Wavelets Applied to CMB Analysis

- CMB non-Gaussianity
- Constraining the Primordial Power Spectrum

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The Universe ..

- is a perturbed Robertson Walker spacetime with dynamics governed by the Einstein's equations
- is described globally by a set of parameters
- has initial conditions given by Inflation
- CMB and LSS are powerful tools
- the current cosmological model is very succesful in explaining observations

Why test for Gaussianity?

- To discriminate between different scenarios for the generation of cosmological perturbations.
- A key underlying assumption of cosmological data analysis which can be tested.
- Non-Gaussianity can also be associated with secondary anisotropies, foreground contamination and systematic effects.

Why use Wavelets?

The signal on the sky can be studied on different scales, with simultaneous position localization so that

- ▷ the obscuring effects of the central limit theorem are reduced
- ▷ the nature and source of a detection may be better determined

Wavelet based methods have been compared to Fourier and pixel based methods in Hobson et al. 1999, Aghanim et al. 2003, Cabella et al. 2004, Starck et al. 2003 ....

Wavelets and WMAP non-Gaussianity

Using the Spherical Mexican Hat Wavelet (SMHW) Transform on the 1st year WMAP data, non-Gaussianity was reported in the form of kurtosis of wavelet coefficients in the southern Galactic hemisphere on 4° scales (10° on the sky) $> 99\%$ significance (Vielva et al. 2004).

We (Mukherjee & Wang, ApJ, 2004):

- confirm the feature; also find that the signal shows up more strongly in an extremum analysis
- find that the signal shows up as scale-scale correlations amongst wavelet coefficients
- establish the robustness of the signal under wider ranging assumptions about the Galactic mask and the noise model
- use this method of SMHW transform to constrain f_{NL}

The Sperical Mexican Hat Wavelet

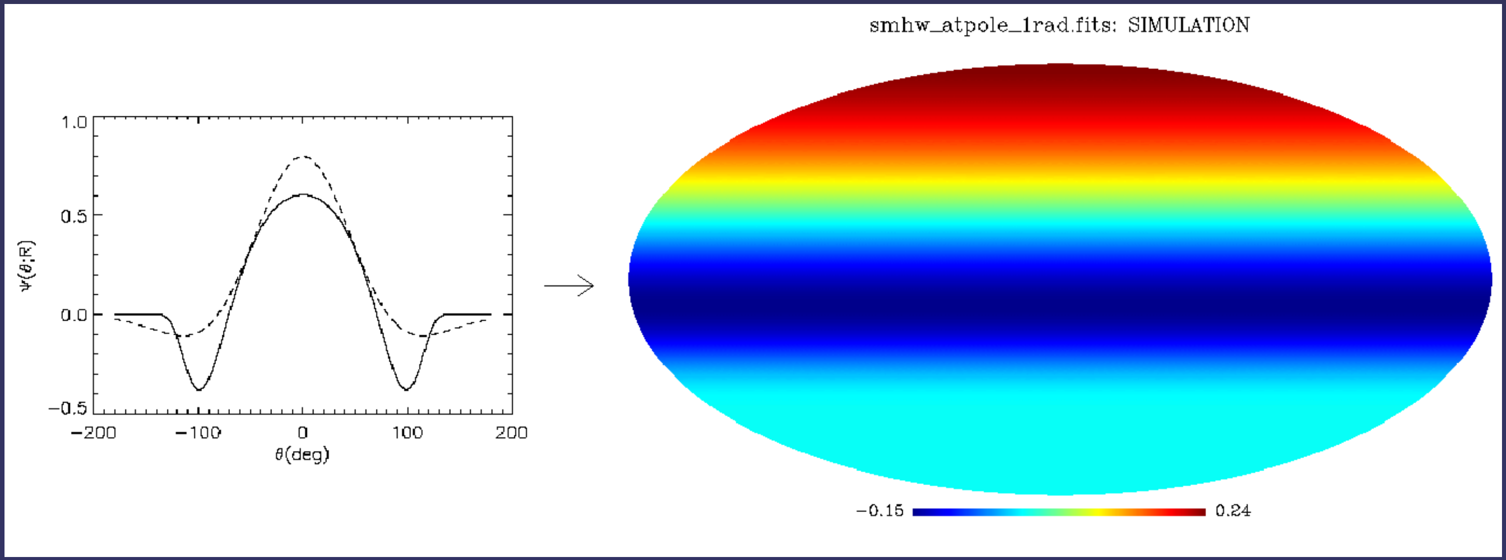
Continuous wavelet:

$$\Psi_R(\theta) = \frac{1}{\sqrt{2\pi}N(R)}\left[1+(\frac{y}{2})^2\right]^2\left[2-(\frac{y}{R})^2\right]e^{-y^2/2R^2},\ y = 2tan\frac{\theta}{2}$$

The wavelet coefficients are

$$w_R(\theta', \phi') = \int d\Omega\, T(\theta,\phi)\, \Psi_R(|\theta - \theta'|)$$

The stereographic projection preserves the compensation ($\int dx\psi(x) = 0$), admissibility ($C_\psi \equiv (2\pi)^2 \int dq q^{-1}\psi^2(q) < \infty$), normalization ($\int dx \Psi^2_R(x) = \frac{1}{R^2}\int dx\psi^2(x) = 1$), and the dilations and translations properties of wavelets.



SMHW Analysis

- Apply SMHW transform to foreground cleaned Q-V-W coadded WMAP data map masked by the Kp0 mask

- apply extended Galactic mask to further exclude the wavelet coefficients contaminated by the mask

