Multiscale Geometric Image Compression using Wavelets and Wedgelets

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Multiscale Geometric Compression of Piecewise Smooth Images using Wavelets and Wedgelets

Textures

Cartoons
Computational Harmonic Analysis

- **Representation**
  \[ f = \sum_k a_k b_k \]
  
  coefficients \quad basis, \, frame

- **Analysis**
  study \( f \) through structure of \( \{a_k\} \)
  \( \{b_k\} \) should extract features of interest

- **Approximation**
  \( \hat{f}_N \) uses just a few terms \( N \)
  exploit sparsity of \( \{a_k\} \)
Nonlinear Approximation

\[ f = \sum_k a_k b_k \]

- \( N \)-term approximation: use largest \( a_k \) independently

\[ \hat{f}_N := \sum_{k'=1}^{N} a_{k'} b_{k'} \]

- Greedy / thresholding

few big

\[ |a_{k'}| \]

sorted index \( k' \)
Error Approximation Rates

\[ f = \sum_k a_k b_k \]

\[ \hat{f}_N = \sum_{k'=1}^{N} a_{k'} b_{k'} \]

\[ \| f - \hat{f}_N \|_2^2 < C N^{-\alpha} \quad \text{as } N \to \infty \]

- Optimize asymptotic error decay rate \( \alpha \)
From Approximation to Compression

- Quantize coefficients \( \{a_k\} \), approximate with \( R \) bits

\[
f = \sum_k a_k b_k
\]

\[
\hat{f}_R = \sum_k a_k^q b_k
\]

- \( R \) bits must specify both value \( a_k^q \) and location \( k \)

- Optimize Rate/Distortion decay rate \( \alpha \)

\[
D(R) = \|f - \hat{f}_R\|_2^2 < CR^{-\alpha} \quad \text{as } R \to \infty
\]
1-D Piecewise Smooth Signals

- \( f \) smooth except for singularities at a finite number of 0-D points

Fourier sinusoids: suboptimal greedy approximation and extraction

Wavelets: \textit{optimal} greedy approximation extract singularity structure
2-D Piecewise Smooth Signals

- $f$ smooth except for singularities along a finite number of smooth 1-D curves

- Challenge: analyze/approximate geometric structure

- Challenge: analyze/approximate geometric structure
Wavelet-based Image Processing

- Standard 2-D tensor product wavelet transform

\[ f = \sum_{k} a_k b_k \]
Wavelet Challenges

- Geometrical info not explicit

- Inefficient - large number of large wc’s cluster around edge contours, no matter how smooth
The JPEG2000 Disappointment

• Current wavelet methods do not improve on decay rate of JPEG!

\[ D(R) = \| f - \hat{f}_R \|_2^2 < C R^{-\alpha} \]

• WHY?
neither DCT nor wavelets are the right transform
2-D Wavelets: Poor Approximation

- Even for a smooth $C^2$ contour, which straightens at fine scales...

- **Too many wavelets required!**

\[
\hat{f}_N \quad : \quad N\text{-term wavelet approximation}
\]

\[
\|f - \hat{f}_N\|_2^2 < C N^{-1} \quad \text{not} \quad N^{-2}
\]
Solution 1: Upgrade the *Transform*

- Introduce *anisotropic transform*
  - curvelets, ridgelets, contourlets, ...

- Optimal error decay rates for cartoons +
Solution 2: Upgrade the Processing

- Replace coefficient thresholding by a new \textit{wc model} that captures \textit{anisotropic spatial correlations}
Our Goal: New Image Models

- Wavelet coefficient models that capture both \textit{geometric} and \textit{textural} aspects of natural images
- Optimal error decay rates for some image class
Part I

Geometric Modeling of Cartoons with Wedgelets
Geometry Model for *Cartoons*

- **Toy model:** flat regions separated by smooth contours

- **Goal:** representation that is
  - sparse
  - simply modeled
  - efficiently computed
  - extensible to texture regions

\[ C^2 \text{ boundary} \]
2-D Dyadic Partition

- **Multiscale** analysis
- *Partition*; not a basis/frame
- Zoom in by factor of 2 each scale
2-D Dyadic Partition = \textit{Quadtree}

- \textit{Multiscale} analysis
- \textit{Partition}; not a basis/frame
- Zoom in by factor of 2 each scale
- Each \textit{parent} node has \textit{4 children} at next finer scale
Wedgelet Representation

• Build a cartoon using *wedgelets* on dyadic squares

wedgelet = atomic geometric element
Wedgelet Representation

- Build a cartoon using **wedgelets** on dyadic squares

- Choose orientation \((r, \theta)\) from finite dictionary (toroidal sampling)

- **Quad-tree** structure
  
  deeper in tree \(\Rightarrow\) finer curve approximation
Wedgelet Representation

- *Prune* wedgelet quadtree to approximate local geometry (adaptive)

