Discrete Geometrical Image Processing using the Contourlet Transform

Minh N. Do

Department of Electrical and Computer Engineering
University of Illinois at Urbana-Champaign

Joint work with **Martin Vetterli** (EPFL and UC Berkeley), **Arthur Cunha**, **Yue Lu**, **Jianping Zhou** (UIUC), and **Duncan Po** (Mathworks).

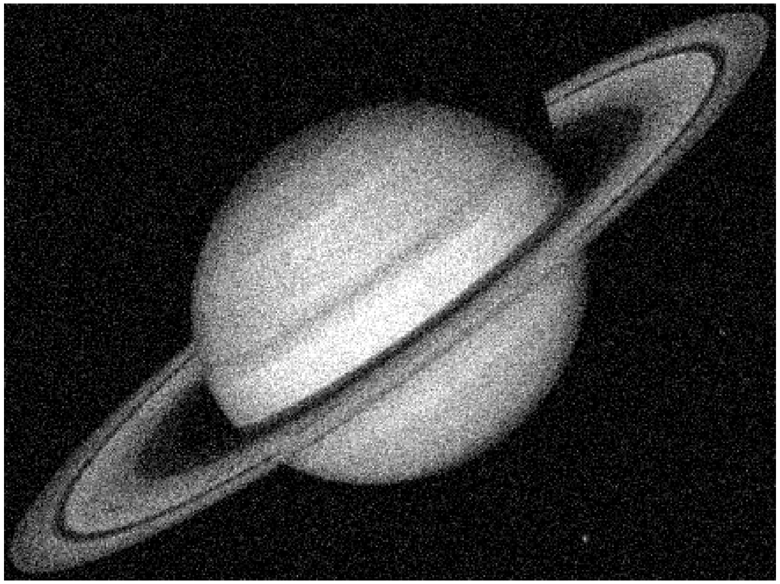
Outline

What Do Image Processors Do for Living?

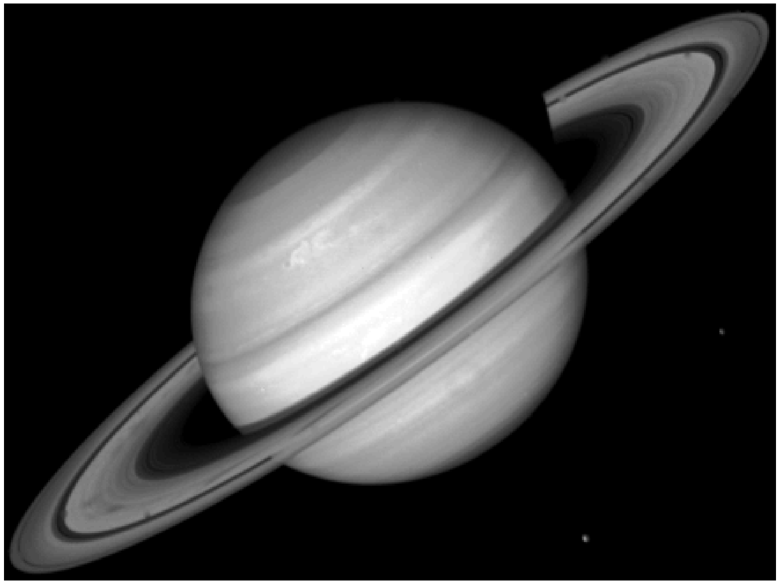
Compression: At 158:1 compression ratio...

What Do Image Processors Do for Living?

Denoising (restoration/filtering)



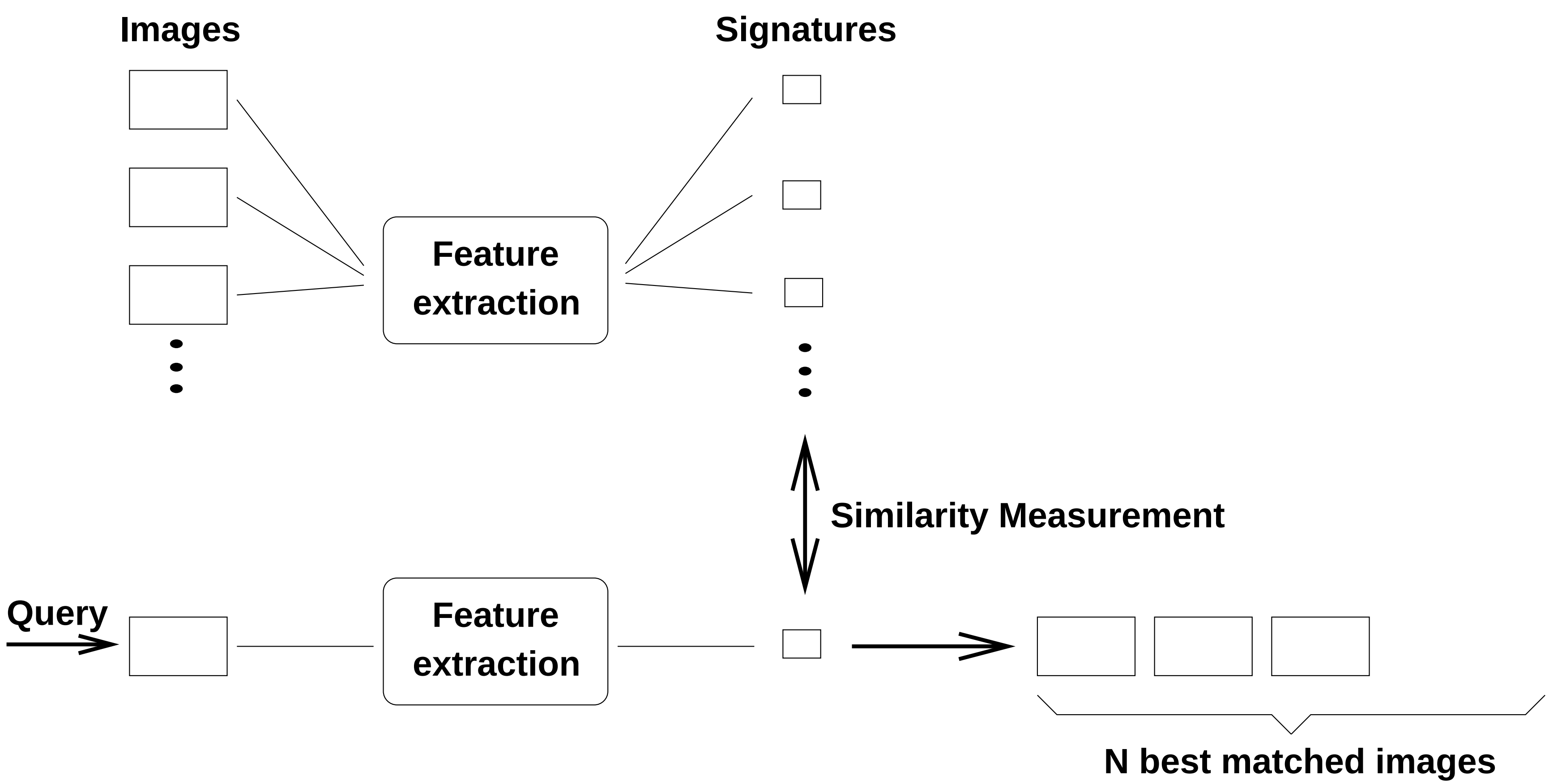
Noisy image



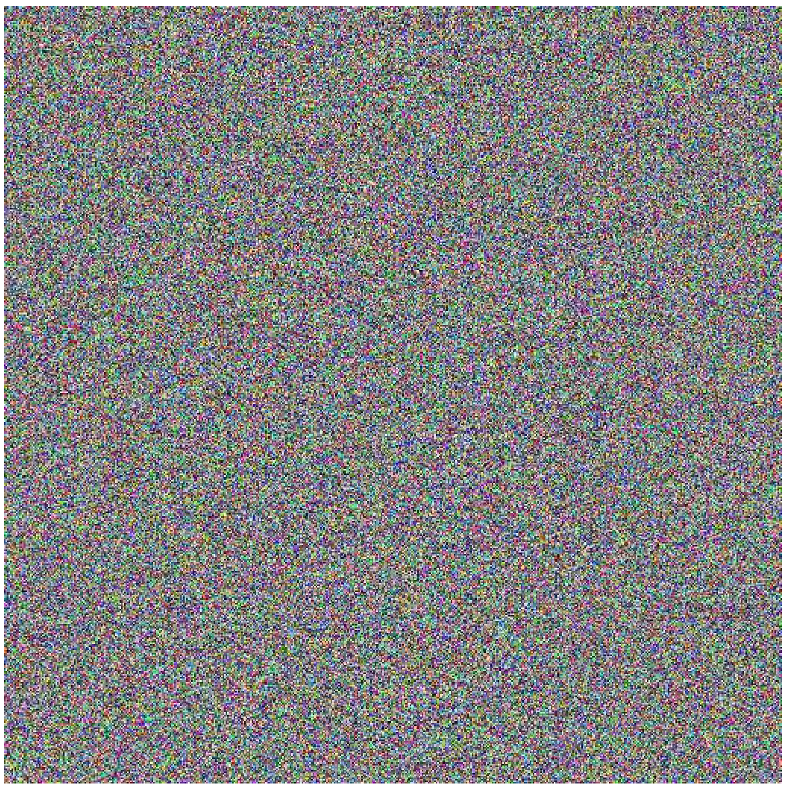
Clean image

What Do Image Processors Do for Living?

Feature extraction (e.g. for content-based image retrieval)



Fundamental Question: Efficient Representation of Visual Information



A random image

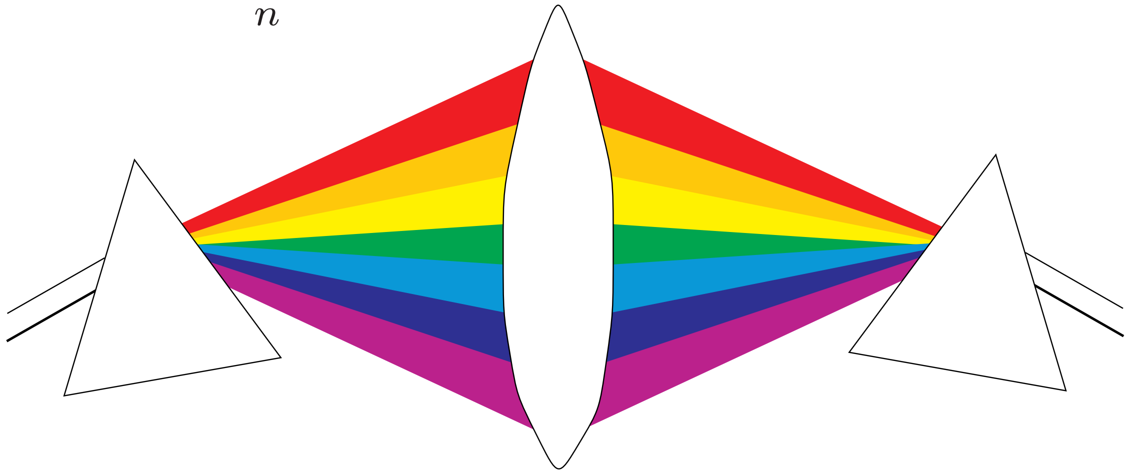
A natural image

Natural images live in a very tiny part of the huge "image space" (e.g. $\mathbb{R}^{512\times512\times3}$)

Mathematical Foundation: Sparse Representations

Fourier, Wavelets... = construction of bases for signal expansions:

$$f = \sum_n c_n \psi_n, \quad \text{where} \quad c_n = \langle f, \psi_n \rangle.$$

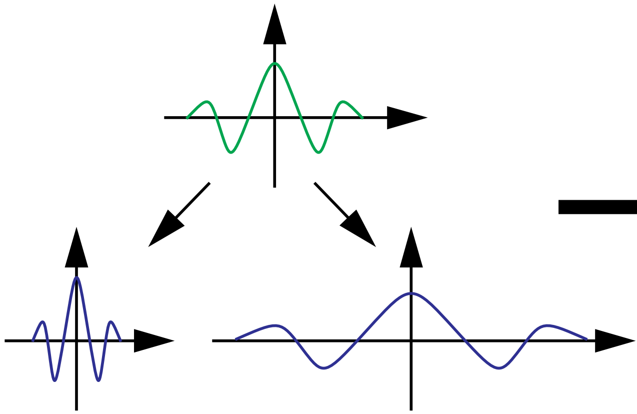


Non-linear approximation:

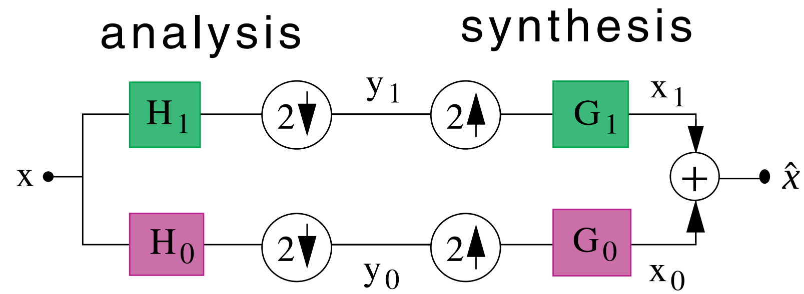
$$\hat{f}_M = \sum_{n \in I_M} c_n \psi_n, \quad \text{where } I_M = \text{indexes of } M \text{ most significant components.}$$

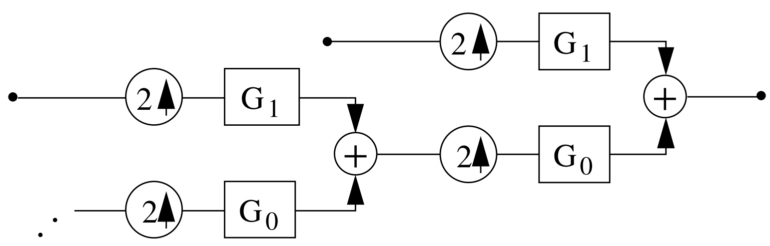
Sparse (efficient) representation: $f \in \mathcal{F}$ can be (nonlinearly) approximated with few components; e.g. $\|f - \hat{f}_M\|_2^2 \sim M^{-\alpha}$.

Wavelets and Filter Banks

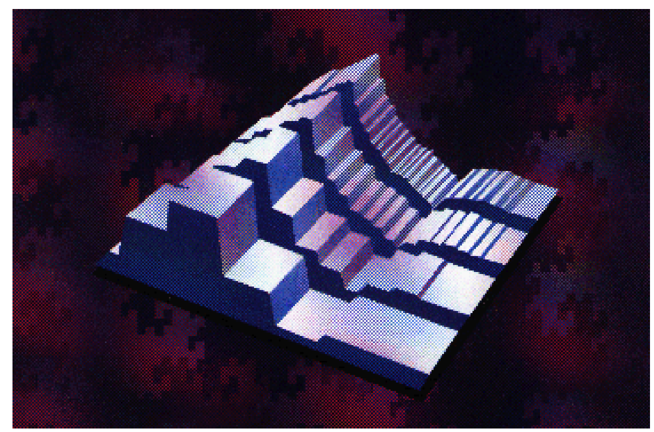


Mallat



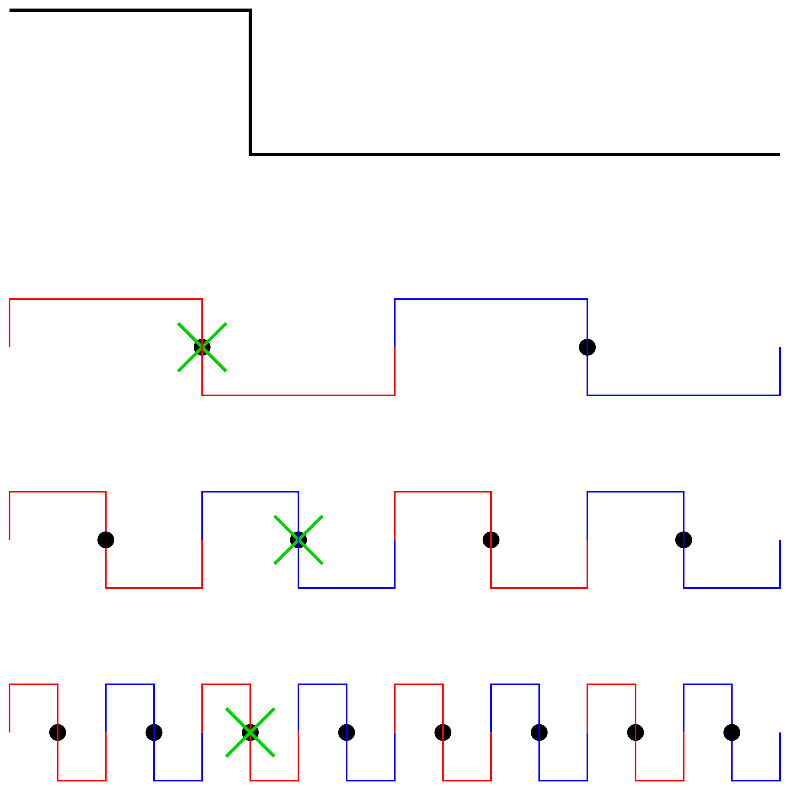


Daubechies



The Success of Wavelets

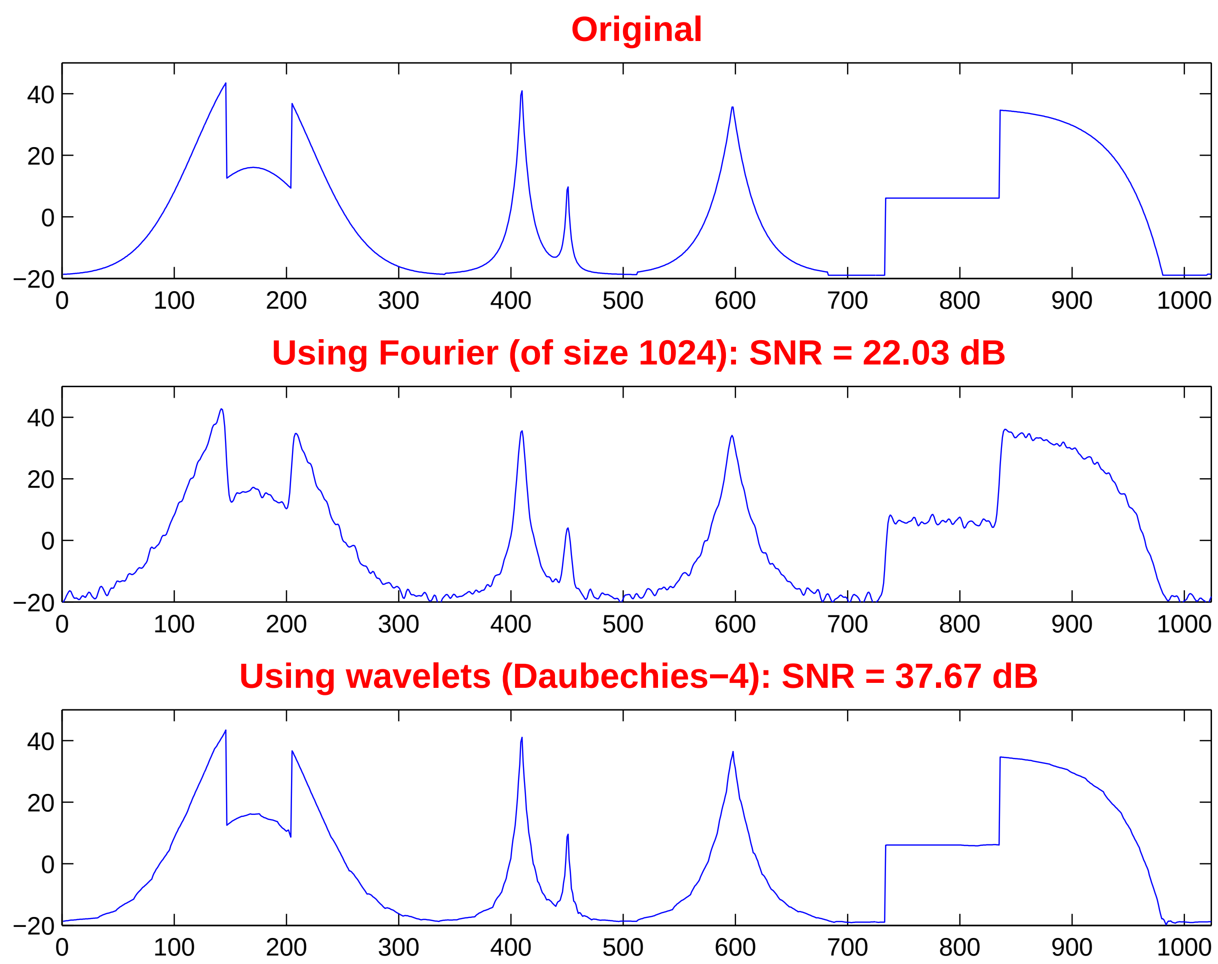
- Wavelets provide a sparse representation for piecewise smooth signals.



