Morse Theory: 3D Object Representation for Classification…

Hamid Krim,
ECE Dept.
NCSU, Raleigh,
Thanks due to NSF and AFOSR
Acknowledge; S. Baloch, A. Ben Hamza, A. Gunduz
Collaborators:M. Genton, I. Kogan, W. Mio, A. Srivastava, T. Yezzi , D. Zenkov

Outline

- Motivation/Background
- 3D Shape Modeling
  - Topological encoding/Graph
    - Height function/Reeb graph
    - Distance functions
  - Geometric encoding
    - Graph weighting
    - Curve modeling
- Conclusions/Perspectives

VISSTA

NC STATE UNIVERSITY

Motivation

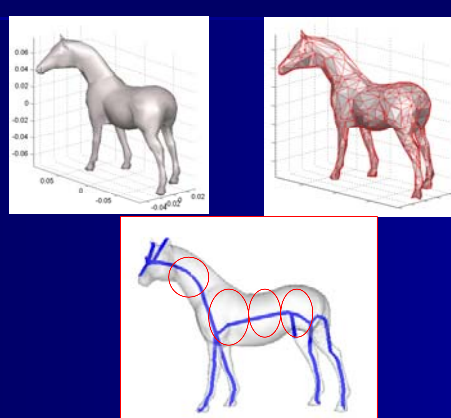
- Generalized framework for classification and recognition
- Biomedical imaging (..surgery assistance)
- Compression of objects for storage/retrieval
- CAD applications, Art archival, terrain modeling

[Shinagawa et. al., Schroeder, Edelsbrunner, Schmidt et. al., Andres et. al., …]

VISSTA

NC STATE UNIVERSITY

Central Idea….

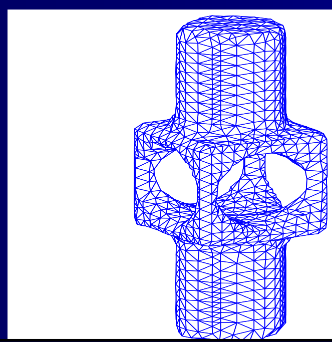


VISSTA

NC STATE UNIVERSITY

Object 3D Representation

- Challenge of 3D representation
  - Sheer size of object (mesh representation)
  - Intertwined information { Geometry / Topology }

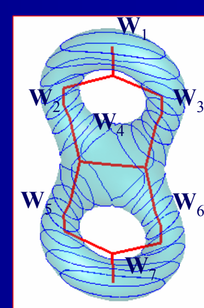


VISSTA

NC STATE UNIVERSITY

Proposed approach

- Capture both topology and geometry
  - Topology through a skeletal graph
  - Geometry via weight assignment
- Weighted graphs
  - For recognition/storage
  - Complete representation of shape
  - A compact object representation (Compression)



VISSTA

NC STATE UNIVERSITY

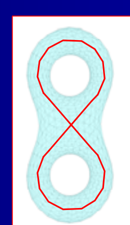
Previous work

- Topology modeling
  - Reeb graph
    - Height function
      - Shinagawa *et al.* 1991
      - Ben Hamza *et al.* 2002
    - Geodesic distance
      - Lazarus *et al.* 1999
      - Hilaga *et al.* 2001
      - Schmidt *et al. 2004*
- Shape distributions
  - Osada *et al.* 2002
  - Ben Hamza *et al.* 2003
- Reflective symmetry descriptors
  - Kazhdan *et al.* 2002

VISSTA NC STATE UNIVERSITY

Topology

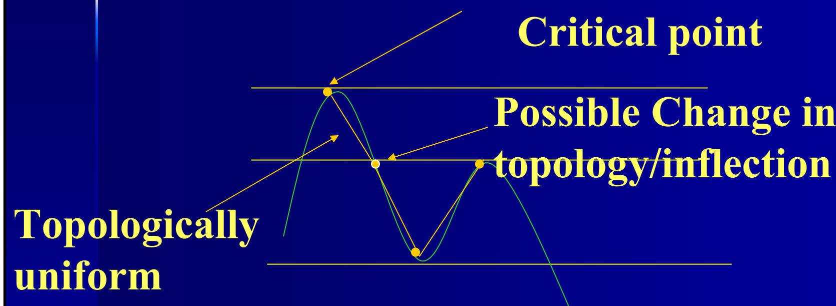
- ***Goal**: Represent a surface/manifold in subparts which may be glued together*



- Information in Topology
- How to capture topology?
  - *Critical* points

VISSTA NC STATE UNIVERSITY

Critical Points of a Curve



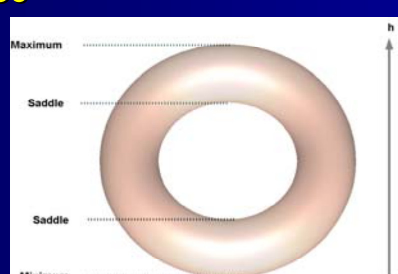
- Theory Dynamical Systems

VISSTA

NC STATE UNIVERSITY

Characterization of topology

- Interest in detecting topological changes
- Tantamount to localizing critical points
- Fast and simple means of exploring surface



