# algorithmic question

undirected, finite graph $G=(V,E)$

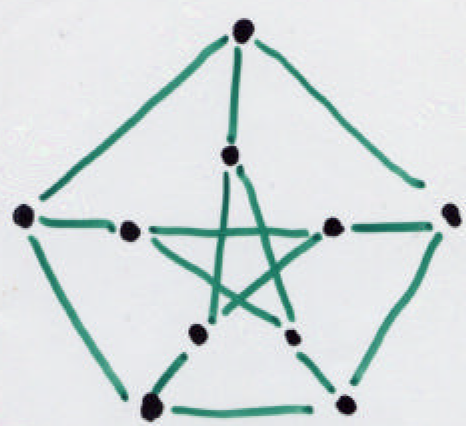
$e(X,Y)$ set of edges between $X,Y$

a **cut** is a set of nodes $\emptyset \neq S \subsetneq V$

**sparsity** of cut $\dfrac{|e(S,V\setminus S)|}{|S|\cdot|V\setminus S|}$

a cut is **c-balanced** iff $\lfloor c\cdot|V|\rfloor \leq |S| \leq |V| - \lfloor c\cdot|V|\rfloor$

**bisection**: $\frac{1}{2}$-balanced cut



# Quasisymmetric Embeddings, the Observable Diameter, and Expansion Properties of Graphs

Assaf Naor (Microsoft)
Yuval Rabani (Technion)
Alistair Sinclair (UC Berkeley)

(cont.)

sparsest cut
minimum bisection } NP-hard

approximability?

bisection representation:

$f: V \longrightarrow \{0,1\}$

$$\sum_{x,y \in V} (f(x)-f(y))^2 \geq \frac{1}{2}\binom{|V|}{2}$$

relaxation:

$f: V \longrightarrow R^{|V|}$

$$\max_{x,y \in V} \|f(x)-f(y)\|_2^2 \leq 1$$

$$\sum_{x,y \in V} \|f(x)-f(y)\|_2^2 \geq \frac{1}{2}\binom{|V|}{2}$$

$$\|f(x)-f(y)\|_2^2 + \|f(y)-f(z)\|_2^2 \geq \|f(x)-f(z)\|_2^2 \quad \forall x,y,z$$

## graph-theoretic question

shortest path metric $d_G$

$N(X)$ set of neighbors of nodes in $X$

vertex expansion $h(G) = \min_{1 \le |S| \le \frac{|V|}{2}} \left\{ \frac{|N(S)|}{|S|} \right\}$

edge expansion $\alpha(G) = \min_{1 \le |S| \le \frac{|V|}{2}} \left\{ \frac{|e(S, V \setminus S)|}{|S|} \cdot \frac{|V|}{|E|} \right\}$

$G^k = (V, E^k)$ $\quad xy \in E^k$ iff $d_G(x,y) \le k$

$$h\left(G^{\frac{2}{h(G)}}\right) \ge \frac{1}{2}$$

what is min. $k$ s.t. $\exists F \subseteq E^k$ s.t. $\alpha(V,F) \ge \frac{1}{2}$?

$G^{O\left(\frac{\log |V|}{h(G)}\right)}$ is a complete graph

# geometric question

(discussion restricted to finite metrics)

metric space $(X, d)$

$A \subseteq X$, $\varepsilon > 0$ $\quad A_\varepsilon = \{x \in X : d(x, A) < \varepsilon\}$

**isoperimetric function** $\quad i : (0, \infty) \to \mathbb{N}$

$$i(\varepsilon) = \max_{|A| \geq \frac{|X|}{2}} \{ |X \setminus A_\varepsilon| \}$$

**observable diameter**

$$\mathrm{Obs}(X, d; k) = \sup\{\varepsilon : i(\varepsilon) \geq k\}$$

<u>example</u>: $(S^{d-1}, \|\cdot\|_2)$

$i(\varepsilon) \leq \sqrt{\frac{\pi}{8}}\, e^{-d\varepsilon^2/2}$ (Levy's isoper. ineq.)

$$\mathrm{Obs}(S^{d-1}, \|\cdot\|_2; k) = O\left(\sqrt{\frac{\log(2/k)}{d}}\right)$$

