Morse Theory: 3D Object
Representation for
Classification...

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Outline

- Motivation/Background
  - 3D Shape Modeling
    - Topological encoding/Graph
      - Height function/Reeb graph
      - Distance functions
    - Geometric encoding
      - Graph weighting
      - Curve modeling
- Conclusions/Perspectives
Motivation

- Generalized framework for classification and recognition
- Biomedical imaging (surgery assistance)
- Compression of objects for storage/retrieval
- CAD applications, Art archival, terrain modeling

[Shinagawa et. al., Schroeder, Edelsbrunner, Schmidt et. al., Andres et. al., …]

Central Idea…..
Object 3D Representation

- Challenge of 3D representation
  - Sheer size of object (mesh representation)
  - Intertwined information

Proposed approach

- Capture both topology and geometry
  - Topology through a skeletal graph
  - Geometry via weight assignment

- Weighted graphs
  - For recognition/storage
  - Complete representation of shape
  - A compact object representation (Compression)
Previous work

- **Topology modeling**
  - **Reeb graph**
    - Height function
      - Shinagawa et al. 1991
      - Ben Hamza et al. 2002
    - Geodesic distance
      - Lazarus et al. 1999
      - Hilaga et al. 2001
      - Schmidt et al. 2004
  - Shape distributions
    - Osada et al. 2002
    - Ben Hamza et al. 2003
  - Reflective symmetry descriptors
    - Kazhdan et al. 2002

Topology

- **Goal**: Represent a surface/manifold in subparts which may be glued together

- Information in Topology
- How to capture topology?
  - Critical points
Critical Points of a Curve

- Critical point
- Possible Change in topology/inflection
- Topologically uniform

- Theory Dynamical Systems

Characterization of topology

- Interest in detecting topological changes
- Tantamount to localizing critical points
- Fast and simple means of exploring surface
**Morse theory**

- Consider a smooth real value function defined on $M$ 
  \[ f : M \rightarrow \mathbb{R} \]
- $p_0 = (u_0, v_0)$ is a critical point of $f$ if 
  \[ \nabla (f \circ x(u_0, v_0)) = 0 \]
- $f(p_0)$ is called a critical value of $f$

- Analogy with a control system

**Morse Function**

- Existence of such a function is generically guaranteed by unity partition theorem, i.e.
  \[ A \cap B = \emptyset, A, B \subset \mathbb{R}^n, \exists \phi \text{ on } \mathbb{R}^n / \]
  \[ \phi(\{ x : x \in A \}) = 1 \text{ and } \phi(\{ x : x \in B \}) = 0 \]
  and $0 \leq \phi(x) \leq 1$ elsewhere

**Definition:** A Morse function is a smooth function on a smooth manifold and its critical points are non-degenerate
Morse function

- A critical point \( p_0 = (u_0, v_0) \) is degenerate if the Hessian of \( f \) is singular
  - Degenerate critical points are unstable
- A smooth function \( f \) defined on a smooth manifold \( \mathcal{M} \) is Morse if all of its critical points are non-degenerate

\[
y^2 - x^2 = 0
\]

Height Function

- A height function \( h : \mathcal{M} \to \mathbb{R} \) on smooth manifold is a real valued function such that
  \[
h(x, y, z) = z, \forall (x, y, z) \in \mathcal{M}
\]
Handle decomposition

\[ M = \{ p \in \mathcal{M} : f(p) \leq t \} \]

\[ L_0 = \{ p \in \mathcal{M} : f(p) = t \} \]

Reeb Graph
Reeb graph may alternatively be described as a quotient space $M / \sim$ where the equivalence relation $\sim$ is defined as:

- $p \sim q$ iff $h(p) = h(q)$  $\Rightarrow$ $p \in \text{ConnComp}(\text{Levelset}(q)) = h^{-1}(h(q))$

$$[p] = \{q \in M : q \sim p\}$$

Not invariant to rotation.

About height function...