VISSTA

NC STATE UNIVERSITY

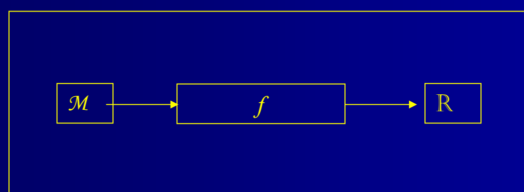
Morse theory

- **Consider a smooth real value function defined on M**

$$f:\mathcal{M}\rightarrow\mathbb{R}$$

- $p_0=(u_0,v_0)$ **is a critical point of f if** $\nabla\left(f\circ\mathbf{x}(u_0,v_0)\right)=0$
- $f(p_0)$ **is called a critical value of f**

- **Analogy with a control system**



VISSTA NC STATE UNIVERSITY

Morse Function

- **Existence of such a function is generically guaranteed by unity partition theorem, *i.e.***

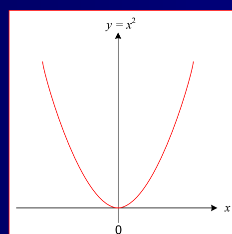
$$A\cap B=\varnothing, A,B\subset\mathbb{R}^n, \exists\phi \text{ on } \mathbb{R}^n /$$
$$\phi(\{x:x\in A\})=1 \text{ and } \phi(\{x:x\in B\})=0$$
$$\text{and } 0\le\phi(x)\le 1 \text{ elsewhere}$$

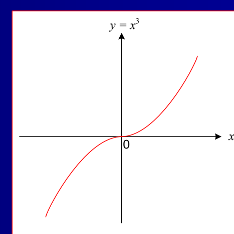
Definition: **A Morse function is a smooth function on a smooth manifold and its critical points are non-degenerate**

VISSTA NC STATE UNIVERSITY

Morse function

- A critical point $p_0=(u_0,v_0)$ is degenerate if the Hessian of $f \circ \mathbf{x}$ is singular
  - ➤ ***Degenerate critical points are unstable***
- A smooth function f defined on a smooth manifold $\mathcal{M}$ is *Morse* if all of its critical points are non-degenerate





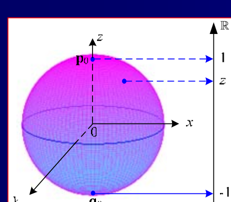
VISSTA

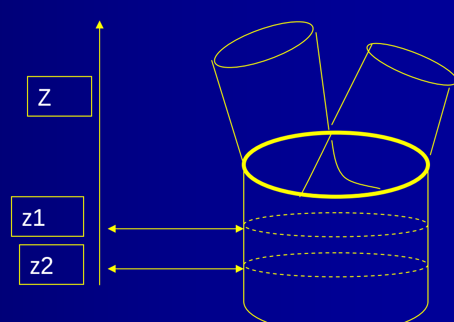
NC STATE UNIVERSITY

Height Function

- A height function $h : \mathcal{M} \rightarrow \mathbb{R}$ on smooth manifold is a real valued function such that

$$h(x, y, z) = z, \forall (x, y, z) \in \mathcal{M}$$





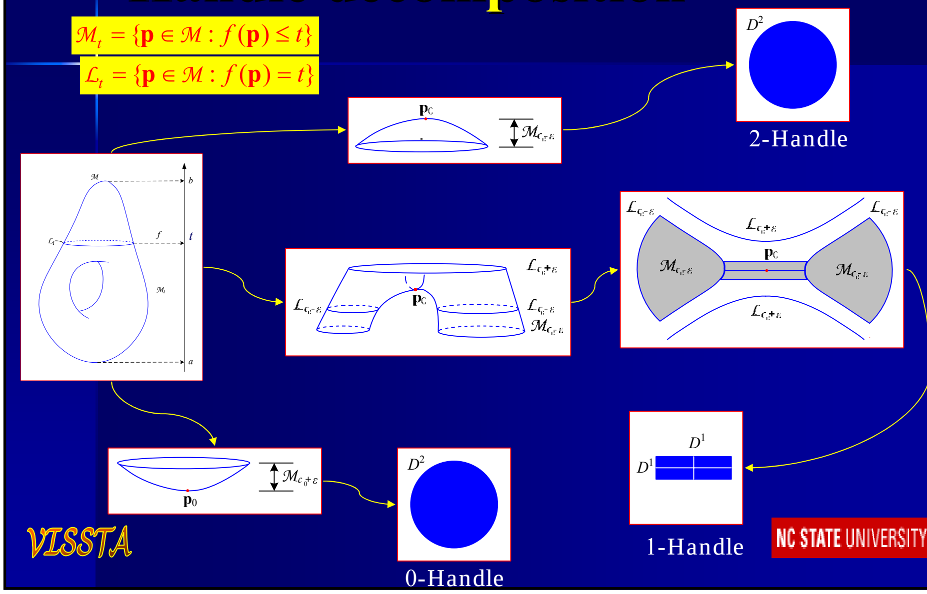
VISSTA

NC STATE UNIVERSITY

Handle decomposition

$\mathcal{M}_t = \{\mathbf{p} \in \mathcal{M} : f(\mathbf{p}) \leq t\}$

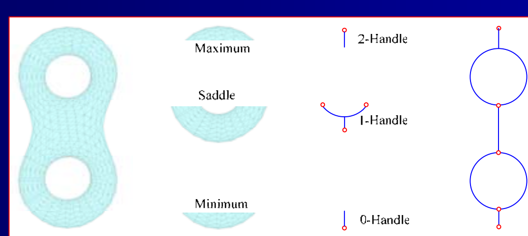
$\mathcal{L}_t = \{\mathbf{p} \in \mathcal{M} : f(\mathbf{p}) = t\}$