(cont.)

two metric spaces $(X, d_X)$ $(Y, d_Y)$

$\eta : [0,\infty) \longrightarrow [0,\infty)$ strictly increasing

a 1-1 $f : X \hookrightarrow Y$ is a quasisymmetric embedding with modulus $\eta$ iff

$\forall x, y, z \in X$ s.t. $x \neq z$,

$$\frac{d_Y(f(x), f(y))}{d_Y(f(x), f(z))} \leq \eta\left(\frac{d_X(x,y)}{d_X(x,z)}\right)$$

<u>example</u>: $\|\cdot\|_2^2$ with $\Delta$-ineq. $\hookrightarrow \ell_2$ with modulus $\eta(s) = \sqrt{s}$

bound $\mathrm{Obs}(X, d; k)$ when $(X, d)$ embeds quasisymmetrically into $\ell_2$

# theorems

<u>Thm 1</u> (ARV) $\exists c, C > 0$ s.t. $\forall G = (V, E)$ $\forall f : V \to \mathbb{R}^{|V|}$ satisfying bisection relaxation's conditions, $\exists$ $c$-balanced cut $S$ s.t.

$$|e(S, V \setminus S)| \leq C \cdot \sqrt{\log |V|} \cdot \sum_{x,y \in E} \|f(x) - f(y)\|_2^2$$

(cont.)

<u>Thm 2</u> (edge replacement NRS) $\exists c, C > 0$ s.t. $\forall G = (V, E)$ with $h(G) \geq \frac{1}{2}$, $\exists F \subseteq E^{C\sqrt{\log |V|}}$ s.t. $\alpha(V, F) \geq c$

<u>Thm 3</u> (NRS) $\forall (X, d)$ s.t. $f : X \hookrightarrow \ell_2$ quasisymmetric with modulus $\eta$, $\exists A, B \subset X$ s.t. $|A|, |B| \geq \frac{\delta}{16}|X|$ and

$$d(A, B) \geq \frac{\gamma \cdot \mathrm{diam}(X)}{\sqrt{\log |X|}}$$

$$\delta = \frac{1}{\binom{|X|}{2}} \sum_{x,y \in X} d(x,y) \Big/ \mathrm{diam}(X) \qquad \gamma = \gamma(\delta, \eta)$$

<u>claim:</u> $2 \Rightarrow 3 \Rightarrow 1$

$3 \Rightarrow 1$

let $f$ be a solution to the relaxation

identity map, $\eta(s) = \sqrt{s}$

$\delta = \frac{1}{2}$

so: $\exists\, A, B \subset V$ $\quad |A|, |B| \geq \frac{1}{32}|V|$ s.t.

$$\min_{x \in A, y \in B} \|f(x) - f(y)\|_2^2 \geq \frac{\gamma \cdot \max \|f(u) - f(v)\|_2^2}{\sqrt{\log |V|}}$$

$$\geq \frac{\gamma'}{\sqrt{\log |V|}}$$

$S$ = cut minimizing $|e(S, V \setminus S)|$ s.t.

$A \subseteq S,\ B \subseteq V \setminus S$

(cont.)