- Morse function
- Easily computed
- Rotationally varying
- Scale dependent
- Non unique graph
Geodesic Distance

- Given a mesh $M=(v,T)$ as a set of vertices and triangles
- Characterize surface by an intrinsic feature
[Osada et. al. (00)]
- Compute cumulative “geodesic” distance of each vertex to all other vertices

$$L(\gamma) = \int_a^b \|\gamma'(t)\|\,dt$$
$$\gamma(a) = v_1 \text{ and } \gamma(b) = v_2$$

MR Reeb Graph

- Morse!
- Geodesic is rotationally invariant
- Graph characteristic of object
- May be computationally intensive (e.g. remeshing)
- Lost notion of sampling
A new approach to topological representation

- Distance function
  \[ d : \mathcal{M} \rightarrow \mathbb{R}, \quad d(p) \cdot p \mapsto \| p \| \]
- Rotation, translation and scale invariance
  \[ d_r(p) = \frac{d(p) - d_{\text{min}}}{d_{\text{max}} - d_{\text{min}}} \]
- Evolving sphere from \(d_{\text{min}}\) to \(d_{\text{max}}\) in \(K\) steps
  (resolution)

Localization of Critical Points
Evolving/exploring spheres
Example

Mesh Models
Example

Object Representation

- Geometric Encoding ↔ Curve modeling
Geometric Modeling

- Different techniques have been attempted
  - Node labeling and homotopy modeling
    [Shinagawa et. al., 92, 00]
  - Distance distribution
    [Ben Hamza et. al., 2002]

Geometry of Curves and Modeling

- All geometric information along arc captured by topologically homogeneous curves
- For object archiving applications, the fewer the curves, the more efficient the representation
- For object reconstruction, the larger the number of curves, the better the reconstruction
How to Capture Geometry?

- Submanifold along an arc is topologically homogeneous
- Model level curves on the submanifold independently to learn weights
- Assign unique weights to a graph arc to capture geometry of the submanifold corresponding to the arc

Interpolation

- Any curve may be viewed as a point in high dimensional space [Mio et. al., 03], [K-Mio-et. al., 04]
- A set of curves lies on a manifold
- Evolution between two curves
  \[ \alpha : [0, L] \rightarrow \mathbb{R}^n \]
Modeling Geometry

- A level curve at level $r$ lies in $\Lambda \times \mathbb{R}$.
  - Level curves are therefore in $\Lambda = [-\pi, \pi] \times [-\pi/2, \pi/2]$
  - Associate a distance field to a curve $C$, to get $\rho = (\rho_1, \ldots, \rho_n)$

\[
\rho(\theta, \phi) = \begin{cases} 
+D((\theta, \phi), C), & \text{if } (\theta, \phi) \in [C'] \\
-D((\theta, \phi), C), & \text{if } (\theta, \phi) \in [C']^c 
\end{cases} \\
\forall (\theta, \phi) \in \Lambda
\]

\[
D((\theta, \phi), C) = \min_{(u, v) \in C} \| (\theta, \phi) - (u, v) \|
\]

Vectorize $\{ \rho(\theta, \phi) \}$ to get $\rho = (\rho_1, \ldots, \rho_n)$

Complete Object Representation

- Evolution of level curves on a graph arc modeled by a trajectory in $\mathbb{R}^3$
  - Geometry completely captured by the trajectory

- Elastic coefficients uniquely determine the trajectory
  - Assign the coefficients as weights to the graph arc

- Alternatively, parameterize the trajectory as $a(t) : t \mapsto (\alpha(t), \alpha(t), \alpha(t))$
  - $t \in [a, b]$

- $r \in [r_1, r_2]$ is mapped to $t \in [a, b]$

- Model $\alpha(i), i = 1, 2, 3$ with their respective Taylor series

- Fewer coefficients (weights)
Modeling

- Given $m$ curves $C_1, \ldots, C_m$, at levels $r_1, \ldots, r_m$
  \[ p(C_i) = \rho_i \quad i = 1, \ldots, m \]

- Optimal trajectory $p(t) \in \mathbb{R}$ minimizes some energy functional subject to
  \[ p(0) = \rho_0 \]
  \[ p(1) = \rho_1 \]
  \[ p(0) = v_0 \]
  \[ p(1) = v_1 \]