- *Decorate* leaves with $(r, \theta)$ parameters
Wedgelet Inference

- Find representation / prune tree to balance a **fidelity** vs. **complexity** trade-off

\[
\min_{W} \| f - \hat{f}_W \|_2^2 + \lambda \text{Comp}(W)
\]

- For \( \text{Comp}(W) \) – need a **model** for the wedgelet representation \( W \)
  (quadtree + \((r, \theta)\))

- Donoho: \( \text{Comp}(W) = \#\text{leaves} \)
Wedgelet Inference

- Find representation / prune tree to balance a **fidelity** vs. **complexity** trade-off

\[ \min_W \| f - \hat{f}_W \|_2^2 + \lambda [\# \text{leaves}] \]

- **O(N)** dynamic programming solution (bottom-up recursion)

- Optimal *approximation*
  \[ L_2 \text{ error } \sim (\# \text{leaves})^{-2} \]

- *Near-optimal* rate/distortion decay
  \[ D(R) \sim (\log R)^2 R^{-2} \]
#Leaves Complexity Penalty

- Accounts for wedgelet partition *size*, but not wedgelet *orientation*

\[
\text{\#leaves} \quad = \quad \text{\#leaves}
\]

- "smooth" "simple"
- "rough" "complex"
Multiscale Geometry Model (MGM)

- Decorate *each* tree node with orientation \((r, \theta)\) and then *model dependencies thru scale*

- Insight: Smooth curve \(\Rightarrow\) *Geometric innovations small at fine scales*

- Model: Favor small innovations over large innovations \((\text{statistically})\)
Multiscale Geometry Model (MGM)

- Wavelet-like geometry model: \textit{coarse-to-fine prediction}
  - model parent-to-child transitions of orientations
MGM

- **Wavelet-like geometry model:**
  - *coarse-to-fine prediction*
  - model parent-to-child transitions of orientations

- **Markov-1 statistical model**
  - state = \((r, \theta)\) orientation of wedgelet
  - parent-to-child state transition matrix
    \[ P_{mn} = P(\text{child orientation} = n | \text{parent orientation} = m) \]

\[ P_{m,1} \]

\[ P_{m,2} \ll P_{m,1} \]
MGM

- Markov-1 statistical model
  \[ P_{mn} = P(\text{child orientation} = n | \text{parent orientation} = m) \]

- **Joint wedgelet Markov probability model:** \( P(W) \)

- **Complexity** = *Shannon codelength* = \(- \log P(W)\) = number of bits to encode \( W \)
MGM and Edge Smoothness

\[ P(W_1) \]
\[ - \log P(W_1) \]

“smooth”
“simple”

\[ \gg \]

\[ P(W_2) \]
\[ - \log P(W_2) \]

“rough”
“complex”
MGM Inference

- Find representation / prune tree to balance the **fidelity** vs. **complexity** trade-off

\[
\min_W \| f - \hat{f}_W \|^2_2 + \lambda [- \log P(W)]
\]

- Efficient \(O(N)\) solution via *dynamic programming*
MGM Approximation

- Find representation / prune tree to balance the **fidelity** vs. **complexity** trade-off

\[
\min_{W} \| f - \hat{f}_W \|_2^2 + \lambda \left[ - \log P(W) \right]
\]

- **Optimal** $L^2$ error decay rate for cartoons $\left( \#W \right)^{-2}$
Wedgelet *Coding* of Cartoon Images

- Choosing wedgelets = *rate-distortion* optimization

$$\min_W \left\| f - \hat{f}_W \right\|_2^2 + \lambda \left[ - \log P(W) \right]$$

Shannon code length

$$\min_W \text{Distortion} + \lambda \text{Rate (bits)}$$

to encode \((r, \theta, m_1, m_2)\)
Wedgelet Coding of Cartoon Images

• Choosing wedgelets = rate-distortion optimization

\[
\min_W \| f - \hat{f}_W \|_2^2 \quad \text{s.t. } (#\text{bits}) \leq R^* 
\]

~130 bits coded independently
~25 bits coded with MGM
Predictive Wedgelet Coding is Optimal

*Optimal* rate-distortion performance

\[ D(R) \sim R^{-2} \]

compared to

\[ (\log R)^2 \ R^{-2} \]

for leaf-only encoding

\( C^2 \) contour (1-d)
Joint Texture/Geometry Modeling

- Dictionary $D = \{\text{wavelets}\} \cup \{\text{wedgelets}\}$

- Representation tradeoff: \text{texture} vs. \text{geometry}

- Test case: \text{approximation / compression}
Subtracting Doesn’t Work

wedgelet approximation  residual = \( f - w.a. \)

* ridge artifacts just as hard to approx/code as edges
Part II

Wedding Wedgelets with Wavelets
Wavelet Representation

- Standard 2-D tensor product wavelet transform

\[ f = \sum_{k} a_k b_k \]
Wavelet Quadtrees

- Wavelet coefficients structured on *quadtrees*
  - each *parent* has 4 *children* at next finer scale
Wavelet Persistence