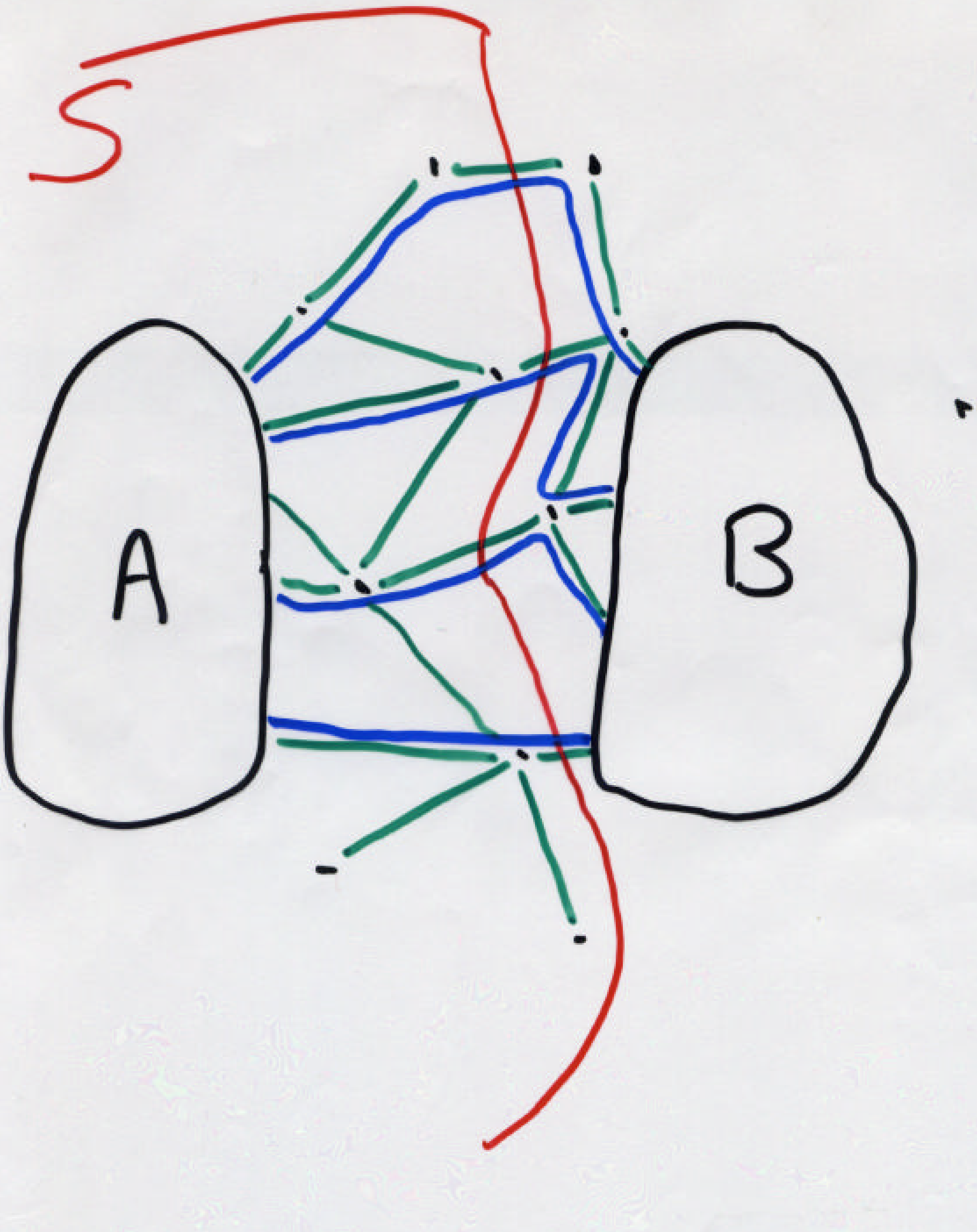
$\exists\ |e(S, V\setminus S)|$ edge-disjoint paths between $A, B$ in $G$ (min-cut/max-flow duality).

by $\Delta$-ineq., $\|\cdot\|_2^2$ length of each path $\geq \frac{\gamma'}{\sqrt{\log|V|}}$.

so: $\sum_{xy \in E} \|f(x) - f(y)\|_2^2 \geq \sum$ path lengths $\geq$

$\frac{\gamma'}{\sqrt{\log|V|}} \cdot |e(S, V\setminus S)|$.

also: $S$ is $\frac{1}{32}$-balanced.


S
A
B

$2 \Rightarrow 3$

fact: $\forall G=(V,E)$, $\forall f: V \hookrightarrow \ell_1$,

$$\frac{1}{|E|} \sum_{x,y \in E} \|f(x)-f(y)\|_1 \geq \alpha(G) \cdot \frac{1}{|V|^2} \sum_{x,y \in V} \|f(x)-f(y)\|_1$$

Lemma (*) $\forall G=(V,E)$ s.t. $h(G) \geq \frac{1}{2}$, $\forall f: V \hookrightarrow \ell_2$, $\exists x,y \in V$ s.t. $d_G(x,y) \leq C\sqrt{\log |V|}$ and $\|f(x)-f(y)\|_2 \geq C \cdot \frac{1}{|V|^2} \cdot \sum_{x',y' \in V} \|f(x')-f(y')\|_2$