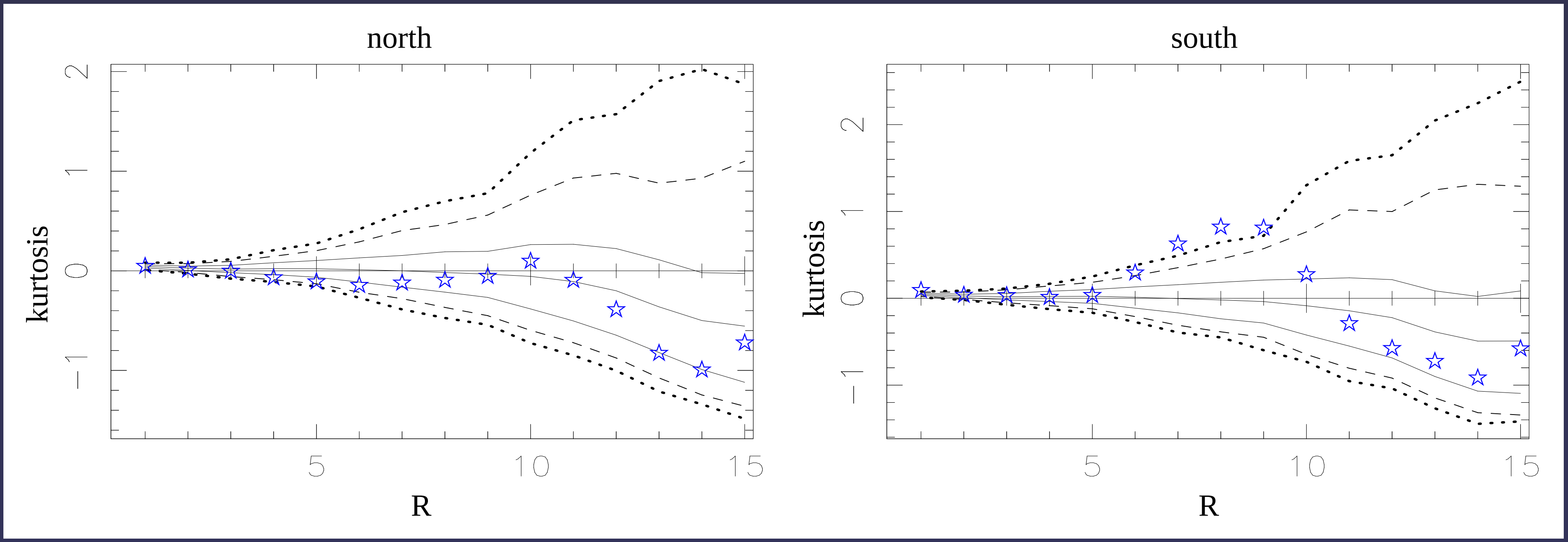
- compute skewness and kurtosis of remaining wavelet coefficients
  $S(R) = \frac{1}{N_R}\sum_{i=1}^{N_R} \frac{w_R(x_i)^3}{\sigma_R^3}(symmetry)$
  $K(R) = \frac{1}{N_R}\sum_{i=1}^{N_R} \frac{w_R(x_i)^4}{\sigma_R^4} - 3(peakedness), \sigma_R$:dispersion of $w_R(x_i)$
  or another statistic

- repeat for different R

- repeat for large number of Gaussian simulations to obtain confidence contours

- compare results from data and simulations to obtain significance of any observed deviations

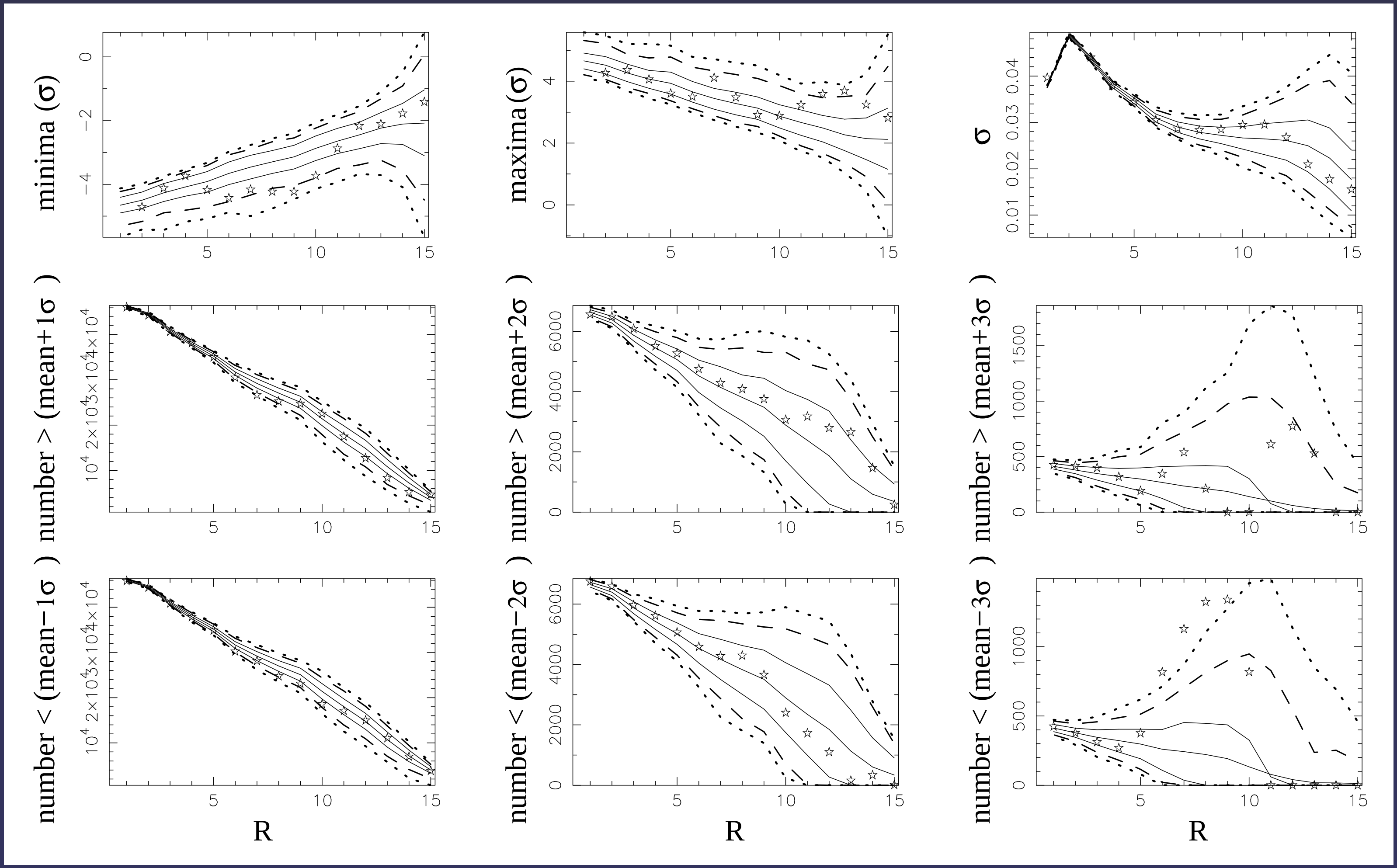
Results: Kurtosis



R_1 to R_{15} => 0.5° to 30° on the sky.
Signal shows no frequency dependence in the 23GHz to 94GHz range.

