Image Analysis and Approximation via Generalized Polyharmonic Local Trigonometric Transforms

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Outline

- Motivations: Issues of Localized Fourier Analysis
- Polyharmonic Local Sine Transform (PHLST) for Rectangular Domains
- Extension to 3D
- Extension to General Shape Domain
- Summary and Future Plan

Motivations

- Want sparse data representation (compression) $\Longrightarrow$ Fast decay of the expansion coefficients
- Want a data representation as less statistically dependent among blocks as possible $\Longrightarrow$ No overlaps among blocks
- Want to develop a local signal analysis tool that can distinguish intrinsic singularities from the artificial discontinuities created by local windowing $\Longrightarrow$ many potential applications
- Want to efficiently represent regions of more general shapes other than rectangular blocks

Life on an Interval (or a Domain)

If data are periodic and smooth, the Fourier basis is very efficient:

- Fast decay of the Fourier coefficients
- Smoothness estimate (e.g., the Lipschitz/Hölder exponents)

However, most signals and images of interest have compact supports, and are neither periodic nor smooth. What to do?

Life on an Interval (or a Domain) . . .

- Design special wavelets (“wavelets on intervals”);
- Use “multiwavelets” of Alpert and Rokhlin (segmented orthogonal polynomials);
- Use “Prolate Spheroidal Wave Functions”;
- Use the “Continuous Boundary Local Fourier Transform” (CBLFT) that periodizes the signal nicely and expand it into a periodic basis; or
- Use the “Polyharmonic Local Trigonometric Transform” (PHLTT) and its generalizations

Basic Ideas of PHLTT

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- Sines and cosines are eigenfunctions of the Laplacian on a rectangular box

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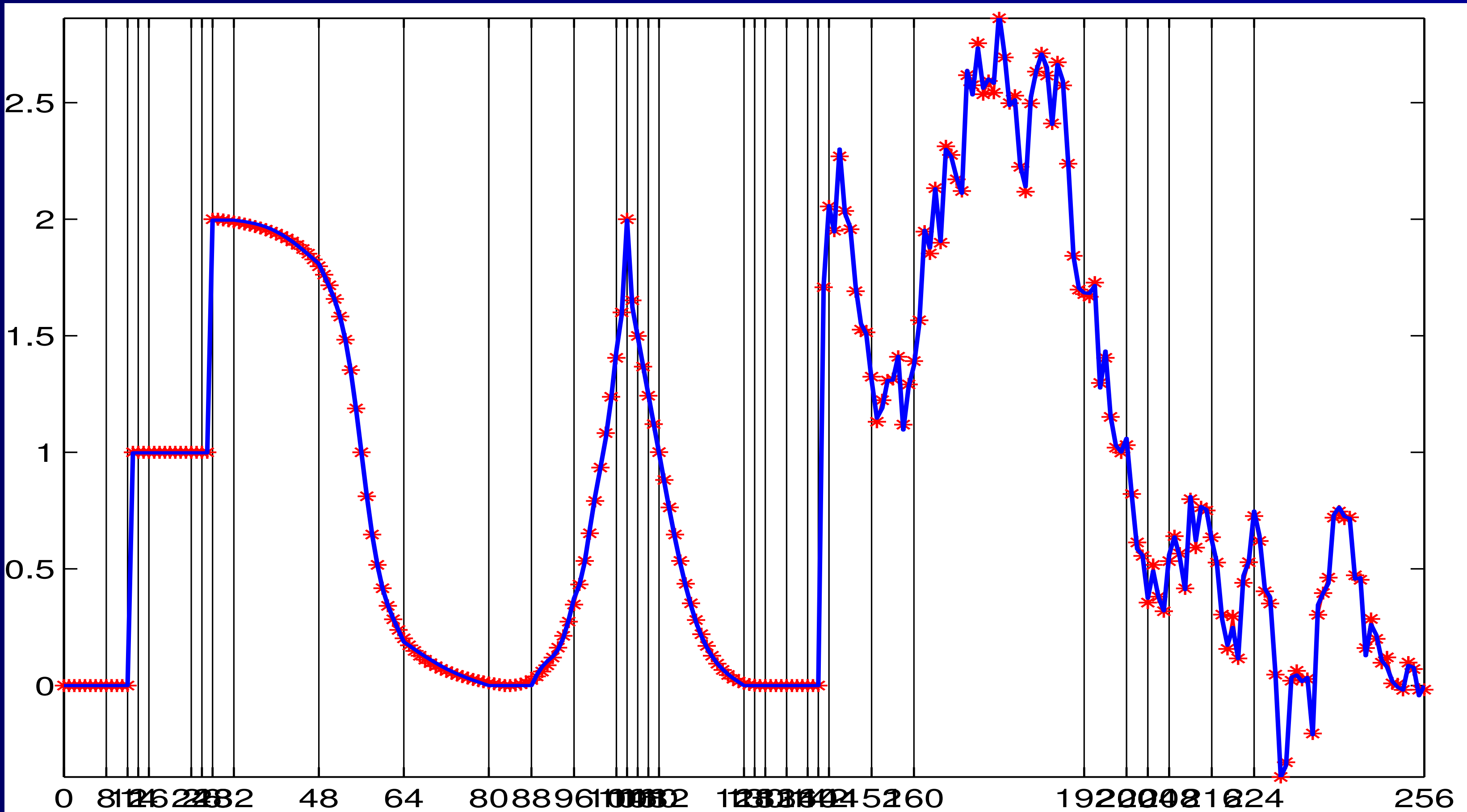
- High order algebraic polynomial $\Longrightarrow$ Runge phenomenon
- Trigonometric polynomial on an interval $\Longrightarrow$ Gibbs phenomenon

1D Example: Compression Ratio ≈ 6



Polyharmonic Global Sine Transform on a Rectangle

- Consider a function $f \in C^{2m}(\overline{\Omega})$, where $m = 1, 2, \ldots$, and $\Omega \subset \mathbb{R}^n$ (e.g., $\Omega = [0, 1]^n$), but not periodic.

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- Decompose this function into the following two components:

$$f(\boldsymbol{x}) = u(\boldsymbol{x}) + v(\boldsymbol{x}),$$

- $u(\boldsymbol{x})$ satisfies the following polyharmonic equation:

$$\Delta^m u = 0 \quad \text{in } \Omega.$$

PHGST on a Rectangle . . .

- The boundary condition for u is:

$$\frac{\partial^{p_\ell} u}{\partial\, \nu^{p_\ell}} = \frac{\partial^{p_\ell} f}{\partial\, \nu^{p_\ell}} \quad \text{on } \Gamma = \partial\Omega, \qquad \ell = 0, \ldots, m-1,$$

where p_ℓ is the order of the normal derivatives to be specified ($p_0 \equiv 0 \Leftrightarrow u = f$ on Γ).

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- Now set $v(\boldsymbol{x}) = f(\boldsymbol{x}) - u(\boldsymbol{x})$, which we will call the residual component with

$$\frac{\partial^{p_\ell} v}{\partial\, \nu^{p_\ell}} = 0 \quad \text{on } \Gamma, \qquad \ell = 0, \ldots, m-1.$$

A Specific Example: $m = 1$ (Laplace) Case

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = f & \text{on } \Gamma. \end{cases}$$

Variational formulation $\Longrightarrow$ minimum gradient interpolation:

$$\min_{u \in H^1(\Omega)} \int_\Omega |\nabla u|^2 \, \mathrm{d}\boldsymbol{x} \quad \text{subject to the above boundary condition.}$$

Note that in 1D, this is simply a line.

A Specific Example: $m = 2$ (Biharmonic) Case

$$\begin{cases} \Delta^2 u = 0 & \text{in } \Omega, \\ u = f,\ \dfrac{\partial^2 u}{\partial\, \nu^2} = \dfrac{\partial^2 f}{\partial\, \nu^2} & \text{on } \Gamma. \end{cases}$$

Variational formulation $\Longrightarrow$ minimum curvature interpolation:

$$\min_{u \in H^2(\Omega)} \int_\Omega \left(\Delta u + 2 \sum_{j \neq k} \partial_j \partial_k u \right)^2 \mathrm{d}\boldsymbol{x},$$

subject to the above boundary condition.

Note that in 1D, this is simply a cubic polynomial.

PHGST on a Rectangle . . .

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- The u component can be represented only by the boundary values $f|_\Gamma$. $\Longrightarrow$ No need to store the whole u.
- The residual component v becomes 0 at the boundary Γ.

PHGST on a Rectangle . . .

- Therefore, the v component is suitable for Fourier analysis. In fact, if $\Omega = [0,1]^n$ and $p_\ell = 2\ell$, $\ell = 0, \ldots, m-1$, then the Fourier sine analysis should be used to get the matching normal derivatives up to order $2m-1$ by odd reflection at the boundaries.

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- The frequency content (in particular, mid to high frequency range) of the original is retained in the residual $\Longrightarrow$ textures remain in v; shading is captured by u.

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- The frequency content (in particular, mid to high frequency range) of the original is retained in the residual $\Longrightarrow$ textures remain in v; shading is captured by u.
- We can get the decay rate as follows:

Theorem

Let $\Omega = [0,1]^n$, and $f \in C^{2m}(\overline{\Omega})$, but non-periodic. Assume further that $\partial_i^{2m+1} f$, $i = 1, \ldots, n$, exist and are of bounded variation. Furthermore, let $f = u + v$ be the PHLST representation where the polyharmonic component u is the solution of the polyharmonic equation of order m with the boundary condition

$$\frac{\partial^{2\ell} u}{\partial\, \nu^{2\ell}} = \frac{\partial^{2\ell} f}{\partial\, \nu^{2\ell}} \quad \text{on } \Gamma, \quad \ell = 0, \ldots, m-1.$$

Then, the Fourier sine coefficient $b_{\boldsymbol{k}}$ of the residual v is of $O\left(\|\boldsymbol{k}\|^{-2m-1}\right)$ for all $\boldsymbol{k} \neq \boldsymbol{0}$, where $\boldsymbol{k} = (k_1, \ldots, k_n)$, and $\|\boldsymbol{k}\|$ is the usual Euclidean (i.e., ℓ^2) norm of $\boldsymbol{k}$.

