Functions on the Sphere
Spherical Harmonics
Polynoms and vectors

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Outline

- Reminder : Fonctions one the sphere, the Laplacian, eigenmodes...
- Eigenmodes are harmonic polynomials of $\mathbb{R}^3$
- Multipole and Harmonic decompositions
- Vector multipoles

The 2-sphere S^2

a (2-dimensional Riemanian) manifold.

isometrically embedded in $\mathbb{R}^3$,
as the surface of equation
$G(X,X) \equiv X \cdot X = 1$

(with the usual Euclidean metric G).

This provides the **spherical coordinates** θ, φ

$$X^1 = \cos\theta, \quad (1)$$

$$X^2 = \sin\theta \cos\varphi, \quad (2)$$

$$X^3 = \sin\theta \sin\varphi. \quad (3)$$

The **restriction** of G to S^2 gives its metric

$$g = ds^2 = d\theta^2 + \cos^2\theta \; d\varphi^2.$$

Isometries of S^2 = Rotations of $\mathbb{R}^3$=
. **= group SO(3)**

Simplest representation: on vectors of $\mathbb{R}^3$.
The rotation $R(\mathbf{u},\alpha)$ $\in$SO(3) is represented by a real orthogonal matrix of order 3:

$$R(\mathbf{u},\alpha): \quad X \quad \mapsto X' = RX \tag{4}$$

$$M: \quad (X^A) \quad \mapsto (X'^A \equiv R^A_B \; X^B). \tag{5}$$

Thus, SO(3) = group of orthogonal matrices
$(R \; R^T = \mathbb{I}_3)$,
with the matrix multiplication (isomorphism).

Functions

$C(S^2) =$ set of smooth functions on S^2.

Any function on $S^2 \subset \mathbb{R}3$ may be seen as the **restriction** of a function on $\mathbb{R}^3$.

Smooth functions on $\mathbb{R}^3$ can be approximated by infinite Polynomials of $\mathbb{R}^3$ (the set of polynomials is dense).

$Pol =$ set of Polynoms in $\mathbb{R}^3$
Pol is a vector space.

Rotation of functions

A rotation $R: \ X \mapsto RX$ of has a natural action T_R on a function:

$$T_R : f \mapsto T_R f :$$

$$T_R f(X) \equiv f(R^{-1}X).$$

Since $C(S^2)$ is a vector space,
$R \mapsto T_R$ is a **representation** of SO(3).

There are some **invariant subspaces**: this representation is **reductible**.

The invariant subspaces will allow to construct the **irreductible representations**.

Multipole Decomposition for a function on S^2.

$$C(S^2) = H^0 \oplus H^1 \oplus ... \oplus H^\ell \oplus ...$$

(direct sum).

For each function on the sphere,

$$f = \sum_\ell f_{(\ell)}; \;\; f_{(\ell)} \in H^\ell.$$

$f \mapsto f_{(\ell)}$ is the projection of C onto H^ℓ.
$f_{(\ell)} \in H^\ell$ is the ℓ - multipole.
($\approx f$ seen at angular scale $2\pi/\ell$).

$f_{(1)}$= dipole, $f_{(2)}$= quadrupole, etc.

Each H^ℓ

- is invariant under $SO(3)$.
- constitutes an IUR of SO(3).
- is an (Hilbert) vector space, of dimenson $2\ell + 1$.
- (is the space of eigenfunctions of Δ_{S^2} with the same eigenvalue $\lambda_\ell = \ell\ (\ell + 1);\ \ell \in \mathbb{N}$.)
- ... has many other properties

Multipole decomposition
and
Eigenmodes

Eigenmodes of S^2.

Laplacian on S^2

$$\Delta_{S^2} := \frac{1}{\sin\theta}\ \frac{\partial}{\partial\theta}\ (\sin\theta\ \frac{\partial}{\partial\theta}) + \frac{1}{\sin^2\theta}\ \frac{\partial^2}{\partial\phi^2}$$

Helmoltz equation on the Sphere :
$\Delta_{S^2} f = -\lambda f,\ \ f \in C(S^2)$.

Solutions = **eigenfunctions** of Δ_{S^2}.
$\equiv$ **eigenmodes** of S^2.

Theorem : Solutions of Helmoltz equation (= eigenfunctions) exist for the eigenvalues

$$\lambda = \lambda_\ell \equiv \ell\ (\ell+1);\ \ \ell \in \mathbb{N}.$$

Theorem :
The vector space of ℓ- eigenmodes of S^2 is H^ℓ.