- 9 of 1000 Gaussian realizations have larger χ^2's than the data => signal significant at 99%
- Only 1 of 1000 simulations have a larger kurtosis on scales R_7 and R_8 => signal on these particular scales significant at $>> 99\%$
- the number of simulations that have kurtosis values outside the the 99% confidence region in any 2 of the 15 scales indicates a significance of $at\ least$ 97%

Too many cold pixels in the southern hemisphere



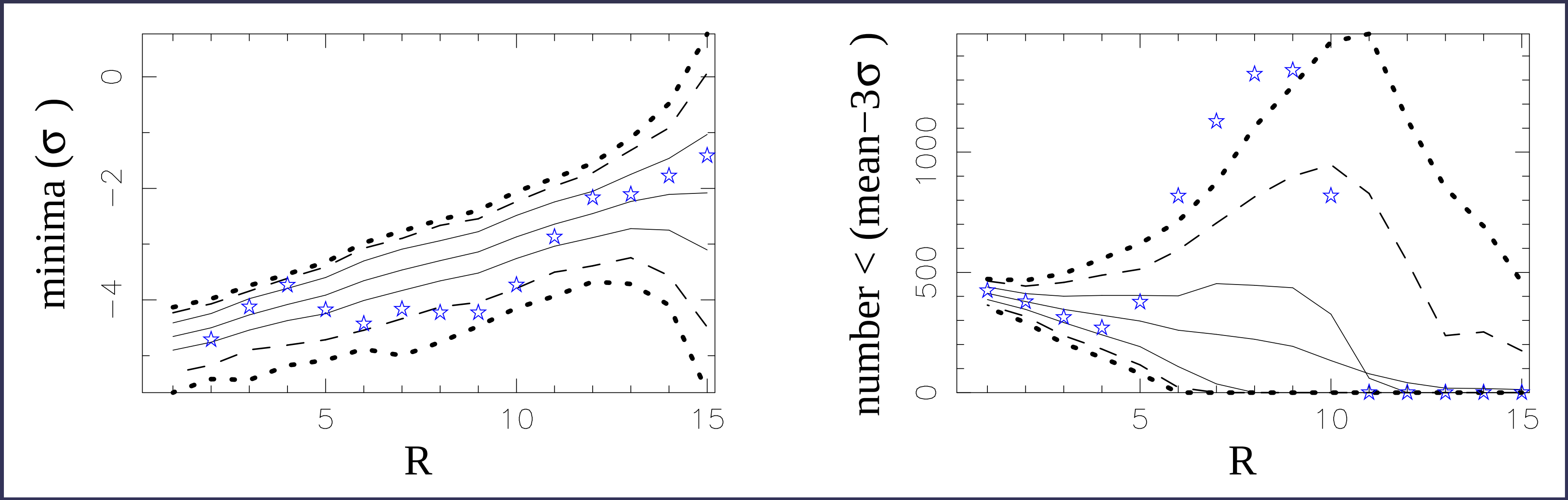
statistics relating to

top panel: the minima, maxima, and σ of wavelet coefficients

middle panel: the # of coefficients that are larger than (mean+1σ), (mean+2σ) and (mean+3σ)

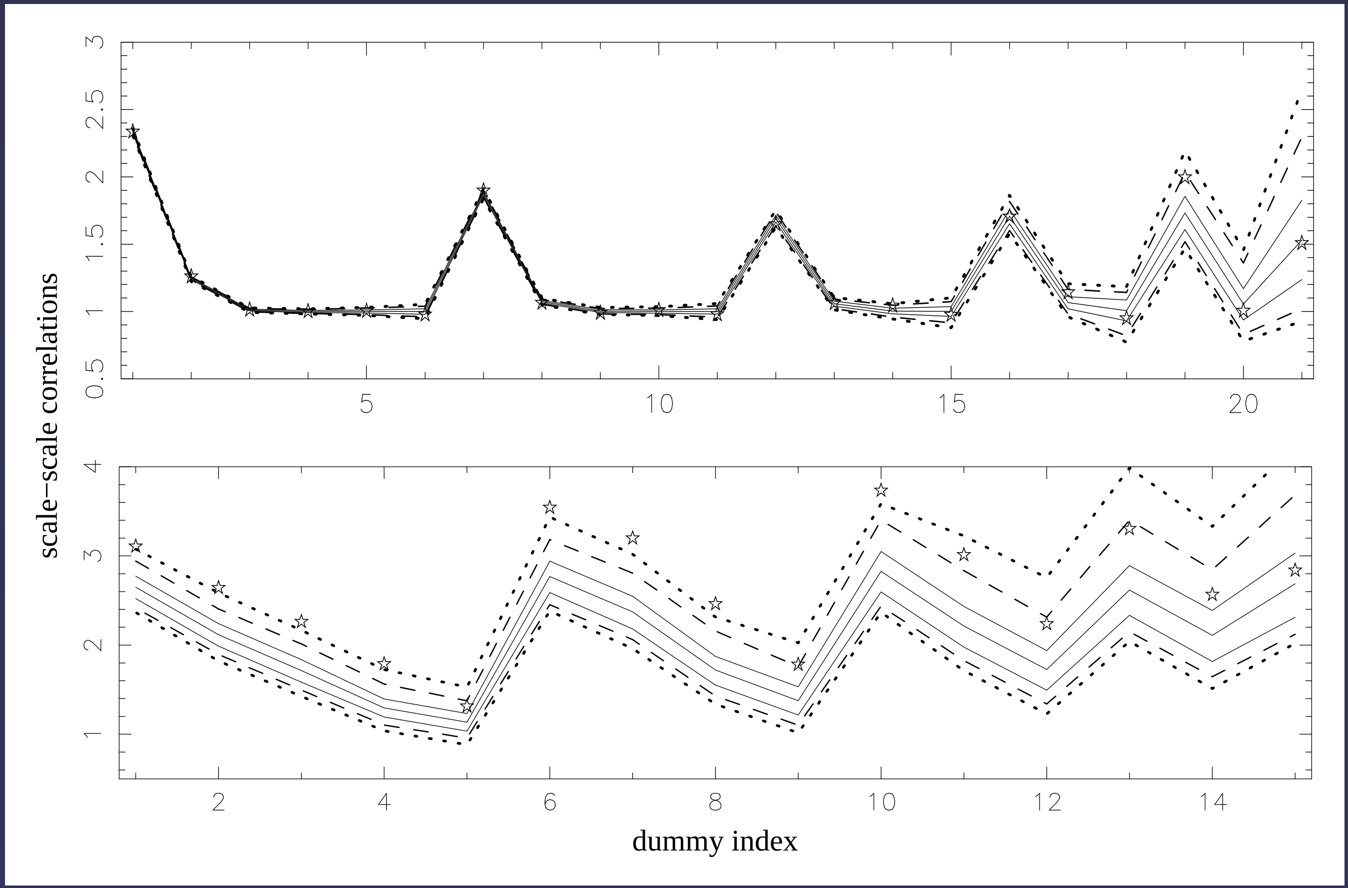
bottom panel: the # of coefficients that are smaller than (mean-1σ), (mean-2σ) and (mean-3σ) of their distribution

Too many cold pixels ...