VISSTA

NC STATE UNIVERSITY

Reeb Graph



VISSTA

NC STATE UNIVERSITY

Reeb gr

Not invariant to
•Rotation

- Reeb graph may [illegible]ernatively be described as a quotient space $\mathcal{M}/\sim$ where the equivalence relation $\sim$ is defined as:
- p ~ q iff
  - $h(p) = h(q)$

$p \in \text{ConnComp}(\text{Levelset}(q)) = h^{-1}(h(q))$

$\mathcal{M}/\sim := \{[p] : p \in \mathcal{M}\}$

$[p] = \{q \in \mathcal{M} : q \sim p\}$

VISSTA

NC STATE UNIVERSITY

About height function…

- Morse function
- Easily computed
- Rotationally varying
- Scale dependent
- Non unique graph

VISSTA

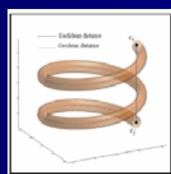
NC STATE UNIVERSITY

Geodesic Distance

- Given a mesh $M=(v,T)$ as a set of vertices and triangles
- Characterize surface by an intrinsic feature

[Osada *et. al.* (00)]

- Compute cumulative "geodesic" distance of each vertex to all other vertices



$$L(\gamma) = \int_a^b \| \gamma'(t) \| \, dt$$

$$\gamma(a) = v_1 \quad and \quad \gamma(b) = v_2$$

VISSTA

NC STATE UNIVERSITY

MR Reeb Graph

- Morse!
- Geodesic is rotationally invariant
- Graph characteristic of object
- May be computationally intensive (e.g. remeshing)
- Lost notion of sampling

VISSTA

NC STATE UNIVERSITY

A new approach to topological representation

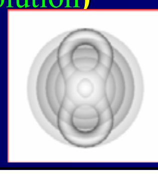
- **Distance function**

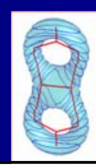
$$d : \mathcal{M} \to \mathbb{R}_+$$
$$d(\mathbf{p}) : \mathbf{p} \mapsto \|\mathbf{p}\|$$

- **Rotation, translation and scale** invariance

$$d_\mu(\mathbf{p}) : \|\mathbf{p} - \mu\|$$
$$\tilde{d}_\mu(\mathbf{p}) = \frac{d_\mu(\mathbf{p}) - d_{\min}}{d_{\max} - d_{\min}}$$