Polyharmonic Local Sine Transform

Now, consider a decomposition of Ω into a disjoint set of subdomains $\{\Omega_j\}$, i.e., $\overline{\Omega} = \cup_{j=1}^{J} \overline{\Omega}_j$. A typical example is $\Omega = (0,1)^n$, and Ω_j is a dyadic subcube. Then, restrict f on Ω_j, i.e., for each j, we decompose f locally as follows:

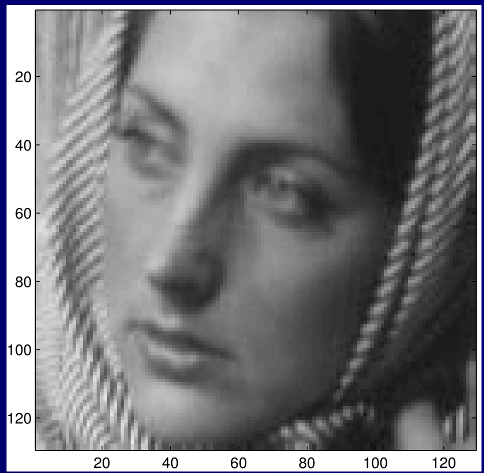
$$f\chi_{\Omega_j} = f_j = u_j + v_j,$$

where we follow the same recipe locally as in the global case. We call this decomposition of f into $\{u_j, v_j\}$ Polyharmonic Local Sine Transform (PHLST).

For $m = 1$: Laplace Local Sine Transform (LLST);

For $m = 2$: Biharmonic Local Sine Transform (BLST).

Polyharmonic Local Sine Transform ...



(a) Original

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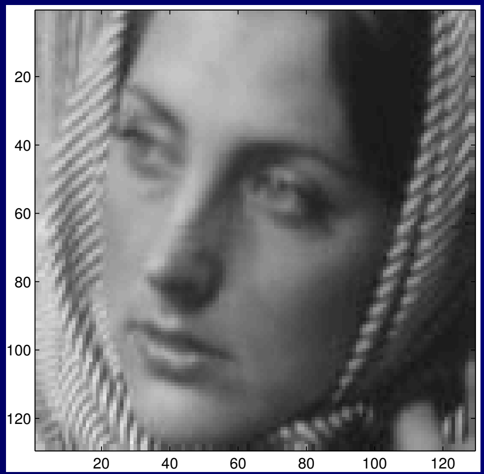

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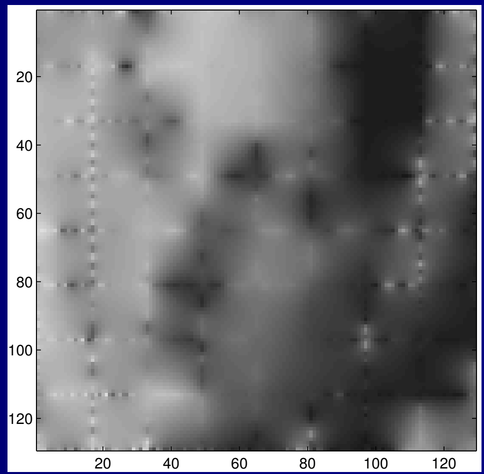

(b) $\cup_j u_j$

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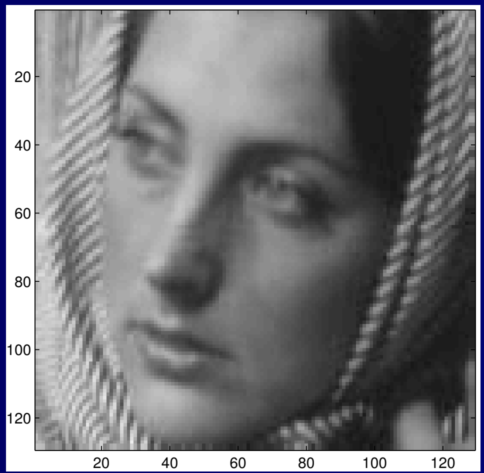

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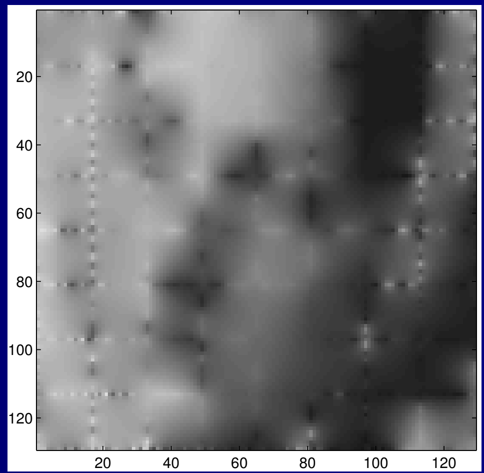

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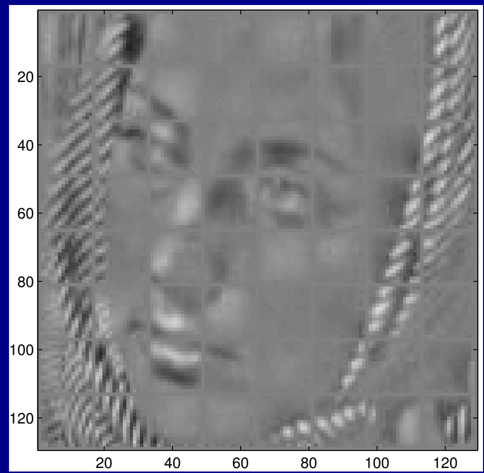

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Remarks on the Laplace Solver

- A fast and accurate (no finite difference approximation of the Laplace operator) algorithm (for both 2D and 3D) exists via FFT if Ω and Ω_j are dyadic cubes (Averbuch, Braverman, Israeli, and Vozovoi, 1998)

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- A fast and accurate (no finite difference approximation of the Laplace operator) algorithm (for both 2D and 3D) exists via FFT if Ω and Ω_j are dyadic cubes (Averbuch, Braverman, Israeli, and Vozovoi, 1998)
- The u component can be computed as:

$$u(x,y) = p(x,y) + \sum_{k \geq 1} \{ b_k^{(1)} h_k(x, 1-y) + b_k^{(2)} h_k(y, 1-x) + b_k^{(3)} h_k(x,y) + b_k^{(4)} h_k(y,x) \},$$

where . . .

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- $p(x, y)$ is a harmonic polynomial that agrees with $f(x, y)$ at the four corner points of the domain, e.g.,

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- The computational cost of u is $O(4N^2 \log N)$.

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- Can be generalized to other geometries (e.g., discs, spheres, star shapes, general smooth boundaries)
- Useful for interpolation and local feature computation (e.g., gradients, directional derivatives, etc.)

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- The residual component v is what is unpredictable using the boundary information.
- Yet another interpretation is: u: trend, v: fluctuation.

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- Other inverse boundary problems . . .

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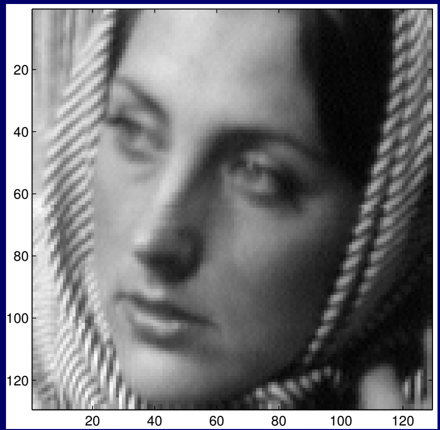
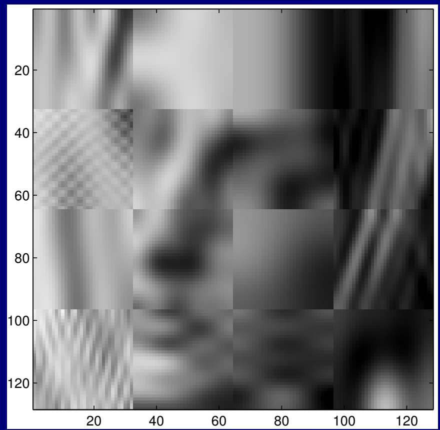
- Select the largest 1% and 5% of coefficients.
- Compression using quantization is in progress.