- minima on $\sim 4^\circ$ scales is significant (only 1% of simulations show a stronger minima)
- the number of cold pixels in the southern hemisphere on $\sim 4^\circ$ scales is too large at $>> 99\%$. On scales R_6 and R_7 there is contribution from more than one cold spot, on larger scales it is mainly the 1 cold spot near $b = -57^\circ,\ l = 209^\circ$. That spot is also present on scales R_6 and R_7
- all this for the northern hemisphere is consistent with Gaussianity

Results: Scale-scale correlations

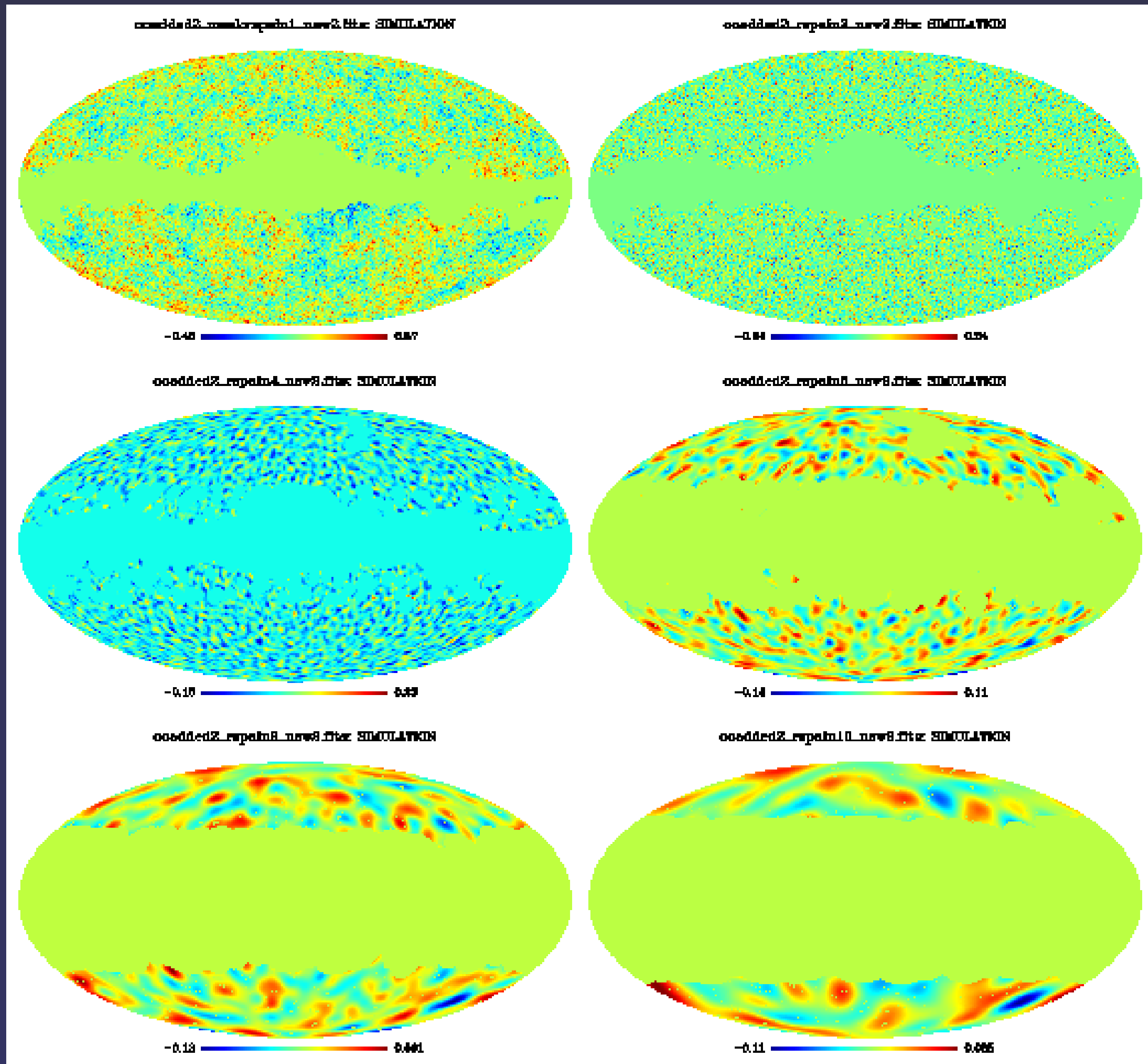


top panel: correlations between coefficients of scales R_1, R_2, R_3, R_5, R_7, R_{10} and R_{14}, all sky
bottom panel: R_6, R_7, R_8, R_9, R_{10}, R_{11}, southern hemisphere

$$C_{R_i,R_j} = \frac{N \sum_x w_{R_i}(x)^2 w_{R_j}(x)^2}{\sum_x w_{R_i}(x)^2 \sum_x w_{R_j}(x)^2}$$

Find significant scale-scale correlations on scales that contain the cold spot

The wavelet coefficients and the cold spot



Constraining f_{NL}

Parametrizing the leading order non-linear correction to primordial perturbations:

$$\Phi(x) = \Phi_L(x) + f_{NL}\left[\Phi_L^2(x) - \langle\Phi_L^2(x)\rangle\right],$$

motivation to constrain it:

- how much non-Gaussianity of this form is allowed by data
- can compare the sensitivity of different data sets and estimators to this form of non-Gaussianity
- can be estimated for different inflationary scenarios using 2nd order calculations

$|f_{NL}| < 1000$ from COBE and MAXIMA

WMAP gave $-58 < f_{NL} < 134$ at 95% using an optimized statistic based on the bispectrum (Komatsu et al.)

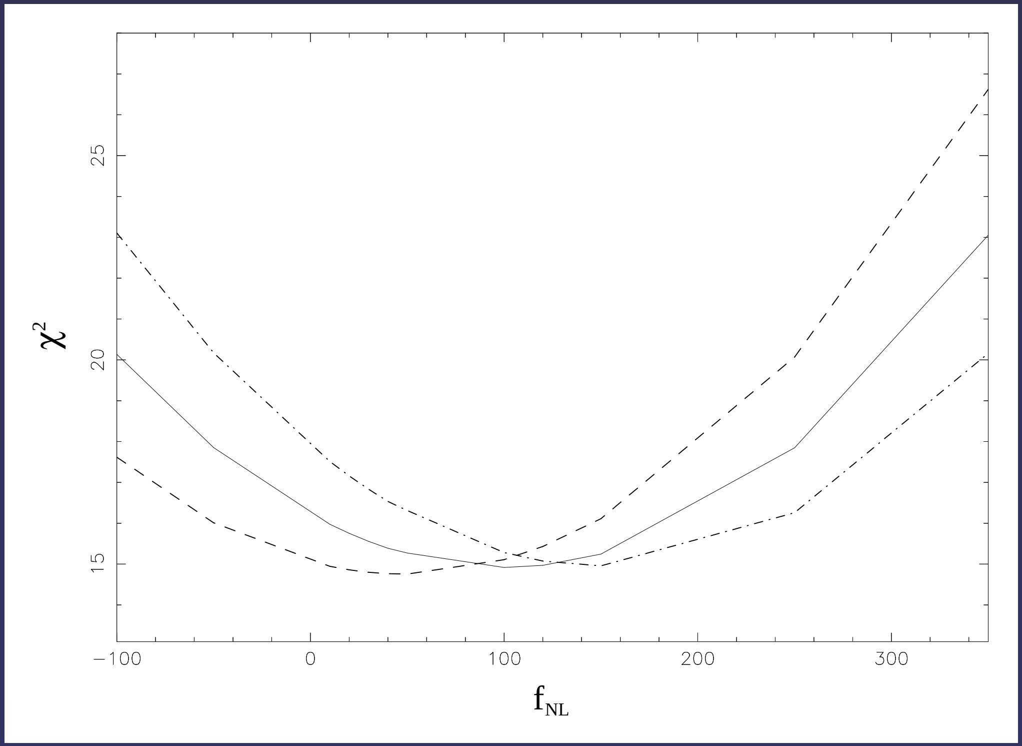
We use the non-Gaussian simulations of Komatsu et al. 2003.