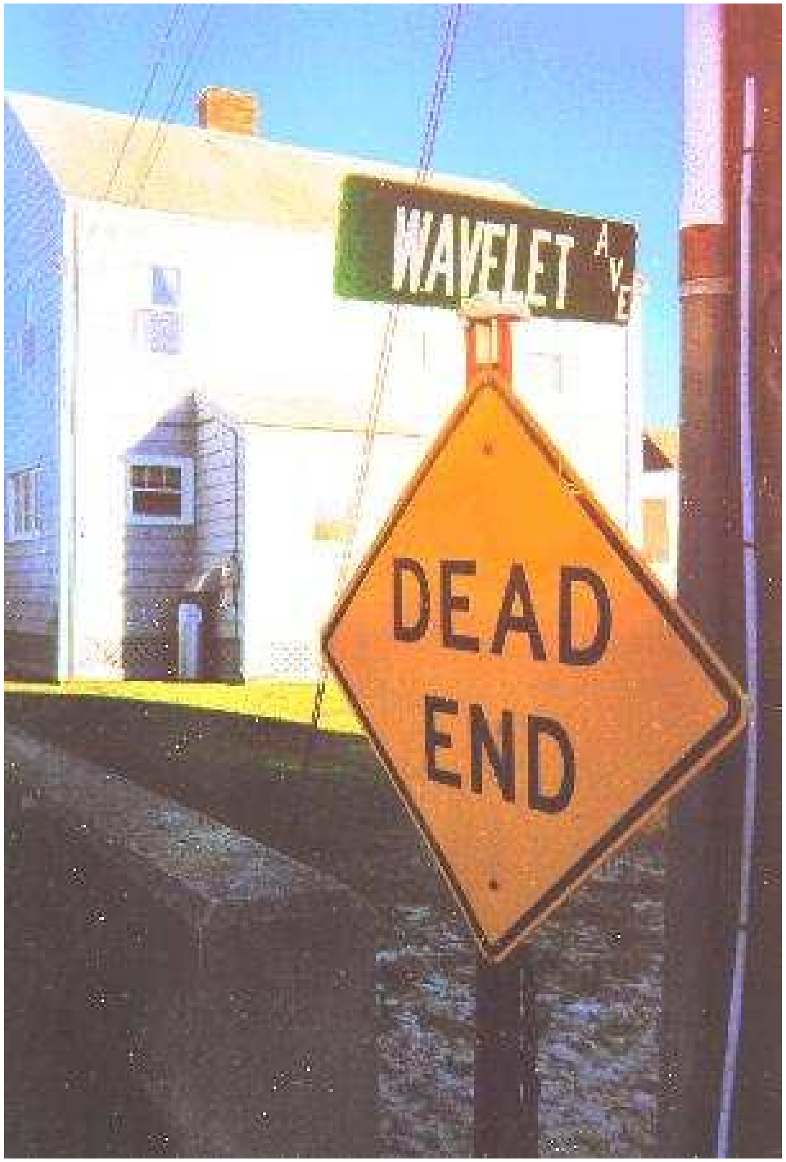
- Multiresolution, tree structures, fast transforms and algorithms, etc.
- Unifying theory $\Rightarrow$ fruitful interaction between different fields.

Fourier vs. Wavelets

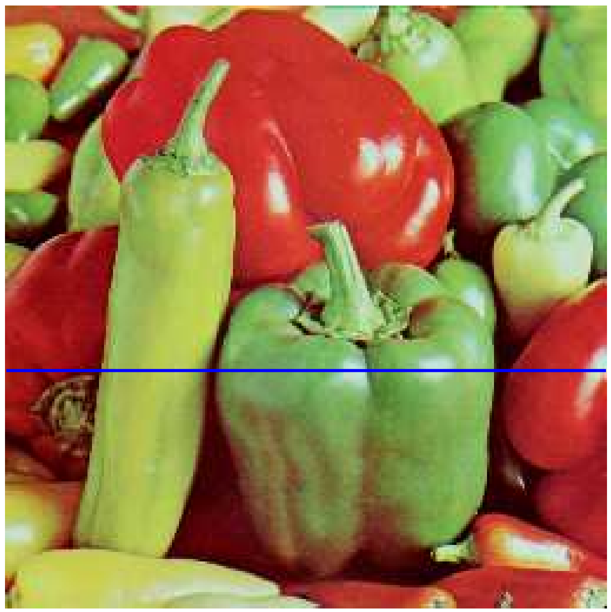
Non-linear approximation: $N = 1024$ data samples; keep $M = 128$ coefficients

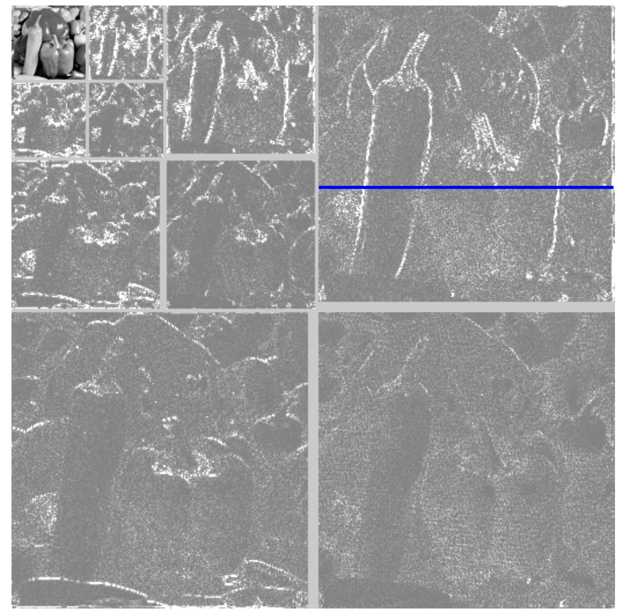


Is This the End of the Story?



Wavelets in 2-D

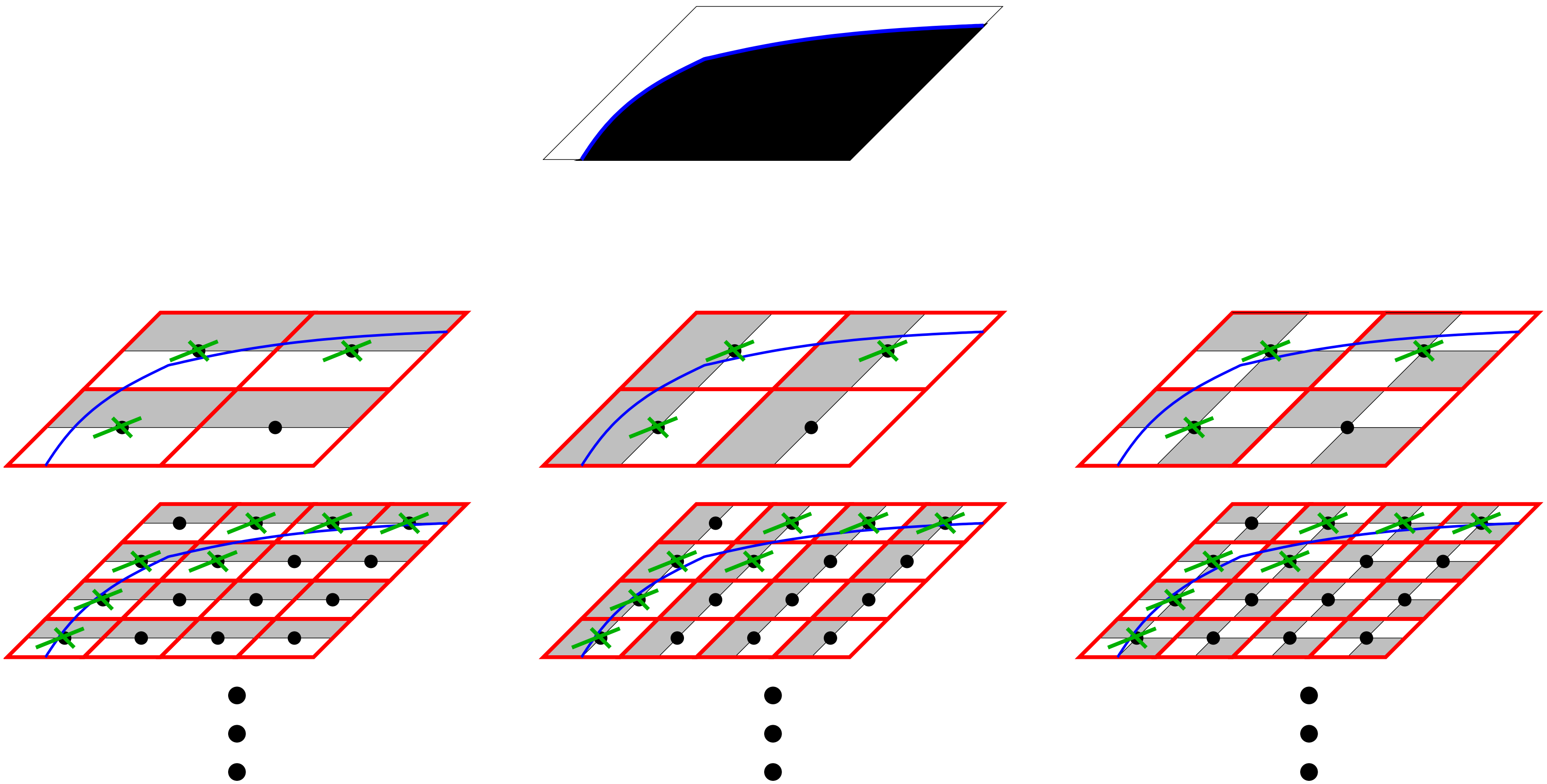




- **In 1-D:** Wavelets are well adapted to abrupt changes or singularities.
- **In 2-D:** Separable wavelets are well adapted to *point*-singularities (only).

  But, there are (mostly) *line*- and *curve*-singularities...

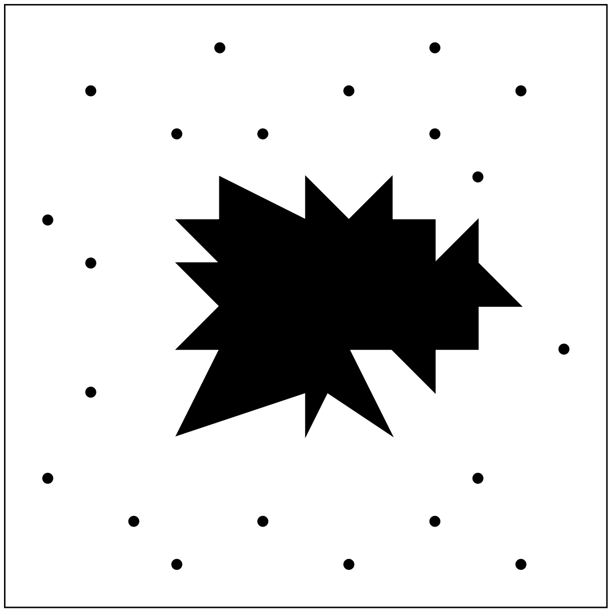
The Failure of Wavelets

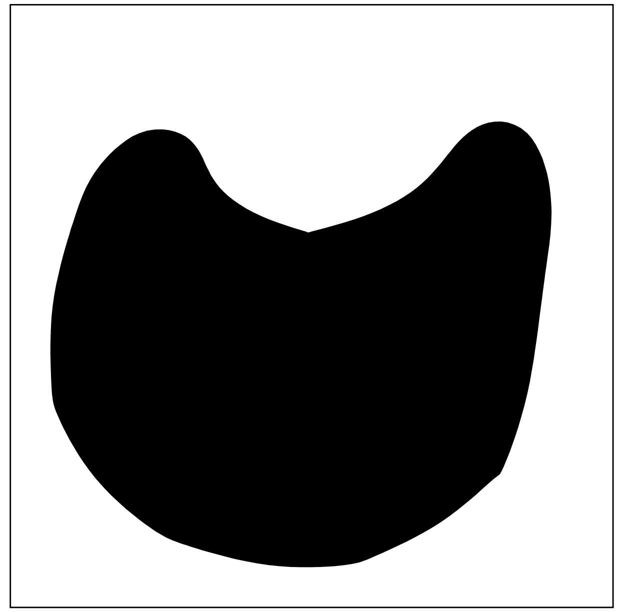


Wavelets fail to capture the geometrical regularity in images.

Edges vs. Contours

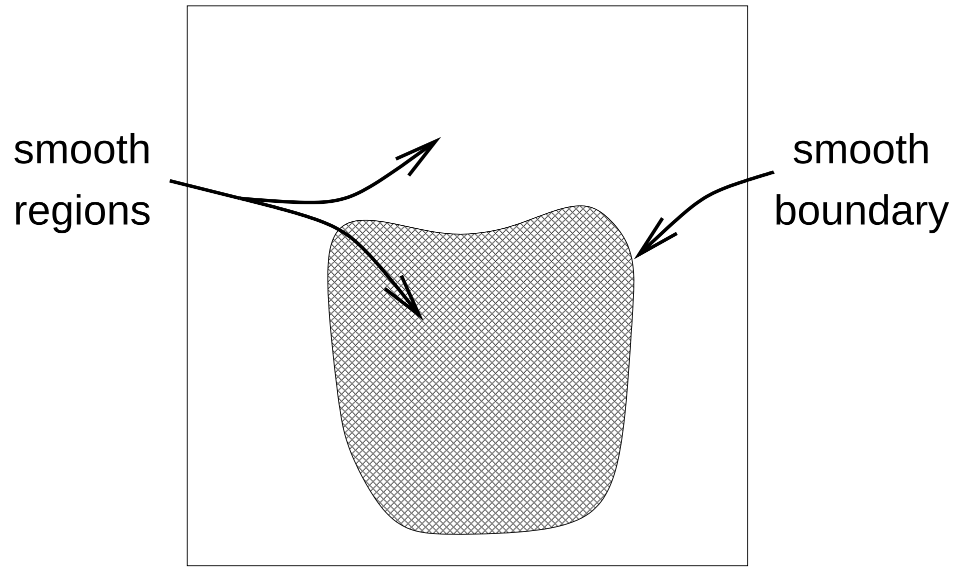
Wavelets cannot "see" the difference between the following two images:





- **Edges**: image points with discontinuity
- **Contours**: edges with localized and regular direction [Zucker et al.]

Goal: Efficient Representation for Typical Images with Smooth Contours

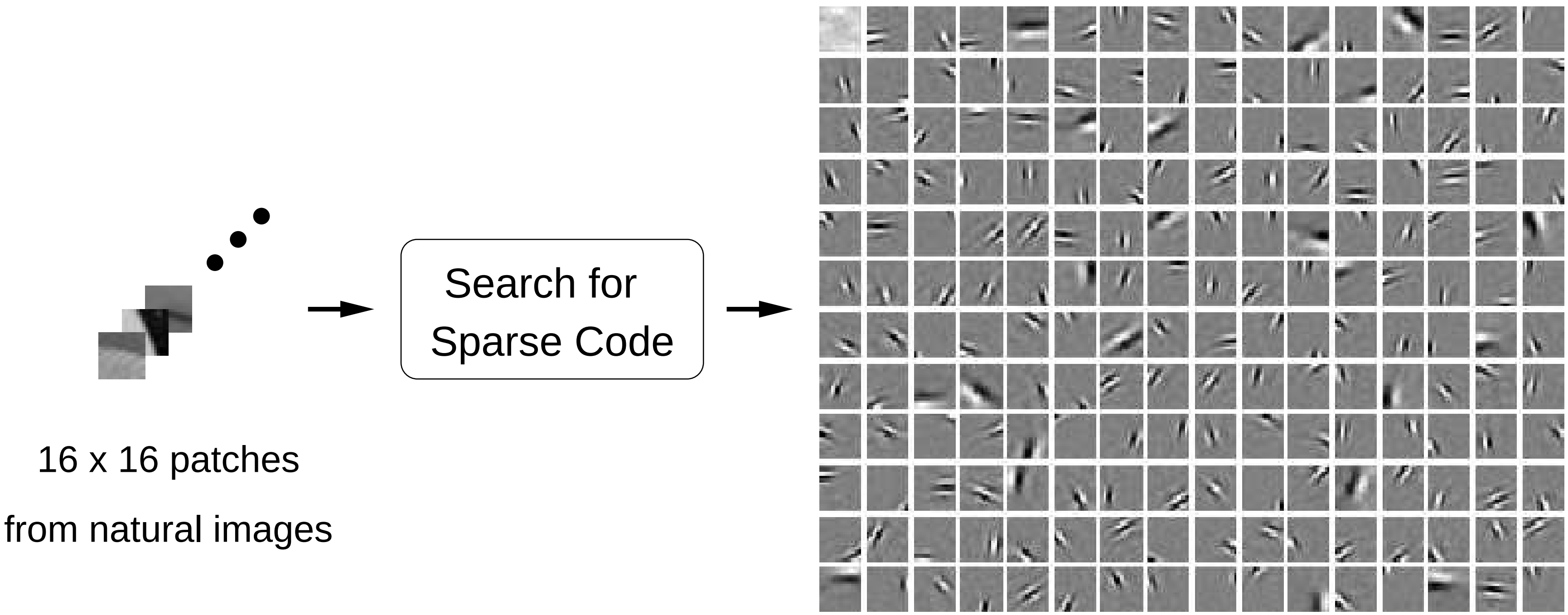


Key: Exploring the intrinsic geometrical structure in natural images.

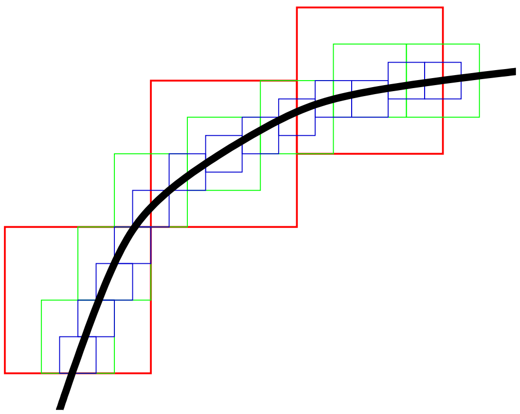
$\Rightarrow$ Action is at the edges!

And What The Nature Tells Us...