Eigenspaces

All eigenmodes of S^2 with the same eigenvalue λ_ℓ ($\equiv \ell$ - eigenmodes) form the vector space $H^\ell \subset C(S^2)$.

H^ℓ is the eigen[vector]space corresponding to the eigenvalue λ_ℓ,
of dimension $2\ell + 1$.

As seen above, H^ℓ is stable under the rotations T_R:

$$\forall f \in H^\ell \Rightarrow \ T_R f \in H^\ell, \ \forall R \in SO(3).$$

H^ℓ is stable under SO(3) : the representation $T_{R,\ell}$ limited to H^ℓ is **irreductible**.

Multipole Decomposition

$$C(S^2) = \quad H^0 \oplus H^1 \oplus \ldots \quad \oplus H^\ell \oplus \ldots \tag{6}$$
$$f = \quad f_{(0)} + f_{(1)} + \ldots \quad + f_{(\ell)} + \ldots \tag{7}$$

$f_{(\ell)} \in H^\ell$ is the multipole.

For instance, $f_{(1)}$ is the dipole,
$f_{(2)}$ is the quadrupole, etc. This is an eigenmode:

$$\Delta_{S^2}\ f_{(\ell)} = -\lambda_\ell f_{(\ell)}.$$

A basis for H^ℓ

H^ℓ is of dimension $2\ell + 1$.

All functions of H^ℓ are eigenvalues of Δ_{S^2} with the same eigenvalue λ_ℓ.

To find a basis, we will use the kinetic momentum operator.

Kinetic momentum operator

$J_i \equiv \epsilon_{ijk}\ x_j\ \frac{\partial}{\partial x^k}$ = components of the *kinetic momentum operator* applied to the functions
(i.e., the generators of the rotations).
(i.e., the generator of the Lie algebra).

We have $\Delta_{S^2} = J^i\ J_i = J^2$. (Casimir operator)
Thus, H^ℓ is an **eigenspace of** J^2.

Kinetic momentum operator

Are there some eigenfunctions of the component J_3 in H^ℓ ?

Yes:

the possible eigenvalues are
$m = -\ell ... \ell$ (in number $2\ell + 1$).
The $2\ell + 1$ corresponding eigenfunctions are called the *spherical harmonics* $Y_{\ell m}$.
They form a basis of H^ℓ.

Decomposition in spherical harmonics

I recall the multipole decomposition :

$$C(S^2) = H^0 \oplus H^1 \oplus ...H^\ell \oplus ...$$

$$f = f_{(0)} + f_{(1)} + ..f_{(\ell)} + ...$$

From the basis of H^ℓ,

$$f_{(\ell)} = \sum_{m=-\ell}^{\ell} f_{\ell m}\, Y_{\ell m},$$

usual decomposition in spherical harmonics

$$f = \sum_{\ell=0} \sum_{m=-\ell}^{\ell} f_{\ell m}\, Y_{\ell m}.$$

The function f is characterized by all its coefficients $f_{\ell m} = a_{\ell m}$.

Why harmonic ?

Approximation

Approximation of f, at the resolution L:

$$f_{(<L)} = \sum_{\ell=0}^{L} \sum_{m=-\ell}^{\ell} f_{\ell m} \, Y_{\ell m}.$$

This is at the basis of the present analyses of the CMB (and of data on a sphere in general).

Why harmonic ?

Eigenmodes
and
Harmonic polynomials

(We associate functions on S^2 to polynomials in $\mathbb{R}^3$)

Laplacian operators

Laplacian in $\mathbb{R}^3$:

$$\Delta_{\mathbb{R}^3} \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

$$= \Delta_{\mathbb{R}^3} = \frac{1}{r^2}\ \partial_r\ (r^2\ \partial_r) + \frac{1}{r^2}\Delta_{S^2}$$

(in spherical coordinates).

Laplacian on S^2

$$\Delta_{S^2} := \frac{1}{\sin\theta}\ \frac{\partial}{\partial\theta}\ (\sin\theta\ \frac{\partial}{\partial\theta}) + \frac{1}{\sin^2\theta}\ \frac{\partial^2}{\partial\phi^2}$$

Harmonic functions in $\mathbb{R}^3$

Definition: $\Delta_{\mathbb{R}^3} F = 0.$

((Any function $f(\theta, \varphi)$ on S^2 is the reduction of a function on $\mathbb{R}^3$))

Homogeneous Polynomials of degree ℓ
(in $\mathbb{R}^3$)

$$P(X) = \sum_{\alpha+\beta+\gamma=\ell} P_{\alpha,\beta,\gamma}\ x^\alpha\ y^\beta\ z^\gamma.$$

Hereafter ℓ - Homogeneous Polynomials .