Proof: Thm 2 + fact.

proof of Thm 3

(just for $\|\cdot\|_2^2$ metrics, $\eta(s)=\sqrt{s}$)

define $G=(X,E)$ by

$$E=\left\{xy:\ x,y\in X \wedge d(x,y)<\frac{c^2\delta^3\cdot\operatorname{diam}(X)}{32\sqrt{\log|X|}\cdot C}\right\}$$

for contradiction, assume: $\forall A,B\subseteq X$ s.t. $|A|,|B|\geq\frac{\delta}{16}|X|$, $d_G(A,B)\leq 1$.

$\Rightarrow \exists Y\subseteq X$ s.t. $|Y|\geq(1-\frac{\delta}{8})|X|$ and $h(G[Y])\geq\frac{1}{2}$.

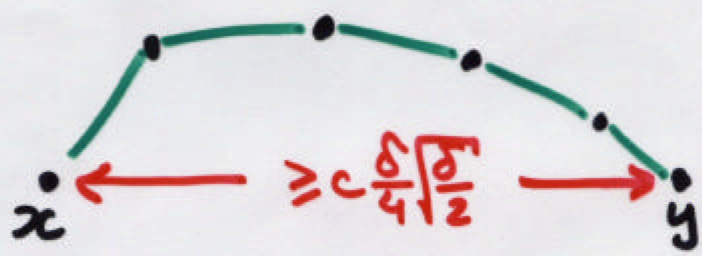
also: $\frac{1}{|Y|^2}\sum_{x,y\in Y}\|f(x)-f(y)\|_2\geq\frac{\delta}{4}\cdot\sqrt{\frac{\delta}{2}}\cdot\operatorname{diam}(f(X))$

(cont.)

by Lemma (*), $\exists x, y \in Y$ s.t.

$d_{G[Y]}(x,y) \leq C \cdot \sqrt{\log |Y|}$ and

$\|f(x) - f(y)\|_2 \geq c \cdot \frac{\delta}{4} \sqrt{\frac{\delta}{2}} \cdot \mathrm{diam}(f(X))$




So: $d(x,y) \geq \dfrac{c^2 \delta^3 \cdot \mathrm{diam}(X)}{32}$

on shortest path $x \rightsquigarrow y$ $\exists x'y' \in E$

s.t. $d(x', y') \geq \dfrac{c^2 \delta^3 \cdot \mathrm{diam}(X)}{32 C \sqrt{\log |X|}}$

a contradiction!

proof of Thm 2

taking all close pairs fails!

follows from proof in ARV:

(**) $\forall \delta > 0 \ \exists c(\delta), C(\delta) > 0$ s.t. $\forall G = (V, E)$ with $h(G) \geq \frac{1}{2}$, $\forall f : V \to B_2^d$ s.t. $\frac{1}{|V|^2} \sum_{x,y \in V} \|f(x) - f(y)\|_2 \geq \delta$, $\exists x, y \in V$ s.t. $d_G(x,y) \leq C(\delta)\sqrt{\log |V|}$ and $\|f(x) - f(y)\|_2 \geq c(\delta)$