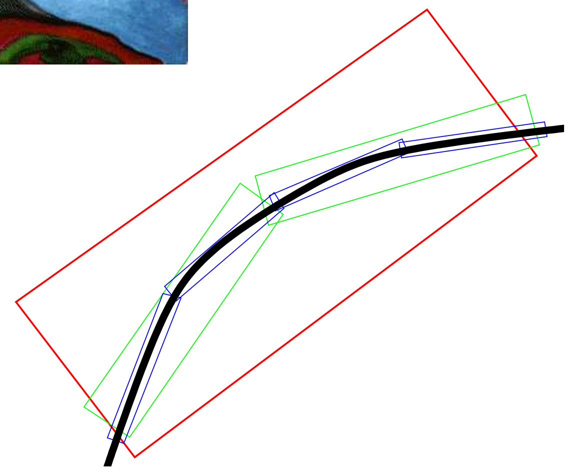
- **Human visual system**:
  - Extremely efficient: 10^7 bits $\longrightarrow$ 20-40 bits (per second).
  - Receptive fields are characterized as localized, multiscale and oriented.
- Sparse components of natural images (Olshausen and Field, 1996):



Wavelet vs. New Scheme



Wavelets



New scheme

For images:

- Wavelet scheme... see edges but not smooth contours.
- New scheme... requires challenging non-separable constructions.

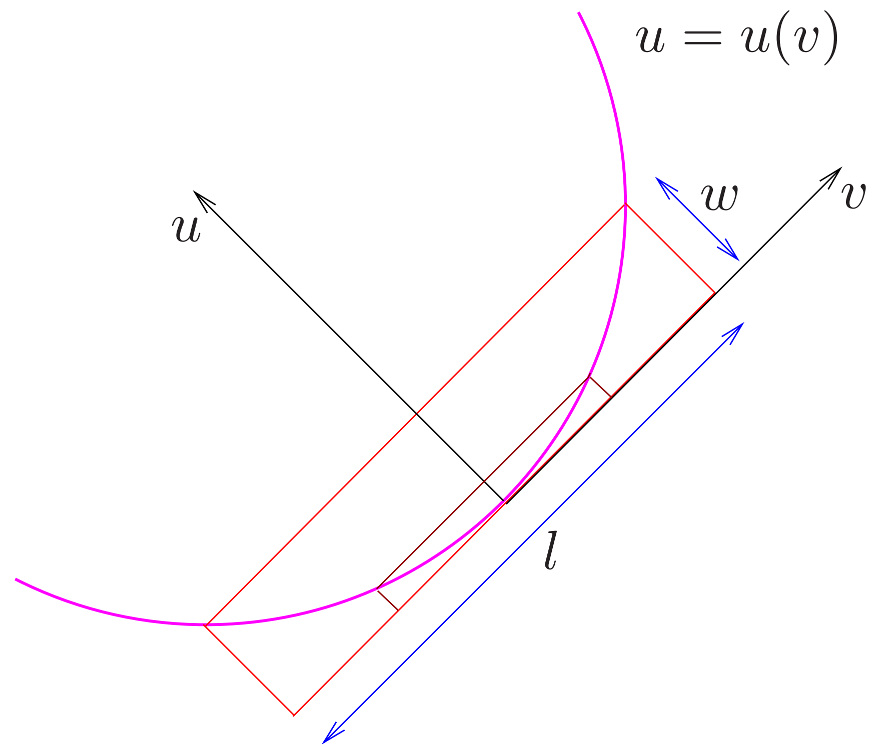
Recent Breakthrough from Harmonic Analysis: Curvelets [Candès and Donoho, 1999]

- Optimal representation for functions in $\mathbb{R}^2$ with curved singularities.

$$\text{For } f \in C^2/C^2: \quad \|f - \hat{f}_M\|_2^2 \sim (\log M)^3 M^{-2}$$

- **Key idea:** parabolic scaling relation for C^2 curves:

$$width \propto length^2$$



“Wish List” for New Image Representations

- Multiresolution ... successive refinement
- Localization ... both space and frequency
- Critical sampling ... correct joint sampling
- Directionality ... more directions
- Anisotropy ... more shapes

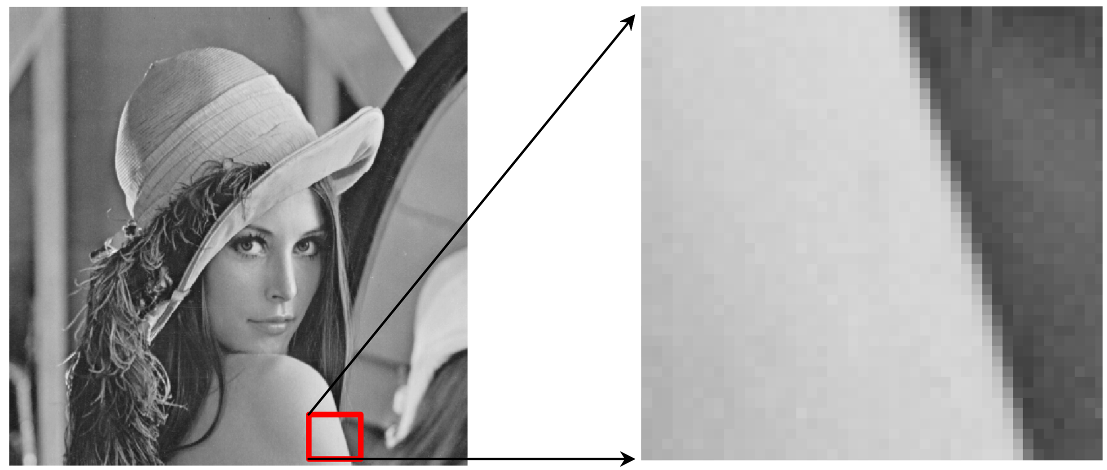
Our emphasis is on discrete framework that leads to algorithmic implementations.

Outline

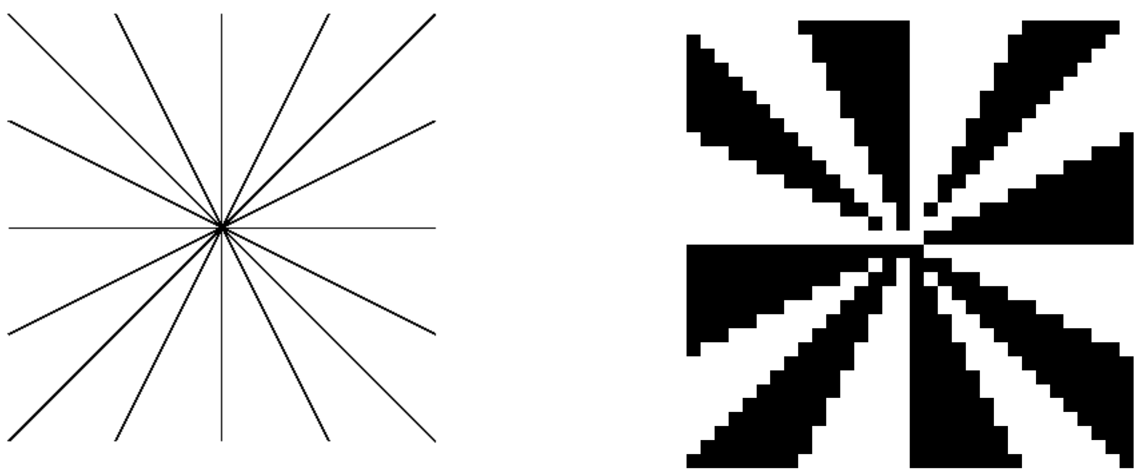
1. Motivation
2. Discrete-domain construction using filter banks
3. Contourlets and directional multiresolution analysis
4. Contourlet approximation and directional vanishing moments
5. Applications

Challenge: Being Digital!

Pixelization:



Digital directions:



Proposed Computational Framework: Contourlets

In a nutshell: contourlet transform is an efficient directional multiresolution expansion that is digital friendly!

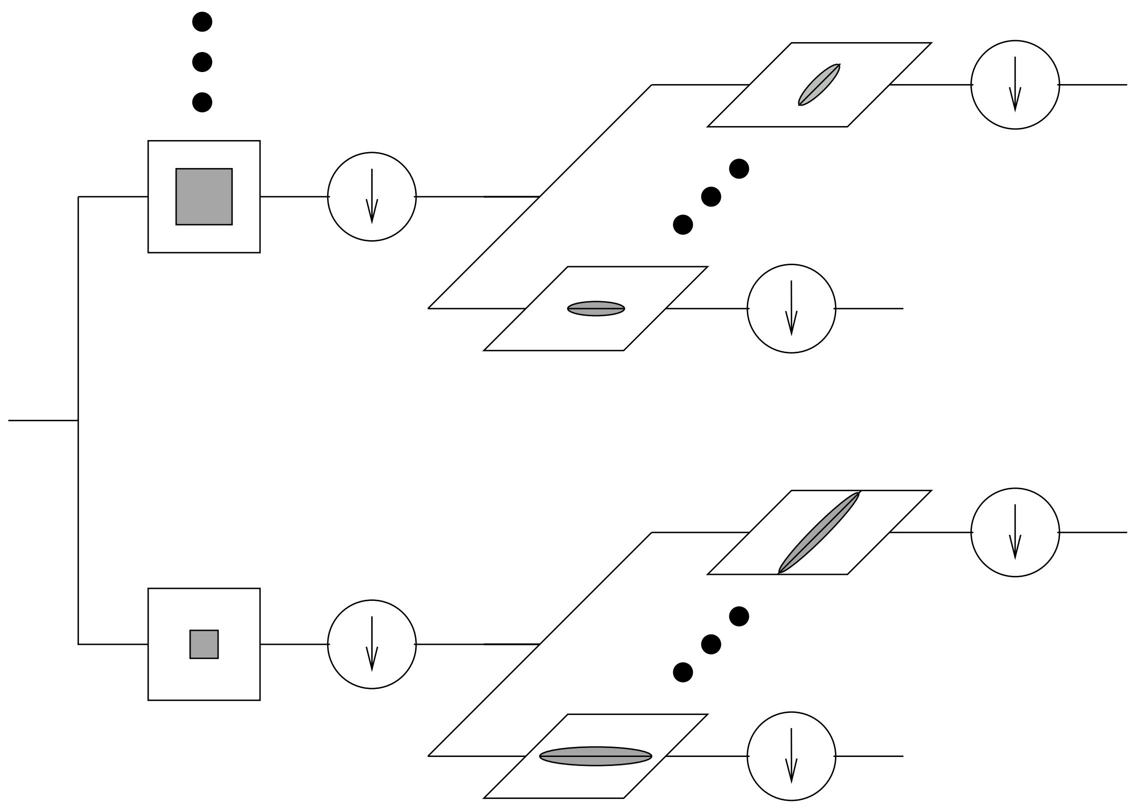
$\boxed{\textbf{contourlets} = \text{multiscale, local and directional contour segments}}$

- Contourlets are constructed via filter banks and can be viewed as an extension of wavelets with directionality
  $\Rightarrow$ Inherit the rich wavelet theory and algorithms.

- Starts with a discrete-domain construction that is amenable to efficient algorithms, and then investigates its convergence to a continuous-domain expansion.

- The expansion is defined on rectangular grids
  $\Rightarrow$ Seamless transition between the continuous and discrete worlds.

Discrete-Domain Construction using Filter Banks

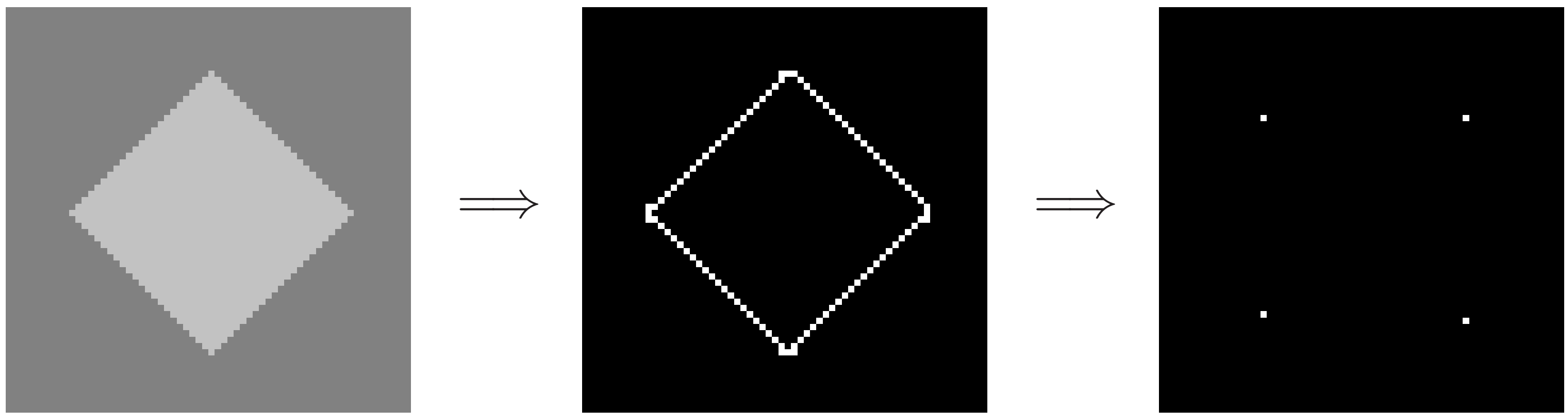


Idea: Multiscale and Directional Decomposition

- Multiscale step: capture point discontinuities, followed by...
- Directional step: link point discontinuities into linear structures.

Analogy: Hough Transform in Computer Vision

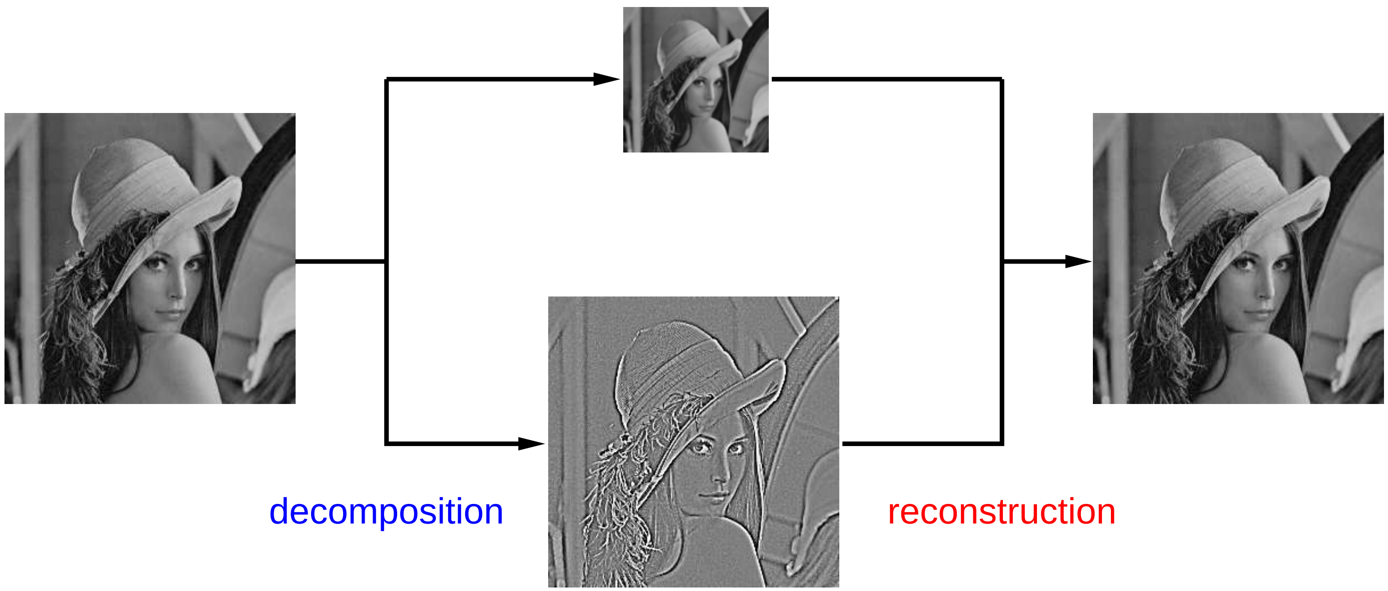
Input image Edge image “Hough” image



Challenges:

- Perfect reconstruction.
- Fixed transform with low redundancy.
- Sparse representation for images with smooth contours.

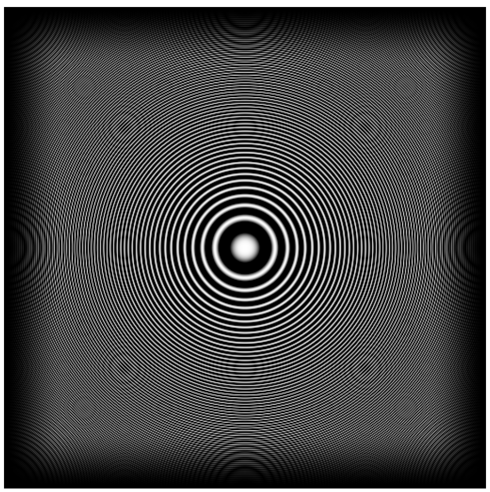
Multiscale Decomposition using Laplacian Pyramids

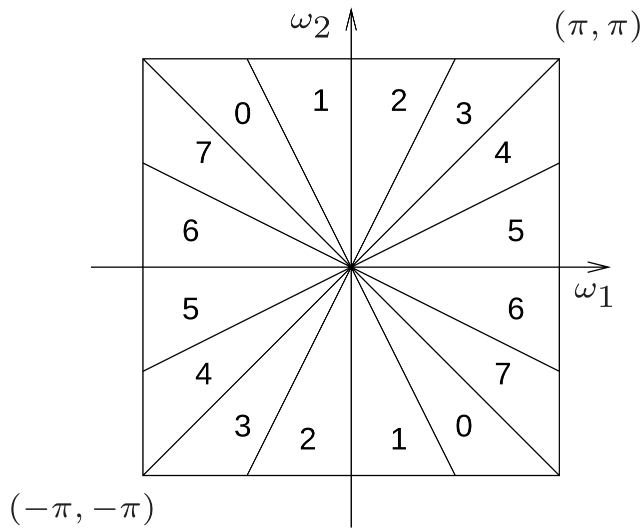


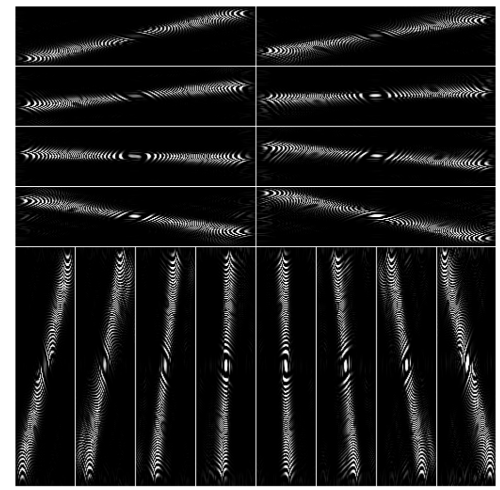
- **Reason:** avoid "frequency scrambling" due to $(\downarrow)$ of the HP channel.
- Laplacian pyramid as a frame operator $\rightarrow$ tight frame exists.
- New reconstruction: efficient filter bank for *dual* frame (pseudo-inverse).

Directional Filter Banks (DFB)

- **Feature:** division of 2-D spectrum into fine slices using tree-structured filter banks.







- **Background:** Bamberger and Smith ('92) cleverly used quincunx FB's, modulation and shearing.