Their set $\equiv$ HOM(ℓ).

And we call HARM(ℓ) $\subset$ HOM(ℓ) the vector space of these polynomials which are harmonic.

Eigenfunctions are harmonic Polynomials

Theorem:
If f is an ℓ - eigenmode of S^2, $r^\ell\ f(\theta, \varphi)$ is an harmonic polynomial, homogeneous of degree ℓ.

In other words, $H^\ell \approx \mathrm{HARM}(\ell)$.
ℓ - Harmonic Polynomials $\approx$ ℓ-Eigenmodes.

$$r^\ell\ f(\theta, \varphi) = P(X) = \sum_{\alpha+\beta+\gamma=\ell} P_{\alpha,\beta,\gamma}\ x^\alpha\ y^\beta\ z^\gamma,$$

with P harmonic.

Harmonic projection

$H^\ell \approx \mathrm{HARM}(\ell)$.

By definition, $\mathrm{HARM}(\ell) \subset \mathrm{HOM}(\ell)$.

There is a projection map:

$$\mathrm{HOM}(\ell) \mapsto \mathrm{HARM}(\ell) \tag{8}$$

$$P \mapsto \Pi P \text{ (harmonic),} \tag{9}$$

the **harmonic projection**.

To each $\ell-$ homogeneous polynomial is associated its unique harmonic projection.

Direct sums

HARM(ℓ) $\subset$ HOM(ℓ).

Theorem

$$\begin{aligned} \mathrm{HOM}(\ell) = &\quad \mathrm{HARM}(\ell) &&+ r^2\ \mathrm{HOM}(\ell-2) \\ P = &\quad \Pi P &&+ r^2\ Q. \end{aligned} \tag{10}$$

P = any $\ell-$ homogeneous polynomial,
ΠP is $\ell-$ harmonic homogeneous (harmonic projection),
Q is $(\ell-2)-$ homogeneous :

Note, one may write

$$\mathrm{HOM}(\ell) = \quad \mathrm{HARM}(\ell) + r^2\ [\mathrm{HARM}(\ell-2) \tag{11}$$
$$+ r^2[\mathrm{HARM}(\ell-4) + r^2 ...]] . \tag{12}$$

Tensorial form

The general $\ell-$homogeneous polynomial ,

$$P(X) = \sum_{\alpha+\beta+\gamma=\ell} P_{\alpha,\beta,\gamma}\ x^{\alpha}\ y^{\beta}\ x^{\gamma} \in \mathrm{HOM}(\ell),$$

can be written in the tensorial form:

$$P(X) = \sum_{a_1,a_2,\ldots a_\ell} P_{(a_1,a_2,\ldots a_\ell)}\ X^{a_1}\ X^{a_2}\ \ldots X^{a_\ell}.$$

The parenthesis means symmetry in all the indices.

Thus, $\mathrm{HOM}(\ell) = \{$symmetric tensors of order $\ell\}$

If P is harmonic, then the tensor $P_{(a_1,a_2,\ldots a_\ell)}$ is traceless (exercise !).

$\mathrm{HARM}(\ell) =$
$\{$symmetric traceless tensors of order $\ell\}$

Harmonic projection $\Pi : P \mapsto \Pi P$:
$[\Pi P]_{(a_1,a_2,\ldots a_\ell)}$ = the traceless part of $P_{(a_1,a_2,\ldots a_\ell)}$.

DATA ON THE SPHERE

Analysis of functions (on the sphere)

The temperature T_{CMB} is a function on the sphere.

Multipole expansion:

$$f = f_{(0)} + f_{(1)} + ..f_{(\ell)} + ...$$

$f_{(\ell)}$ represents the temperature fluctuation at angular scale $\approx 2\pi/\ell$.

$$f_{(\ell)} = \sum_{m=-\ell}^{\ell} f_{\ell m}\, Y_{\ell m}.$$

$$f = \sum_{\ell=0} \sum_{m=-\ell}^{\ell} f_{\ell m}\, Y_{\ell m}.$$

Finite resolution: one works with the approximation of f at some (angular) scale

$$f_{\leq L} = \sum_{\ell=0}^{L} \sum_{m=-\ell}^{\ell} f_{\ell m}\, Y_{\ell m}.$$

Density modes

Fluctuations of T are caused by fluctuations in the density, velocities, gravitational potential at the period of recombination (when the CMB was emitted).