- **Evolving sphere from $d_{\min}$ to $d_{\max}$ in K steps (**resolution**)**

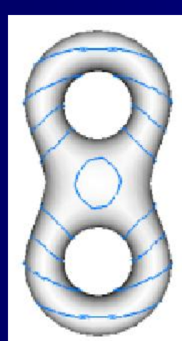


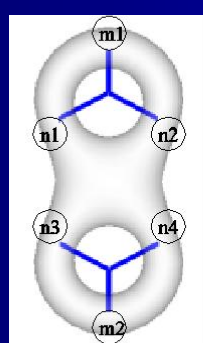


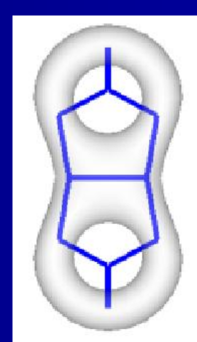
VISSTA

NC STATE UNIVERSITY

Localization of Critical Points



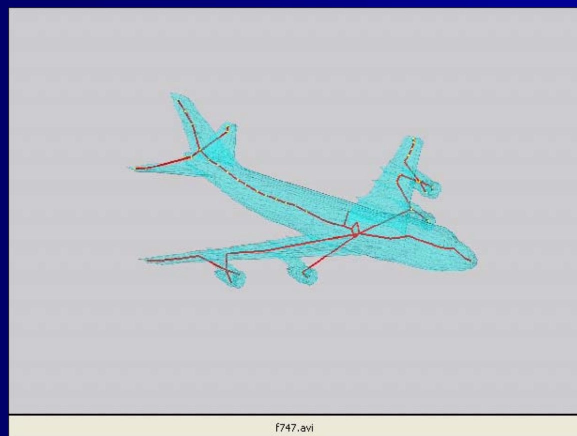




VISSTA

NC STATE UNIVERSITY

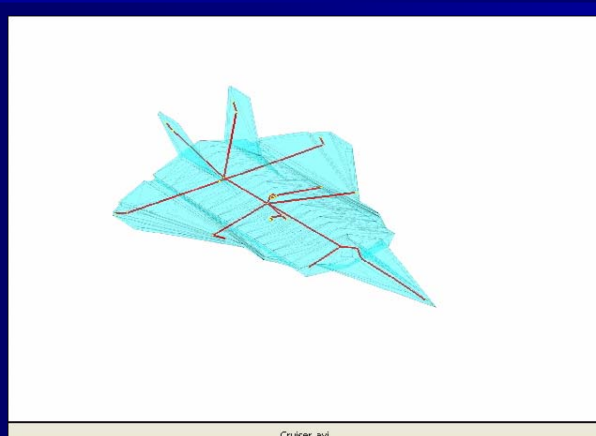
Evolving/exploring spheres



VISSTA

NC STATE UNIVERSITY

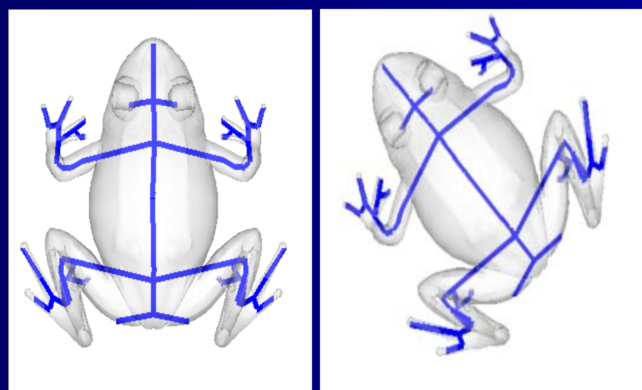
Evolving/exploring spheres



VISSTA

NC STATE UNIVERSITY

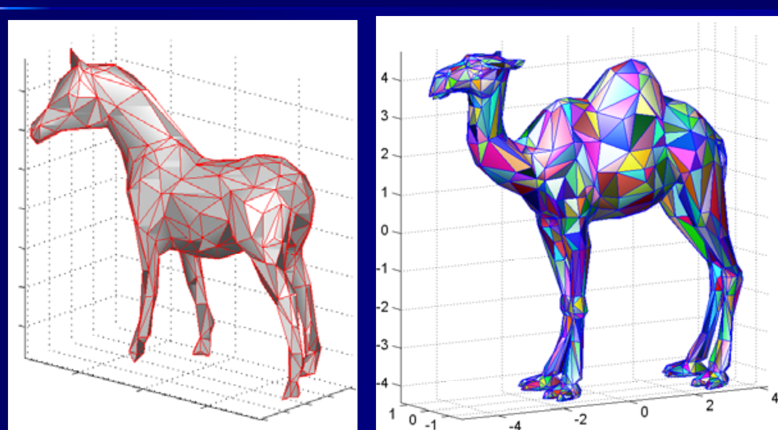
Example



VISSTA

NC STATE UNIVERSITY

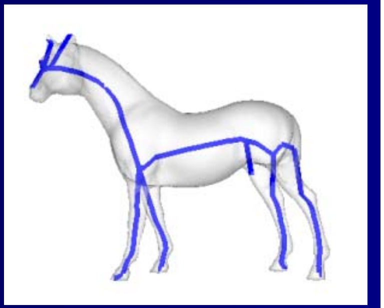
Mesh Models

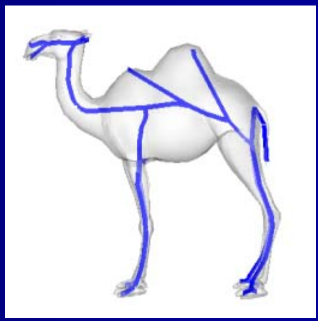


VISSTA

NC STATE UNIVERSITY

Example



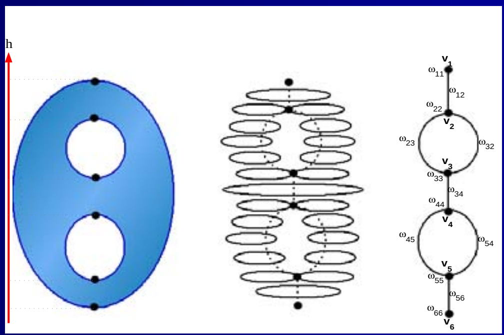


VISSTA

NC STATE UNIVERSITY

Object Representation

- Geometric Encoding ⟷ Curve modeling



VISSTA

NC STATE UNIVERSITY

Geometric Modeling

- Different techniques have been attempted
  - Node labeling and homotopy modeling [Shinagawa et. al., 92, 00]
  - Distance distribution [Ben Hamza et. al., 2002]

VISSTA

NC STATE UNIVERSITY

Geometry of Curves and Modeling

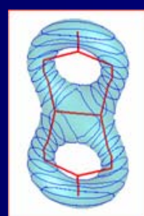
- All geometric information along arc captured by topologically homogeneous curves
- For object archiving applications, the fewer the curves, the more efficient the representation
- For object reconstruction, the larger the number of curves, the better the reconstruction

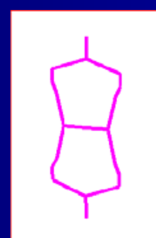
VISSTA

NC STATE UNIVERSITY

How to Capture Geometry?

