L^p BOUNDS ON SPECTRAL CLUSTERS FOR BOUNDED PLANAR DOMAINS

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Abstract. These notes are a summary of the author's talk given in the IPAM MGA2004 Workshop V, November 15-19, 2004.

1. Introduction

In this talk we describe recent work of the author and Chris Sogge concerning bounds on the L^p norm of approximate eigenfunctions on two dimensional compact Riemannian manifolds with boundary, where we will assume that eigenfunctions satisfy Dirichlet conditions on the boundary.

We more generally consider a second order self-adjoint differential operator on a two dimensional compact manifold M with boundary, which in local coordinates takes the form:

$$\big(Pf\big)(x) = \rho(x)^{-1} \sum_{i,j=1}^{n} \partial_i \Big(\rho(x)\, \mathrm{g}^{ij}(x)\, \partial_j f(x)\Big)\,. \tag{1.1}$$

We assume that the function $\rho(x)$ and matrix function g^{ij} are real and uniformly positive on M. Then P is self-adjoint with respect to the measure $\rho(x)\,dx$, and has negative spectrum, thus we can enumerate the eigenvalues in decreasing order $\{-\lambda_j^2\}$, where $\lambda_j \to \infty$. We will call λ_j the *frequency* of the associated eigenfunction ϕ_j.

A *spectral cluster* of frequency λ is a function f belonging to the span of ϕ_j with $\lambda_j \in [\lambda, \lambda+1]$. Alternately, if we let Π_λ be the orthogonal projection onto eigenfunctions in this range, then the spectral clusters of frequency λ are the image of Π_λ.

The question of interest is comparing the L^p norm of a spectral cluster f of frequency λ to its L^2 norm. Specifically, we seek the best possible exponent $\delta(p)$ for $p \geq 2$ so that, for general f,

$$\|\Pi_\lambda f\|_{L^p(M)} \leq C\, \lambda^{\delta(p)}\, \|f\|_{L^2(M)}\,.$$

In the case that M does not have a boundary, and the coefficients of P are C^∞ functions, the sharp estimates were established by Sogge [8], who showed that in general dimension n

$$\big\|\Pi_\lambda f\big\|_{L^p(M)} \leq C\, \lambda^{\frac{n-1}{2}(\frac{1}{2}-\frac{1}{p})}\, \|f\|_{L^2(M)}\,, \qquad 2 \leq p \leq p_n = \tfrac{2(n+1)}{n-1}\,. \tag{1.2}$$

The author was supported in part by NSF grant DMS-0140499.

$$\big\|\Pi_\lambda f\big\|_{L^p(M)} \le C\,\lambda^{n(\frac12-\frac1p)-\frac12}\,\|f\|_{L^2(M)}\,, \qquad p_n \le p \le \infty\,, \tag{1.3}$$

Thus, there are two regimes to the estimates. The bounds for $2 \le p \le p_n$ can be considered as placing a lower bound on the volume of a region in which a spectral cluster can be concentrated. Precisely, if a function f is concentrated in a region of volume V, then by Holder's inequality

$$\frac{\|f\|_{L^p(M)}}{\|f\|_{L^2(M)}} \ge V^{\frac1p-\frac12}\,.$$

Consequently, the first estimate (1.2) above shows that a spectral cluster must occupy a volume of at least $\lambda^{-\frac{n-1}{2}}$.

The second estimate (1.3) is related to the Sobolev embedding theorem, and loosely says that the eigenfunctions in the range $[\lambda, 2\lambda]$ are evenly distributed over the λ disjoint subintervals of length 1.

Recently, the first author in [4] has extended Sogge's estimates to the case of metrics which are only twice differentiable, which is the lowest level of regularity (in the Holder classes, at least) under which they can be proven, according to the counterexamples of Smith-Sogge [6]. This fact plays a key role in the estimates for manifolds with boundary.

It is worth considering the case of $M = \mathbb{R}^n$ to see the nature of the examples that show the above estimates are best possible. On $\mathbb{R}^n$, a spectral cluster is simply a function whose Fourier transform $\widehat{f}(\xi)$ is supported in the set $\lambda \le |\xi| \le \lambda + 1$.

The example which shows that (1.2) is sharp is the simplest. Consider a Schwartz function $\phi(x)$ which has Fourier transform supported in the ball $|\xi| \le \frac12$, intersected with the region $\xi_1 \ge 0$. Let

$$f_\lambda = e^{i\,\lambda\,x_1}\phi(x_1, \lambda^{\frac12}x')\,,$$

where $x' = (x_2, \ldots, x_n)$. Then the Fourier transform $\widehat{f_\lambda}(\xi)$ is contained in the set $\lambda \le \xi_1 \le \lambda + \frac12$ and $|\xi'| \le \frac12\lambda^{\frac12}$. This region is in turn contained in the set $\lambda \le |\xi| \le \lambda + 1$. On the other hand

$$\|f_\lambda\|_{L^p} = \lambda^{-\frac{n-1}{2p}}\|\phi\|_{L^p}\,,$$

and comparing $p = p$ to $p = 2$ yields the sharpness of (1.2).