Posterior (secondary) fluctuations are neglected here.

Ex.: Sachs-Wolfe effect: scalar fluctuations in the potential (itself generated by density fluctuations): it results that T is proportional to the density ρ.
(or its relative fluctuation $\delta \equiv \delta\rho / < \rho >$)

The density ρ is a function in space. When space is flat ($= \mathbb{R}^3$), ρ is decomposed in Fourier modes:

$$\delta(X) = \int d^3k \ \delta_k \ (e^{ik\cdot X}), \ k \in \mathbb{R}^3$$

(3-dimensional Fourier transform).

When space is non flat, similar decomposition but the ($e^{ik\cdot X}$) are replaced by more complicated (but well known) modes. Here I consider the flat case for illustration.

Theorem

$$e^{ik\cdot X} = \sum_{\ell} \; (2\ell+1)\; i^{\ell}\; j_{\ell}(K)\; j_{\ell}(\mid X \mid)\; P_{\ell}(\hat{k}\cdot\hat{X}),$$

where the hat means unit vector: $\hat{k} \equiv k/K$, $K \equiv \mid k \mid$,
j_{ℓ} is a Bessel function,
and P_{ℓ} a Legendre Polynomial.

In non flat space, we have similar decomposition of the modes (no more exponentials) with different functions than Bessel functions.

The restriction of the previous formula to the sphere gives the decomposition of an unique Fourier mode in spherical harmonics :

For $X \in S^2$ $(\hat{X} = X)$,

$$e^{iK \cdot X} = \sum_{\ell} \ (2\ell + 1) \ i^{\ell} \ j_{\ell}(\mid K \mid) \ P_{\ell}(\hat{K} \cdot X) =$$

$$= 4\pi \sum_{\ell} i^{\ell} \ j_{\ell}(\mid K \mid) \sum_{m} Y^*_{\ell m}(\hat{K}) \ Y_{\ell m}(X),$$

where the star means complex conjugation.

This is for one mode.
The density = a distribution of modes.

Distribution of modes

Prejudice for the mode distribution :
gaussianity and isotropy :
$\delta(X) = \int d^3K \ \delta_K \ e^{iK\cdot X}$ has a Gaussian distribution.

This implies that
δ_K is a random variable with

$< \delta_K >= 0$,

$< \delta_K \ \delta_{K'} >= \delta^{Dirac}_{KK'} \ \mathcal{P}(\mid K \mid)$,

where $\mathcal{P}$ is the **Power Spectrum** .
(Fourier transform of correlation function).

(**Gaussianity and isotropy have to be tested**).

gaussianity and isotropy imply that the $a_{\ell m}$ are also random variable with Gaussian distribution:

$< a_{\ell m} >= 0$,

$< a_{\ell m}\ a_{\ell' m'} >= \frac{\delta_{\ell\ell'}^{Dirac}\ \delta_{mm'}^{Dirac}}{2\ell+1}\ C_\ell^2$.

Thus the C_ℓ^2 are the expression of the power spectrum.

One goal is to measure them (WMAP).

The next goal is to check gaussianity and isotropy.

Anisotropy and non gaussianity would appear as

- non random distribution of the $a_{\ell m}$ for given ℓ
- correlations between the $a_{\ell m}$ for different ℓ, m

One example of non isotropy is given by multi-connected models (non trivial topology).

Non gaussianity is predicted by non standard physics.

Example : multi-connected models

Non trivial topology has mainly two effects:
-1 **Large scales do not exist physically**
$\Rightarrow$ loss of power at small ℓ in the Power spectrum .
This is exactly what sees WMAP !
(but this can be noise, or intrinsic shape of spectrum ...)

-2 **Loss of isotropy** at large scale.
maybe qualitatively apparent in CMB data.

Checking with spherical harmonics implies to measure non diagonal terms in the correlation matrix, and compare to predictions.
This requires to know the eigenmodes of M/Γ.
This has just been done for spherical spaces : ($\star\star$ work in progress $\star\star$) .

Spherical harmonic decomposition is not well adapatated to check gaussianity and isotropy .
Other methods ?