- Submanifold along an arc is topologically homogeneous
- Model level curves on the submanifold independently to learn weights
- Assign unique weights to a graph arc to capture geometry of the submanifold corresponding to the arc





VISSTA

NC STATE UNIVERSITY

Interpolation

- Any curve may be viewed as a point in high dimensional space [Mio *et. al.*, 03], [K-Mio-*et. al.*, 04]
- A set of curves lies on a manifold
- Evolution between two curves

$$\alpha : [0, L] \rightarrow \mathbb{R}^n$$

VISSTA

NC STATE UNIVERSITY

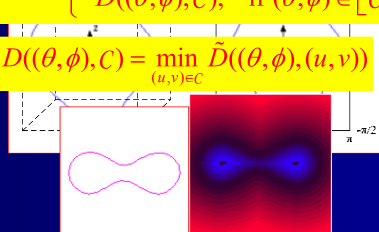
Modeling Geometry

- A level curve at level *r* lies in $\Lambda \times \mathbb{R}_+$
  - Level curves are therefore in $\Lambda = [-\pi, \pi] \times [-\pi/2, \pi/2]$

$\rho : \Lambda \to \mathbb{R}$

- Associate a $\rho(\theta,\phi) = \begin{cases} +D((\theta,\phi),C), & \text{if } (\theta,\phi) \in \lceil C \rceil \\ -D((\theta,\phi),C), & \text{if } (\theta,\phi) \in \lfloor C \rfloor \end{cases}, \quad \forall(\theta,\phi) \in \Lambda$

$D((\theta,\phi),C) = \min_{(u,v)\in C} \tilde{D}((\theta,\phi),(u,v))$

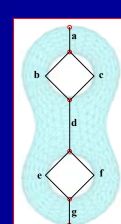


- Vectorize { $\rho(\theta, \varphi)$ } to get $\rho = (\rho_1, \ldots, \rho_n)$ $\boldsymbol{\rho} : \Lambda \to \mathbb{R}^n$

VISSTA NC STATE UNIVERSITY

Complete Object Representation

- Evolution of level curves on a graph arc modeled by a trajectory in $\mathbb{R}^3$
  - Geometry completely captured by the trajectory
- Elasticæ coefficients uniquely determine the trajectory
  - Assign the coefficients as weights to the graph arc
    $\boldsymbol{\alpha} : [a,b] \to \mathbb{R}^3$
- Alternatively, parameterize the trajectory as
  $\boldsymbol{\alpha}(t) : t \mapsto (\alpha_1(t), \alpha_2(t), \alpha_3(t))$
  - $r \in [r_i, r_{i+1}]$ is mapped to $t \in [a,b]$
  - Model $\alpha_i(t), i = 1,2,3$ with their respective Taylor series
  - Fewer coefficients (weights)

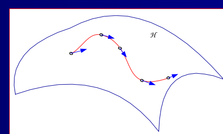


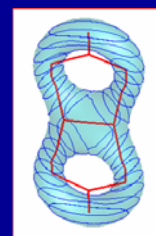
VISSTA NC STATE UNIVERSITY

Modeling

- **Given m curves $C_1,\ldots,C_m$, at levels $r_1,\ldots,r_m$**

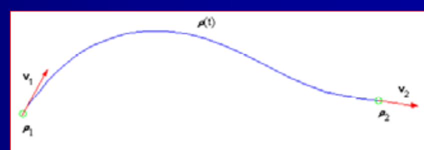
$$\rho(C_i) = \rho_i, \quad i = 1,\cdots,m$$



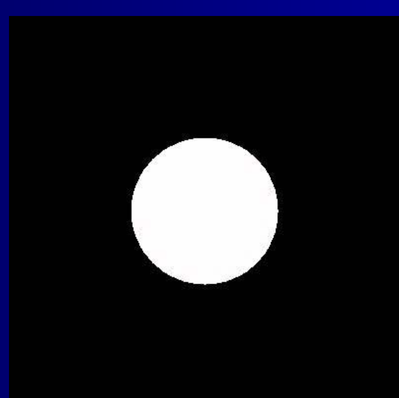


- **Optimal trajectory $\rho(t) \subset \mathbb{R}^n$ minimizes some energy functional subject to**

$$\rho(0) = \rho_1$$
$$\rho(1) = \rho_2$$
$$\dot{\rho}(0) = v_1$$
$$\dot{\rho}(1) = v_2$$



Curve Interpolation



vase_with_hole_elastica_video.avi

Dimension reduction

- Given two curves ρ_1 and ρ_2, respective tangent vectors v_1 and v_2
  - Displacement vector $d(\rho_1, \rho_2) = \rho_2 - \rho_1$
  - Assume $\{v_1, v_2, d\}$ form a linearly independent set
  - Orthogonalize $\{v_1, v_2, d\}$ to get $\{b_1, b_2, b_3\} \subset \mathbb{R}^n$
  - Project to $\{v_1, v_2, d\}$ to $\mathbb{R}^3$

$$\mathbf{w}_i = \langle \mathbf{v}_i, \mathbf{b}_1 \rangle \mathbf{e}_1 + \langle \mathbf{v}_i, \mathbf{b}_2 \rangle \mathbf{e}_2 + \langle \mathbf{v}_i, \mathbf{b}_3 \rangle \mathbf{e}_3, \qquad i = 1,2,3$$