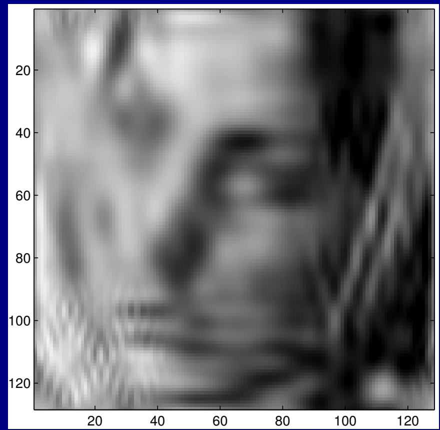
LLST Approximation Experiment: Top 1%



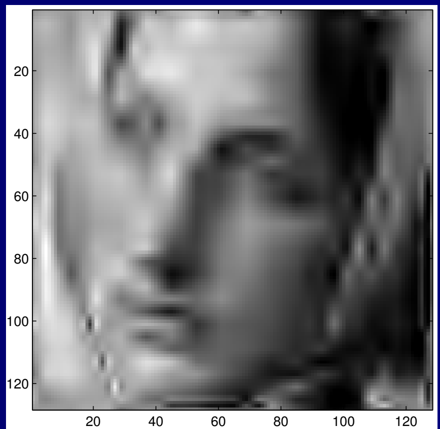
(a) Original



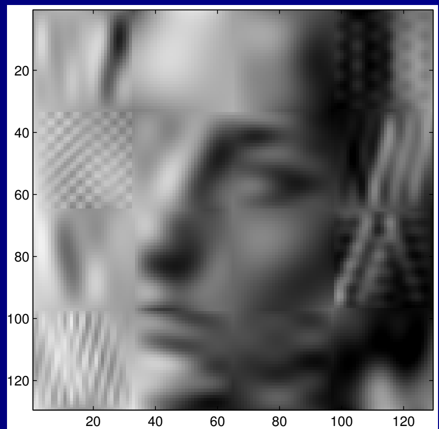
(b) DCT



(c) LCT

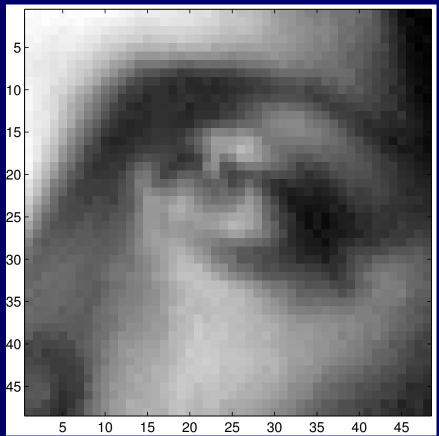


(d) C12

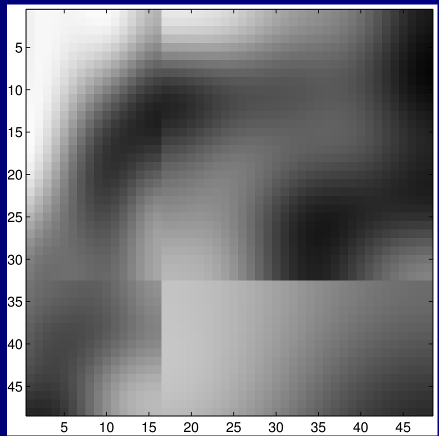


(e) LLST

Zoomed up images: Top 1%



(a) Original



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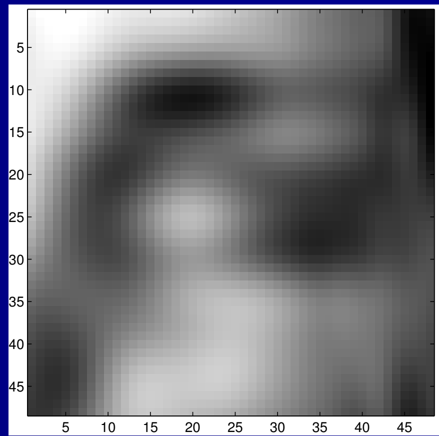

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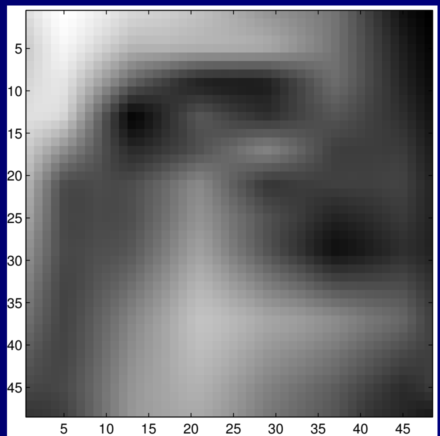

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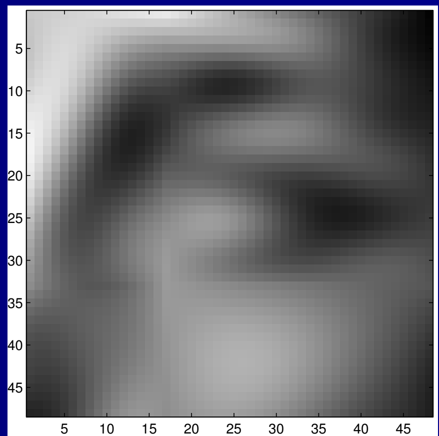
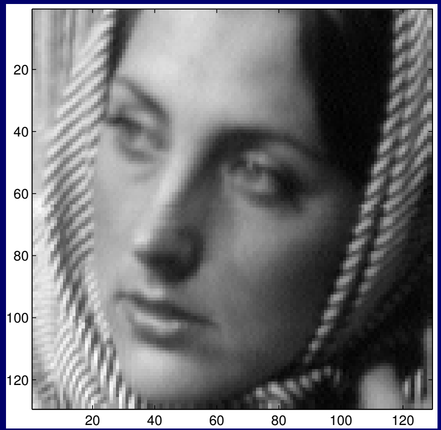
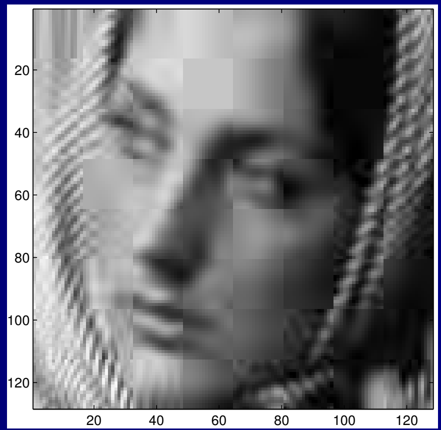


(e) LLST

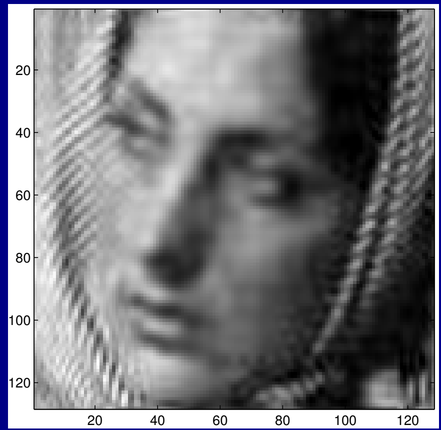
LLST Approximation Experiment: Top 5%



(a) Original



(b) DCT



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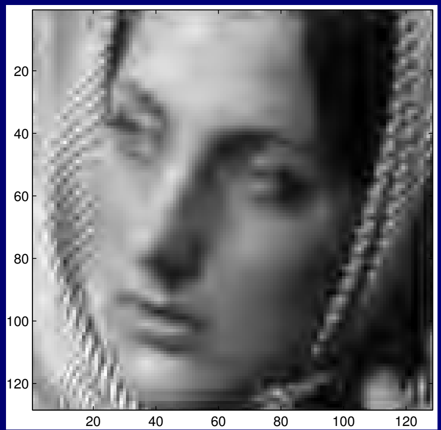

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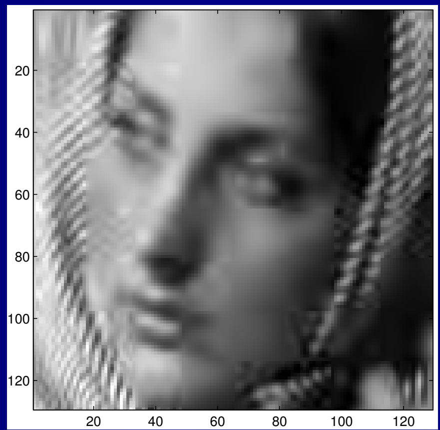

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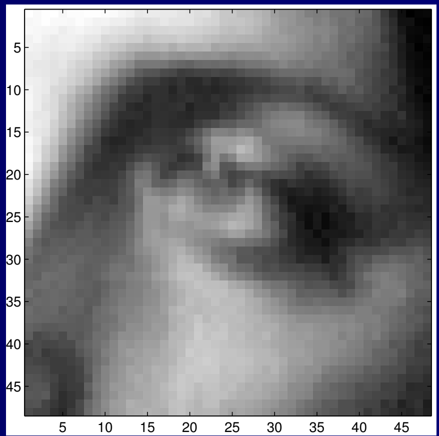
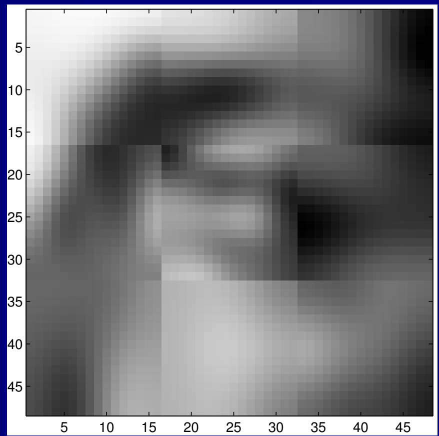
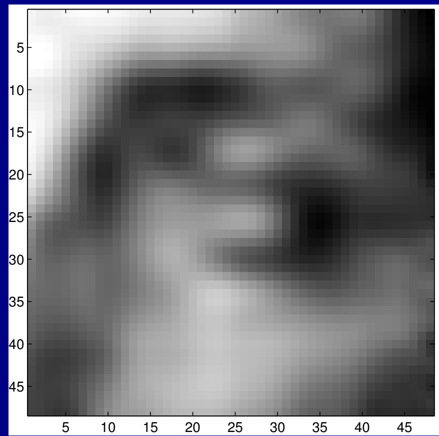
Zoomed up images: Top 5%