1. wavelets ?

2. Multipole vectors (...)

Multipole vector decomposition: An alternative to spherical harmonics ?

Notations

$\mathrm{HOM}^R(\ell)$ is the set of homogeneous polynomials of degree ℓ (ℓ - homogeneous) with real coefficients.
$\mathrm{HOM}(\ell)$ is the set of ℓ - homogeneous polynomials with real or complex coefficients.
$\mathrm{HARM}^R(\ell)$ is the set of ℓ - homogeneous and harmonic polynomials with real coefficients.
$\mathrm{HARM}(\ell)$ is the set of ℓ - homogeneous and harmonic polynomials with real or complex coefficients.

References

CHS = C.J. Copi, D. Huterer and G.D. Starkman 2003,
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MLR = M. Lachieze-Rey 2004, Harmonic projection and multipole Vectors (astro-ph/0409081),

KW = G. Katz and J. Weeks 2004
Polynomial Interpretation of Multipole Vectors,
(astro-ph/0405631)

The real case

CHS : To any real harmonic polynomial
$H \in \mathrm{HARM}^R(\ell)$
corresponds the unique polynomial $vecH$ such that

$$vecH(X) = A\ (X \cdot V_1)...(X \cdot V_\ell), \tag{13}$$

where the V_i are unit (real) vectors of $\mathbb{R}^3$
= points on the sphere.

(They explicit the correspondence in the tensorial notation):

$$H(X) = \sum_{i_1,i_2,\ldots,i_\ell} (F_{i_1,i_2,\ldots,i_\ell})_0\ x^{i_1} x^{i_2} \ldots x^{i_\ell},$$

where $()_0$ means traceless part.
(quite complicated procedure)

This implies
$H(X) = vecH(X) + X^2\ Q$,
with $Q \in \mathrm{HOM}^R(\ell - 2)$
an $(\ell - 2)$-homogeneous polynomial.

This reminds the harmonic projection.

$$\begin{array}{rcl} \mathrm{HOM}(\ell) = & \mathrm{HARM}(\ell) & \oplus r^2\ \mathrm{HOM}(\ell-2) : \\ P = & \prod P & +r^2\ Q. \end{array}$$

where
Q is $(\ell - 2)$-homogenous
$\prod P$ ℓ-homogenous and harmonic (*harmonic projection*)
$\prod P$ is effectively a projection operator (not inversible).

CHS result as Harmonic Projection

MLR :

$$H = \prod(vecH)$$

(H is an harmonic polynomial)

Unicity of the correspondence allows to invert the operator $\prod$ in $\mathrm{Harm}(\ell)$

Deprojection

I define $\mathrm{VEC}^R(\ell) \subset \mathrm{HOM}^R(\ell)$ the set of ℓ -homogeneous polynomials of the form $Cte\ (X \cdot V_1)...(X \cdot V_\ell)$ (real coefficients).

The harmonic projector $\prod$ establishes a one to one correspondence $\mathrm{HARM}^R \mapsto \mathrm{VEC}^R$.

This allows to invert the relation:

$$\prod{}^{-1} : \mathrm{HARM}^R \mapsto \mathrm{VEC}^R.$$

$$A\ (X \cdot V_1)...(X \cdot V_\ell) = \prod{}^{-1} H(X)$$

My intepretation of CHS's result : some reciprocal of the harmonic decomposition:

$$vecH = \prod{}^{-1} H.$$

Extension to homogeneous polynomials KW have extended this result to $\mathrm{HOM}^R(\ell)$:
any real homogeneous polynomial $P \in \mathrm{HOM}^R(\ell)$ can be uniquely decomposed as

$$P(X) = vecP(X) + X^2\ Q, \tag{14}$$

with $vecP(X)$ of the form above (the V_i unit and real)
and $Q \in \mathrm{HOM}^R(\ell - 2)$.

The decomposition is unique.

MLR: The KW result as Harmonic Projection

For $P \in \mathrm{HOM}(\ell)$,
$A\ (X \cdot V_1)...(X \cdot V_\ell) = \Pi^{-1}\Pi P(X)$.

MLR interpretation of KW's result.

The Polydipole decomposition

(= KW corollary)

The previous decomposition was for one multipole (one scale ℓ) only (= homogeneous polynomial).