- Find elasticæ $\boldsymbol{\alpha} : [0,1] \to \mathbb{R}^3$ that minimizes *bending energy*

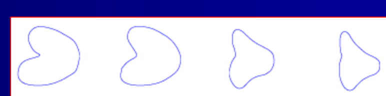
$$E_\alpha = \int_0^1 \kappa_\alpha^2(s)\,ds \quad \text{subject to} \quad \begin{aligned} &\boldsymbol{\alpha}(0) = 0, \boldsymbol{\alpha}(1) = \mathbf{w}_3, \\ &\dot{\boldsymbol{\alpha}}(0) = \mathbf{w}_1, \dot{\boldsymbol{\alpha}}(1) = \mathbf{w}_2 \end{aligned}$$

VISSTA

NC STATE UNIVERSITY

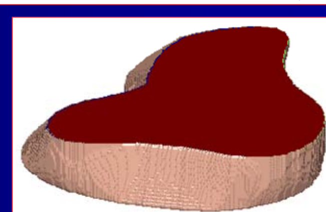
Examples

Given curves



Evolution of curves

Reconstructed surface

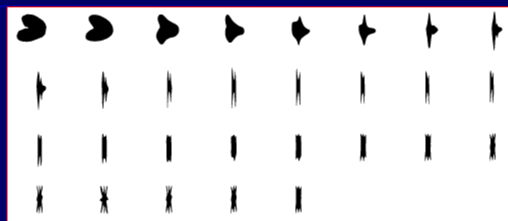


VISSTA

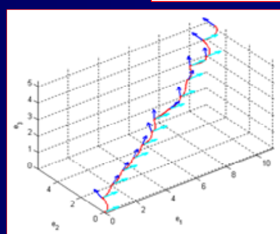
NC STATE UNIVERSITY

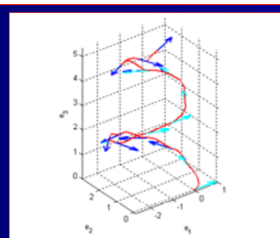
Example

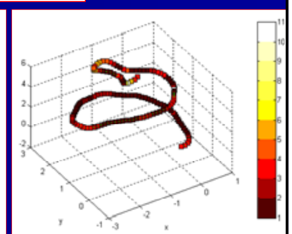
Level curves



Curvature profile







VISSTA

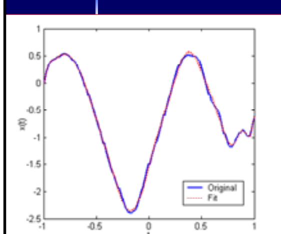
Trajectory

With basis rotation

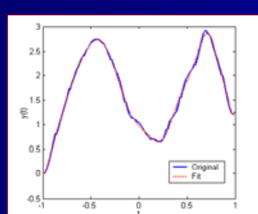
NC STATE UNIVERSITY

Example

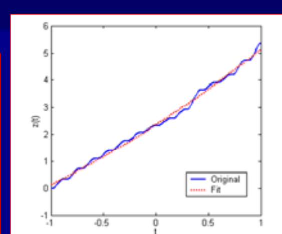
Taylor series representation of the trajectory



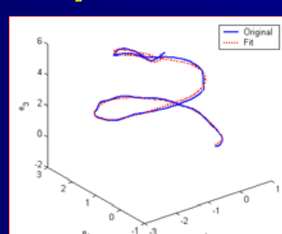
$\alpha_1(t)$ – 20 coefficients



$\alpha_2(t)$ – 20 coefficients



$\alpha_3(t)$ – 5 coefficients



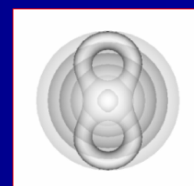
$\boldsymbol{\alpha}(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t))$

VISSTA

NC STATE UNIVERSITY

Redundancy in Geometric Information

- Sphere evolved from r_{min} to r_{max} in K steps
- K determines resolution of Reeb graph
  - Large K
    - Critical points captured well
    - Redundant geometry information
- Two approaches for removing redundancy
  - Curvature minimization
  - Correlation

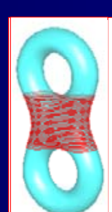


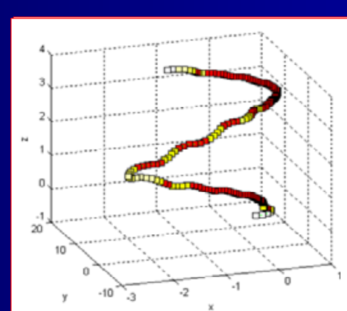
VISSTA

NC STATE UNIVERSITY

Experimental Results

- Example 1



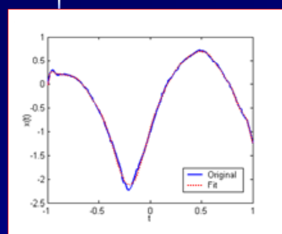


VISSTA

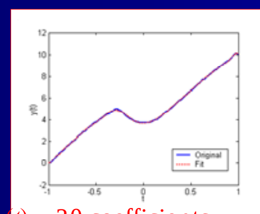
NC STATE UNIVERSITY

Experimental Results

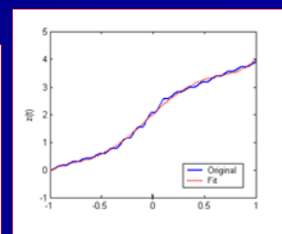
- **Example** Taylor series representation of the trajectory



