(x,r)$ for r small enough.

If we look closely at the argument, (and apply it also to neighbors of x in E_Y), we get that E_Y is Reifenberg-flat of dimension 1 away from E_T.

We use all this and the regularity of E near points of type P, and get that E lies very close to a P or a Y in all the balls B such that $CB \subset B(x, r/2)$.

A modification of Reifenberg's theorem now says that E is locally biHölder-equivalent to a Y near x.

The proof extends to the case when E is an almost-minimizer, with small enough gauge function $\varepsilon(r)$.

And a similar result holds near a point of type T, (for almost-minimizers with small $\varepsilon(r)$): near such a point, E has a biHölder parameterization by a T.

[End of the discussion of J. Taylor's regularity result]

11. What are the minimal sets in $\mathbb{R}^3$?

We already know about cones. So suppose E is not a cone. Then $\theta(x) < \theta_\infty$ for all $x \in E$.

If $\theta_\infty = 3/2$, every point of E is of type P. But E looks like a Y on big balls, and the argument of Section 9 shows that there is a point of type Y (contradiction).

Remains the case when $\theta_\infty \sim 1.82$. Since E looks like a T on big balls, there are also places where it looks like a Y, and so E has points of type Y. But there is no point of type T.

So we know that locally, E looks like a Y or a plane (with biHölder parameterizations).

It is tempting to say that this local description is impossible when E looks like a T on big balls.

This is false: there are sets E that fit the local description and coincide with a T out of $B(0,1)$. (But these sets that are not minimal). So we need another proof.

12. Mumford-Shah minimal sets in $\mathbb{R}^3$

For MS-minimizers, there is a simple proof by topology.

Suppose E is very close to a T (centered at 0) in $B(0, 10)$. Call x_i, $i = 1, \dots 4$ the points of $B(0, 1)$ that are furthest from T.

We claim that E separates x_i from x_j for $i \neq j$.

Otherwise, we could remove a big piece of E near the face of T between x_i and x_j, and get a much better competitor. This is allowed here, because all we care about is the topological constraint on competitors.

Since there is no point of type T, E_Y is locally Reifenberg-flat of dimension 1, so it is composed of curves.

Since E is so close to a T, we can apply the local regularity near points of type Y, and get that E_Y has four branches that get out of $B(0, 2)$. So E_Y is composed of two (disjoint) curves.

Near each point $x \in E_Y \cap B(0, 2)$, the local description gives three components of $B(0, 3) \setminus E$ that touch x. And this collection of components is locally constant on each curve.

But this collection is different on each of the four branches, because E separates x_i from x_j for $i \neq j$. A contradiction.

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