- **We propose:**
  - a simplified DFB with fan FB's and shearing
  - using DFB to construct directional bases

Multidimensional Sampling

Example. Downsampling by $\boldsymbol{M}$: $\quad x_d[\boldsymbol{n}] = x[\boldsymbol{M}\boldsymbol{n}]$

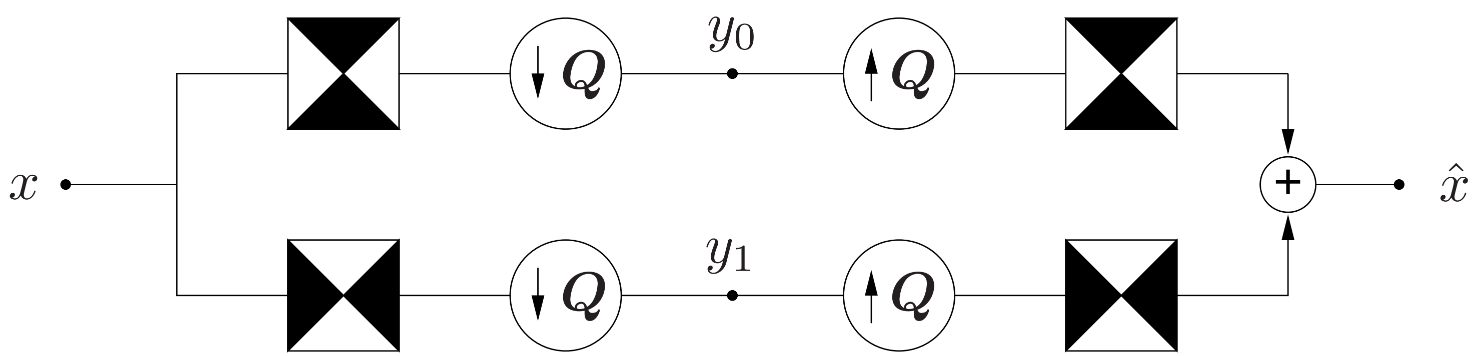
$\downarrow \boldsymbol{R}_3$

$\downarrow \boldsymbol{Q}_1$

$$\boldsymbol{R}_3 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \qquad \boldsymbol{Q}_1 = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \boldsymbol{R}_0 \boldsymbol{D}_0 \boldsymbol{R}_3$$

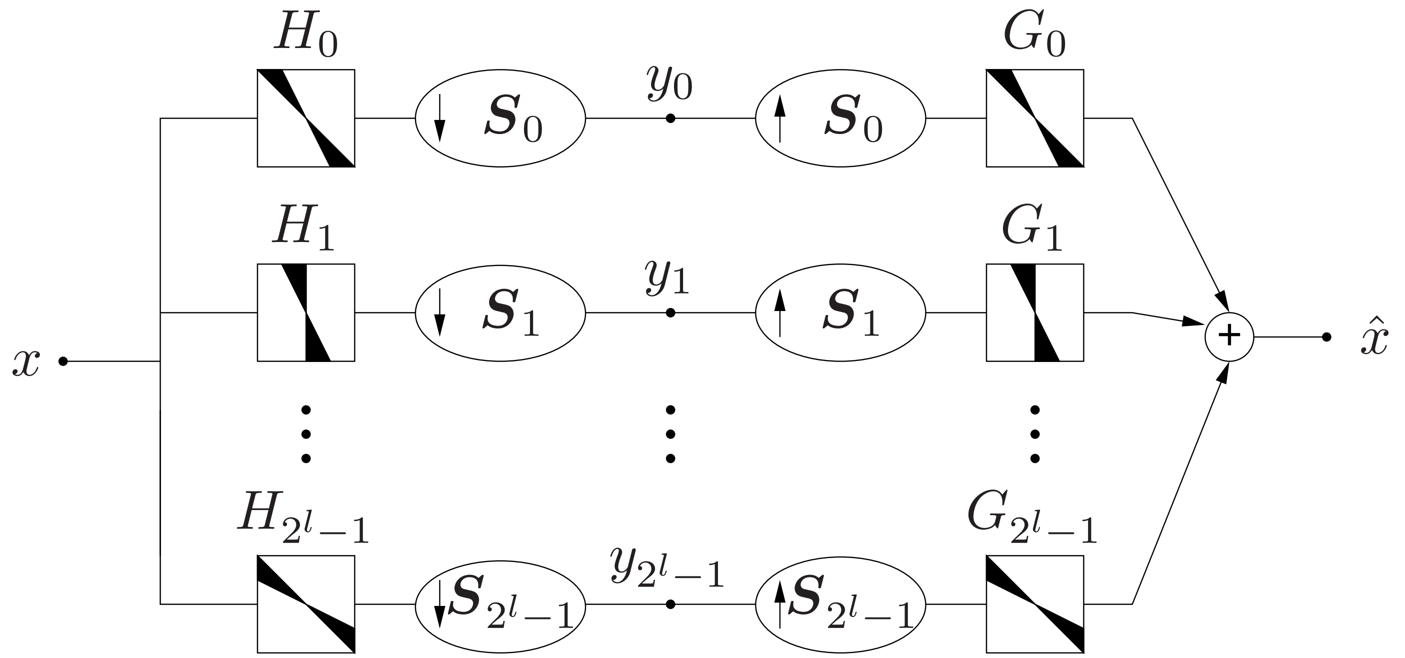
Simplified DFB: Two Building Blocks

- Frequency splitting by the quincunx filter banks (Vetterli'84).



- Shearing by resampling

Multichannel View of the Directional Filter Bank



Use two *separable* sampling matrices:

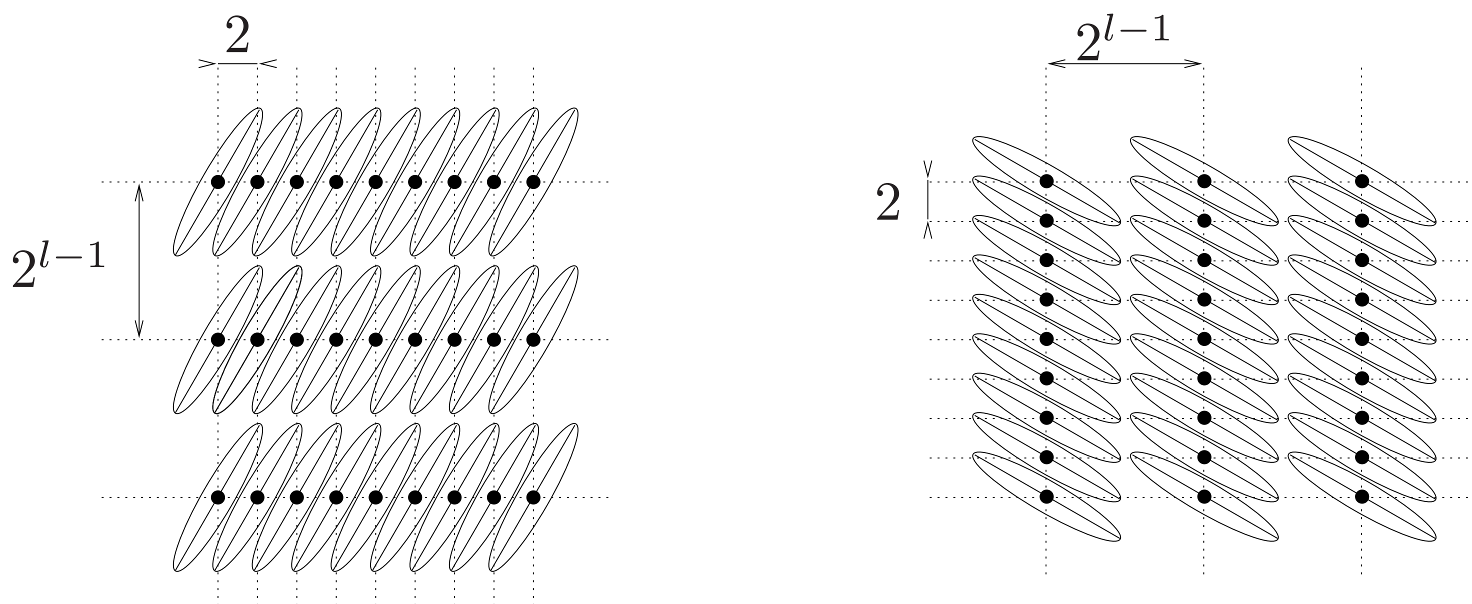
$$\boldsymbol{S}_k = \begin{cases} \begin{bmatrix} 2^{l-1} & 0 \\ 0 & 2 \end{bmatrix} & 0 \le k < 2^{l-1} \quad (\text{“near horizontal” direction}) \\ \begin{bmatrix} 2 & 0 \\ 0 & 2^{l-1} \end{bmatrix} & 2^{l-1} \le k < 2^l \quad (\text{“near vertical” direction}) \end{cases}$$

General Bases from the DFB

An l-levels DFB creates a local directional basis of $l^2(\mathbb{Z}^2)$:

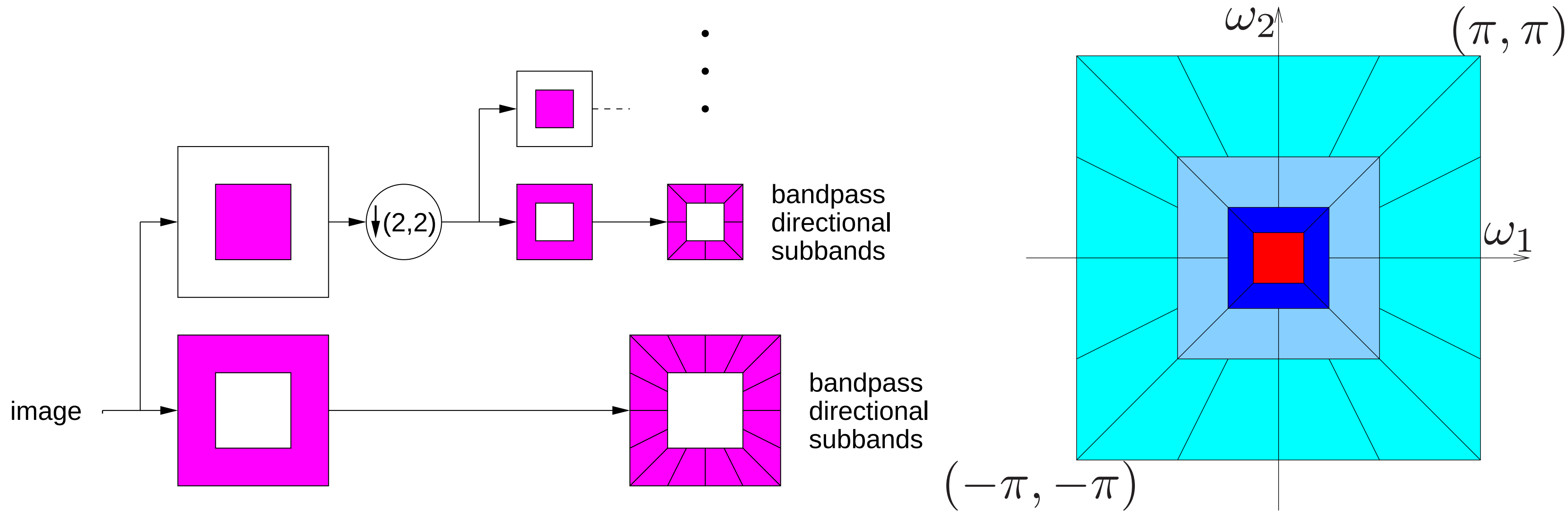
$$\left\{ g_k^{(l)}[\cdot - S_k^{(l)} n] \right\}_{0 \le k < 2^l,\ n \in \mathbb{Z}^2}$$

- $G_k^{(l)}$ are directional filters:
- Sampling lattices (spatial tiling):



Discrete Contourlet Transform

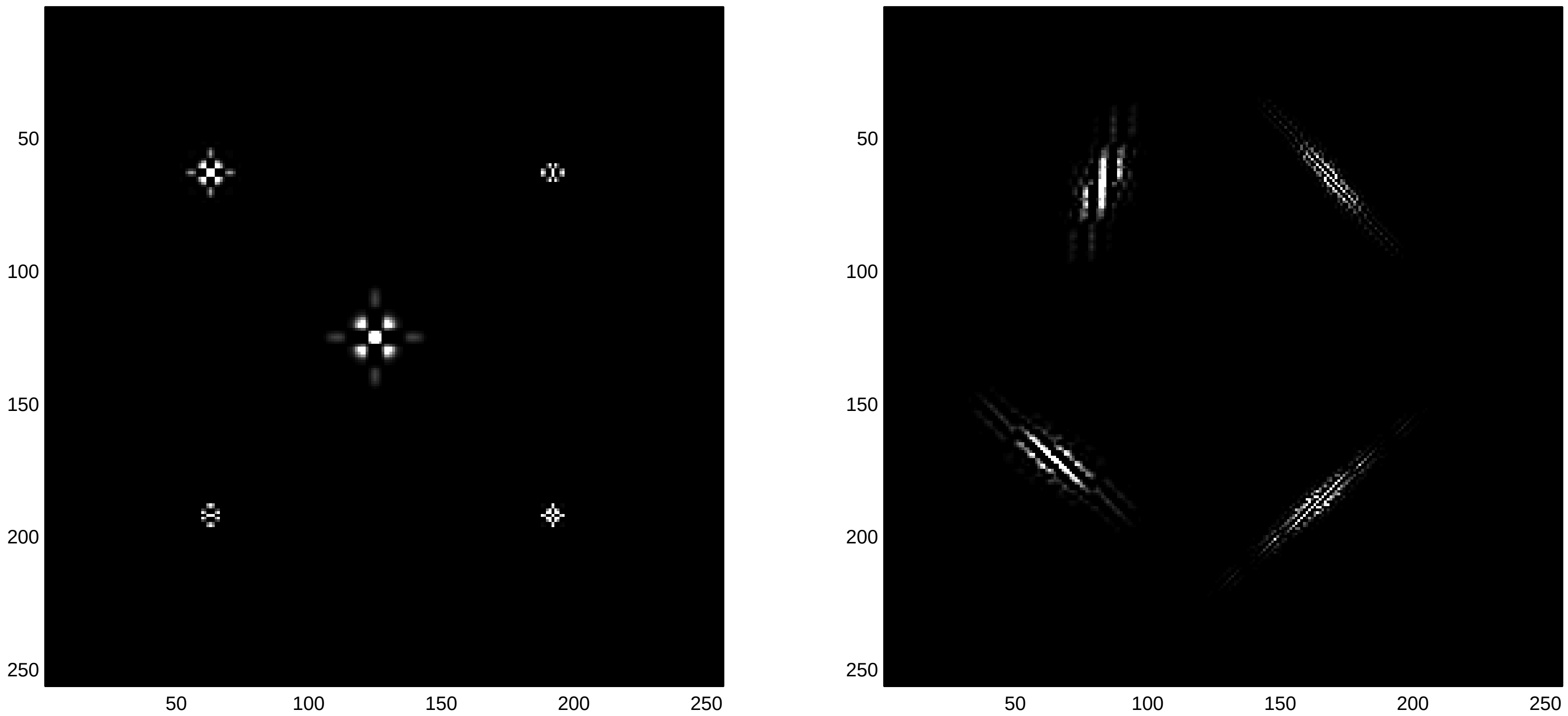
Combination of the Laplacian pyramid (multiscale) and directional filter banks (multidirection)



Properties:

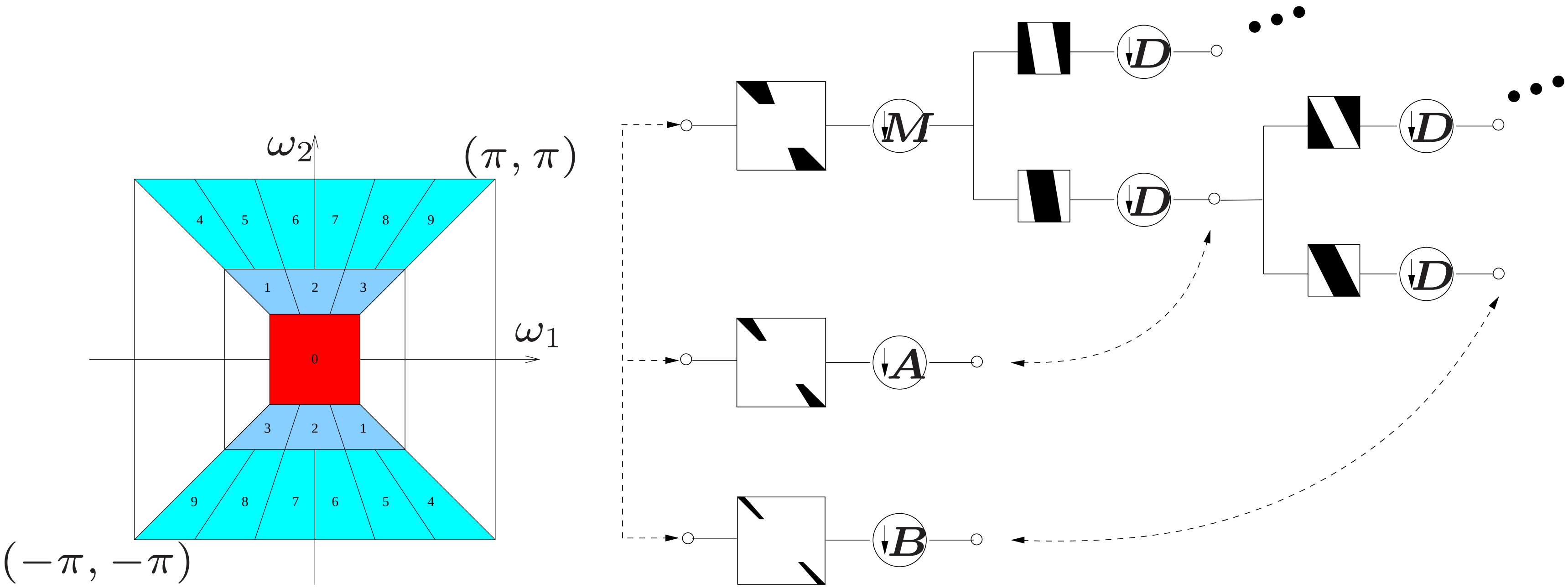
+ Flexible multiscale and directional representation for images (can have different number of directions at each scale!)
+ Tight frame with small redundancy ($< 33\%$)
+ Computational complexity: $O(N)$ for N pixels thanks to the iterated filter bank structures

Wavelets vs. Contourlets



Examples of Discrete Contourlet Transform

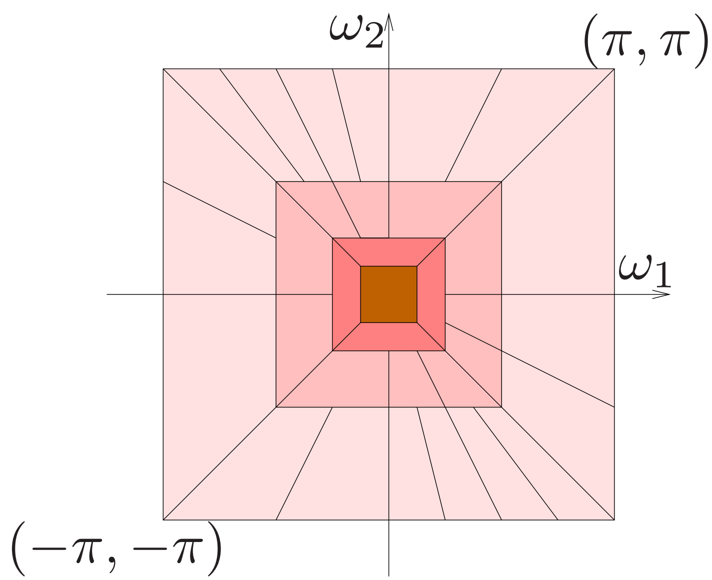
Critically Sampled (CRISP) Contourlet Transform [LuD:03]



- Directional bandpass: 3×2^n $(n = 1, 2, \ldots)$ directions at each level.
- Refinement of directionality is achieved via iteration of two filter banks.