KW have extended to non homogeneous polynomials: any real polynomial P of degree L has the unique decomposition

$$P = \sum_{\ell=0}^{L} P_{V(\ell)},$$

with

$$P_{V(\ell)} = A^{\ell}\ (X \cdot V_1^{\ell})\ (X \cdot V_{\ell}^{\ell}).$$

(with real unit vectors as above).

On the other hand, the polynomial P may be trivially written as a sum of its homogeneous polynomials :

$$P = P_{(0)} + P_{(1)} + \ldots P_{(L)} = \sum_{\ell=0}^{L} P_{(\ell)},\ P_{(\ell)} \in \mathrm{HOM}^{R}(\ell).$$

Note that the $P_{V(\ell)}$'s do not coincide with the $vecP_{(\ell)}$ decomposition above.

This sets the

Stability question.

The stability question

Approximation of f at scale $\approx 2\pi/L$,

$$f_{\leq L} = \sum_{\ell=0}^{L} f_{(\ell)}.$$

Each $f_{(\ell)}$ is a ℓ-homogeneous polynomial.
$f_{\leq L}$ is a polynomial of degree L.
Polydipole decomposition:

$$f_{\leq L} = \sum_{\ell=0}^{L} f_{V(\ell)}.$$

As we said, $f_{V(\ell)} \neq vec f_{(\ell)}$.

For a given function f, we can modify the scale L of approximation.
Stability question (asked by CHS, then by KW): do the $f_{V(\ell)}$ change ?

MLR: YES: The polydipole decomposition is unstable.

Example : the exponential function

$$f(x) = \exp(k \cdot x) = \sum_n \frac{K^n}{n!} (\hat{k} \cdot x)^n. \; (1)$$

Approximate the exponential (on the sphere) as

$$f_{\leq L}(x) = \sum_{\ell=0}^{L} f_\ell(x).$$

Apply the multidipole ecomposition

$$f_{\leq L}(x) = \sum_{\ell=0}^{L} f_{V\ell(x)}.$$

I calculated the higher order term

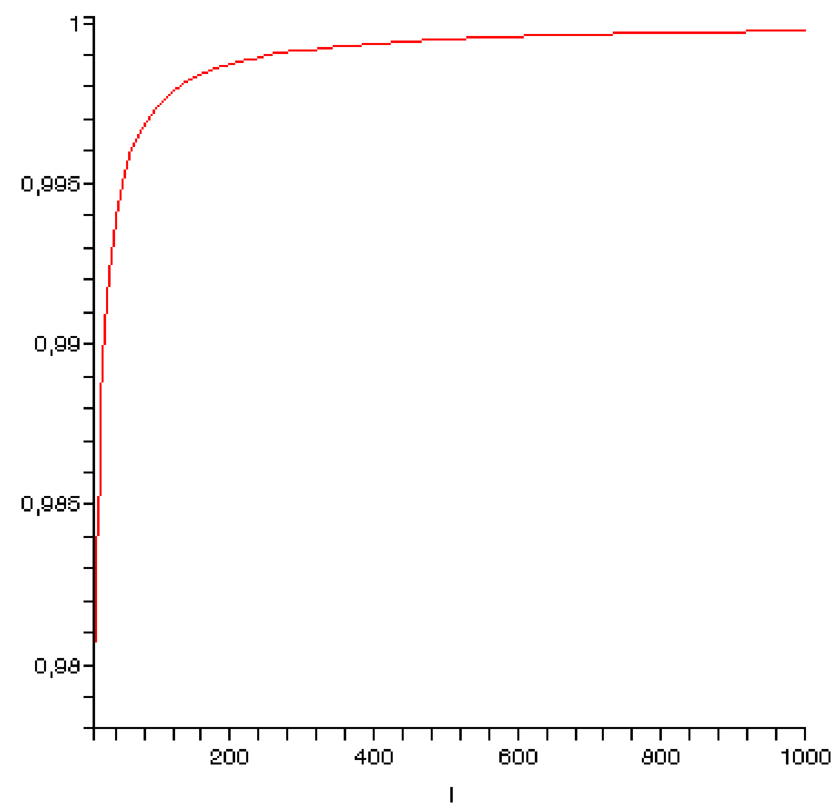
$$f_{V\ L(x)} = (2L+1)\ j_L(K)\ \frac{(2L)!}{2^L\ (L!)^2} (\hat{k} \cdot x)^L.$$