The example to consider for estimate (1.3) is a function f_λ where $\widehat{f_\lambda}(\xi) = 1$ on the set $\lambda \le |\xi| \le \lambda + 1$, and 0 elsewhere. Then

$$\|f_\lambda\|_{L^2} \approx \lambda^{\frac{n-1}{2}}\,.$$

On the other hand, it is easy to see that $|f(x)| \gtrsim \lambda^{n-1}$ for $|x| \le \lambda^{-1}$, hence

$$\|f_\lambda\|_{L^p} \ge \lambda^{n-1-\frac{n}{p}}\,.$$

Taking the ratio shows that (1.3) cannot be improved.

Similar examples can be patched onto any Riemannian manifold using a Fourier integral operator representation of the wave group.

In the case of a manifold with boundary, however, it is known by an example of Grieser [1] that for a range of p the estimates (1.2)-(1.3) can fail. The example concerns the standard Laplace operator on the unit disc $|x| \le 1$ in $\mathbb{R}^2$. The eigenfunctions with Dirichlet condition can be written in the form

$$\phi_\lambda = e^{in\theta}\, J(c\,|x|)\,,$$

where J is a certain Bessel function, depending on n, and $J(c) = 0$. If one takes c to be the first 0 of J, then $\lambda \approx n$, and the asymptotics of Bessel functions show that $J(c|x|)$ is concentrated if the set $1 - |x| \lesssim n^{-\frac{2}{3}}$. Thus, ϕ_λ is concentrated in a set of volume $\lambda^{-\frac{2}{3}}$, which yields that for $p \ge 2$

$$\|\phi_\lambda\|_{L^p} \gtrsim \lambda^{\frac{2}{3}(\frac{1}{2}-\frac{1}{p})}\,\|\phi_\lambda\|_{L^2}\,.$$

This contradicts (1.2) and also (1.3) for $p < 8$.

The recent work of the author and Sogge is to show that this example is in fact the worst possible on manifolds with boundary, in two dimensions.

Theorem 1.1. *Let M be a two dimensional manifold with boundary, and P a differential operator of the form* (1.1). *Let Π_λ be the orthogonal projection onto the span of Dirichlet eigenfunctions with frequencies in the range $[\lambda, \lambda + 1]$. Then*

$$\big\|\Pi_\lambda f\big\|_{L^p(M)} \le C\,\lambda^{\frac{2}{3}(\frac{1}{2}-\frac{1}{p})}\,\|f\|_{L^2(M)}\,, \qquad 2 \le p \le 8,. \tag{1.4}$$

$$\big\|\Pi_\lambda f\big\|_{L^p(M)} \le C\,\lambda^{2(\frac{1}{2}-\frac{1}{p})-\frac{1}{2}}\,\|f\|_{L^2(M)}\,, \qquad 8 \le p \le \infty\,. \tag{1.5}$$

In the next section we outline the main ideas in the proof. It should be noted that the counterexample of Grieser is intimately connected to the convexity of the boundary. Indeed, near any point of strict convexity of a two dimensional Riemannian manifold, one can construct an approximate spectral cluster establishing the lower bound (1.4), using the Melrose and Taylor Airy function solutions of the wave equations near glancing rays. On the other hand, if the boundary of M is everywhere strictly geodesically concave, then the results of Smith-Sogge [6] show that the sharper bounds (1.2) and (1.3) hold.

2. Sketch of the Proof

Since (1.4) follows by interpolation of (1.3) with the trivial $p = 2$ endpoint, we restrict attention to estimate (1.3).