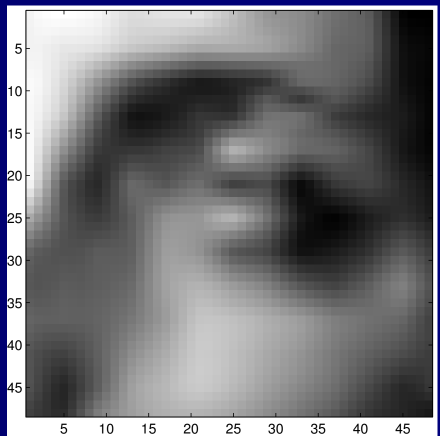
(a) Original



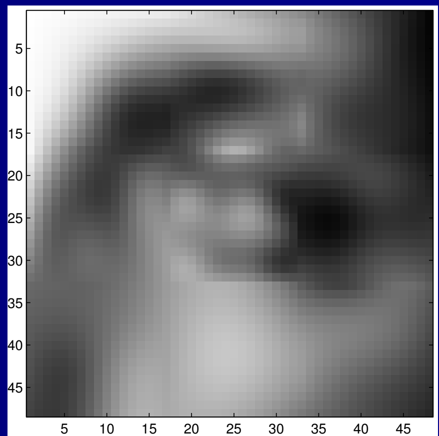
(b) DCT



(c) LCT

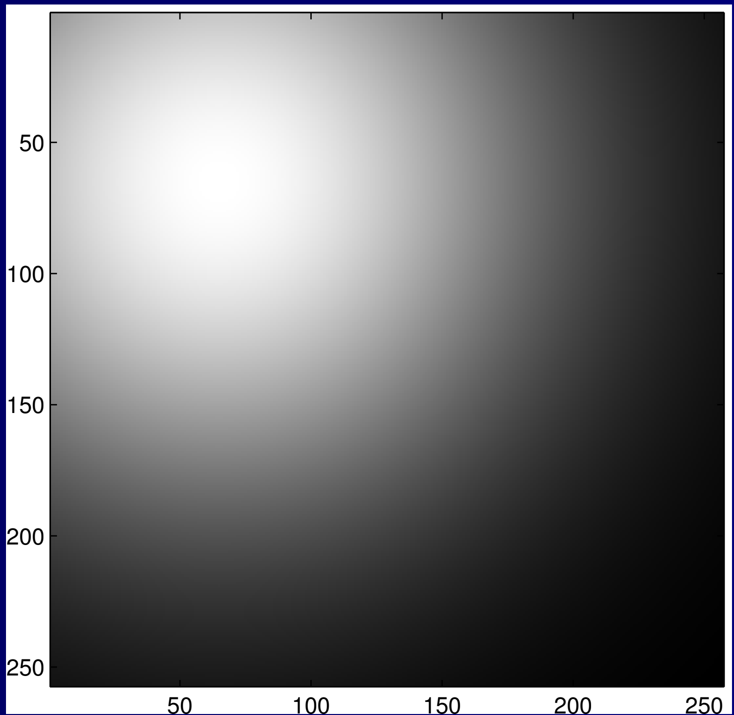


(d) C12

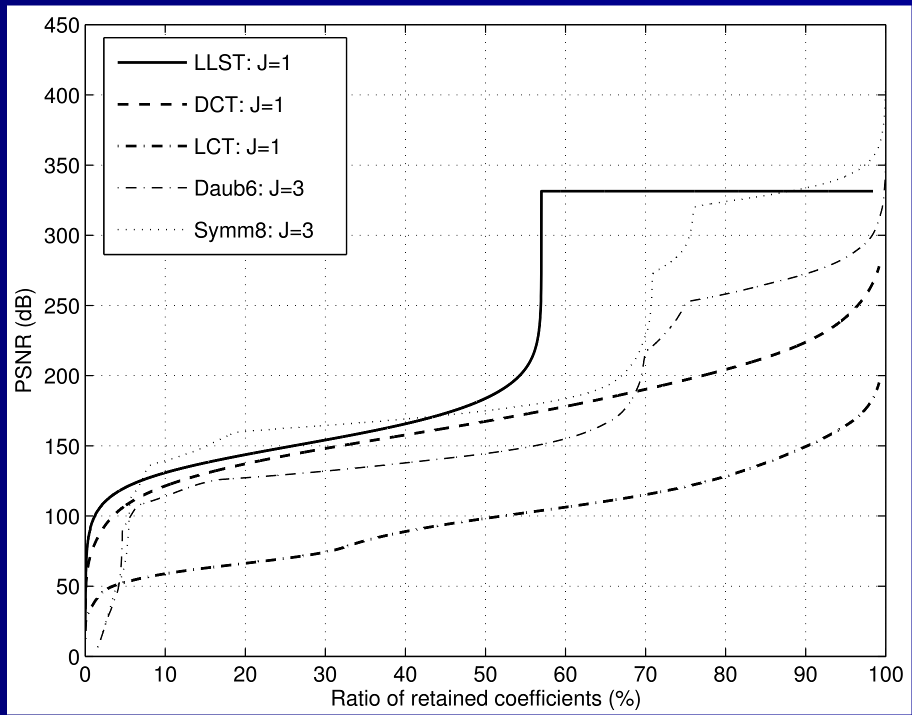


(e) LLST

Approximation Test: Smooth Function

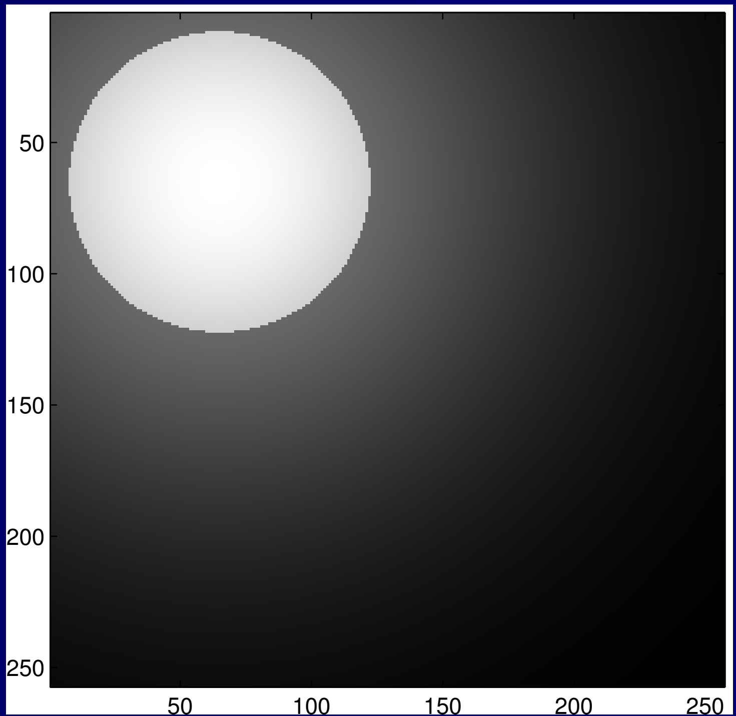


(a) Original

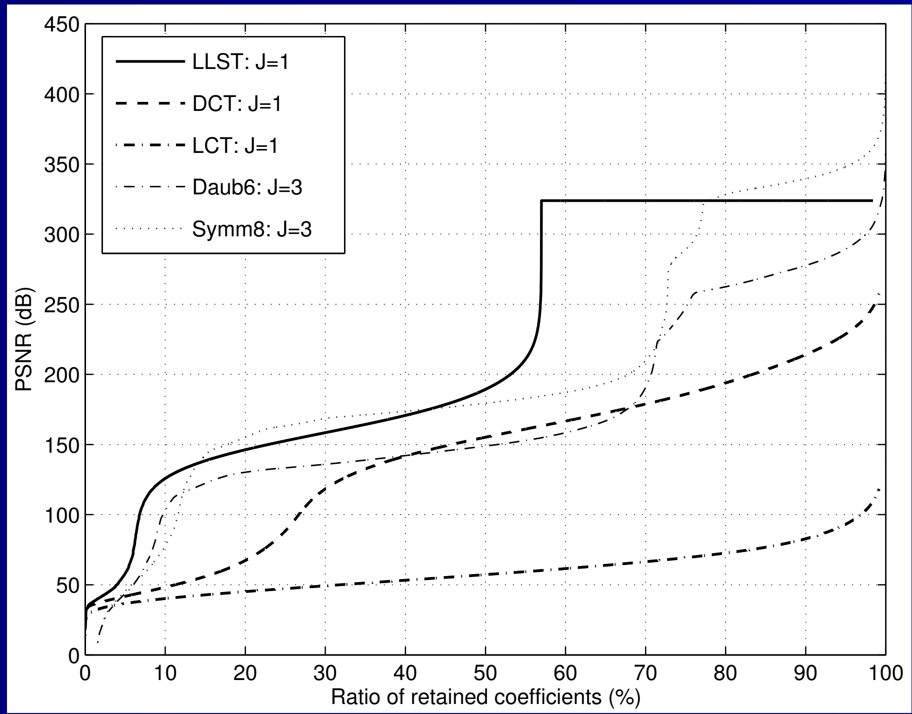


(b) PSNR

Approximation Test: Piecewise Smooth Function

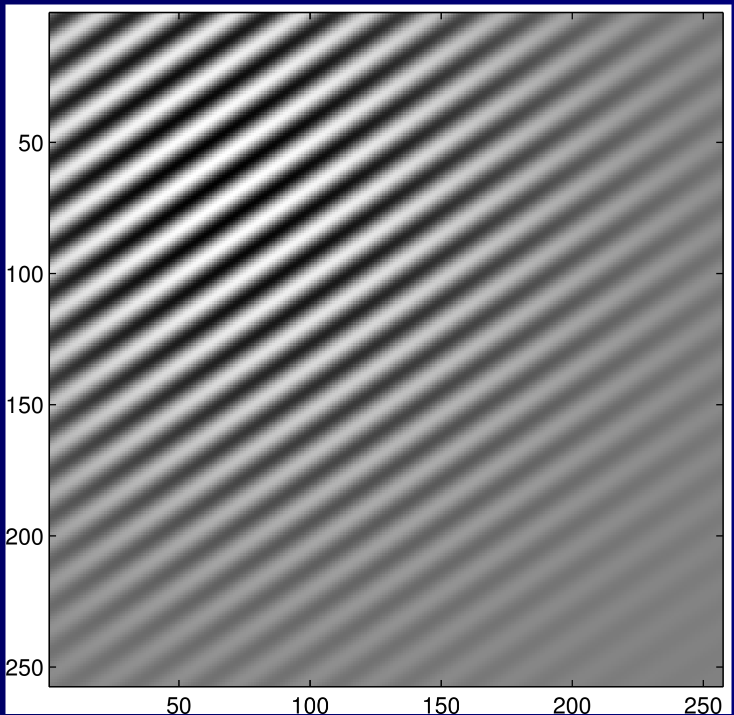


(a) Original

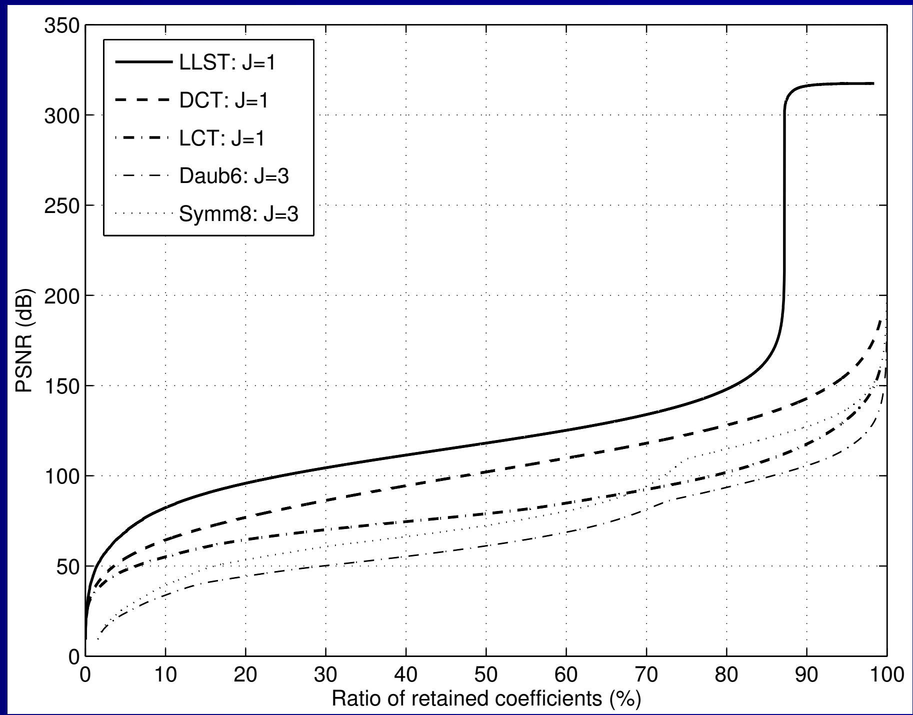


(b) PSNR

Approximation Test: Oscillatory Function

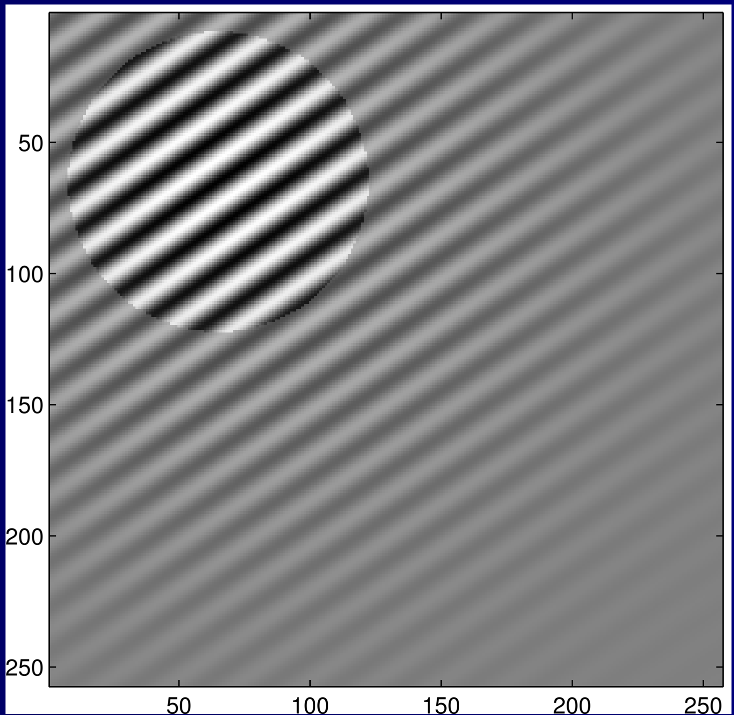


(a) Original

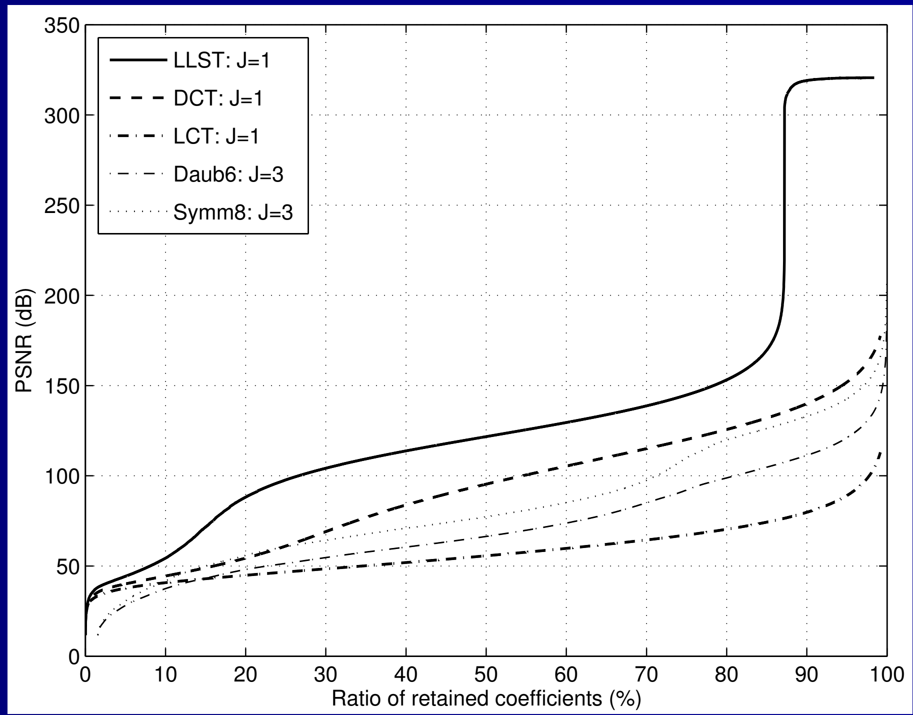


(b) PSNR

Approximation Test: Oscillatory Function with Discontinuity



(a) Original



(b) PSNR

Extension to 3D

- Boundary data contains more information compared to 1D and 2D

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- Promising applications include analysis and compression of 3D seismic, medical, and video data, . . .

Extension to General Shape Domains

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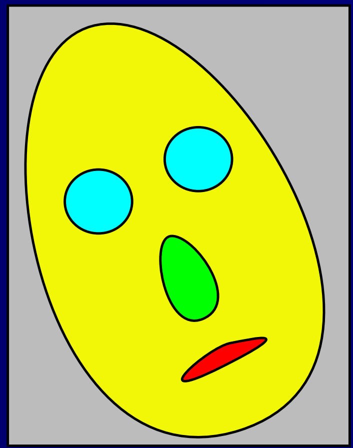
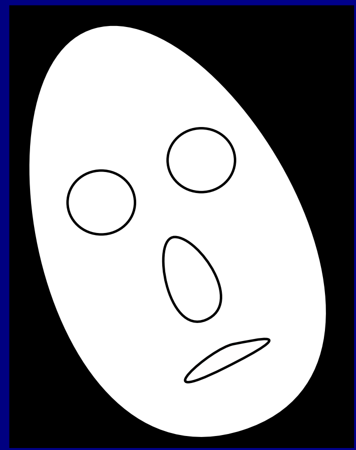
$\Longrightarrow$ Smoothly extend the function defined on Ω to the outside!

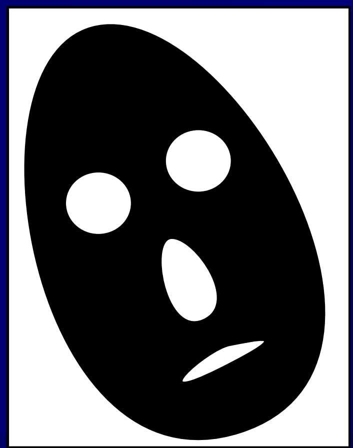
Object-Oriented Image Analysis



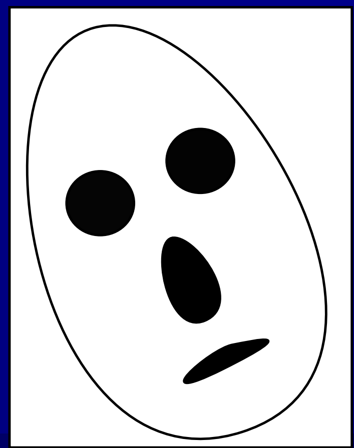
(a) Original



(b) Background



(c) Object



(d) Anomalies

Object-Oriented Image Analysis . . .

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Object-Oriented Image Analysis ...