- **Smooth** region - *small* values down tree
- **Singularity/texture** - *large* values down tree
Zero Tree Approximation

- Idea: *Prune* wavelet subtrees in smooth regions
  - *tree-structured thresholding*
Zero Tree Approximation

- Label pruned wavelet quadtree with 2 states

**zero-tree** - smooth region (prune)

**significant** - edge/texture region (keep)

\[ Z: \text{all wc's below}=0 \]

ie: wc's of

\[ +\text{atom} \]
Enter Geometry

- Label pruned wavelet quadtree with 2 states
  
  **zero-tree** - smooth region (prune)
  **significant** - edge/texture region (keep)

- *Suboptimal* NLA/R-D decay w/ edges \( D(R) \sim R^{-1} \)
Wedgelets for Geometry

• Label pruned wavelet quadtree with 2 states
  - zero-tree: smooth region (prune)
  - significant: edge/texture region (keep)

• Idea: use wedgelets in geometric edge squares

G: wc’s below are wc’s of a wedgelet
Wedgelets for Geometry

- Label pruned wavelet quadtree with 3 states

  - **zero-tree** - smooth region (prune)
  - **geometry** - edge region (prune)
  - **significant** - texture region (keep)

\[ G: wc's \text{ below are } wc's \text{ of a wedgelet} \]
Wedgelets for Geometry

• Label pruned wavelet quadtree with 3 states

  zero-tree - smooth region (prune)
  geometry - edge region (prune)
  significant - texture region (keep)

• Optimize placement of Z, G, S by dyn. programming

G: wc’s below are wc’s of a wedgelet
Wedgelet Trees for Geometry

- Label pruned wavelet quadtree with 3 states
  - zero-tree - smooth region (prune)
  - geometry - edge region (prune)
  - significant - texture region (keep)

- Optimize placement of Z, G, S by dyn. programming

G: wc’s below are wc’s of a wedgelet tree
Image Approximation

\( G \): wc’s below are wc’s of a wedgelet

ie: wc’s of +atom

\( Z \): all wc’s below = 0

ie: wc’s of +atom

\( "smoothprint" \)

\( S \): describe (code) wc’s

\( "wedgeprint" \)
Wedgeprints

- Wedgelet projected onto wavelet subtree yields an *adaptive atom* matched to local *edge geometry*

- *Wedgeprint* collapses many wc’s (entire subtree) into a *single* oriented atom

- Akin to wavelet *vector quantization*
Optimality of Wedgeprints

• **Theorem**

For $C^2 / C^2$ images, *optimal* asymptotic $L^2$ error decay

$$\| f - \hat{f}_N \|_2^2 < C N^{-2}$$

and *near-optimal* rate-distortion

$$D(R) = \log(R)^2 R^{-2}$$
Practical Image Coder

- **Wedgelet-SFQ (WSFQ) coder** builds on SFQ coder [Xiong, Ramchandran, Orchard]

  - At low bit rates, often significant improvement in visual quality over SFQ and JPEG-2k (much sharper edges)

- **Bonus**: WSFQ representation contains *explicit geometry information*
Wet Paint Test Image
SFQ Compressed

PSNR = 29.77dB @ 0.0103bpp
SFQ WC Code Map

green = significant  blue = zero tree
SFQ Zoom
WSFQ Compressed

PSNR = 30.19dB @ 0.0102bpp
WSFQ WC Code Map

green=significant  blue=zero tree  red=contour tree
WSFQ Contour Trees
SFQ vs. WSFQ

SFQ

WSFQ
Extensions

zerotree

DCTprint

barprint

wedgeprint

“Coifman’s Dream”
Bars / Ridges

16x16 image block
Multiple Wedgelet Coding

80 bits to jointly encode 5 wedgelets
“Barlet” Coding

(fat edgelet/beamlet)

22 bits to encode 1 barlet
Periodic Textures
DCTprint

32x32 block = 1024 pixels
DCTprint

32x32 block = 1024 pixels

4x fewer coefficients
Extensions

zerotree

barprint

wedgeprint

DCTprint
Conclusions

• Capturing *geometrical information* in images requires new tools.

- **Wavelets**
  - Basis
  - Wedgeprints

- **Wedgelets**
  - Model

• *Optimal approximation* of $C^2/C^2$ function class
Isotropic Anisotropy?

• Wedgeprints+MGM achieve optimal performance of *anisotropic* curvelets

• But wavelet/wedgelet tiles are *isotropic* (based on dyadic squares, quadtree)

• Ultimately could prove a curse for closely spaced geometrical features
Related Work

• Multiresolution Fourier transform
  [Calway, Pearson, Wilson]

• “Prune and join” quadtree approximation
  [Shukla, Dragotti, Do, Vetterli]

• Platelets
  [Willett and Nowak]

• Wavelet footprints
  [Dragotti, Vetterli]

• Bandelets
  [Le Pennec, Mallat]