Constraining f_{NL}...

Best fit f_{NL}:

$$\chi^2 = \sum_{R_i, R_j} \left[S(R_i) - \bar{S}_{sim}(R_i) \right] \Sigma^{-1}_{R_i, R_j} \left[S(R_j) - \bar{S}_{sim}(R_j) \right],$$

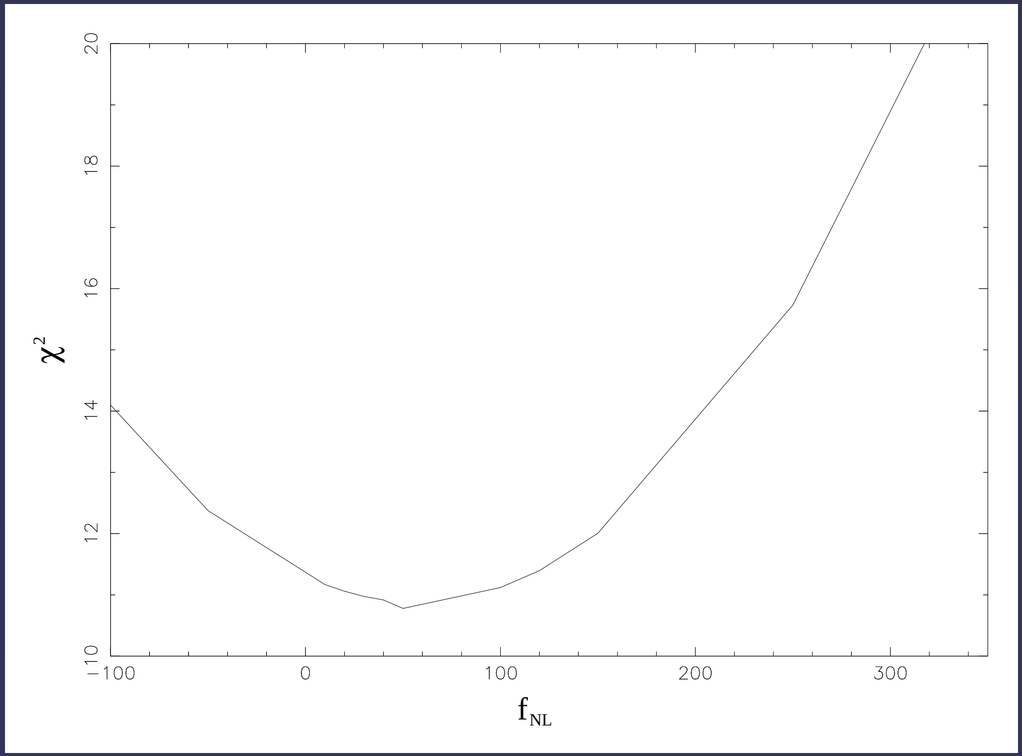
$S(R_i)$ is skewness of WMAP data on scale R_i, $\bar{S}_{sim}(R_i)$ is mean got from Monte Carlo simulations, computed for different values of f_{NL}, Σ_{R_i, R_j} is the scale-scale skewness covariance matrix from simulations.



Test that the χ^2 statistic accurately recovers f_{NL} by using it on simulated maps.

f_{NL} Constraints

$f_{NL} = 50 \pm 80$ at 68% confidence, with 95% (99%) upper limits of 220 (280).



Consistent limits are obtained using fisher discriminants

Limits may be improved by applying method to Wiener reconstructions of the primordial density field

Using wavelets one can obtain scale dependent constraints on f_{NL}

Results from applying other estimators to WMAP data

Non-detections

Minkowski funtionals, (Komatsu et al., Colley & Gott), bispectrum (Komatsu et al., Magueijo & Medeiros), 3-pt statistics (Gaztanaga& Wagg)

Detections

North-south asymmetry and/or non-Gaussianity reported by Park (genus), Eriksen et al. (2 and 3-pt correlations and genus), Hansen et al., Cabella et al. (local curvature, local power spectra), Vielva et al., Mukherjee & Wang, Cruz et al., McEwan et al. (wavelets), Chiang et al., Naselsky et al., Coles et al. (phase analysis, some high l correlations), Copi et al. (multipole vectors, low l correlations), Gurzadyan et al. (ellipticity of spots) Tegmark et al. (alignment of C_2 and C_3), Larson & Wandelt (real space extremum analysis), Land & Magueijo (bispectrum and multipole invariants)

Some detections are at $\geq 99\%$ level

Conclusion

A detection at $\geq 99\%$ significance, more or less in the form of a localized cold spot on $\sim 10^\circ$ scales on the sky.

- independent of frequency
- robust against changes in Galactic mask, and not due to noise

Derived constraints on f_{NL}

Future ...

More stringent limits on (non-)Gaussianity expected as the data improve.

Complimentary estimators can be combined to improve sensitivity.

3D LSS data on large scales can also provide complementary constraints.

Improvement in model predictions and calculation of 2nd order effects can help constrain specific scenarios.

Constraining the Primordial Power Spectrum

$$CMB\ data : C_l \propto \int \frac{dk}{k} P_{in}(k) \left|\Delta_{Tl}(k, \tau = \tau_0)\right|^2$$

$$LSS\ data : P(k) \propto T(k)^2 P_{in}(k)$$

Motivation to reconstruct $P_{in}(k)$ model independently

- primary window to unknown physics during inflation
- determine how constraining the data are to its shape
- test assumptions regarding early universe physics:
  - ▷ scale invariance $P_{in}(k) = A$,
  - ▷ power law $P_{in}(k) = A \left(\frac{k}{k_0}\right)^{n_s - 1}$,
  - ▷ power law with running $P_{in}(k) = A(k_0) \left(\frac{k}{k_0}\right)^{n_s(k_0) - 1 + 0.5(dn_s/dlnk)ln(k/k_0)}$

Method

$$P_{in}(k_i) = \sum_{j=0}^{J-1} \sum_{l=0}^{2^j-1} b_{j,l}\psi_{j,l}(k_i)$$

$\psi_{j,l}$ are discrete, compactly supported and orthogonal wrt both the scale j and position l indices => multiresolution analysis and complete and non-redundant (invertible) representation of a function.