replacing $C(\delta)$ by $C$ and $c(\delta)$ by $c \cdot \delta$

$\Updownarrow$

Thm 2

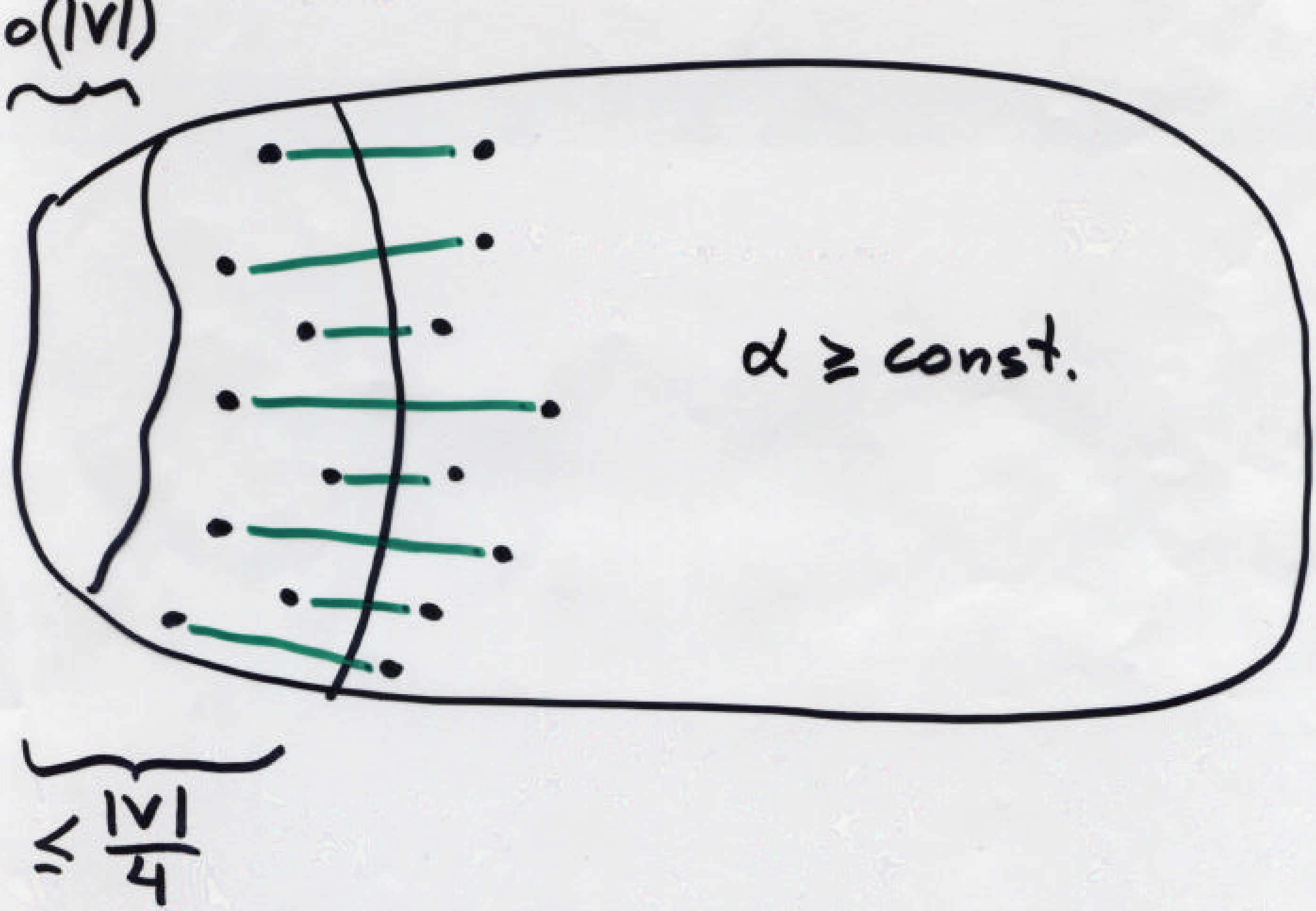
(we don't know a direct proof)

