REMARKS

1. FUNCTIONAL CLASSES
2. WAVELETS & FUNCTIONAL CLASSES
3. APPLICATIONS

OUTLINE

1. SINGULARITIES
2. ANALYSIS OF SINGULARITIES
3. SINGULAR SUPPORT
4. WAVEFRONT SET
5. EXAMPLES
6. DIRECTIONAL WAVELETS
7. WAVE PACKETS

See E.J. Candès & D.L. Donoho
"Continuous Curvelet Transform"

① SINGULARITIES

MISBEHAVIOR AT A POINT

δ - DIRAC

$|x|^{-\alpha}$ POWER

$|x|^{\alpha} \sin(1/x)$ CHIRP

$\mathrm{sgn}(x)$ (HEAVISIDE-LIKE)

MISBEHAVIOR ON A LINE IN $\mathbb{R}^2$

$\nu = \delta(x_1)$

$|x_1|^{-\alpha}$

$\mathrm{sgn}(x_1)$

WITH ONE EXCEPTION ALL VERY NICE AWAY FROM SINGULARITY

2 ANALYSIS OF SINGULARITIES

$\psi(t)$ OSCILLATORY, $\psi : \mathbb{R} \to \mathbb{R}$

$$\psi_{a,b}(t) = \psi((t-b)/a)\, a^{-1/2}$$

$$W_f(a,b) = \langle \psi_{a,b}, f \rangle$$

BEHAVIOR

δ	
$\lvert x\rvert^{-\alpha}$	
$\mathrm{sgn}(x)$	

Key Idea

$$\int \psi\left(\frac{(t-b)}{a}\right) a^{-1/2} \cdot f(t)\, dt$$

If f is a homogeneous distribution

$$\langle \psi_{a,b}, f \rangle = a^{e} \langle \psi_{1,b/a}, f \rangle$$

scaling exponent

Examples

$$\langle \psi_{a,b}, \delta \rangle = \psi_{a,b}(0) = a^{-1/2}\psi(0)$$

$$\langle \psi_{a,b}, |x|^{-\alpha} \rangle = a^{\frac{1}{2}-\alpha} W_{\alpha}(1, b/a)$$

$$\langle \psi_{a,b}, \mathrm{sgn}(x) \rangle = a^{1/2} W_{\sigma}(1, b/a)$$

(3) SINGULAR SUPPORT

$$SS(f) = \{x : \varphi_x \cdot f \notin C^\infty \quad \forall\ \varphi_x(x) = 1,\ \varphi_x \in C^\infty\}$$

i.e. $(\varphi_x \cdot f)^\wedge(\xi)$ not of rapid decay.

$$SS(\delta) = \{0\}$$

$$SS(|x|^{-\alpha}) = \{0\} \quad \text{etc.}$$

IN $\mathbb{R}^2$

δ	DIRAC at $(0,0)$		
ν	DIRAC at $(0, x_2)$		
S	$\mathbb{1}_{\{\|x\|_\infty \le 1\}}$		
B	$\mathbb{1}_{\{\|x\|_2 \le 1\}}$		

Relation to Wavelet Transform

Let $\psi \in C^\infty$, and $\hat{\psi}$ vanish near $\xi = 0$.

Define

$$\mathcal{R} = \{ x : W_f \text{ decays rapidly near } x \text{ as } a \to 0 \}$$

Then

$$SS(f) = \mathcal{R}^c.$$

$$W_f(a,b) = O(a^N) \qquad N > 0, \; a \to 0$$

$O(\cdot)$ uniform in b near x.