If total number of coefficients=2^J, j increases from 0 to $J-1$ and within each scale l runs from 0 to 2^j-1. Coefficients with increasing j represent structure in $P_{in}(k)$ on increasingly smaller scales. We used 16 coefficients to represent $P_{in}(k)$ over $0.0002 \le k/(\text{Mpc}^{-1}) \le 0.2$

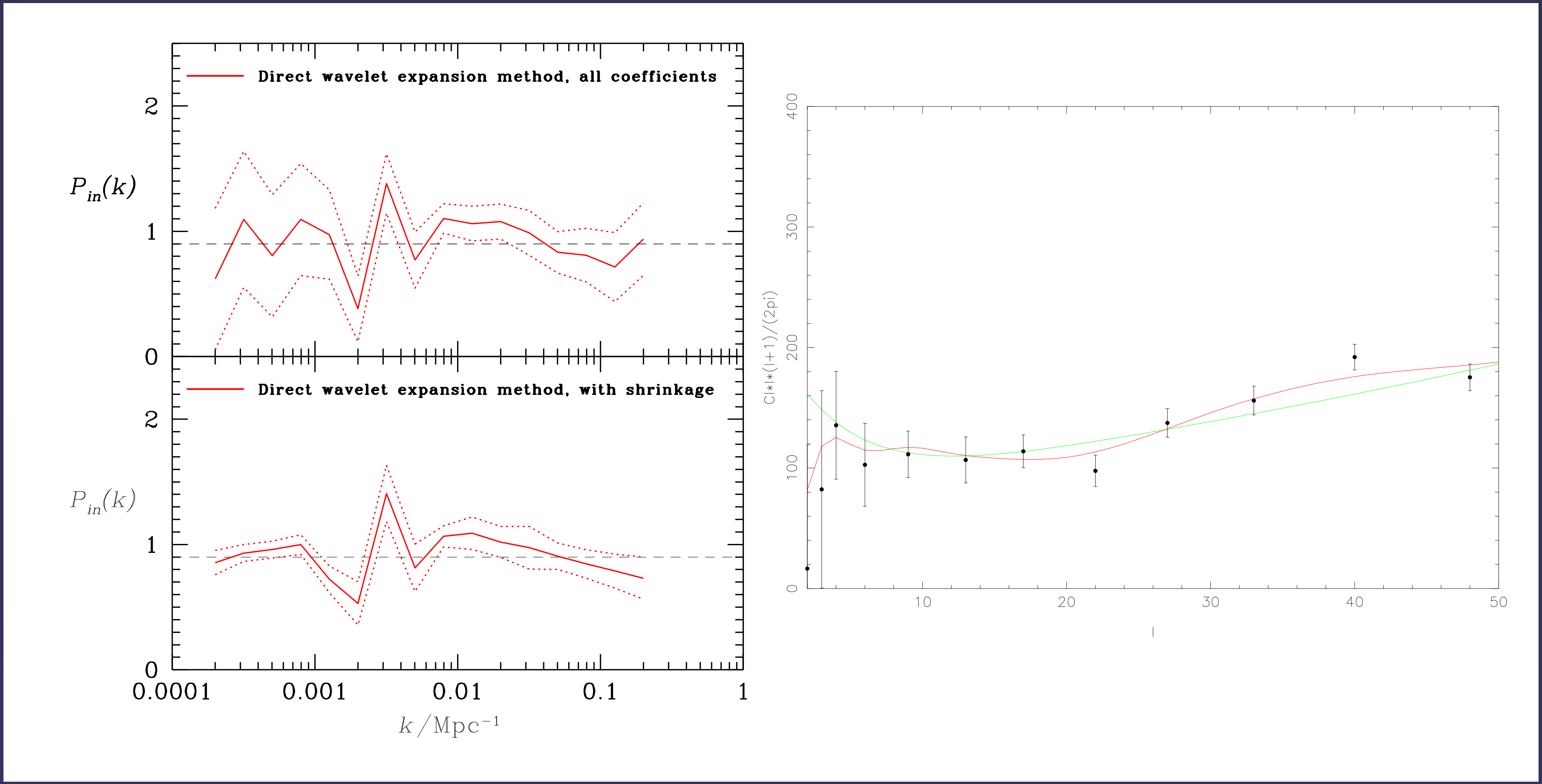
Likelihood(data) = f ($P_{in}(k)$ parameters, cosmological parameters)
Vary all parameters in an MCMC based likelihood analysis to obtain constraints

Advantages:

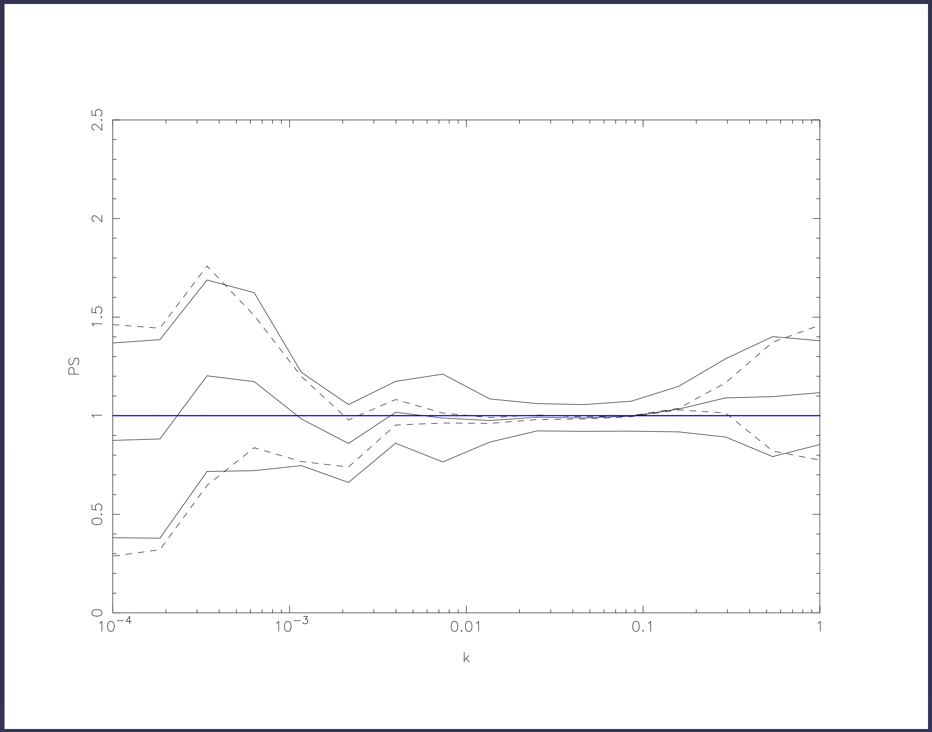
- most signals are sparse in wavelet space; allows for denoising
- the resulting reconstruction suffers less from the correlated errors of binning methods