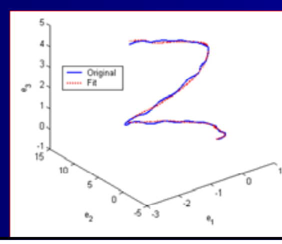
$\alpha_1(t)$ – 20 coefficients



$\alpha_2(t)$ – 20 coefficients



$\alpha_3(t)$ – 5 coefficients



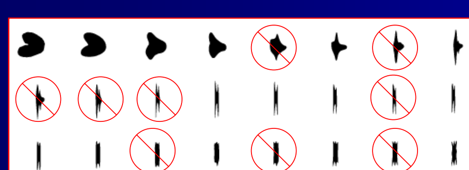
$\boldsymbol{\alpha}(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t))$

VISSTA

NC STATE UNIVERSITY

Experimental Results

- **Example 2**



Level curves

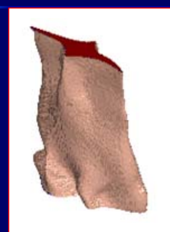
Reconstructed surface

VISSTA

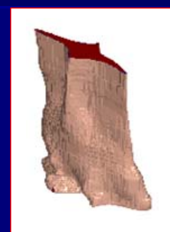
NC STATE UNIVERSITY

Experimental Results

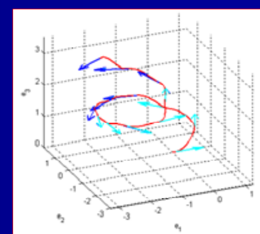
Example 3



Given 108 level curves



Reconstructed from 14 curves



VISSTA

NC STATE UNIVERSITY

Conclusions/Perspectives

- Overview of methodologies
- Applications in classification
- Other applications
  - GIS applications
  - Human tracking (e.g. training and rehabilitation)
  - Aids to physically challenged
  - Data base archiving and retrieval

VISSTA

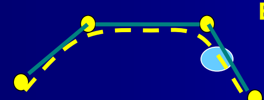
NC STATE UNIVERSITY

Curve Modeling

Let $\alpha : [0, L] \rightarrow R^n$

- *Given points* $p, q \in R^n$ *and unit vectors* $\vec{v}, \vec{w} \in R^n$, *find a unit-speed curve* $\alpha \in R^n$ *of scale-invariant elastic energy (Mumford, and others) with* p, q *as initial and terminal points, and* $\vec{v}, \vec{w}$ *as initial and terminal velocity vectors*

$$\min\left\{\int \kappa^2(s)ds \text{ with } \kappa(s) = \|\alpha''(s)\|\right.$$



Each point is a curve

VISSTA

NC STATE UNIVERSITY

Interpolation of curves

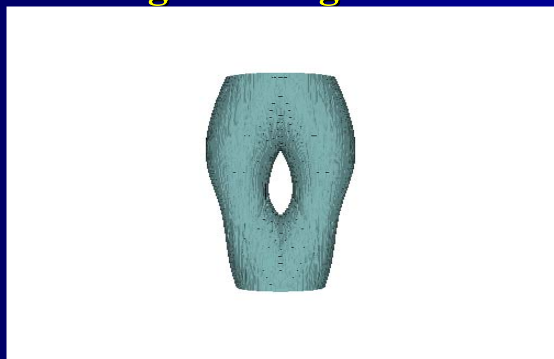
- **Constraints yield a formulation of fitting a smooth curve through two end points**
  - **Minimum curvature**
  - **Satisfying the boundary conditions as described by the two curves**

VISSTA

NC STATE UNIVERSITY

NL Interpolation

- Marching cube Algorithm



VISSTA NC STATE UNIVERSITY

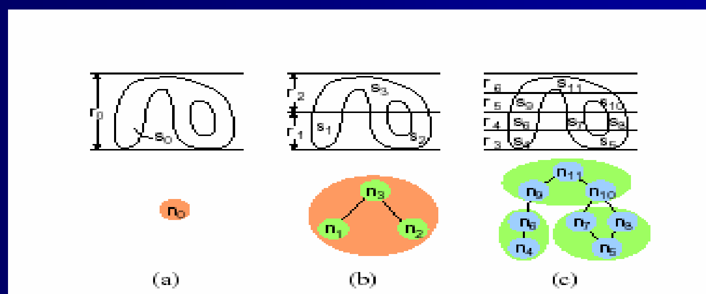
Curve Modeling

- May be best solution to accurately capture geometry
- Classification applications of 3D objects requires representation parsimony
- Other weight optimization under investigation