Estimate (1.3) is established as a consequence of certain squarefunction estimates for solutions to the wave equation $\partial_t^2 u = Pu$ on $\mathbb{R} \times M$. Precisely, let

$$\|u\|_{L^q_x L^2_t([-1,1]\times M)} = \Big(\int\Big(\int_{-1}^{1} |u(t,x)|^2\,dt\Big)^{\frac{q}{2}}dx\Big)^{\frac{1}{q}}\,.$$

Then the estimate that will yield (1.3) is

$$\|u\|_{L^q_x L^2_t([-1,1]\times M)} \le C\,\|f\|_{H^{\delta(q)}(M)}\,, \qquad 8 \le p \le \infty\,, \tag{2.1}$$

where

$$\delta(q) = 2\Big(\frac{1}{2} - \frac{1}{q}\Big) - \frac{1}{2}\,,$$

H^δ is the Sobolev space of order δ, and

$$u(t,x) = \Big(\exp\big(it\sqrt{-P}\big)f\Big)(x)\,.$$

If f is a spectral cluster, then $\|f\|_{H^\delta} \approx \lambda^\delta \|f\|_{L^2}$, and u is essentially periodic in t. Multiplying $u(t,x)$ by $\exp(-it\lambda)$ and integrating over $[-1,1]$ essentially recovers f, and (1.5) follows easily from (2.1). For details, we refer to [4].

The advantage of working with the estimate (2.1) is that it can be localized by a partition of unity to a coordinate patch, since by finite propagation velocity $u(t,x)$ is localized if f is. Consequently, we may assume that we are working in a small coordinate neighborhood near ∂M (away from ∂M the estimate (2.1) holds for $6 \le p \le \infty$ by results of Mockenhaupt-Seeger-Sogge [2].)

We work in geodesic normal coordinates along ∂M, where M is then defined by $x_2 \ge 0$, and the metric takes the form

$$\mathrm{g}^{11}(x_1,x_2)\,dx_1^2 + dx_2^2\,.$$

We next extend the metric function g^{11} by reflecting in an even manner across $x_2 = 0$, and similarly extend the solution $u(t,x)$ in an odd manner, so that

$$\mathrm{g}^{11}(x_1,x_2) = \mathrm{g}^{11}(x_1,|x_2|)\,, \qquad u(x_1,x_2) = \mathrm{sgn}(\mathrm{x}_2)\,\mathrm{u}(\mathrm{x}_1,|\mathrm{x}_2|)\,.$$

Then g is defined on an open subset of $\mathbb{R}^2$, and by extension we may assume it is defined on all of $\mathbb{R}^2$, and equals 1 outside the unit ball. (We similarly extend $\rho(x)$ in an even manner, but for the purpose of these notes we may as well assume it equals 1 everywhere.) The new metric, however, has a Lipschitz singularity at $x_2 = 0$.

It is easily verified that the extended function u satisfies the wave equation $\partial_t^2 u = Pu$, with P extended as above and initial data f extended in an odd manner, and since f is compactly supported we may treat u as a solution on all of $\mathbb{R}^2$. Thus, we see that the boundary problem can be treated as a special case of proving estimates of type (1.5) on boundary-free manifolds, but where the metric is only assumed to be Lipschitz.

Strichartz estimates for metrics of low regularity were extablished in the case of C^2 (in fact $C^{1,1}$) metrics by Smith [3] in dimensions 2 and 3, and subsequently by Tataru [9] in all dimensions. The C^2 result, together with a frequency dependent scaling argument, can be used to establish similar estimates for metrics of Lipschitz regularity, but with a loss in the regularity exponent δ required of the initial data, as shown by Tataru in [9].

In [4], Smith established the estimates (1.3) for metrics of regularity $C^{1,1}$, and used scaling arguments to establish results for Lipschitz metrics with loss in $\delta(p)$ in [5].

Our proof of (1.5) (that is, no loss for $p \ge 8$) depends on a three-step multiscale decomposition of the solution u, which exploits the special nature of the Lipschitz singularity in g, together with the fact that the estimate (1.3) can be improved for $p > 6$ if the solution u is localized in frequency to a cone of small angle. Precisely, suppose that g is a $C^{1,1}$ metric, that u solves $\partial_t^2 u = Pu$ with initial data f, and that $\widehat{u}(t,\xi)$ is supported where $|\xi| \approx \lambda$, and where ξ belongs to some cone Γ of angle θ, with $\theta \ge \lambda^{-\frac12}$. Then modifying the arguments of [4] yields

$$\|u\|_{L^p_x L^2_t([-1,1]\times\mathbb{R}^2)} \le C\,\lambda^{\delta(p)}\theta^{\frac12-\frac3p}\|f\|_{L^2(\mathbb{R}^2)}\,. \tag{2.2}$$

The first decomposition we make of u is a Littlewood-Paley decomposition, after which we may assume that $\widehat{u}(t,\xi)$ is supported where $\lambda \le |\xi| \le 2\lambda$, for some real $\lambda \ge 1$.