The ratio with the term of same order in (1) is

$$R \equiv \frac{j_L(K)}{K^L}\ \frac{(2L+1)!}{2^L\ (L!)} \neq 1.$$

It tends towards 1 when ℓ goes to infinity, as shown in Figure

Figure 1: The ratio R as a function of the multipole index ℓ, for $K = 1$



Maxwell decomposition

Dennis =
M R Dennis 2004,
Canonical representation of spherical functions: Sylvesters theorem, Maxwells multipoles and Majoranas sphere,
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MLR =
M. Lachieze-Rey 2004, Harmonic projection and multipole Vectors,

Maxwell decomposition

Sylvester's theorem : any ℓ-harmonic polynomial with real coefficients can be uniquely written as

$$H(X) = r^{2\ell+1}\ \nabla_{u_1}\nabla_{u_2}...\nabla_{u_\ell}\frac{1}{r},$$

$$\forall H \in \mathrm{HARM}(\ell, \mathbb{R}).$$

with $r^2 \equiv X^2 = X \cdot X$,
the u_i as above
and the directional derivatives $\nabla_{u_i} \equiv u_i \cdot \nabla$.

This is the *Maxwell multipole representation*

Dennis : it implies the unique decomposition

$$H = vecH + r^2\ Q; Q \in \mathrm{HOM}(\ell - 2, \mathbb{R}).$$

This correspondence proves CHS's result.

A new stable decomposition of the multipole expansion of a function f:

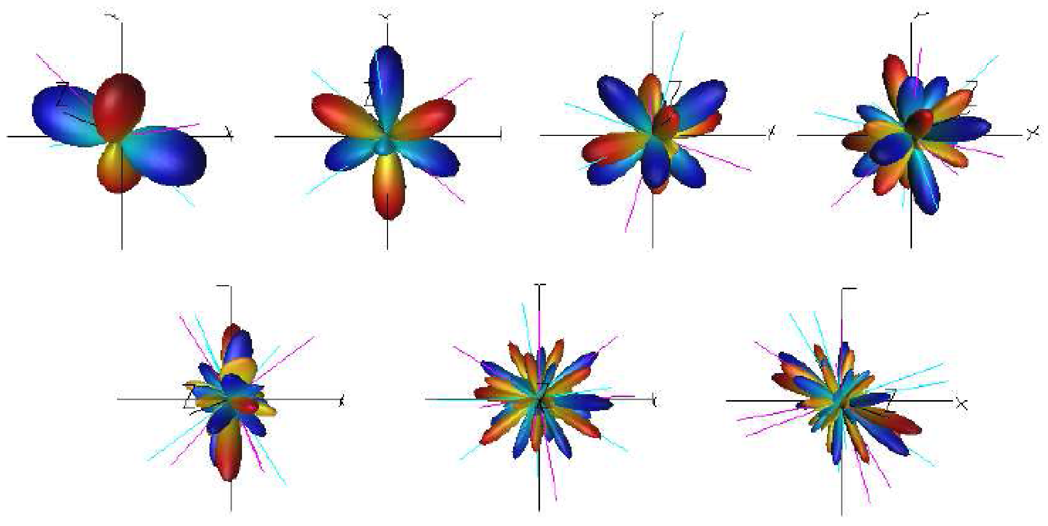
MLR :

$$f_{\leq L}(x) = \big[\lambda_0 \ r \ + \ \lambda_1 \ r^3 \ \nabla_{u_{1,1}} + ...$$
$$+\lambda_\ell \ r^{2\ell+1} \ \nabla_{u_{\ell} \ ,1} ... \nabla_{u_{\ell} \ ,\ell} + ...$$
$$... + \lambda_L \ r^{2L+1} \ \nabla_{u_{L} \ ,1} ... \nabla_{u_{L} \ ,L}\big] \ (\frac{1}{r}), \qquad (15)$$

This decomposition is stable by construction. (thanks to Jeff Weeks for this concise demonstration).

It should be emphasized that this (Maxwell) decomposition differs form the polydipole decomposition (one is stable, the other not, in the sense above).

Figure 2: CHS : The vector multipoles for the first multipoles of WMAP



CHS are led to reject the hypothesis that the vectors of multipole ℓ are uncorrelated with the vectors of multipole ℓ' up to $\ell, \ell' = 8$.

They find high correlations and conclude to inconsistency with the standard assumptions of statistical isotropy and Gaussianity of the $a_{\ell m}$.

For instance,

> astonishing alignment between quadrupole and octopole.

COnfirmed by KW (with different algorithms)