- Suppose its boundary Γ is already detected explicitly.
- Let $f \in C^2(\Omega) \cap C(\overline{\Omega})$.
- Let S be a rectangular domain containing $\overline{\Omega}$.
- Then, we will find a smooth extension $\tilde{v}$ to the entire S from Ω and the harmonic function u in Ω:

$$\Delta \tilde{v} = \begin{cases} \Delta f & \text{in } \Omega, \\ 0 & \text{in } S \backslash \overline{\Omega}, \end{cases}$$

$$\tilde{v} = 0 \quad \text{on } \partial S.$$

Motivations for $\tilde{v}$

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- can also show $\tilde{v}|_{S\backslash\Gamma} \in C^2(S\backslash\Gamma)$
- $\tilde{v}|_{\partial S} = 0 \Longrightarrow$ suitable for Fourier sine series expansion, with decay rate $O(\|\boldsymbol{k}\|^{-3})$.

How to compute $\tilde{v}$

- Introduce the potential function:

$$P(\boldsymbol{x}) \triangleq \int_\Omega \Delta f(\boldsymbol{y}) \Phi(\boldsymbol{x}, \boldsymbol{y})\, \mathrm{d}\boldsymbol{y} \quad \text{for } \boldsymbol{x} \in \mathbb{R}^n.$$

where

$$\Phi(\boldsymbol{x}, \boldsymbol{y}) \triangleq \begin{cases} \frac{1}{2\pi} \log \|\boldsymbol{x} - \boldsymbol{y}\| & \text{if } n = 2, \\ \frac{\|\boldsymbol{x}-\boldsymbol{y}\|^{2-n}}{(2-n)\omega_n} & \text{if } n > 2, \end{cases}$$

is the so-called fundamental solution of the Laplace equation.

How to compute $\tilde{v}$ ...

- Can show that P satisfies the above Poisson equation:

$$\Delta P = \begin{cases} \Delta f & \text{in } \Omega, \\ 0 & \text{in } S \backslash \overline{\Omega}, \end{cases}$$

How to compute $\tilde{v}$ ...