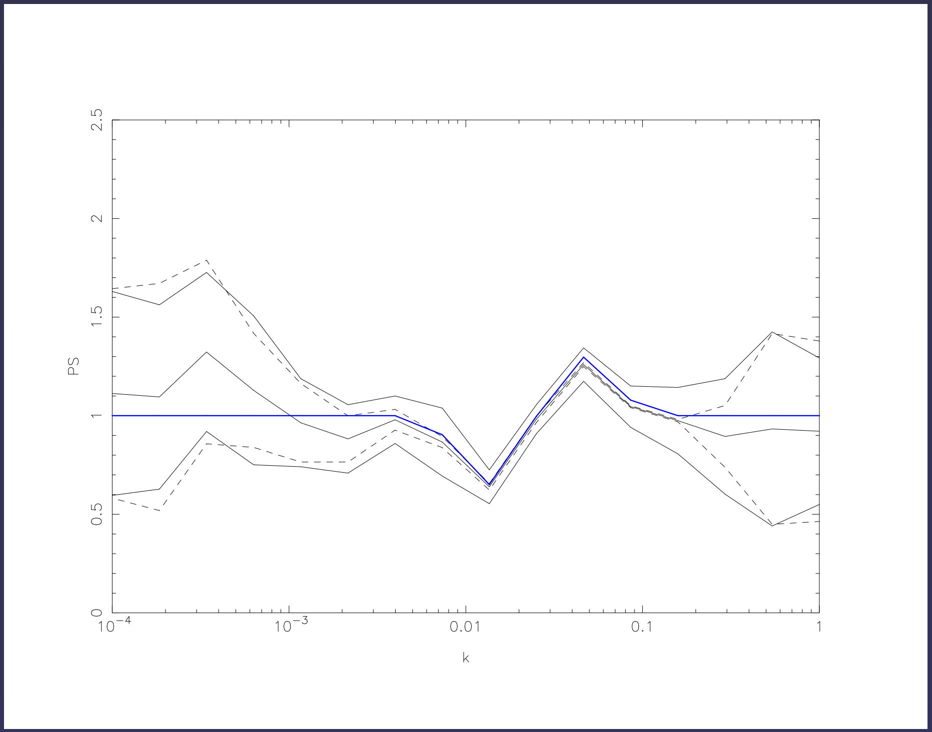
Results

Using CMB data from WMAP, CBI and ACBAR (out to l=2000), and LSS data from2dFGRS and PSCZ galaxy redshift surveys over linear scales, marginalizing over a linear bias, we (Mukherjee & Wang, ApJ, 2003) get



Application to Simulated Data ...