We next make a dyadic decomposition with regards to the angle from the ξ_1 axis. By considering the piece where $|\xi_2| \geq |\xi_1|$ separately, we may accomplish this angular decomposition by taking a dyadic decomposition in the ξ_2 variable. Precisely, for $j \geq 1$ we take

$$\mathrm{supp}\big(\widehat{u_j}\big) \subseteq \big\{(\xi_1,\xi_2) \,:\, |\xi_1| \approx \lambda\,,\ |\xi_2| \approx 2^j \lambda^{\frac{2}{3}}\big\}\,,$$

and take

$$\mathrm{supp}\big(\widehat{u_0}\big) \subseteq \big\{(\xi_1,\xi_2) \,:\, |\xi_1| \approx \lambda\,,\ |\xi_2| \lesssim \lambda^{\frac{2}{3}}\big\}\,.$$

Thus, $\widehat{u_j}$ is localized to a cone of angle $\theta_j = 2^j \lambda^{-\frac{1}{3}}$, where θ_j ranges dyadically between $\lambda^{-\frac{1}{3}}$ and 1.

Finally, for each j we decompose the x_1 axis into disjoint intervals of length θ_j, and taking the corresponding spatially disjoint decomposition of u_j write

$$u = \sum_j \sum_{k=1}^{\theta_j^{-1}} u_{j,k}$$

(we supress the λ since it can be assumed fixed.)

The key estimate is that for each term $u_{j,k}$ we can prove the analog of estimate (2.2)

$$\|u_{j,k}\|_{L^p_x L^2_t} \lesssim \lambda^{\delta(p)} \theta_j^{\frac{1}{2}-\frac{3}{p}} \|f\|_{L^2}\,. \tag{2.3}$$

Since the pieces $u_{j,k}$ are spatially disjoint in x_1, and since there are θ_j^{-1} pieces, adding over k yields

$$\|u_j\|_{L^p_x L^2_t} \lesssim \lambda^{\delta(p)} \theta_j^{\frac{1}{2}-\frac{4}{p}} \|f\|_{L^2}\,.$$

For $p > 8$, the exponent of θ_j is positive, and the dyadic sum over j thus converges, yielding

$$\|u\|_{L^p_x L^2_t} \lesssim \lambda^{\delta(p)} \|f\|_{L^2}\,.$$

For $p = 8$, a more delicate summation argument yields the result.

The key to proving (2.3) is that after rescaling space by a factor of θ_j the metric g behaves like a $C^{1,1}$ metric on the microlocal support of $u_{j,k}$, and thus the techniques used to prove (2.2) can be applied. Since the squarefunction estimates are scale invariant (in that both sides scale by the same factor under dilation) this will imply (2.3).

The statement that $\mathrm{g}(\theta_j x)$ "behaves like a $C^{1,1}$ metric" is best understood in terms of Tataru's work [9]. Tataru showed that, for wave equations with time dependent metrics $\mathrm{g}(t,x)$, one can prove Strichartz estimates provided that

$$\int \|D^2 \mathrm{g}(t,\,\cdot\,)\|_{L^\infty_x}\, dt \leq C\,.$$

This condition is sufficient, for example, to prove well posedness of the geodesic flow, and boundedness of the various remainder terms that arise in the paradifferential techniques of the papers [3] and [9].

In our setting we have a weaker version of this condition, which is that the second derivatives of $\mathrm{g}(\theta_j x)$ are integrable along each bicharacteristic passing through the microlocal support of $u_{j,k}$. Presicely, bicharacteristics γ corresponding to cotangent vectors in the support of $u_{j,k}$ satisfy $\frac{d\gamma_2}{dt} \approx \theta_j$ on the spatial support of $u_{j,k}$, and thus

$$\int_\gamma |D^2 \mathrm{g}(\theta_j \cdot)|\, dt \approx \theta_j^2 \int_\gamma \delta(\theta_j \gamma_2(t))\, dt \lesssim 1\,.$$

Here we use the special nature of the Lipschitz singularity of g in noting that $D^2\mathrm{g} \approx \delta(x_2)$.

3. Future directions

This result is the first success in a program of proving dispersive type estimates for boundary value problems by treating the situation as a special case of a Lipschitz metric. In this case, our estimates are sharp in the worst case (corresponding to a convex domain) but not in the case of a concave boundary.

Obtaining improved results in the case of concave boundaries will require estimates on the amount of time that the energy of the solution u can remain in regions near tangent to the boundary. For strictly convex obstacles, tangent geodesics increase in angle as they move away from the boundary, and thus energy does not remain trapped at small angles. This should lead to an improvement when summing over the variable k in the x_1 decomposition of u_j.

Making rigorous these ideas would free the proof of [6] for dispersive estimates outside convex obstacles from the explicit parametrix construction of Melrose and Taylor, and hopefully permit the extension of these estimates to the setting of (not necessarily strictly) convex obstacles.

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