- Can show that P satisfies the above Poisson equation:

$$\Delta P = \begin{cases} \Delta f & \text{in } \Omega, \\ 0 & \text{in } S \backslash \overline{\Omega}, \end{cases}$$

- But the boundary condition at the outside box is not satisfied, i.e., $P|_{\partial S} \neq 0$.

How to compute $\tilde{v}$ . . .

- Introduce now a boundary correction function Q:

$$\begin{aligned} \Delta Q &= 0 \quad \text{in } S, \\ Q &= -P \quad \text{on } \partial S; \end{aligned}$$

How to compute $\tilde{v}$ . . .

- Introduce now a boundary correction function Q:

$$\Delta Q = 0 \quad \text{in } S,$$
$$Q = -P \quad \text{on } \partial S;$$

- Finally set $\tilde{v}(\boldsymbol{x}) = P(\boldsymbol{x}) + Q(\boldsymbol{x})$, which satisfies the desired conditions.

How to compute $\tilde{v}$ ...

- Can write the solution via single layer and double layer potentials as:

$$P(\boldsymbol{x}) = \tilde{\chi}_\Omega(\boldsymbol{x}) f(\boldsymbol{x}) + \int_\Gamma \left(\frac{\partial f}{\partial \nu}(\boldsymbol{y}) \Phi(\boldsymbol{x}, \boldsymbol{y}) - f(\boldsymbol{y}) \frac{\partial \Phi(\boldsymbol{x}, \boldsymbol{y})}{\partial \nu_{\boldsymbol{y}}} \right) \mathrm{d}s(\boldsymbol{y}),$$

where

$$\tilde{\chi}_\Omega(\boldsymbol{x}) = \begin{cases} 1 & \text{if } \boldsymbol{x} \in \Omega; \\ 1/2 & \text{if } \boldsymbol{x} \in \Gamma; \\ 0 & \text{if } \boldsymbol{x} \in \mathbb{R}^n \backslash \overline{\Omega}. \end{cases}$$

Object-Oriented Image Analysis . . .

- The u component satisfies:

$$\Delta u = 0 \quad \text{in } \Omega,$$

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Object-Oriented Image Analysis ...

- The u component satisfies:

$$\Delta u = 0 \quad \text{in } \Omega,$$

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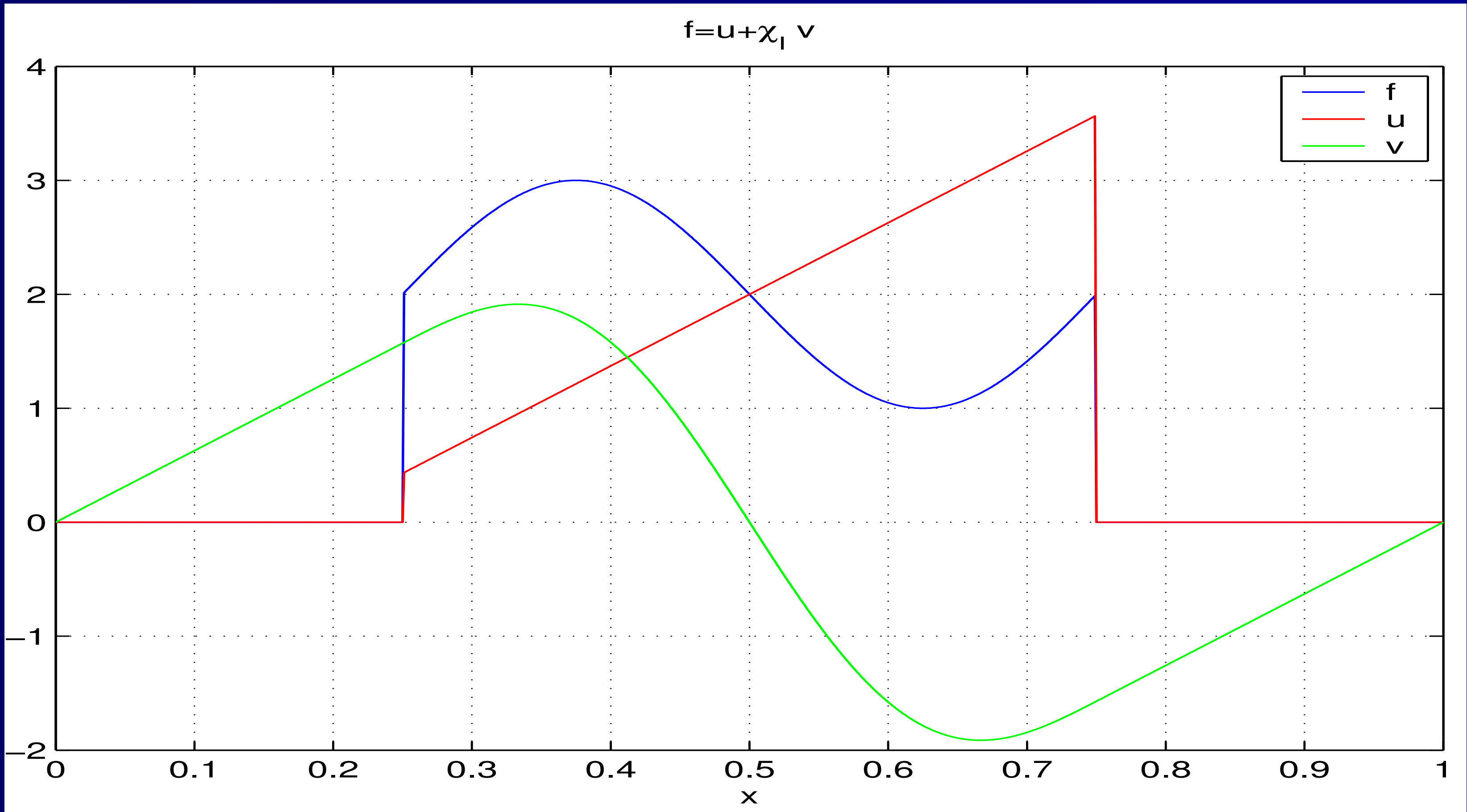
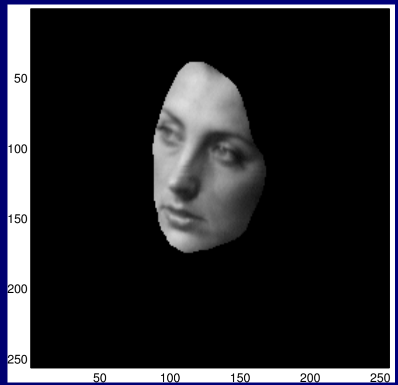
- u can be recovered from $f|_\Gamma$, $\left.\dfrac{\partial f}{\partial \nu}\right|_\Gamma$, and $\tilde{v}$ via Fast Multipole Method (just potential evaluation, no need to solve the Laplace equation).

1D Example

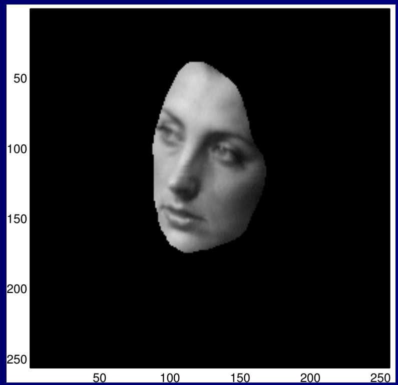


2D Example



(a) Orig.1

2D Example



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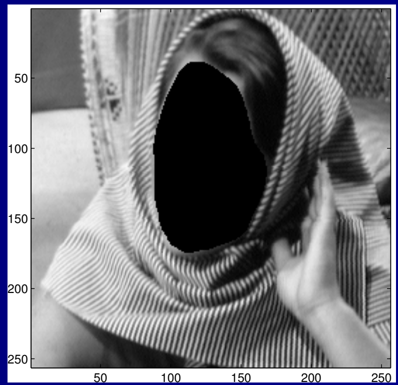
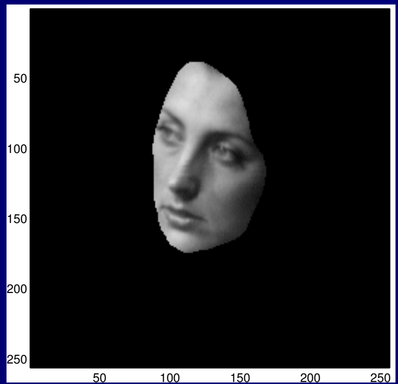


(b) Orig.2

2D Example



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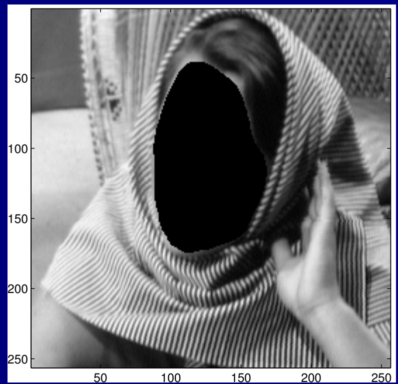
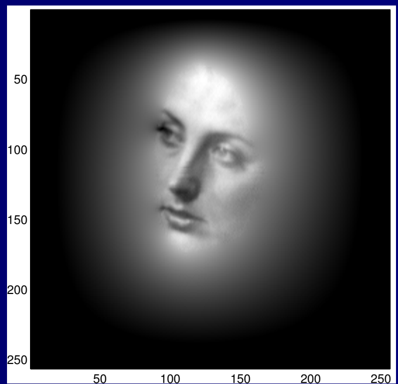
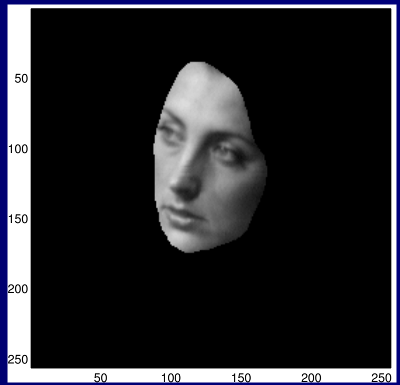