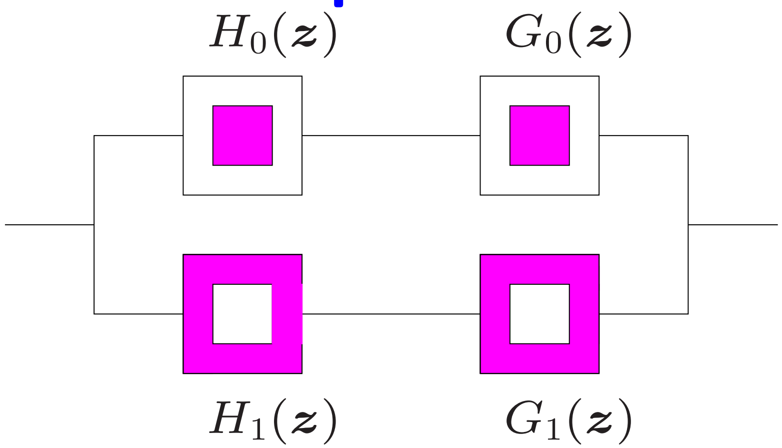
Contourlet Packets

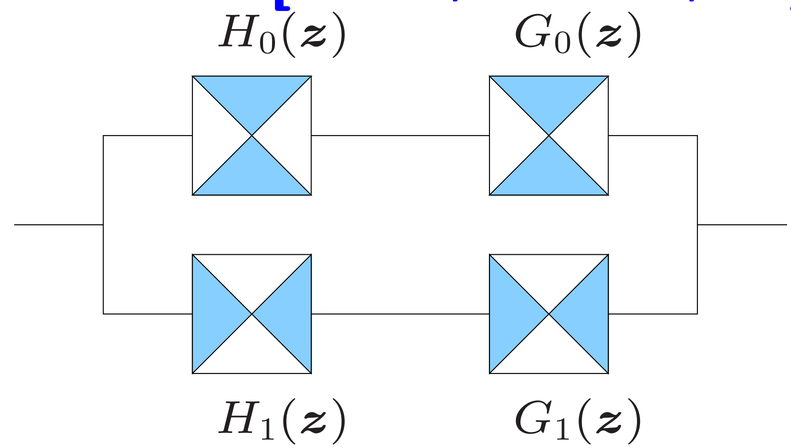
- Adaptive scheme to select the "best" tree for directional decomposition.



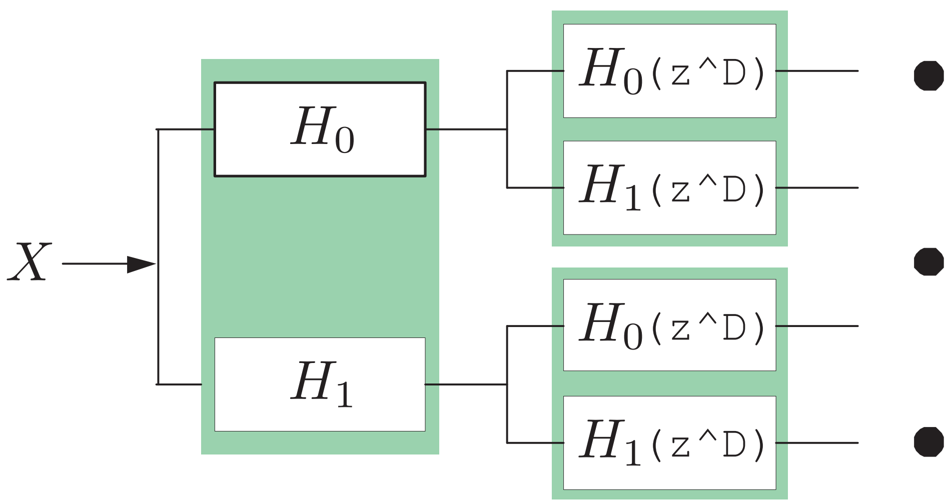
- Contourlet packets: obtained by altering the depth of the DFB decomposition tree at different scales and orientations
  - Allow different angular resolution at different scale and direction
  - Rich set of contourlets with variety of support sizes and aspect ratios
  - Include wavelet(-like) transform

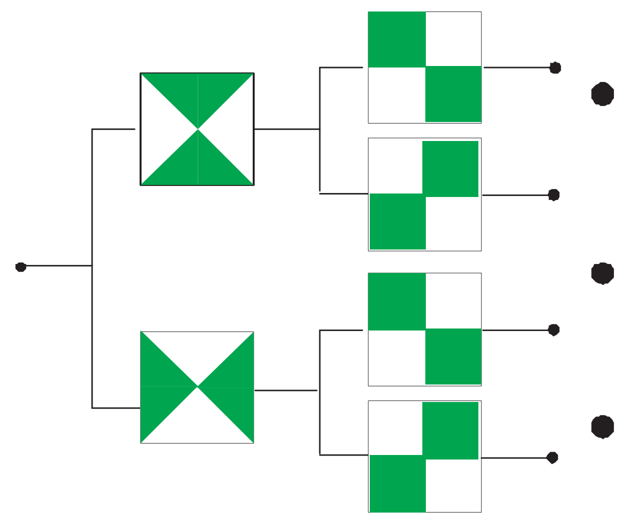
Nonsubsampled Contourlet Transform [Zhou, Cunha, ...]





Less stringent filter bank condition → design better filters.

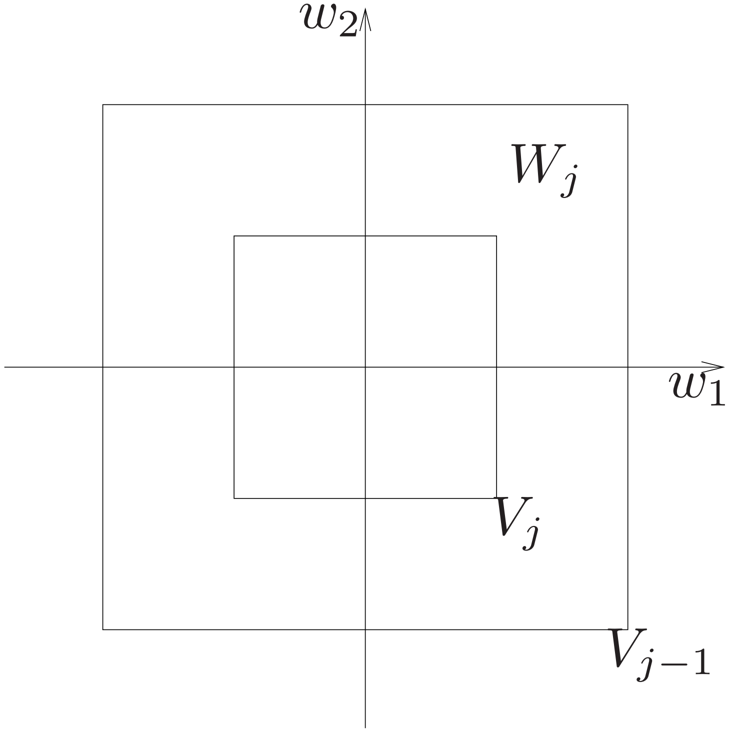




Key: “à trous” algorithm = efficient filtering “with holes” using the equivalent sampling matrices

Outline

Multiresolution Analysis



$$V_{j-1} = V_j \oplus W_j,$$
$$L^2(\mathbb{R}^2) = \bigoplus_{j\in\mathbb{Z}} W_j.$$

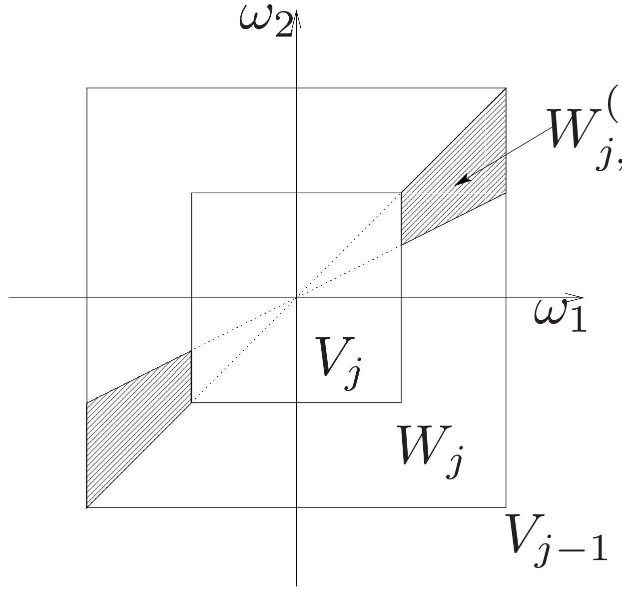
V_j has an orthogonal basis $\{\phi_{j,\boldsymbol{n}}\}_{\boldsymbol{n}\in\mathbb{Z}^2}$, where

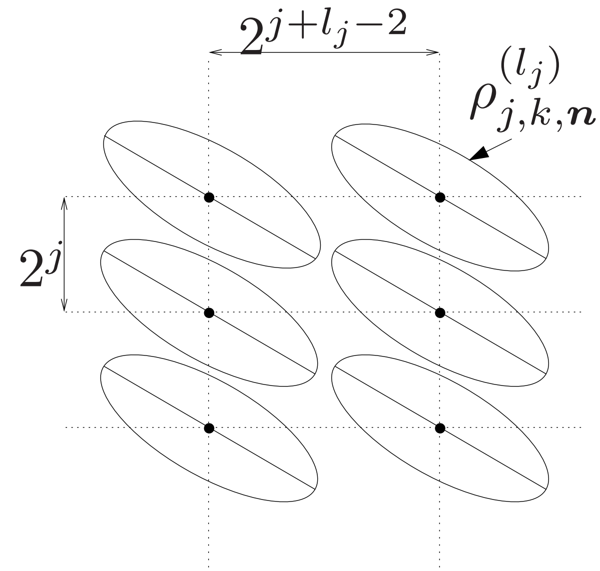
$$\phi_{j,\boldsymbol{n}}(t) = 2^{-j}\phi(2^{-j}\boldsymbol{t} - \boldsymbol{n}).$$

W_j has a tight frame $\{\mu_{j-1,\boldsymbol{n}}\}_{\boldsymbol{n}\in\mathbb{Z}^2}$ where

$$\mu_{j-1,2\boldsymbol{n}+\boldsymbol{k}_i} = \psi^{(i)}_{j,\boldsymbol{n}}, \quad i = 0,\ldots,3.$$

Directional Multiresolution Analysis





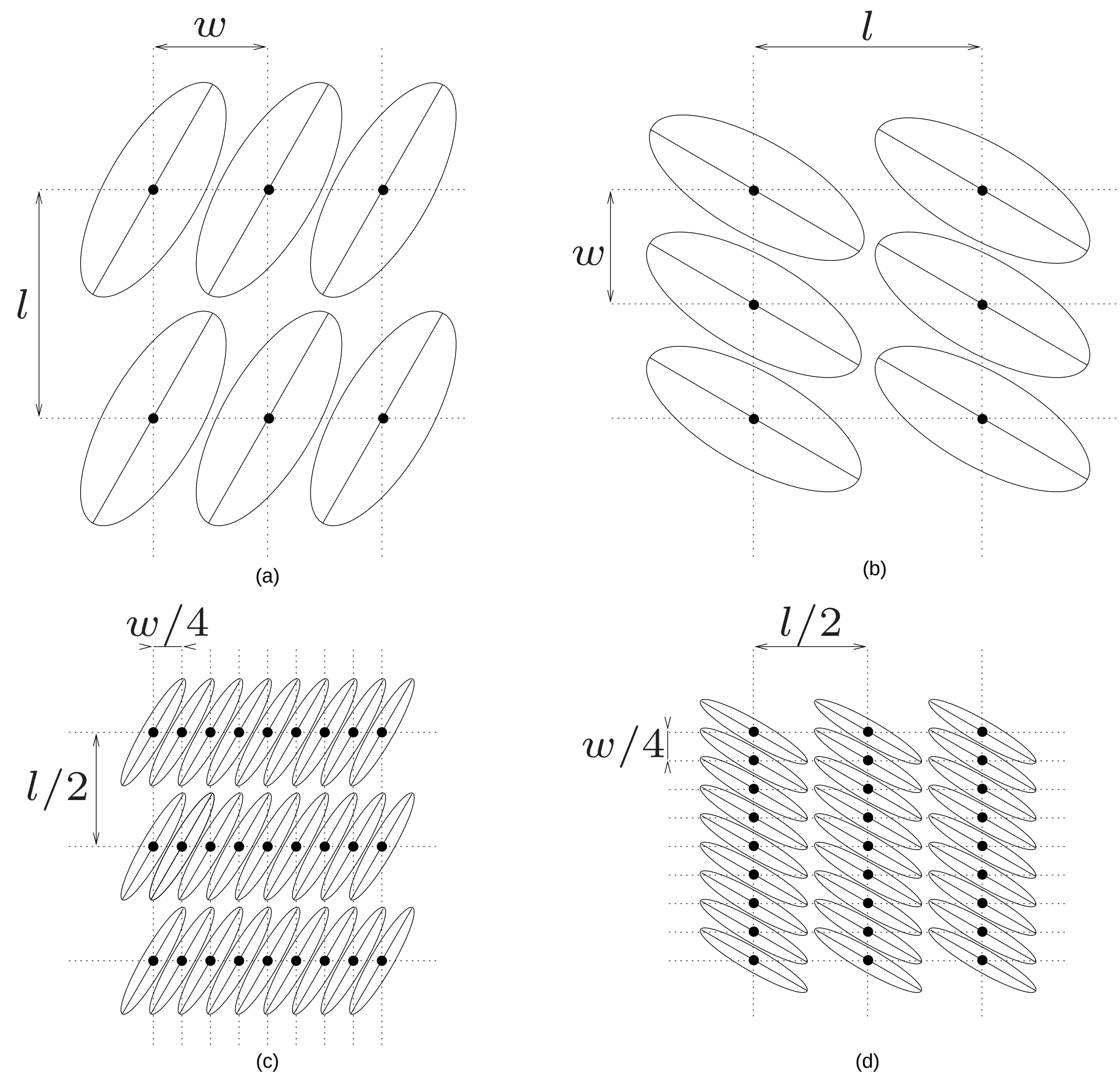
$W_j = \bigoplus_{k=0}^{2^{l_j}-1} W_{j,k}^{(l_j)}$

$W_{j,k}^{(l_j)}$ has a tight frame $\left\{\rho_{j,k,\boldsymbol{n}}^{(l)}\right\}_{\boldsymbol{n}\in\mathbb{Z}^2}$ where

$$\rho_{j,k,\boldsymbol{n}}^{(l)}(\boldsymbol{t}) = \sum_{\boldsymbol{m}\in\mathbb{Z}^2} \underbrace{g_k^{(l)}[\boldsymbol{m} - \boldsymbol{S}_k^{(l)}\boldsymbol{n}]}_{\text{DFB basis}} \underbrace{\mu_{j-1,\boldsymbol{m}}(\boldsymbol{t})}_{\text{LP frame}} = \rho_{j,k}^{(l)}(\boldsymbol{t} - 2^{j-1}\boldsymbol{S}_k^{(l)}\boldsymbol{n}).$$

Theorem (Contourlet Frames) [DoV:03].
$\left\{\rho_{j,k,\boldsymbol{n}}^{(l_j)}\right\}_{j\in\mathbb{Z},\ 0\le k<2^{l_j},\ \boldsymbol{n}\in\mathbb{Z}^2}$ is a tight frame of $L^2(\mathbb{R}^2)$ for finite l_j.

Sampling Grids of Contourlets



Connection between Continuous and Discrete Domains

Theorem [DoV:04] Suppose $x[\boldsymbol{n}] = \langle f, \phi_{L,\boldsymbol{n}} \rangle$, for an $f \in L^2(\mathbb{R}^2)$, and

$$x \xrightarrow{\text{discrete contourlet transform}} \left(a_J,\ d_{j,k}^{(l_j)}\right)_{j=1,\dots,J;\ k=0,\dots,2^{l_j}-1}$$

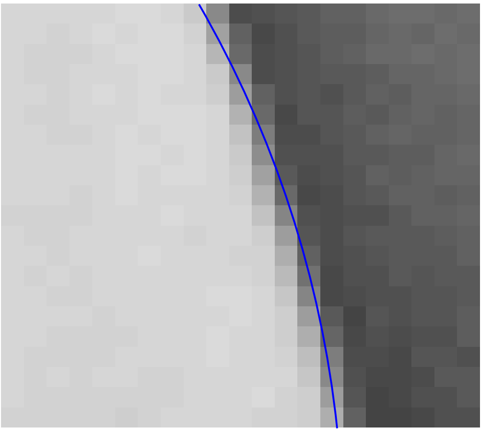
Then

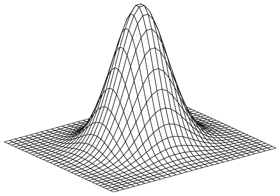
$$a_J[\boldsymbol{n}] = \langle f, \phi_{L+J,\boldsymbol{n}} \rangle \quad \text{and} \quad d_{j,k}^{(l_j)}[\boldsymbol{n}] = \langle f, \rho_{L+j,k,\boldsymbol{n}}^{(l_j)} \rangle$$

Digital images are obtained by:

$$\tilde{x}[\boldsymbol{n}] = (f * \bar{\varphi})(T\boldsymbol{n}) = \langle f, \varphi(\cdot - T\boldsymbol{n}) \rangle$$

For $T = 2^L$, with prefiltering we can get $x[\boldsymbol{n}]$ from $\tilde{x}[\boldsymbol{n}]$





Contourlet Features

- Defined via iterated filter banks $\Rightarrow$ fast algorithms, tree structures, etc.
- Defined on rectangular grids $\Rightarrow$ seamless translation between continuous and discrete worlds.
- Different contourlet kernel functions $(\rho_{j,k})$ for different directions.
- These functions are defined iteratively via filter banks.
- With FIR filters $\Rightarrow$ compactly supported contourlet functions.