Curve Interpolation
**Dimension reduction**

- Given two curves $\rho_1$ and $\rho_2$, respective tangent vectors $v_1$ and $v_2$
  - Displacement vector $d(\rho_1, \rho_2) = \rho_2 - \rho_1$
  - Assume $\{v_1, v_2, d\}$ form a linearly independent set
  - Orthogonalize $\{v_1, v_2, d\}$ to get $\{b_1, b_2, b_3\} \subset \mathbb{R}^3$
  - Project to $\{v_1, v_2, d\}$ to $\mathbb{R}^3$

- Find elastic $\alpha : [0,1] \rightarrow \mathbb{R}^3$ that minimizes bending energy

\[ E_a = \int_0^1 k_0^2(s) ds \quad \text{subject to} \quad \alpha(0) = 0, \alpha(1) = w_1, \]
\[ \alpha(0) = w_1, \alpha(1) = w_2 \]

**Examples**

- Given curves
- Evolution of curves
- Reconstructed surface
Example

Taylor series representation of the trajectory

\[ \alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t)) \]
Redundancy in Geometric Information

- Sphere evolved from $r_{\text{min}}$ to $r_{\text{max}}$ in $K$ steps
- $K$ determines resolution of Reeb graph
  - Large $K$
    - Critical points captured well
    - Redundant geometry information
- Two approaches for removing redundancy
  - Curvature minimization
  - Correlation

Experimental Results

- Example 1
Experimental Results

- Example
  - Taylor series representation of the trajectory

\[ \alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t)) \]

- Example 2

Level curves

Reconstructed surface
Experimental Results

Example 3

Given 108 level curves
Reconstructed from 14 curves

Conclusions/Perspectives

- Overview of methodologies
- Applications in classification
- Other applications
  - GIS applications
  - Human tracking (e.g. training and rehabilitation)
  - Aids to physically challenged
  - Data base archiving and retrieval
Curve Modeling

Let $\alpha : [0, L] \rightarrow \mathbb{R}^n$

- Given points $p, q \in \mathbb{R}^n$ and unit vectors $\tilde{v}, \tilde{w} \in \mathbb{R}^n$, find a unit-speed curve $\alpha \in \mathbb{R}$ of scale-invariant elastic energy (Mumford, and others) with $p, q$ as initial and terminal points, and $\tilde{v}, \tilde{w}$ as initial and terminal velocity vectors.

$$\min \left( \int \kappa^2(s) ds \mid \kappa(s) = \| \dot{\alpha}(s) \| \right)$$

Each point is a curve

Interpolation of curves

- Constraints yield a formulation of fitting a smooth curve through two end points
  - Minimum curvature
  - Satisfying the boundary conditions as described by the two curves
NL Interpolation

- Marching cube Algorithm

Curve Modeling

- May be best solution to accurately capture geometry
- Classification applications of 3D objects requires representation parsimony
- Other weight optimization under investigation
Multiresolution Reeb Graph

- Use a MR technique to construct graph

Comprehensive description

- Statistical characterization of geodesics
  [Ben Hamza et al., 2002]
Information Theoretic Distance

- Jensen-Shannon divergence

\[ D(p, q) = H\left(\frac{p+q}{2}\right) - \frac{H(p) + H(q)}{2} \]

[Y. He, et. al. 2002]

Classification
Example

Visualization of Distance Function

- Easily implemented
Implementation

To isotropically explore a surface, let $v$ be a fixed point in space, define

$$d : M \to \mathbb{R}$$

For any $v$ in space

$$\forall p \in M, d_v(p) = |p - v|^2$$

Let $v$ be the centroid of $M$ and carry it to the origin

[Aysegul et. al., 2003]

$$d(p) = |p|^2$$

Morse Distance Function
Free Form Curve Representation

Let $\psi : \Omega \rightarrow \mathbb{R} \ (R \subseteq \Omega)$

\[
\psi((x,y)) = \begin{cases} 
  d((x,y), C), & \text{if } (x,y) \in R; \\
  -d((x,y), C), & \text{if } (x,y) \notin R 
\end{cases}
\]

Each curve is a field $\Psi_i$ in $\mathbb{R}^n$

Trajectory modeling

- Constraints added to better model trajectory