## $(**) \Rightarrow$ Thm 2

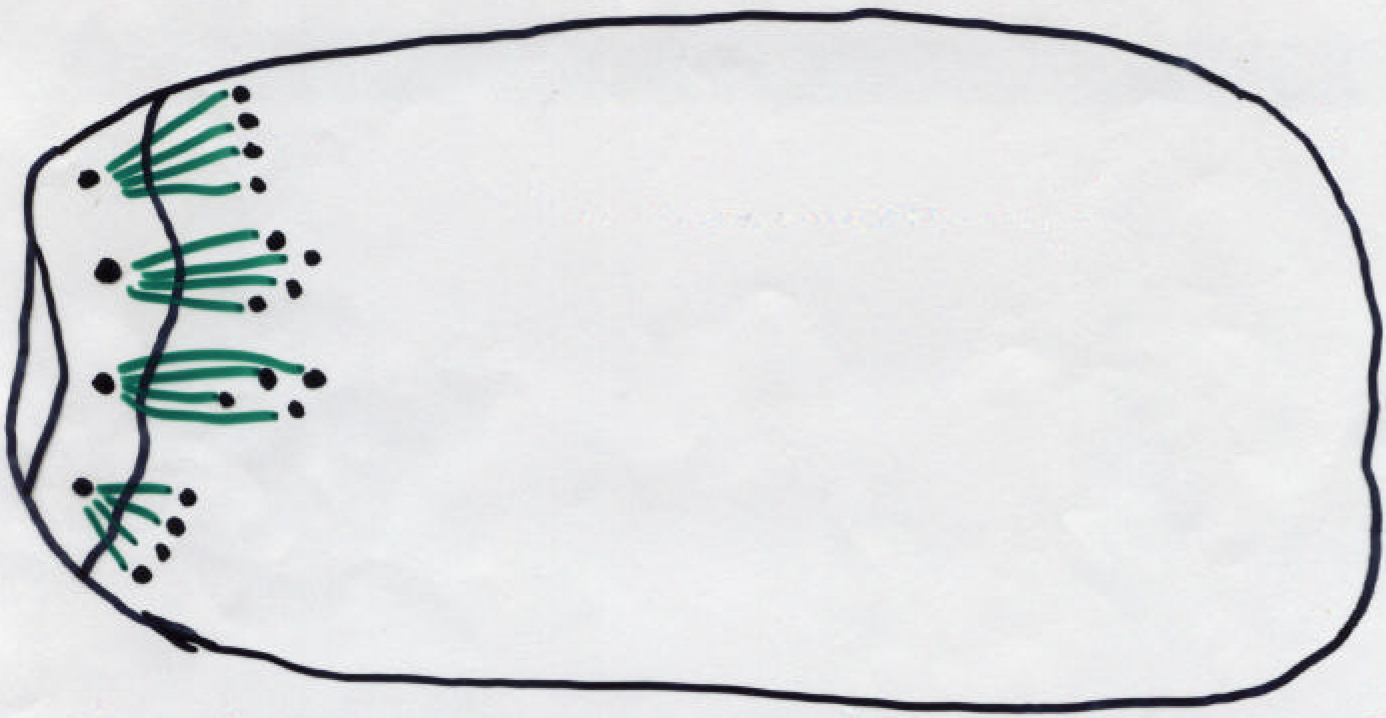
by duality: $\exists$ distrib. $\pi$ over pairs $x, y \in V$ with $d_G(x,y) \le C\sqrt{\log|V|}$ s.t. $\forall \frac{1}{100}$-balanced cut $S$, $\sum_{\substack{x \in S \\ y \notin S}} \pi(x,y) \ge c$.

<u>step 1</u>: take $O(|V|)$ sample from $\pi$ w.h.p. every $\frac{1}{4}$-balanced cut is "hit" by const. fraction of sample.

## step 2!



step 3:



proof of (**)

(follows ARV)

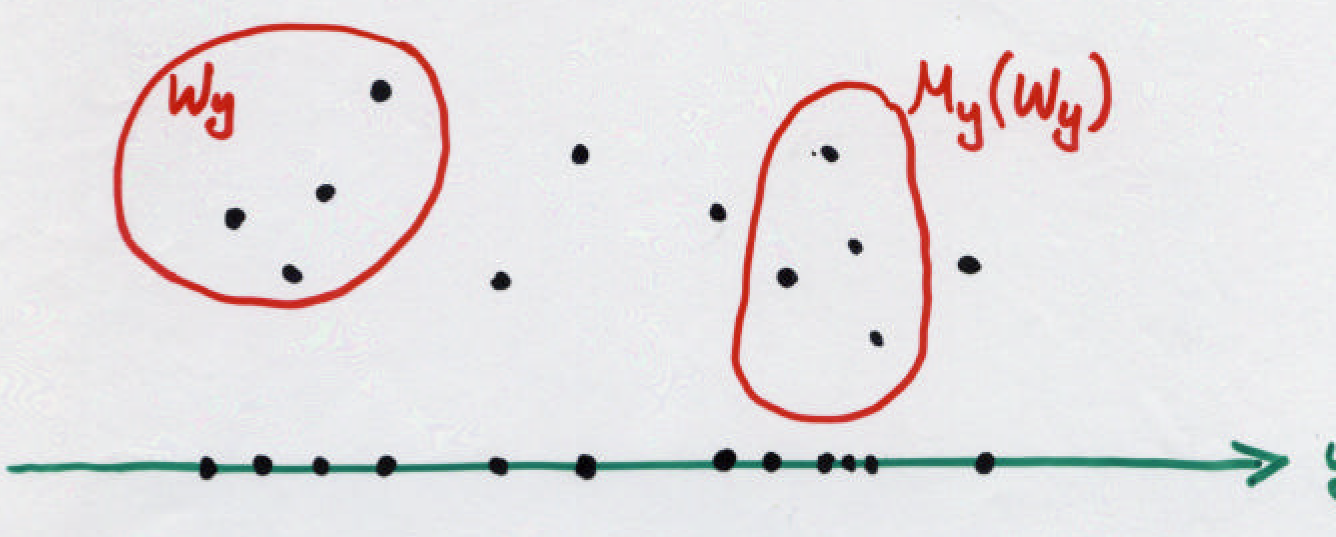
$\sigma$ = normalized Haar measure on $S^{|V|-1}$