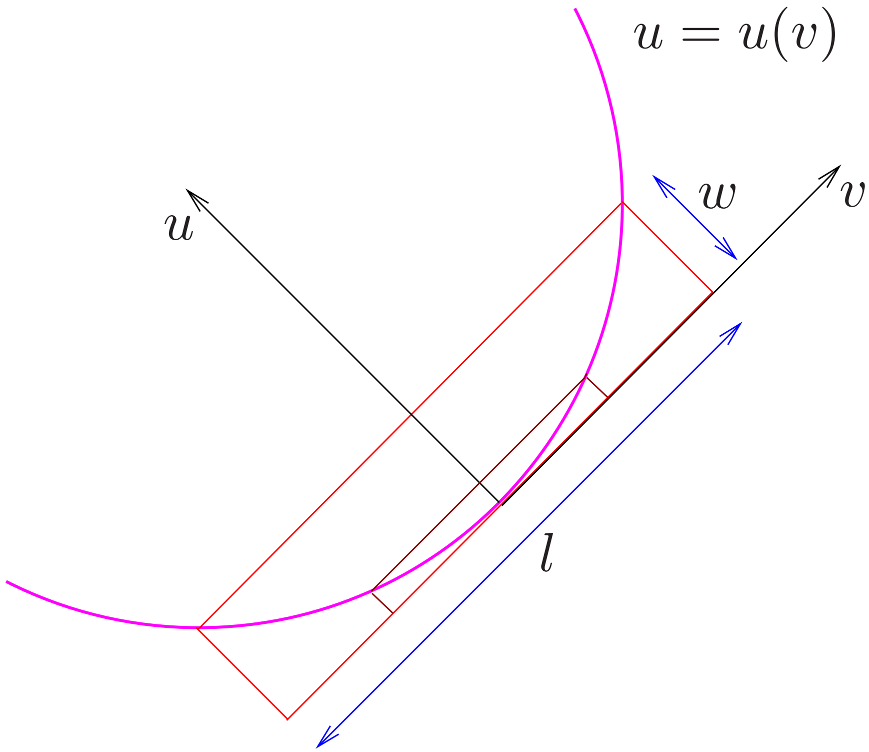
Outline

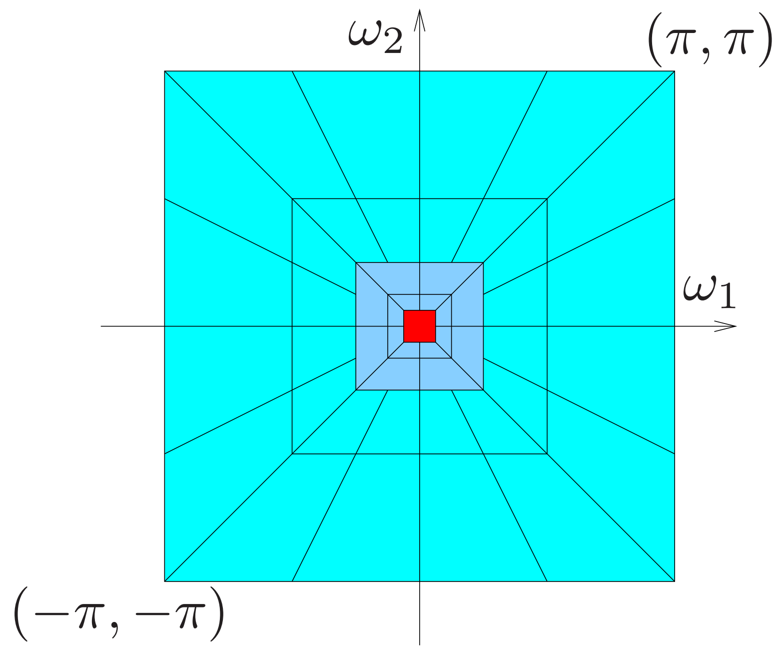
1. Motivation
2. Discrete-domain construction using filter banks
3. Contourlets and directional multiresolution analysis
4. Contourlet approximation and directional vanishing moments
5. Applications

Contourlets with Parabolic Scaling

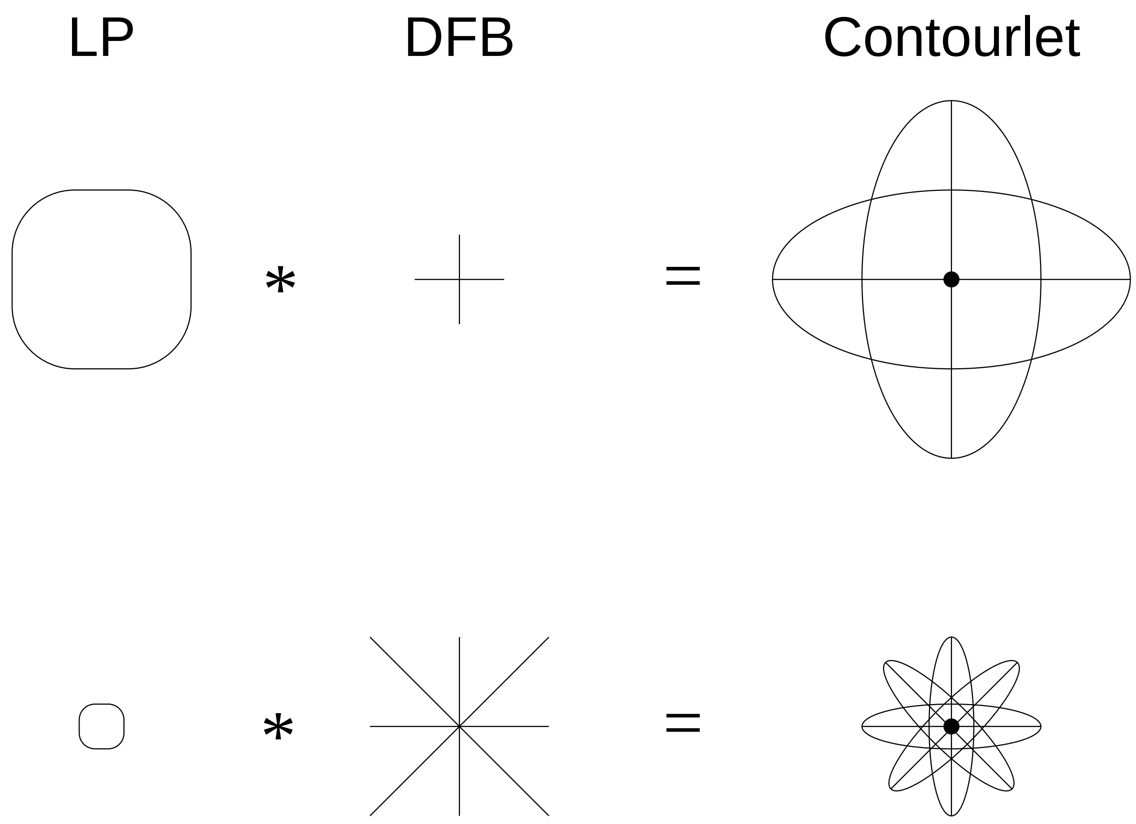
Support size of the contourlet function $\rho_{j,k}^{l_j}$: $width \approx 2^j$ and $length \approx 2^{l_j+j}$

To satisfy the parabolic scaling: $width \propto length^2$, simply set:
the number of directions is doubled at every ***other*** *finer scale*.



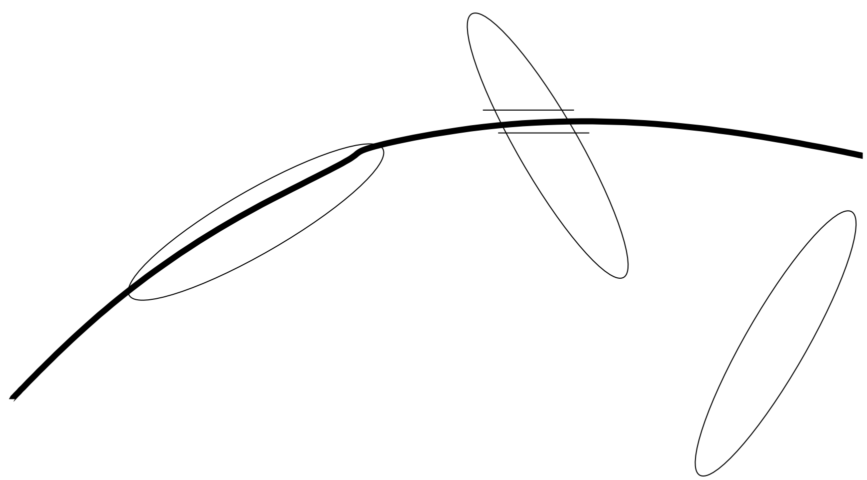


Supports of Contourlet Functions



Key point: Each generation doubles spatial resolution
as well as angular resolution.

Contourlet Approximation



Desire: Fast decay as contourlets turn away from the discontinuity direction

Key: Directional vanishing moments (DVMs)

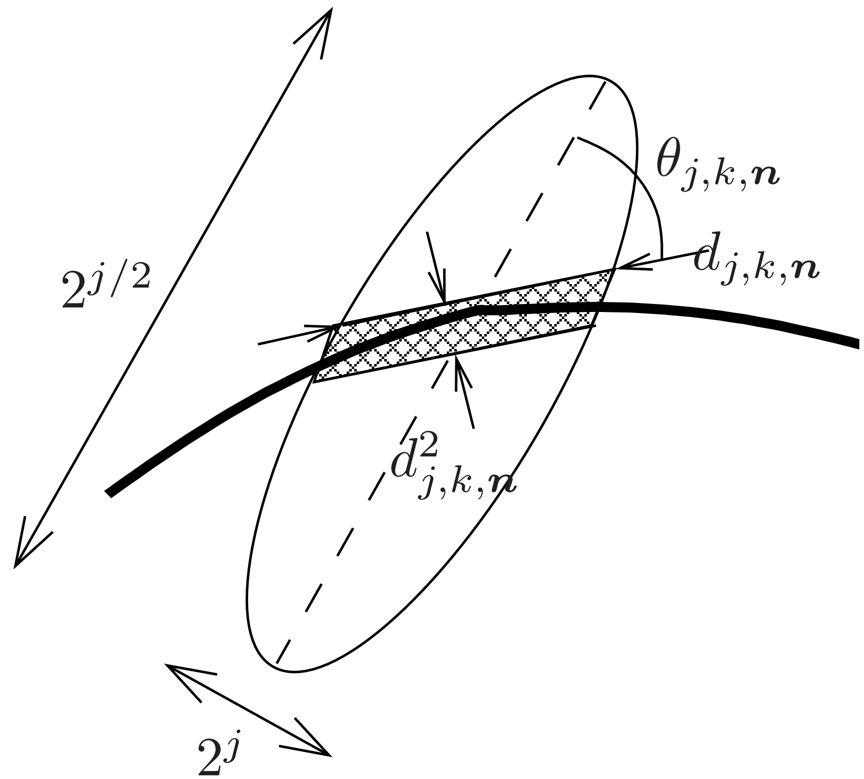
Geometrical Intuition

At scale 2^j $(j \ll 0)$:

$$width \approx 2^j$$

$$length \approx 2^{j/2}$$

$$\#directions \approx 2^{-j/2}$$

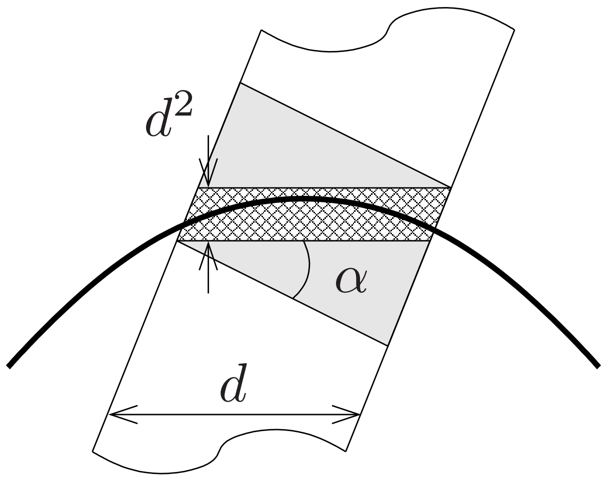


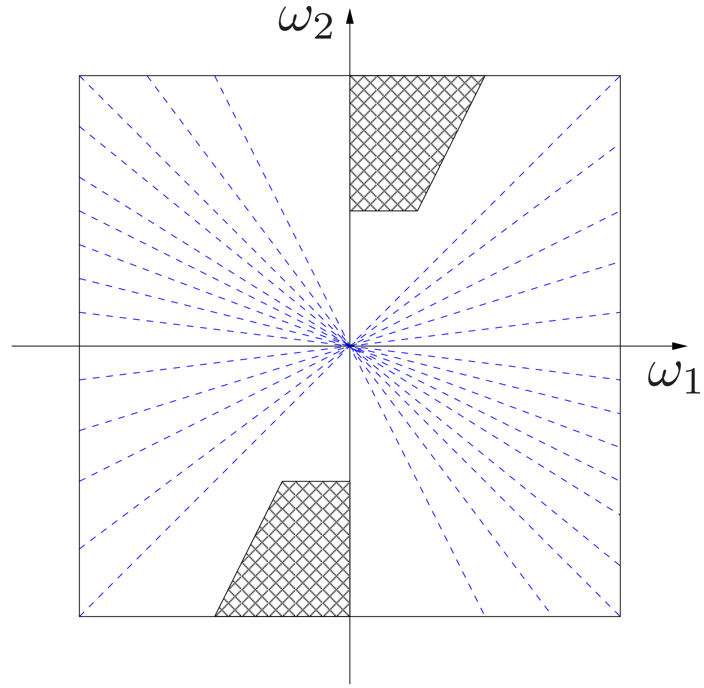
$$|\langle f, \rho_{j,k,\boldsymbol{n}} \rangle| \sim 2^{-3j/4} \cdot d^3_{j,k,\boldsymbol{n}}$$

$$d_{j,k,\boldsymbol{n}} \sim 2^j / \sin\theta_{j,k,\boldsymbol{n}} \sim 2^{j/2}\tilde{k}^{-1} \quad \text{for} \quad \tilde{k} = 1, \ldots, 2^{-j/2}$$

$$\Longrightarrow |\langle f, \rho_{j,\tilde{k},\boldsymbol{n}} \rangle| \sim 2^{3j/4}\tilde{k}^{-3}$$

How Many DVMs Are Sufficient?





Sufficient if the gap to a direction with DVM:

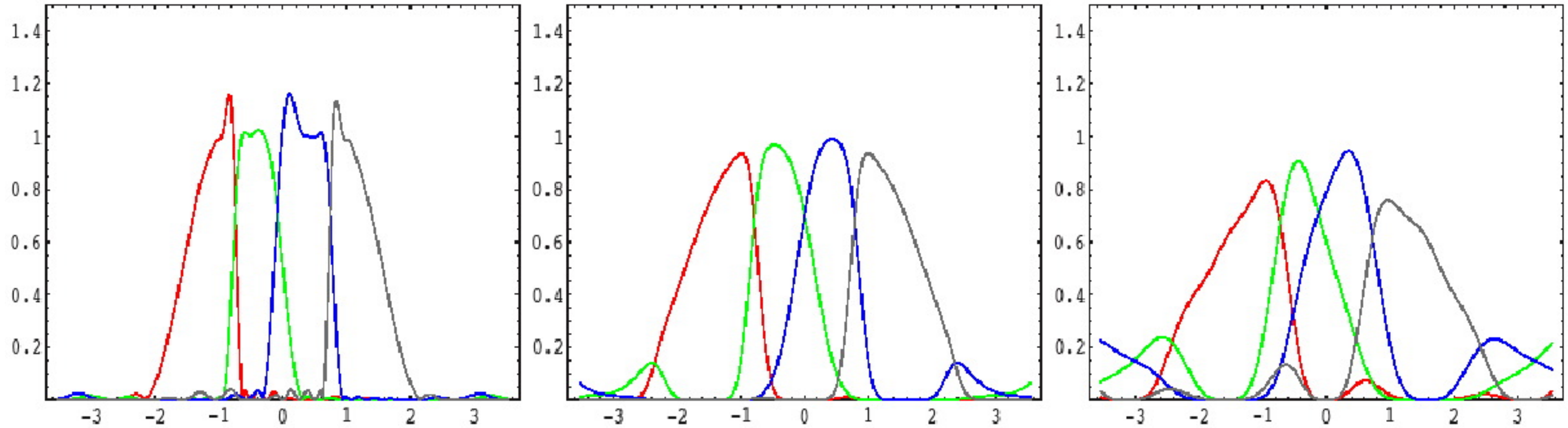
$$\alpha \lesssim d \sim 2^{j/2}\tilde{k}^{-1} \quad \text{for} \quad \tilde{k} = 1, \ldots, 2^{-j/2}$$

This condition can be replaced with fast decay in frequency across directions.

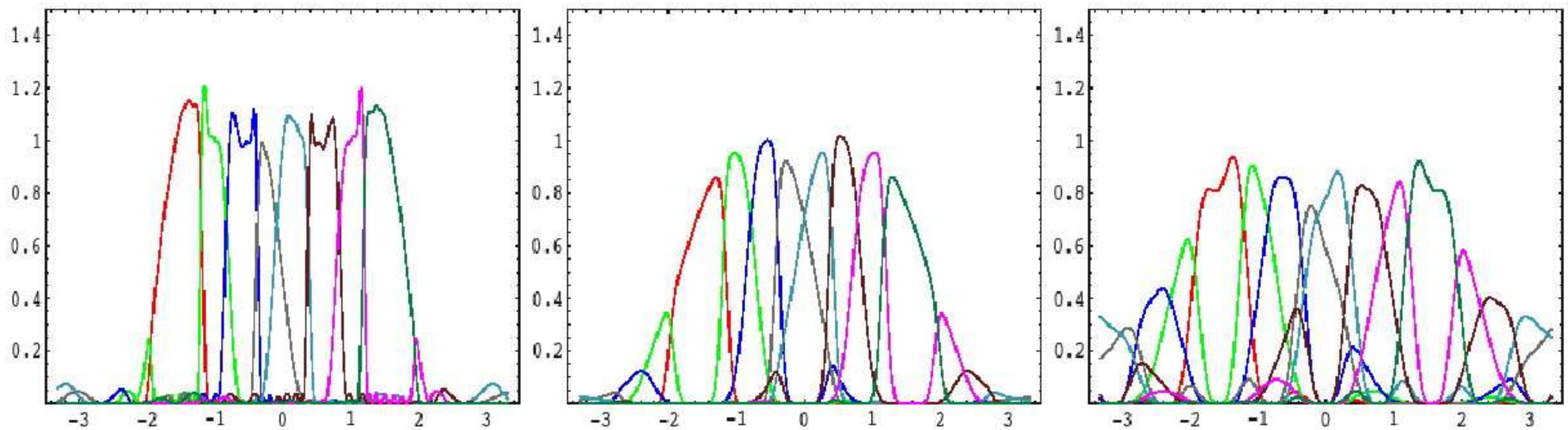
It is still an open question if there is an FIR filter bank that satisfies the sufficient DVM condition

Examples of Frequency Responses

8 directions: along the line $\omega_2 = \pi/2$

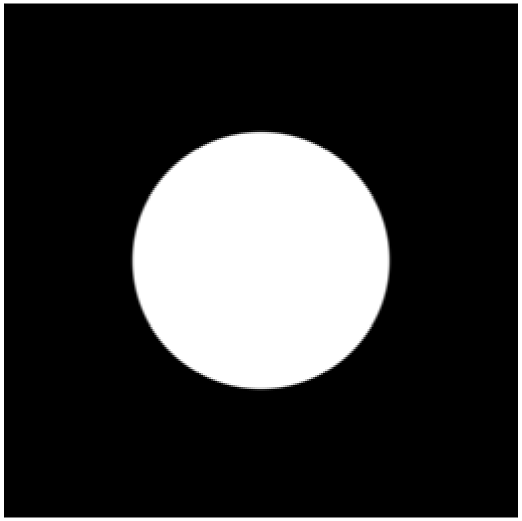


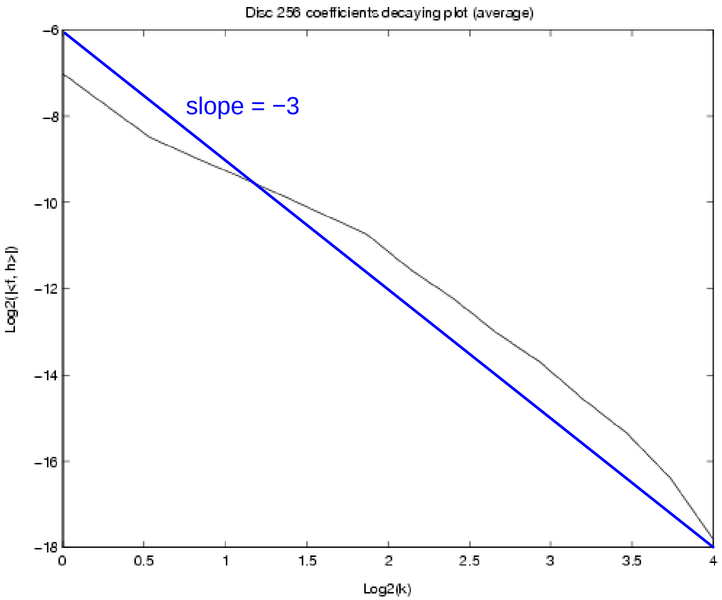
16 directions: along the line $\omega_2 = \pi/2$

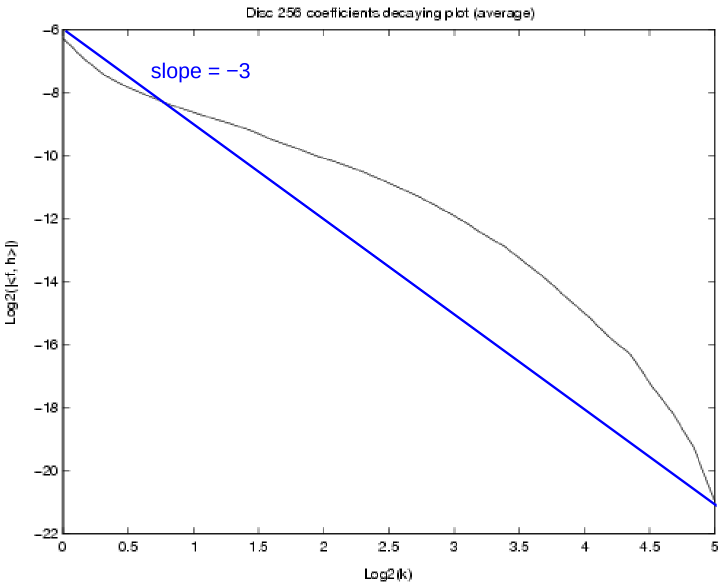


Left: PKVA filters (size 41×41). Middle: New filters (size 11×11). Right: CD filters (size 9×9)