(b) Orig.2



(c) Ext.1

2D Example



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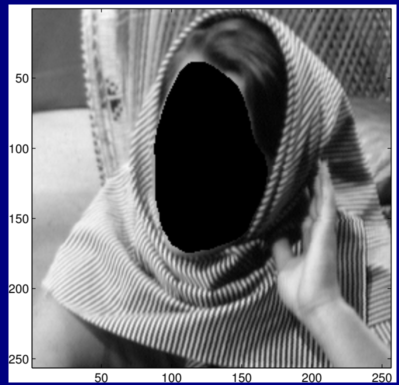

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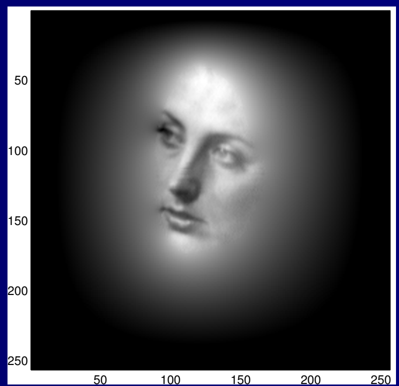

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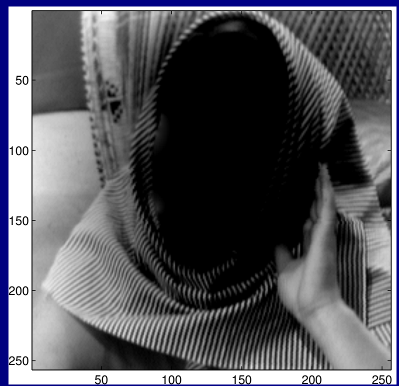

(d) Ext.2

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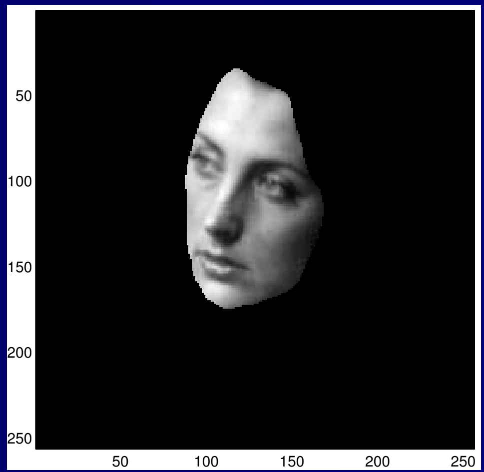

(a) Original

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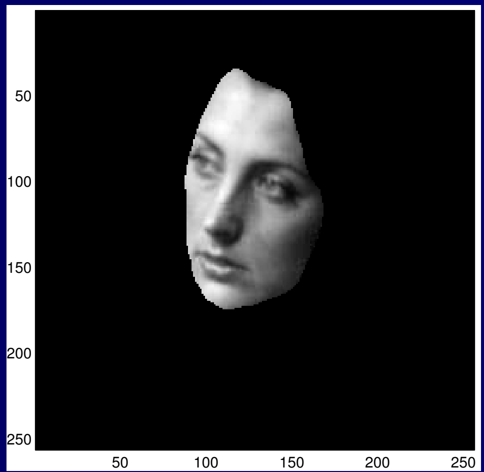

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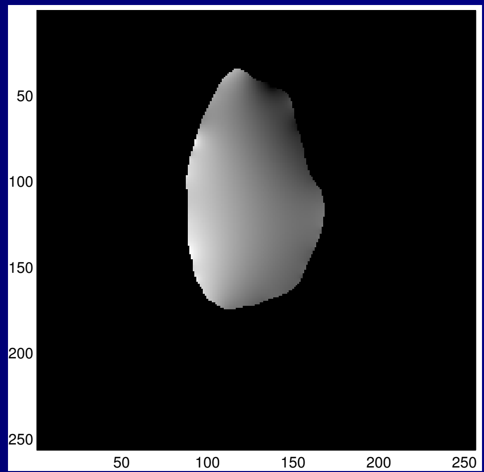
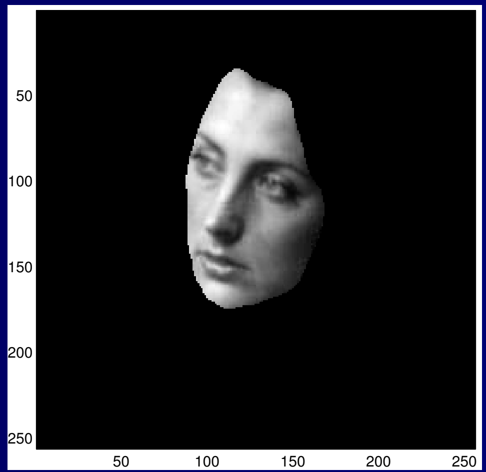
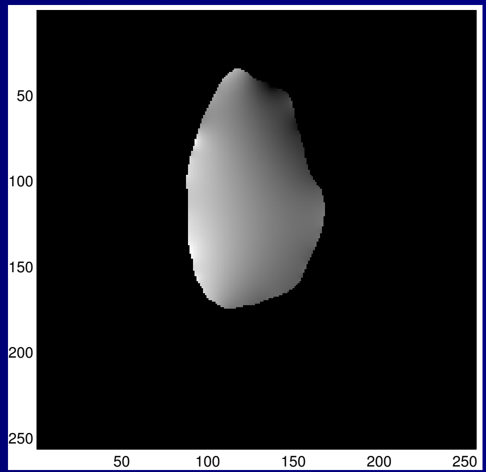


(b) u

2D Example . . .



(a) Original



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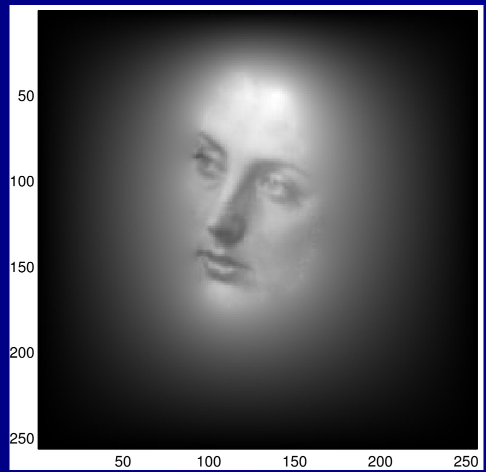
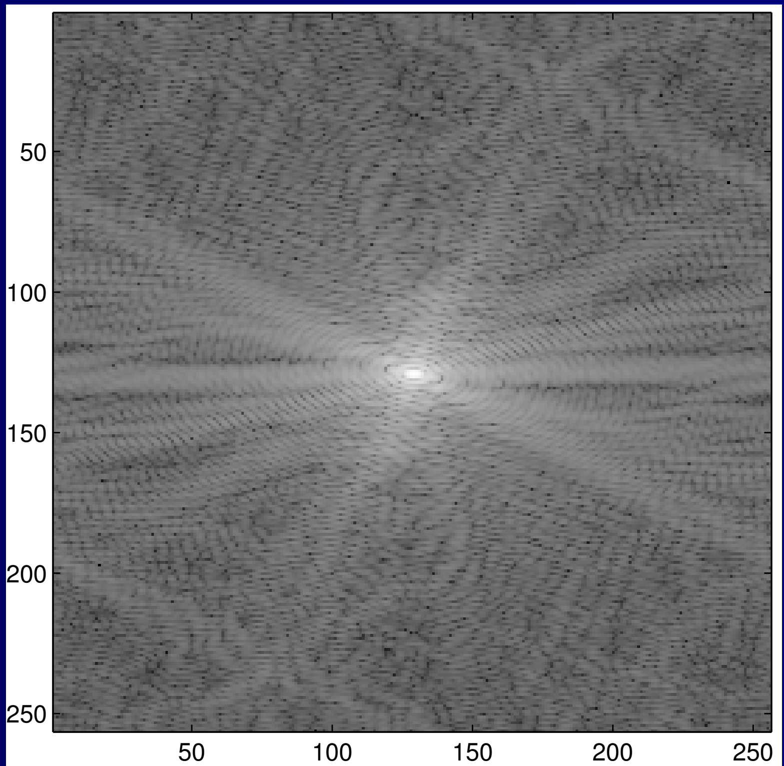


(c) $\tilde{v}$

The Fourier Transform Magnitudes (log-scale)



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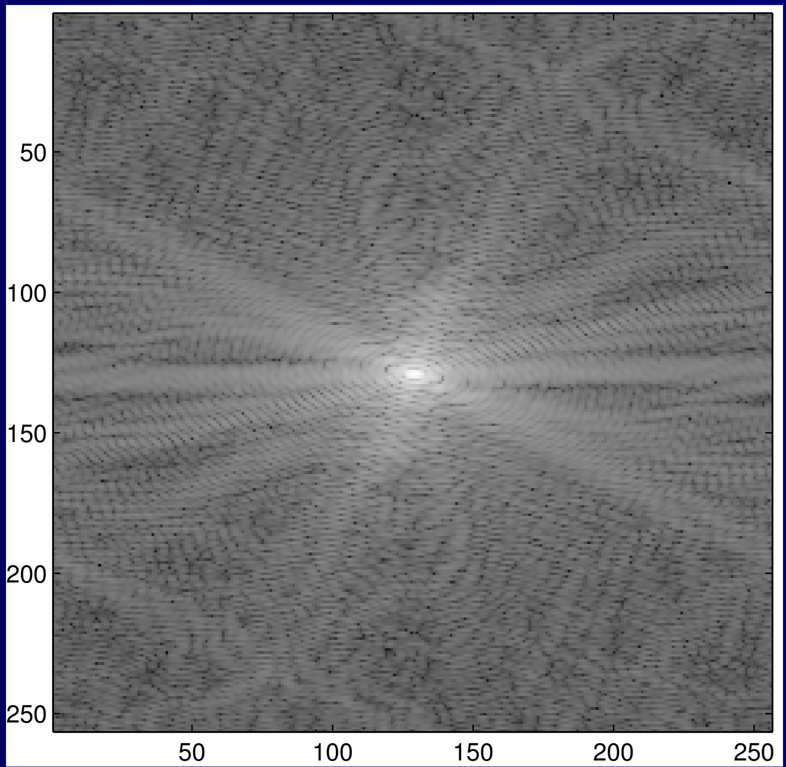