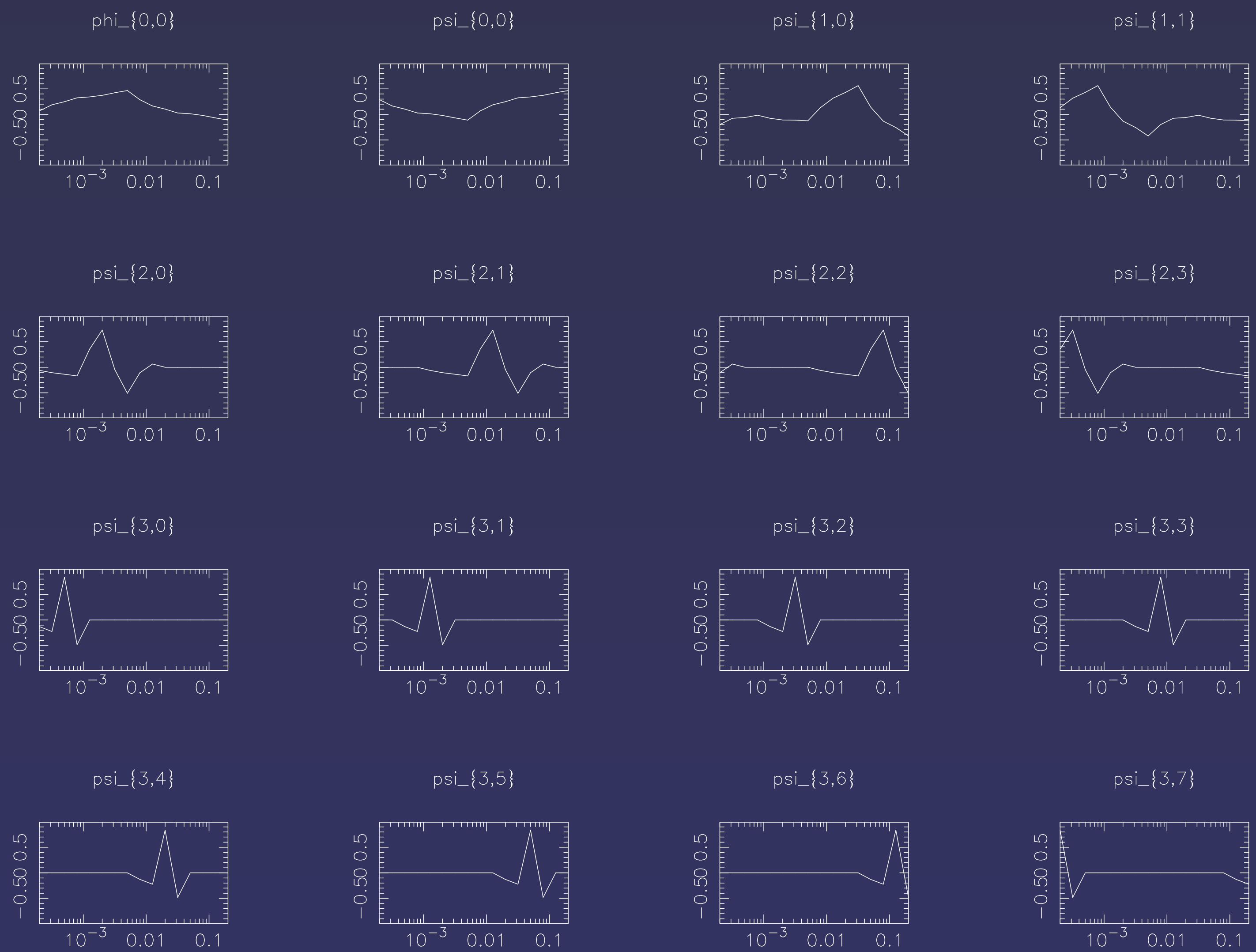




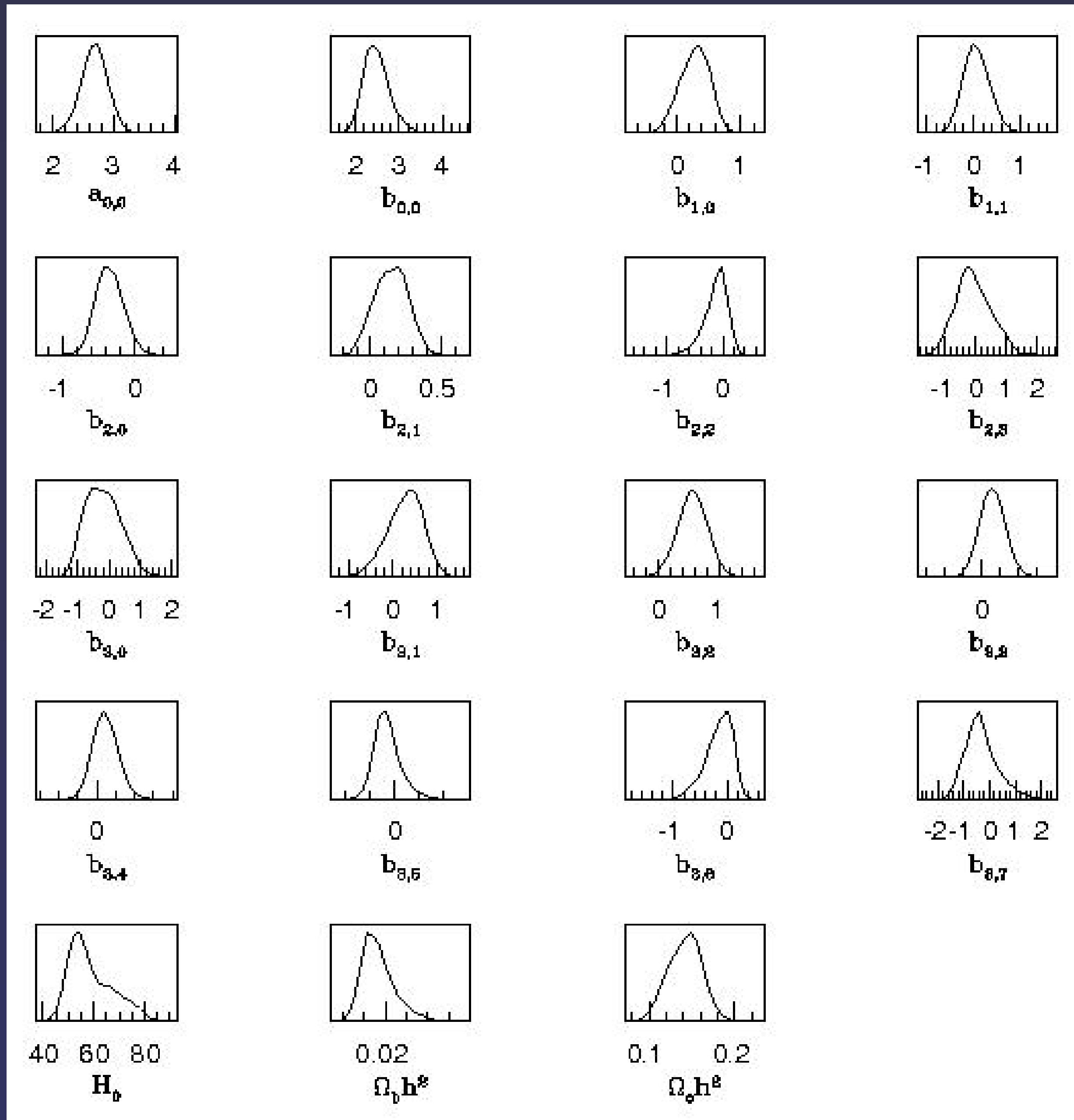
Conclusions

Wavelets are an efficient basis for function reconstruction; They can give reliable reconstructions of the primordial power spectrum, with less correlated errors than binning methods.

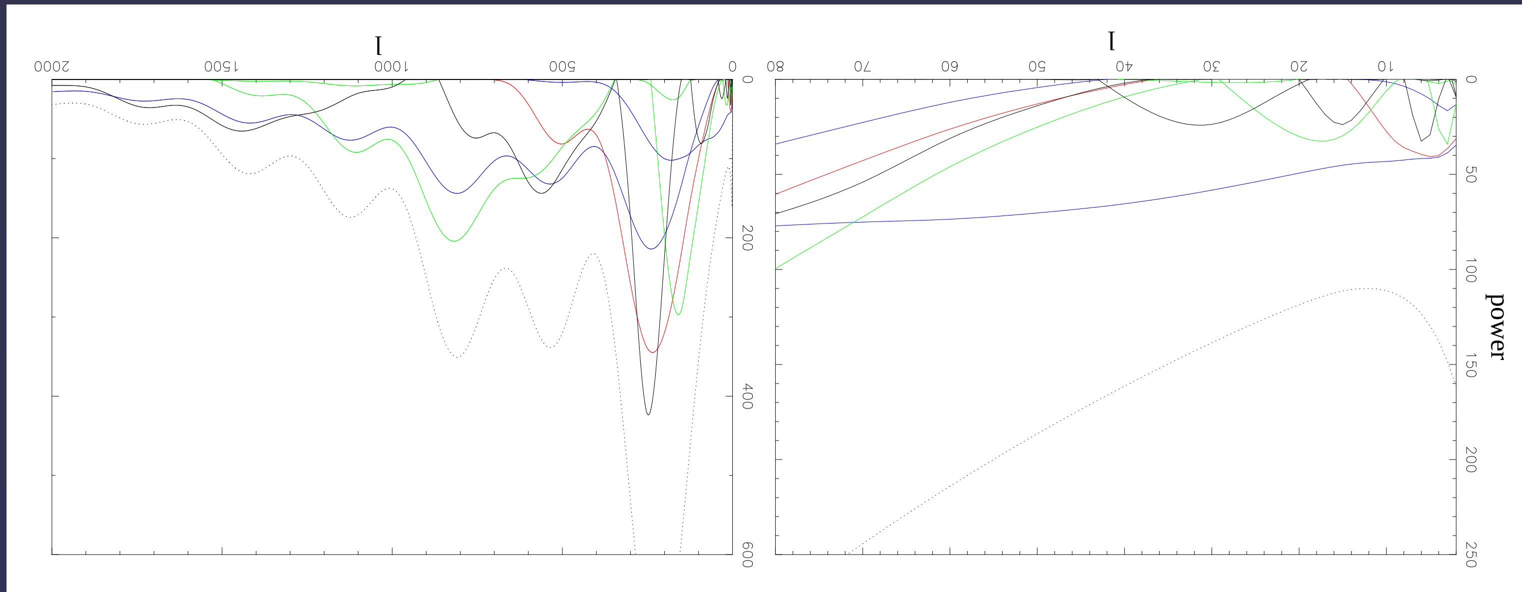
The wavelet basis functions ..



The likelihood curves ..



The wavelet basis functions projected in l space



The Spectrum