VISSTA NC STATE UNIVERSITY

Multiresolution Reeb Graph

- Use a MR technique to construct graph

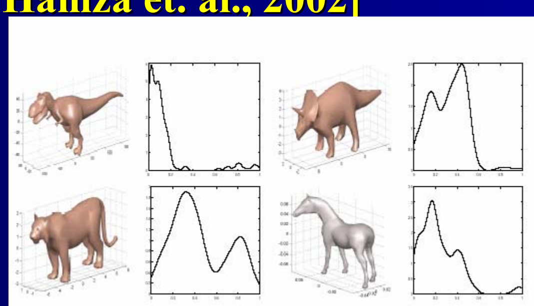


VISSTA

NC STATE UNIVERSITY

Comprehensive description

- Statistical characterization of geodesics [Ben Hamza et. al., 2002]



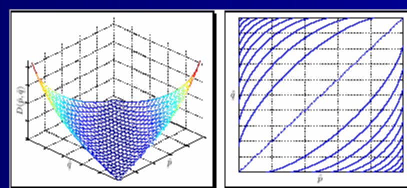
VISSTA

NC STATE UNIVERSITY

Information Theoretic Distance

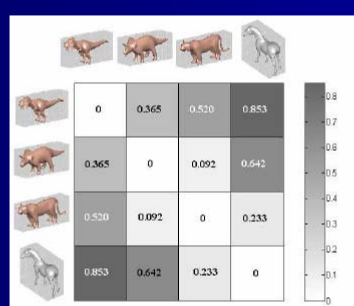
- **Jensen-Shannon divergence**

$$D(p,q) = H\left(\frac{p+q}{2}\right) - \frac{H(p)+H(q)}{2}$$

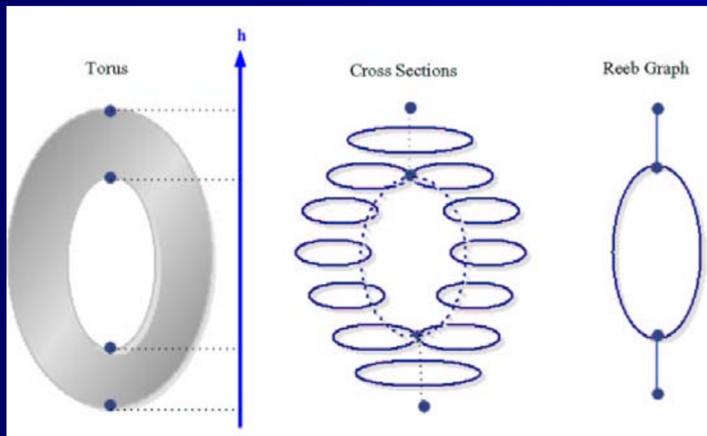


[Y. He, *et. al.* 2002]

Classification



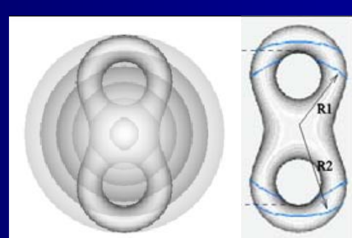
Example



VISSTA NC STATE UNIVERSITY

Visualization of Distance Function

- Easily implemented



VISSTA NC STATE UNIVERSITY

Implementation

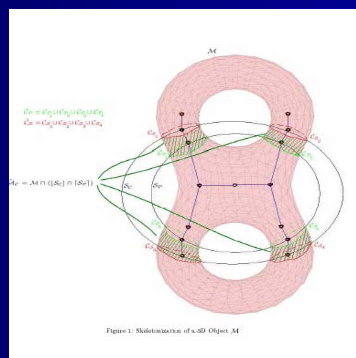


Figure 1: Skeletonization of a 3D Object $\mathcal{M}$

VISSTA

NC STATE UNIVERSITY

Morse Distance Function

- To isotropically explore a surface, let v be a fixed point in space, define

$$d : M \rightarrow \mathbb{R}$$

- For any v in space

$$\forall p \in M, d_v(p) = | p - v |^2$$

- Let v be the centroid of M and carry it to the origin

[Aysegul et. al., 2003]

$$d(p) = | p |^2$$

VISSTA

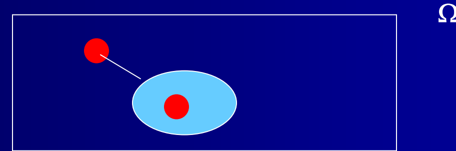
NC STATE UNIVERSITY

Free Form Curve Representation

Let $\psi : \Omega \rightarrow \mathbb{R} \; (R \subseteq \Omega)$

$$\psi((x,y)) = \begin{cases} d((x,y), C), \text{ if } (x,y) \in R; \\ -d((x,y), C), \text{ if } (x,y) \notin R \end{cases}$$

Each curve is a field Ψ_i in R^n

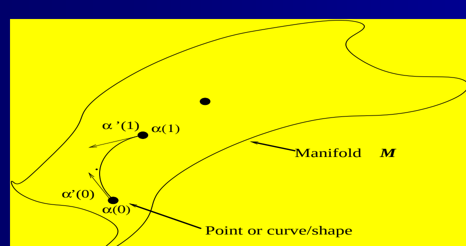


VISSTA

NC STATE UNIVERSITY

Trajectory modeling

- Constraints added to better model trajectory



VISSTA

NC STATE UNIVERSITY