<u>step 1</u>: $\exists\, U \subseteq V$ with $|U| = \Omega(|V|)$ and $\forall y \in S^{|V|-1}$ a matching $M_y : W_y \longrightarrow U \setminus W_y$

s.t. (1) $\forall x \in W_y$, $d_G(x, M_y(x)) = O(1)$;

(2) $\forall x \in W_y$, $y \cdot \left(f(M_y(x)) - f(x)\right) = \Omega\left(\frac{1}{\sqrt{|V|}}\right)$;

(3) $\forall x \in U$, $\sigma(y \in S^{|V|-1} : x \in W_y) = \Omega(1)$.



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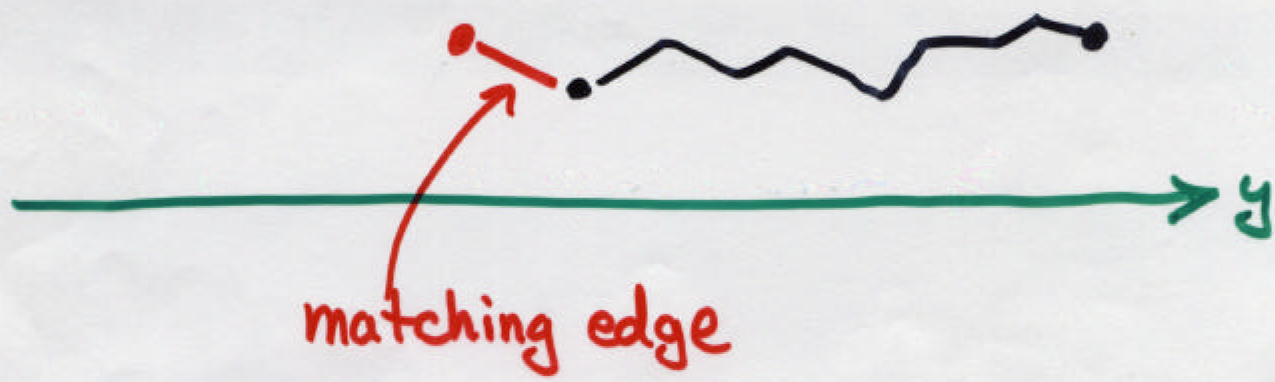
**step 2:** (chaining)




$\exists\, Y_i \subseteq U$ with $|Y_i| = \Omega(|U|)$ s.t.

$\forall x \in Y_i$,

$$\sigma\Big\{ y \in S^{|V|-1} : \exists z \in U \text{ s.t. } d_G(x,z) = O(i) \text{ and } y\cdot\big(f(z)-f(x)\big) = \Omega\Big(\tfrac{i}{\sqrt{|V|}}\Big) \Big\} = \Omega(1).$$

(cont.)

<u>step 3</u>: (boosting of directions)
increase const. measure to close to 1
(concentration of measure)

<u>step 4</u>: (boosting of node set)
increase $|Y_i|$ to $\frac{|U|}{2}$
(expansion of $G$)