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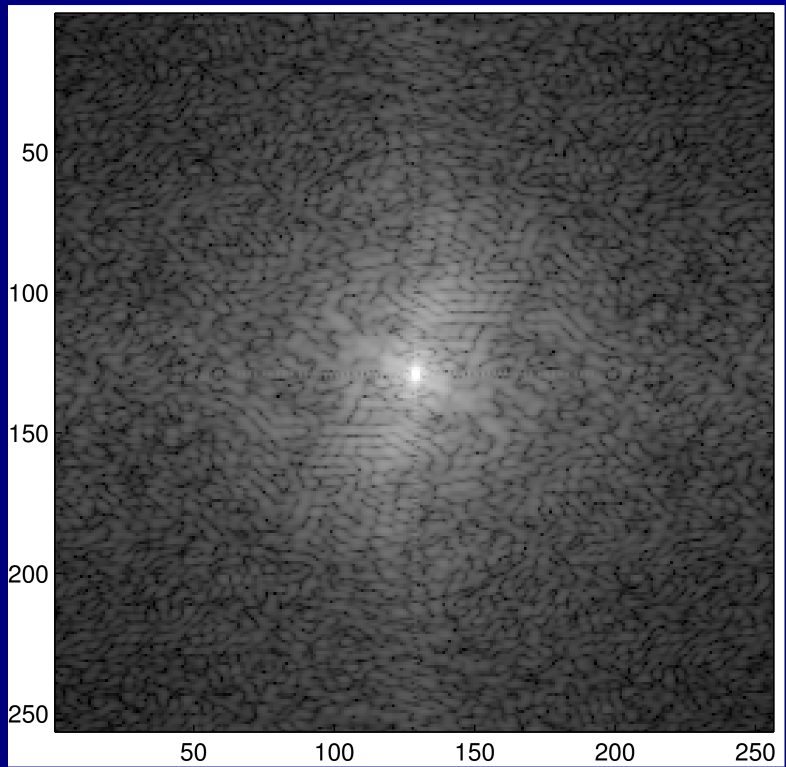
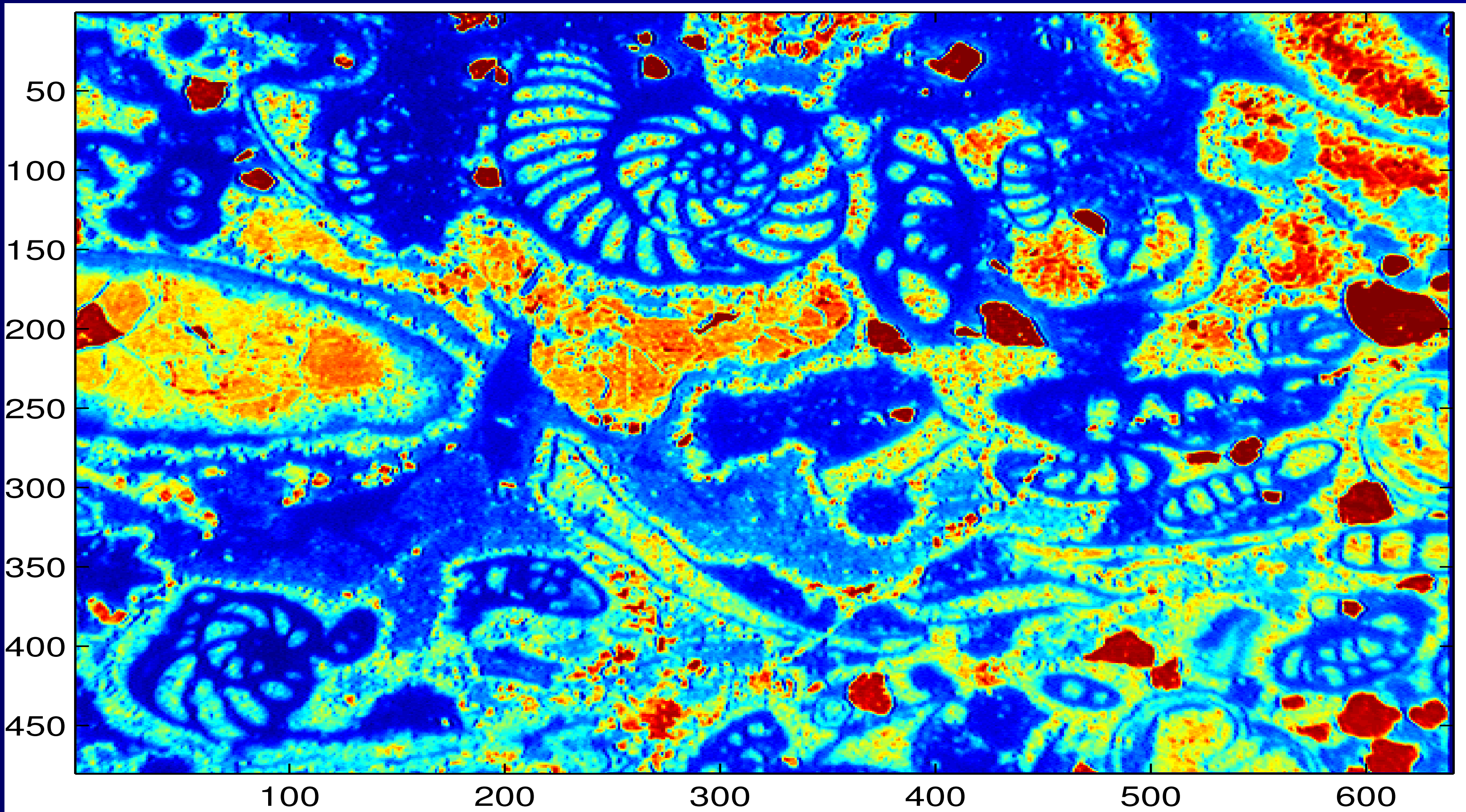


(b) $\tilde{v}$

Another example



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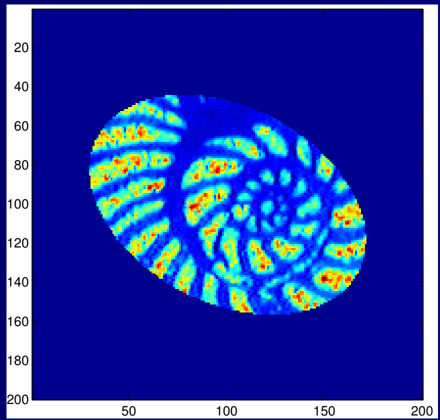
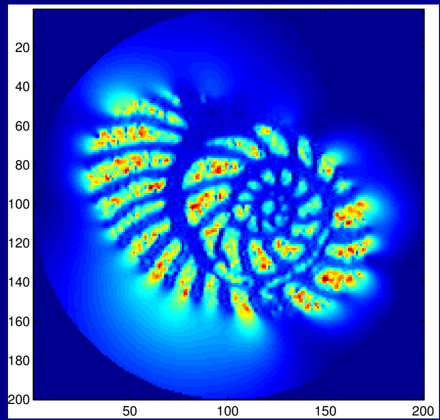
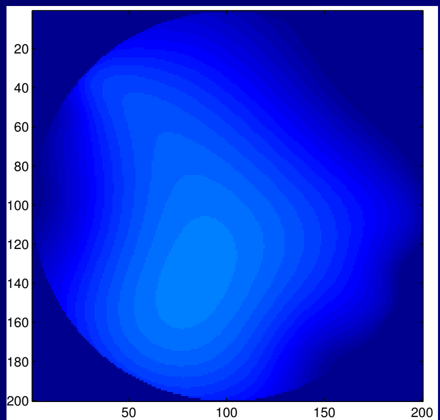


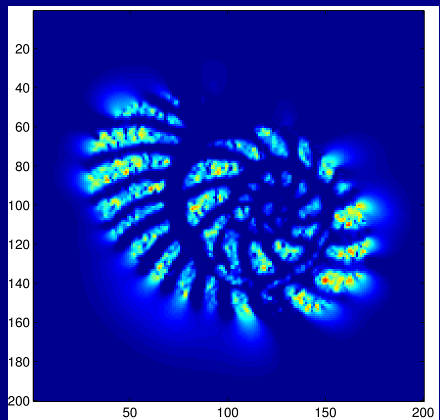
(a) Extracted shell



(b) Extended



(c) u component



(d) $\tilde{v}$ component

Related Previous Work

- Whitney (1934): Smooth extensions of $C^m(\overline{\Omega})$ functions in $\mathbb{R}^n$, and the Whitney decomposition
- Zygmund (1935): Removable discontinuities
- Lanczos (1938,1966): Trigonometric interpolation with linear/polynomial component removal
- Kantrovich & Krylov (1958): Rapidly convergent trigonometric series by polynomial removal
- Briggs (1974): Minimum curvature surface interpolation
- Grimson (1981): Visual surface reconstruction
- Terzopoulos (1983): Multilevel visual surface reconstruction

Related Previous Work . . .

- Leclerc (1989): Image segmentation using MDL and local polynomials
- Wahba (1990): Variational formulation/multivariate splines
- Dimitrov (1996): Polyharmonic functions for quadratures
- Kounchev (1993-2001): Theory of polysplines
- Casas & Torres (1996): Two-stage coding
- Caselles, Morel, & Sbert (1998): Axiomatic interpolation

All of them except Lanczos mainly focus on the u-component. Lanczos explored neither higher dimensions nor multiscale setting.

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PHLTT and its generalizations:

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- Proactive use of boundary information
- Can get faster decay expansion coefficients
- Allow object-oriented image analysis & synthesis
- Variety of applications: segmentation, compression, interpolation, local feature computation, 3D, . . .

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- Anisotropy and many interesting issues
- Reliability of boundary information and noise is an issue

References

- Check http://www.math.ucdavis.edu/˜saito/publications/ periodically for more preprints on this work.
- For preliminary work, see: N. Saito & J.-F. Remy: "A new local sine transform without overlaps: A combination of computational harmonic analysis and PDE," in *Wavelets X* (M. A. Unser, A. Aldroubi, & A. F. Laine, eds.), Proc. SPIE 5207, pp.495–506, 2003.
- See also references therein.

Thank you very much for your attention!