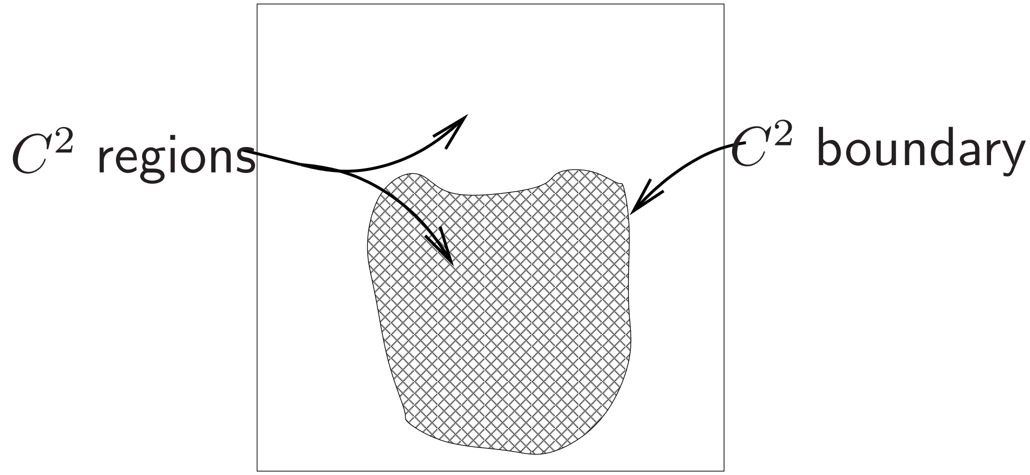
Experiments with Decay Across Directions using Near Ideal Frequency Filters







Nonlinear Approximation Rates



Under the (ideal) sufficient DVM condition

$$|\langle f, \rho_{j,\tilde{k},\boldsymbol{n}} \rangle| \sim 2^{3j/4} \tilde{k}^{-3}$$

with number of coefficients $N_{j,\tilde{k}} \sim 2^{-j/2} \tilde{k}$. Then

$$\|f - \hat{f}_M^{(contourlet)}\|^2 \sim (\log M)^3 M^{-2}$$

Note: $\|f - \hat{f}_M^{(Fourier)}\|^2 \sim O(M^{-1/2})$ and $\|f - \hat{f}_M^{(wavelet)}\|^2 \sim O(M^{-1})$

Non-linear Approximation Experiments

Image size $= 512 \times 512$. Keep $M = 4096$ coefficients.

Original image

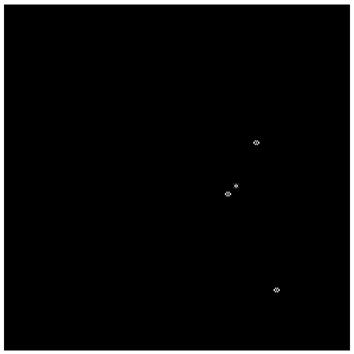
Wavelets:
PSNR = 24.34 dB

Contourlets:
PSNR = 25.70 dB

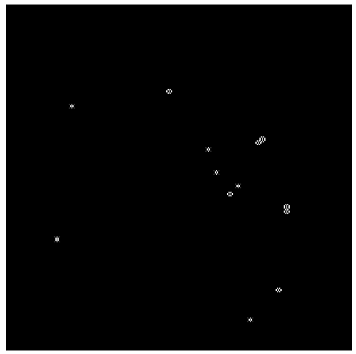
Detailed Non-linear Approximations

Wavelets

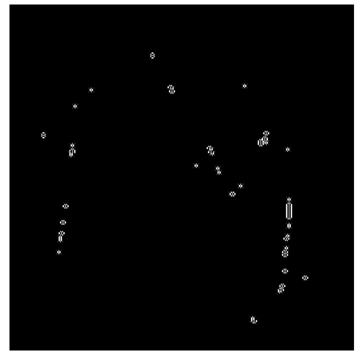
M = 4



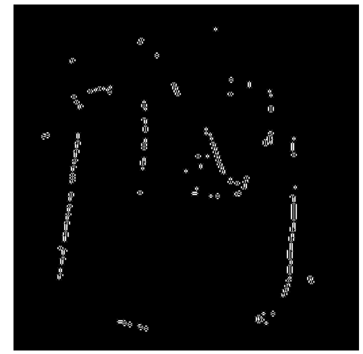
M = 16



M = 64

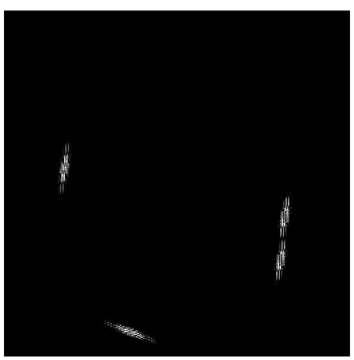


M = 256

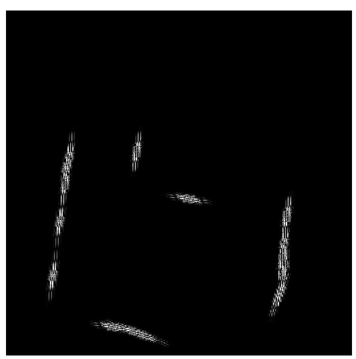


Contourlets

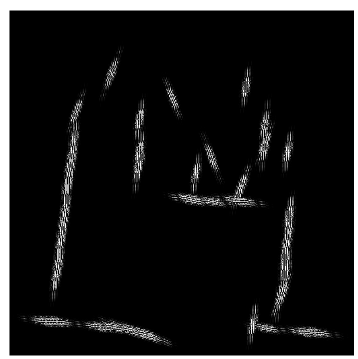
M = 4



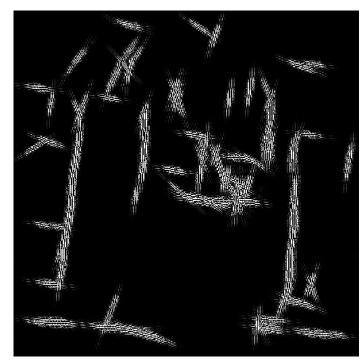
M = 16



M = 64



M = 256

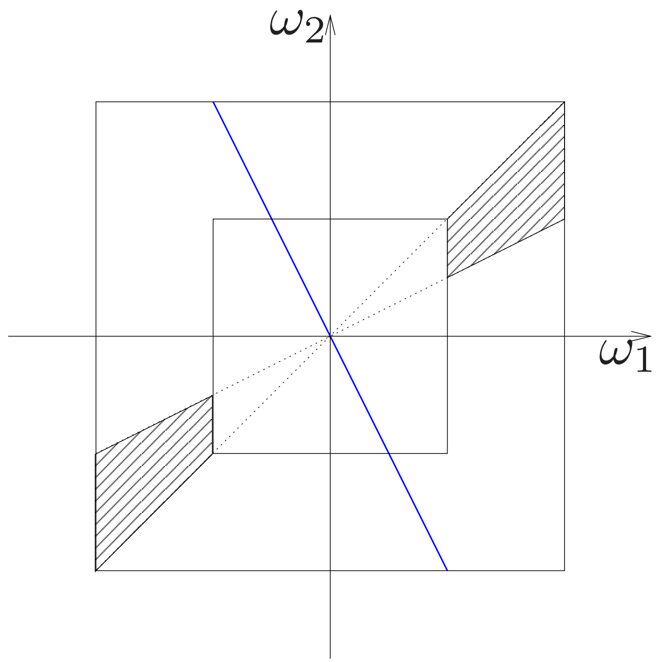


Filter Bank Design Problem

$$\rho_{j,k}^{(l)}(\boldsymbol{t}) = \sum_{\boldsymbol{m} \in \mathbb{Z}^2} c_k^{(l)}[\boldsymbol{m}] \phi_{j-1,\boldsymbol{m}}(\boldsymbol{t})$$

$\rho_{j,k}^{(l)}(\boldsymbol{t})$ has an L-order DVM along direction (u_1, u_2)

$\Leftrightarrow C_k^{(l)}(z_1, z_2) = (1 - z_1^{u_2} z_2^{-u_1})^L R(z_1, z_2)$



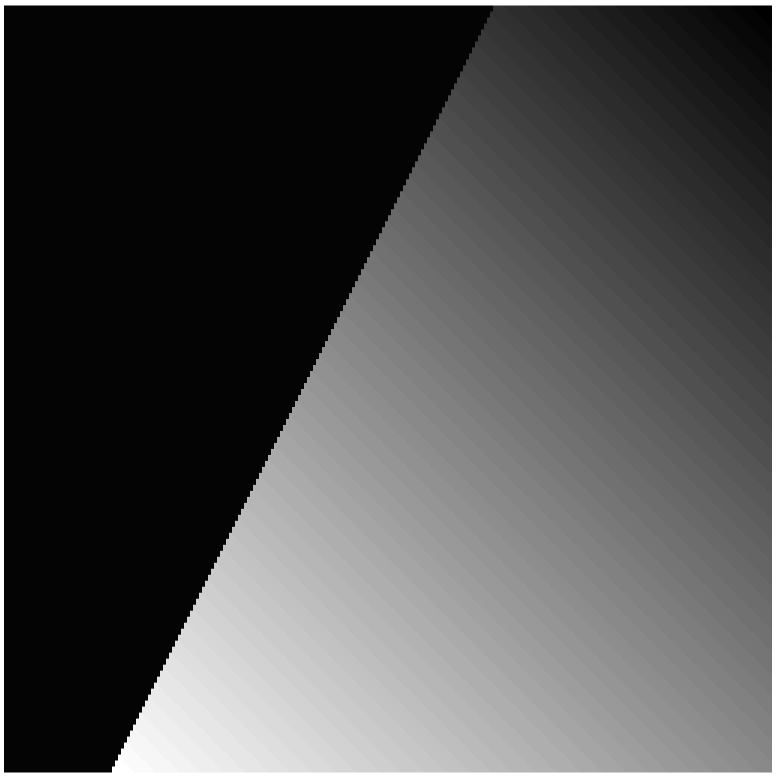
So far: Use good frequency selectivity to approximate DVMs.

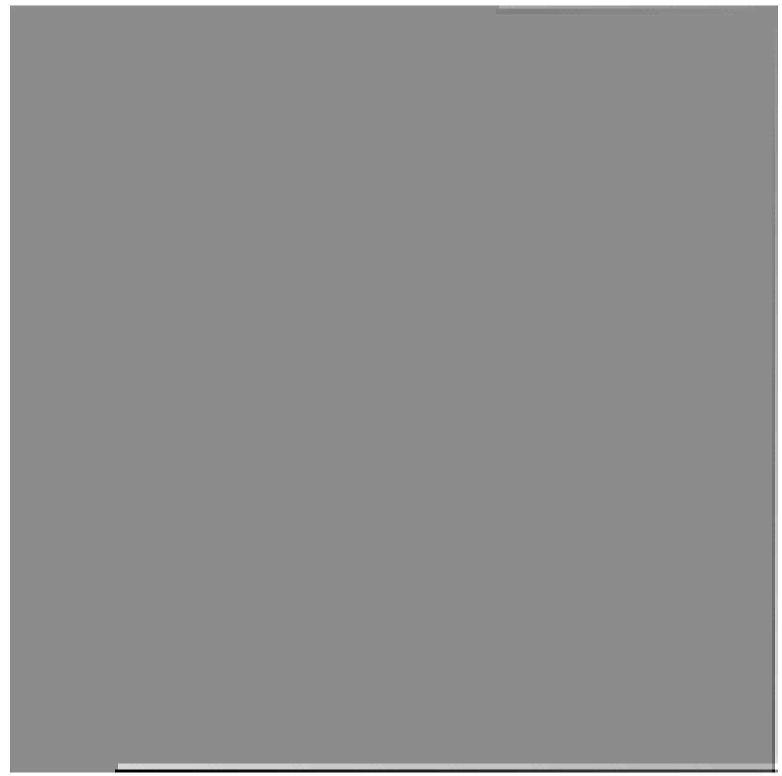
Draw back: long filters...

Next: Design short filters that lead to many DVMs as possible.

Filters with DVMs = Directional Annihilating Filters

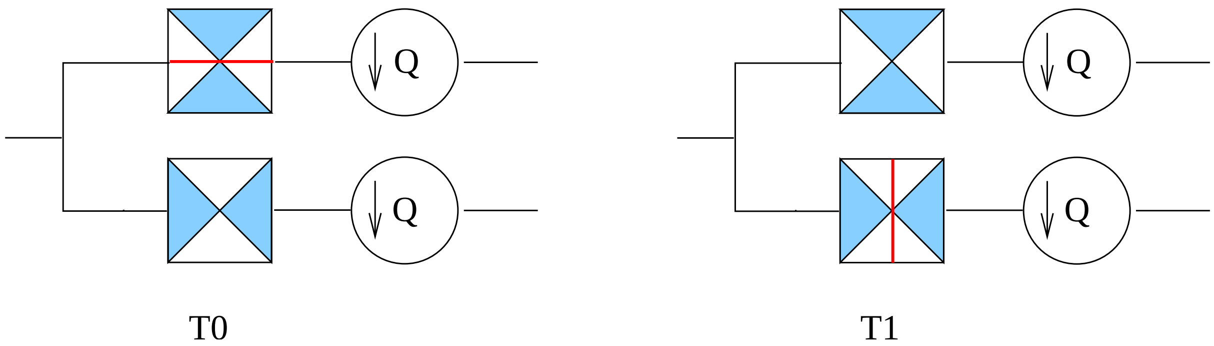
Input image and after being filtered by a directional annihilating filter





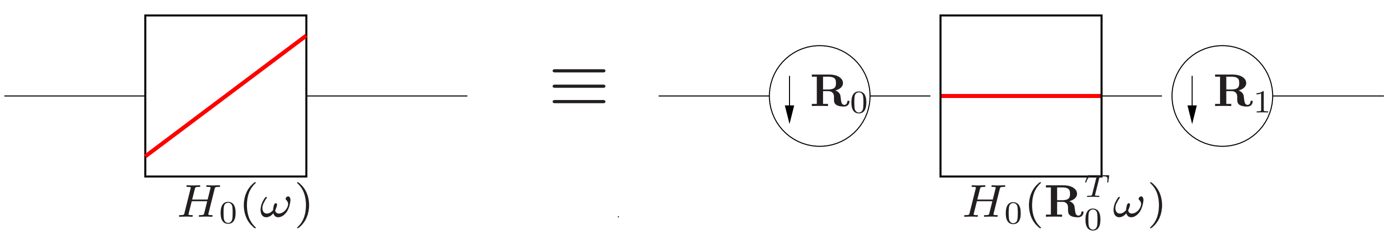
Two-Channel Filter Banks with DVMs

Filter bank with order-L horizontal or vertical DVM:



Filters have a factor $(1 - z_1)^L$ or $(1 - z_2)^L$

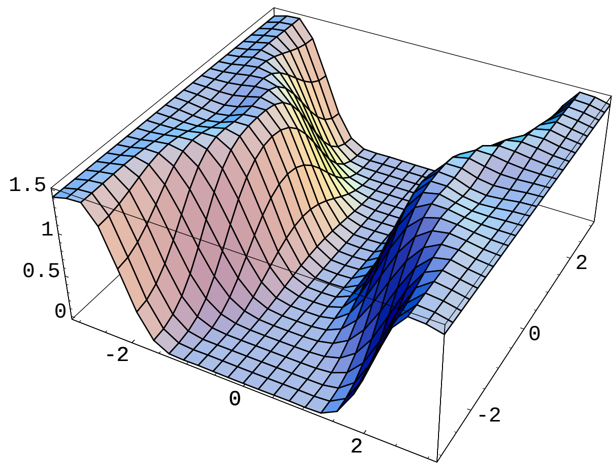
To get DVMs at other directions: Shearing or change of variables



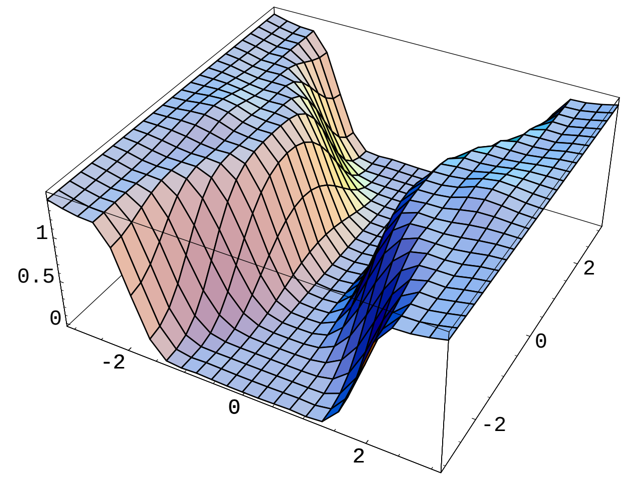
Design Example [CunhaD:04]

"Extended" 9-7 filters:

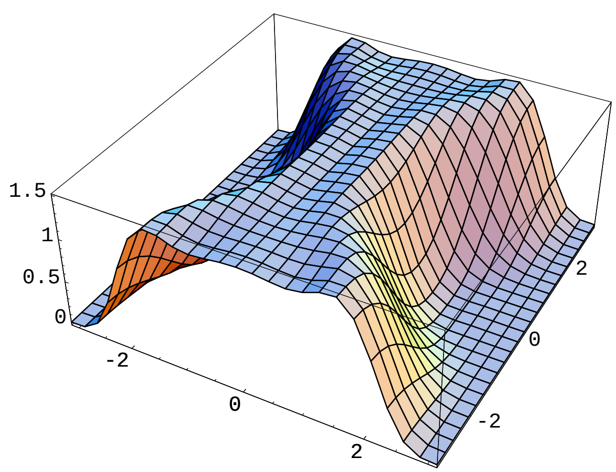
$|H_0(e^{j\boldsymbol{w}})|$



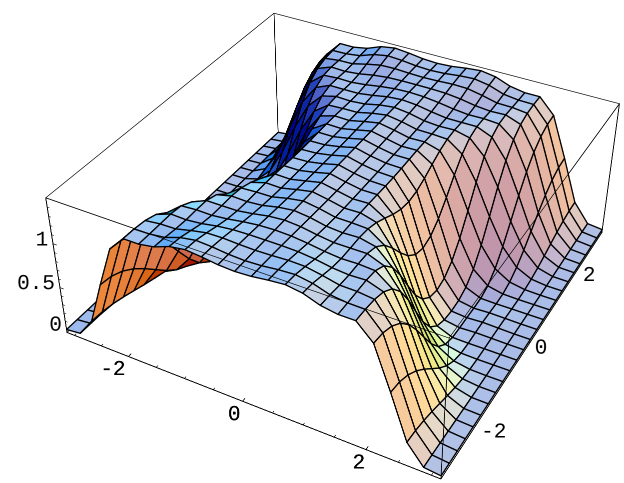
$|G_0(e^{j\boldsymbol{w}})|$



$|H_1(e^{j\boldsymbol{w}})|$

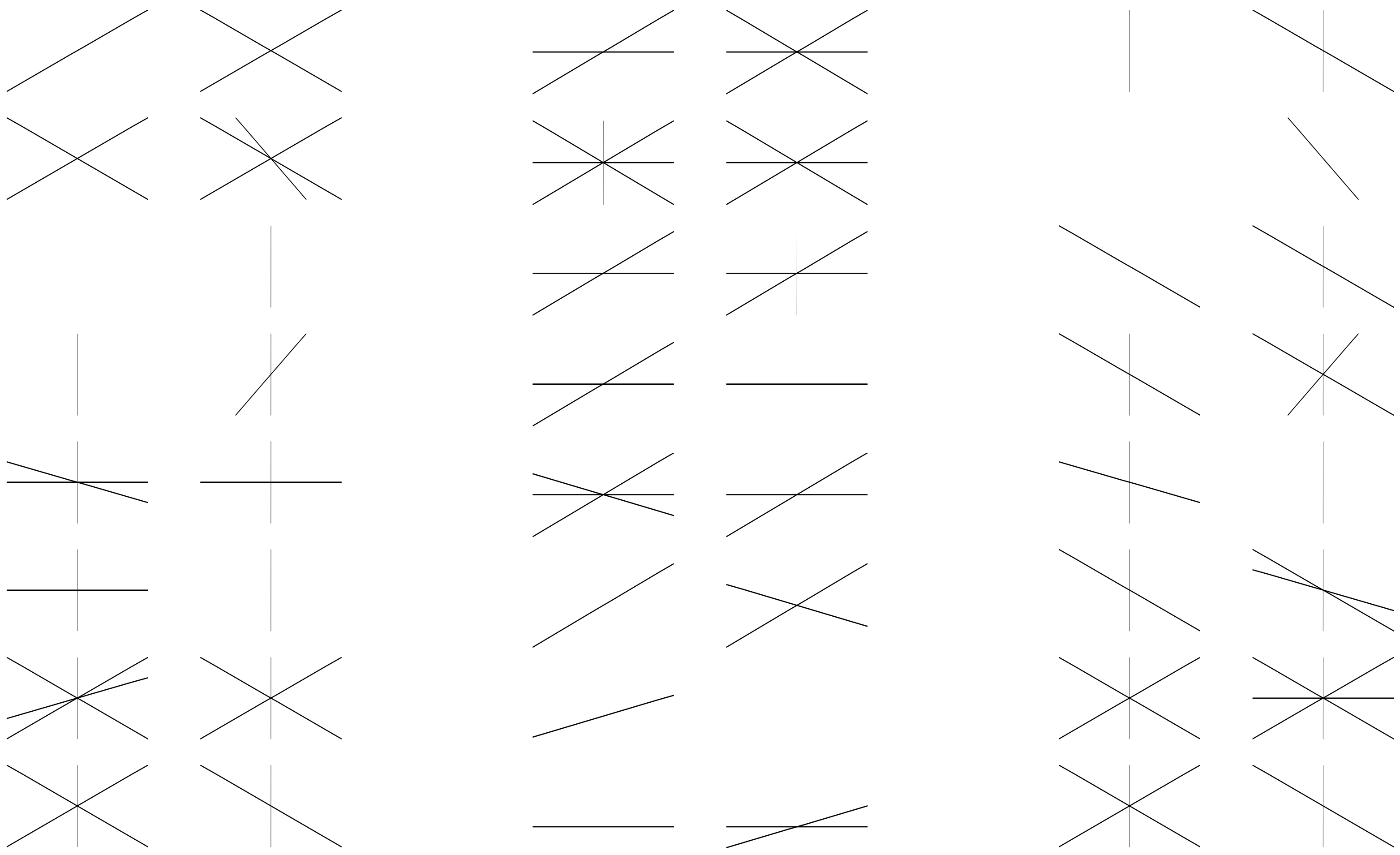


$|G_1(e^{j\boldsymbol{w}})|$

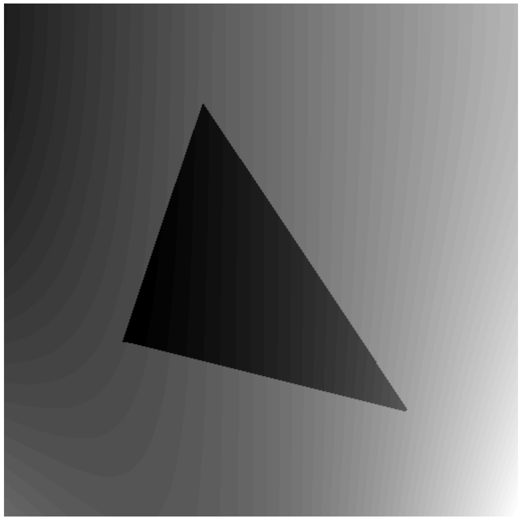


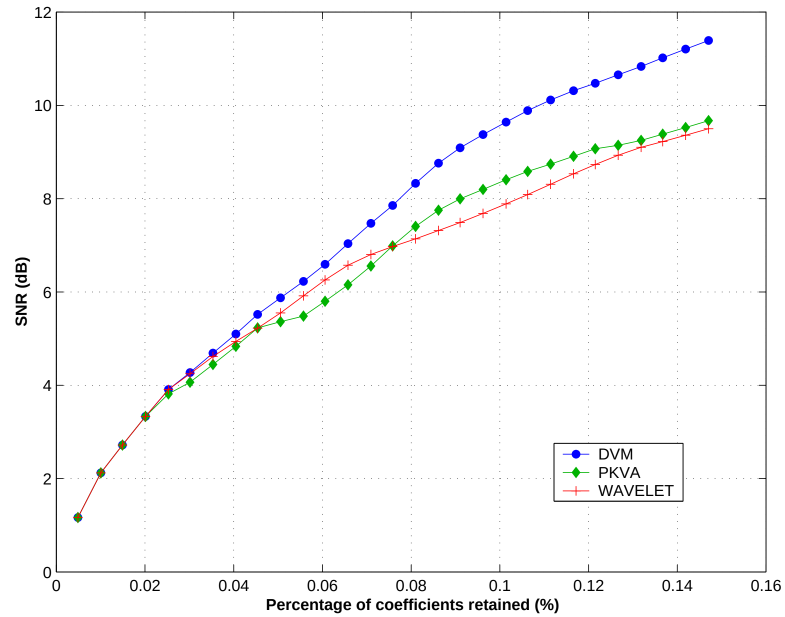
Directional Vanishing Moments Generated in Directional Filter Banks

Different expanding rules lead to different set of directions with DVMs.



Gain by using Filters with DVMs





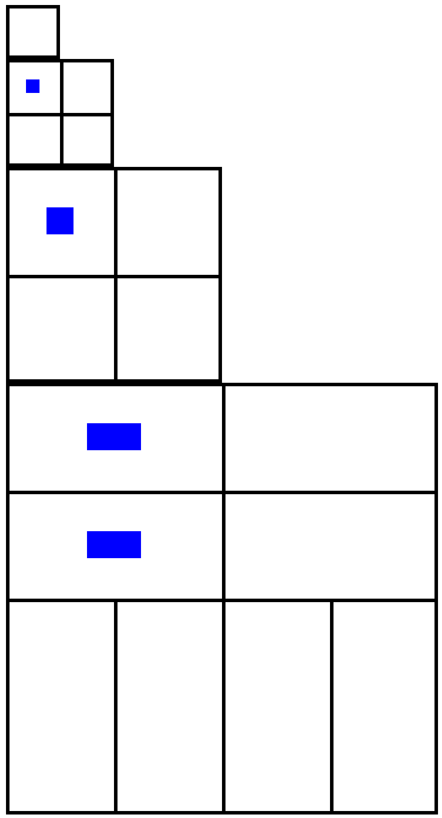
Using PKVA filters

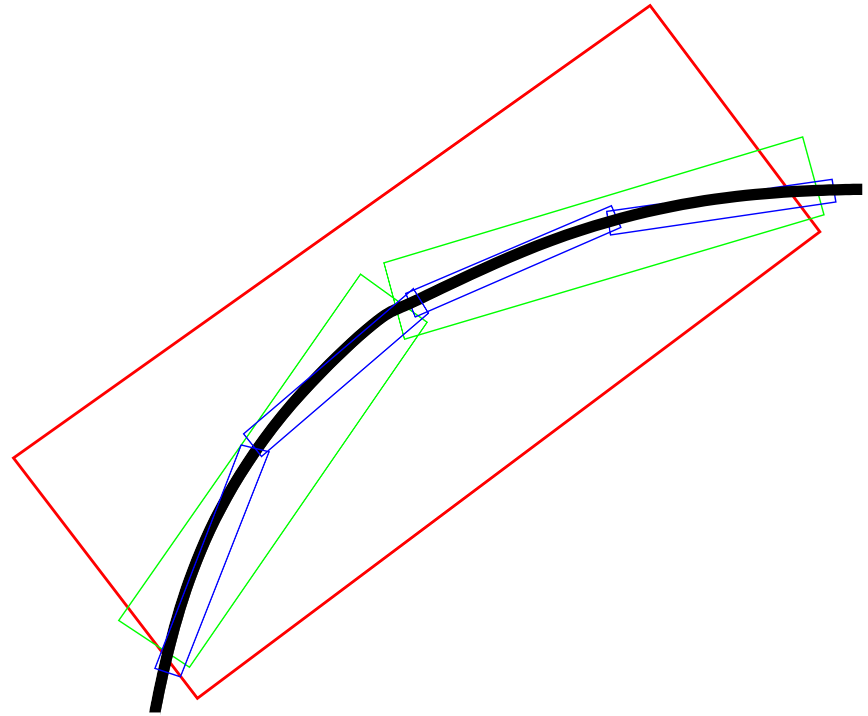
Using DVM filters

Outline

1. Motivation
2. Discrete-domain construction using filter banks
3. Contourlets and directional multiresolution analysis
4. Contourlet approximation and directional vanishing moments
5. Applications

Contourlet Embedded Tree Structure

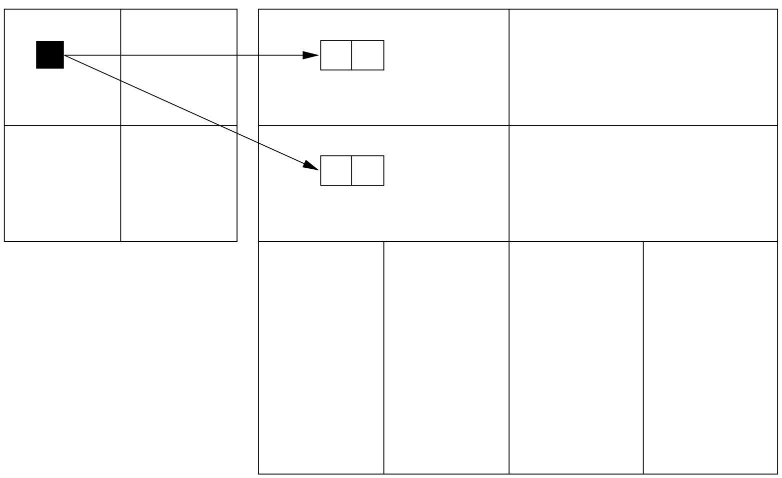


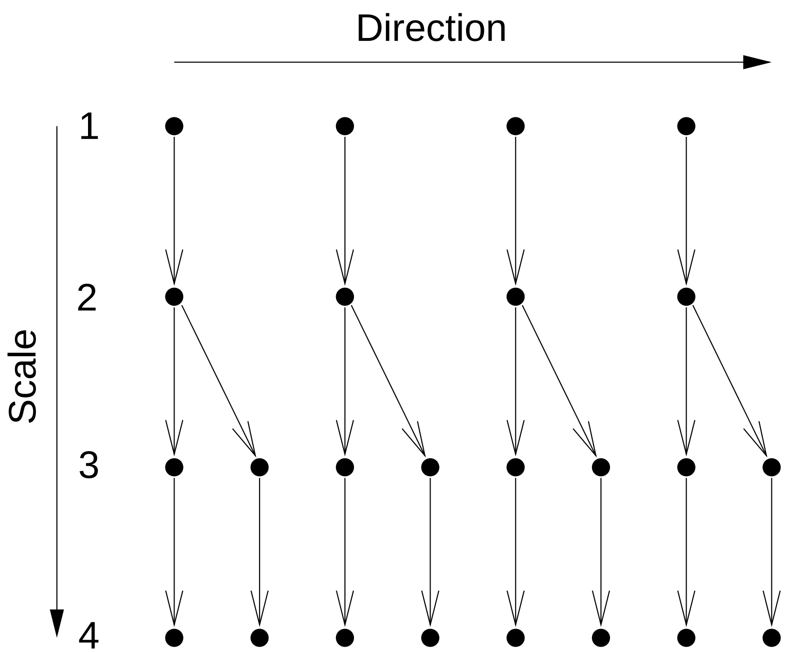


- **Embedded tree data structure** for contourlet coefficients: successively locate the position and direction of image contours.

- **Contourlet tree approximation**: significant contourlet coefficients are organized in trees $\longrightarrow$ low indexing cost for compression.

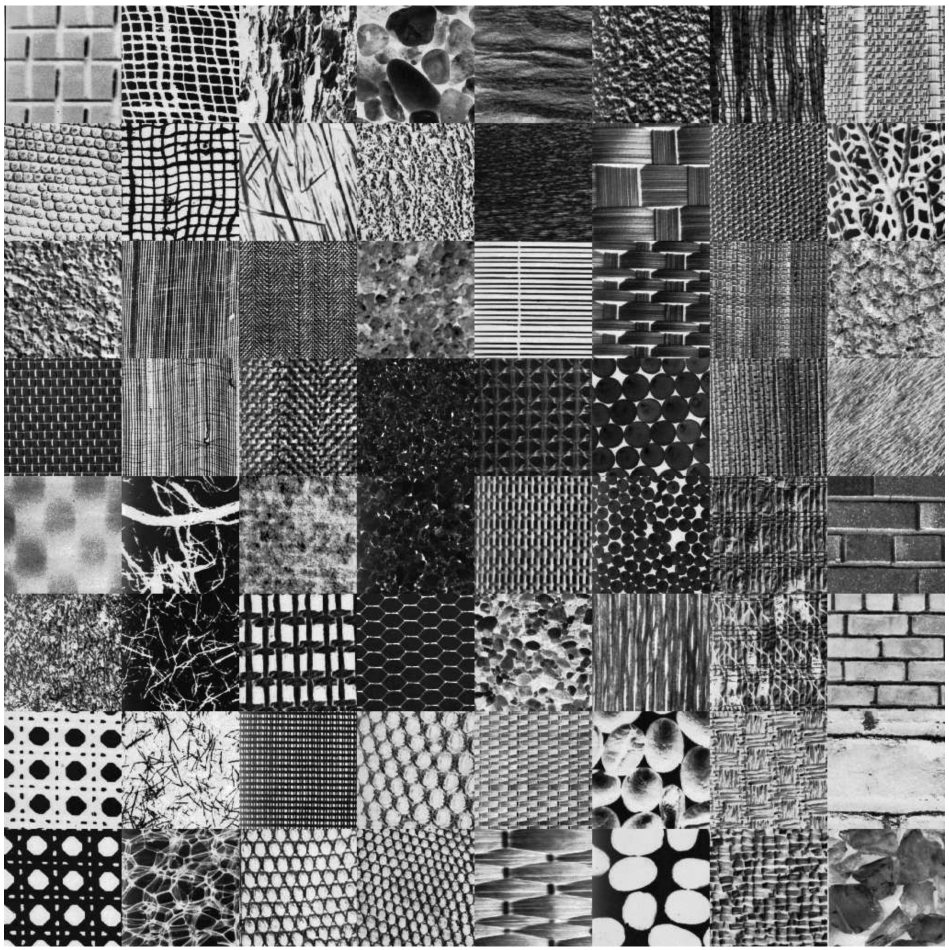
Contourlet-domain Hidden Markov Tree Models [PoD:04]





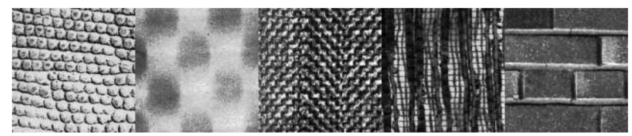
Contourlet HMT models all inter-scale, inter-direction, and inter-location independencies.

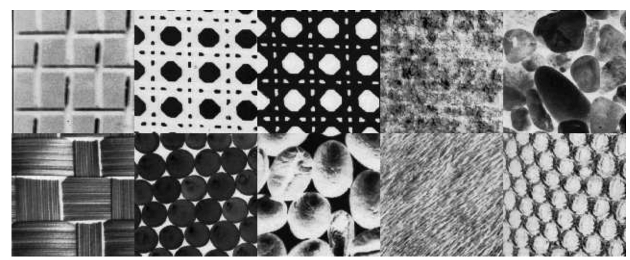
Texture Retrieval Results: *Brodatz Database*



Average retrieval rates

wavelet HMT	contourlet HMT
90.87%	**93.29%**

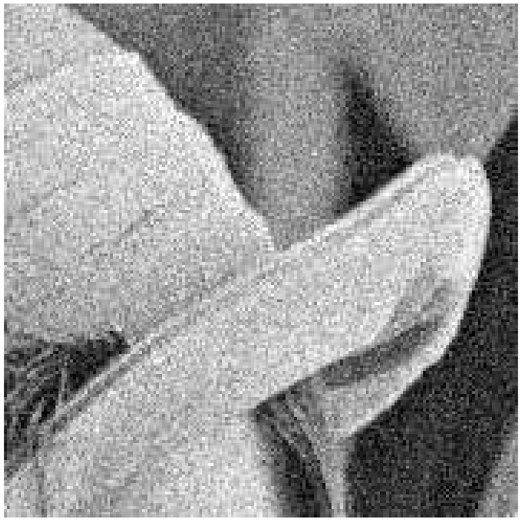




Top: Wavelets do better ($> 5\%$).
Bottom: Contourlets do better ($> 5\%$).

Denoising using Nonsubsampled Contourlet Transform (NSCT) [Cunha]:
"Hat Zoom"

Comparison against SI-Wavelet (nonsubsampled wavelet transform) methods using Bayes shrink with adaptive soft thresholding



Noisy Lena	SI-Wavelet	NSCT Den
PSNR 22.13db	PSNR 31.82dB	PSNR 32.14dB

Denoising Result: *"Peppers"*

Noisy Lena	SI-Wavelet	NSCT Den
PSNR 22.14db	PSNR 31.38dB	PSNR 31.53dB

Image Enhancement [Zhou]

Image Enhancement Algorithm

- Nonsubsampled contourlet decomposition
- Nonlinear mapping on the coefficients
  - Zero-out noises
  - Keep strong edges or features
  - Enhance weak edges or features
- Nonsubsampled contourlet reconstruction

Image Enhancement Result: *Barbara*

(a)

(b)

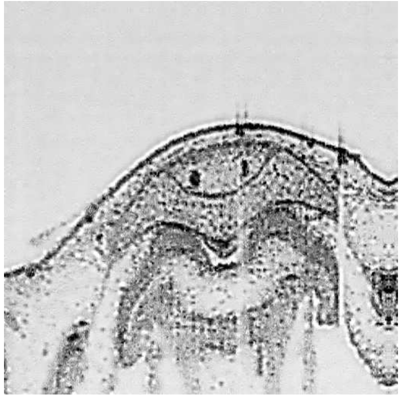
(c)

(a) Original image. (b) Enhanced by DWT. (c) Enhanced by NSCT.

Image Enhancement Result: *OCT Image*

(a)

(b)



(c)

(a) Original OCT image. (b) Enhanced by DWT. (c) Enhanced by NSCT.

Summary

- Image processing relies on *prior information* about images.
  - Geometrical structure is the key!
  - New desideratum beyond wavelets: directionality
- New two-dimensional discrete framework and algorithms:
  - Flexible directional and multiresolution image representation.
  - Effective for images with smooth contours $\Rightarrow$ contourlets.
  - Front-end for hierarchical geometrical image representation.
- Dream: Another fruitful interaction between harmonic analysis, vision, and signal processing.
- **Software and papers**: `www.ifp.uiuc.edu/~minhdo`