④ Wavefront Set in $\mathbb{R}^n$

$WF(f) = \{ (x_0, \theta_0) :$

$x_0 \in SS(f)$,

$\forall \varphi \in C^\infty$, $\varphi(x_0) = 1$,

$\text{supp}(\varphi) \subset B(x_0, \delta)$, $\delta > 0$,

$\widehat{\varphi f}(\xi)$ not of rapid decay

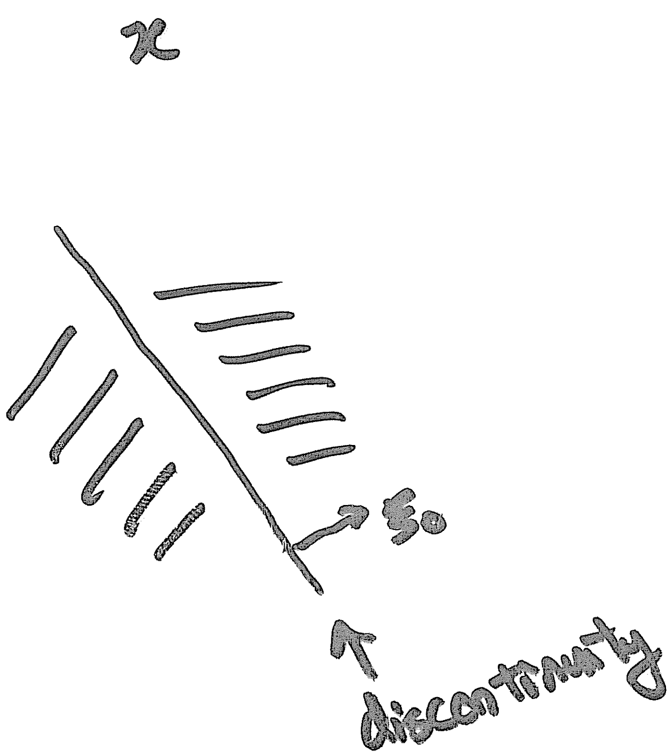
in any wedge of $\xi = (r\cos w, r\sin w)$

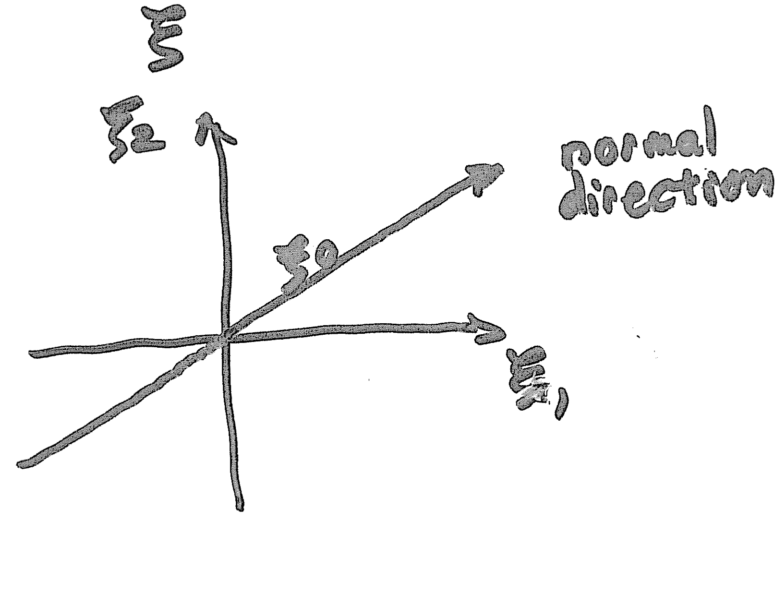
where $|w - \theta_0| < \delta$, $\}$

(WHAT?)

If f is a nice distribution,

$\hat{f}(\xi)$ has asymptotic behavior along a line dictated by singularities of f "aligned" with ξ.





$f(t\xi_0)$ decays slowly

Examples

δ DIRAC

ν $\langle \nu, f \rangle = \int f(0, x_2)\, dx_2$

H HEAVISIDE $1_{\{x_1 \geq 0\}}$

L CORNER $1_{\{x_1 > 0\}} 1_{\{x_2 > 0\}}$

S $1_{\{\|x\|_\infty \leq 1\}}$

B $1_{\{\|x\|_2 \leq 1\}}$

WF

$\delta \qquad \{0\} \times [0, 2\pi)$

$V \qquad \{((0, x_2), 0) : x_2 \in \mathbb{R}\}$

$H \qquad \{((0, x_2), 0) : x_2 \in \mathbb{R}\}$

$L \qquad \{((0, x_2), 0) : x_2 > 0\}$

$\qquad \cup \{((x_1, 0), \pi/2) : x_2 > 0\}$

S

$B \qquad \{((\cos(\theta), \sin(\theta)), \theta) \quad \theta \in (0, 2\pi)\}$

CAN THIS NOTION OF DW
"RESOLVE WAVEFRONT SET"??

RECALL: RESOLVE SS(f):

$$\mathcal{R} = \{x : DW_f(a,b,\theta) \text{ of rapid decay near } b = x\}$$

$$SS(f) = \mathcal{R}^c.$$

HOWEVER, IN GENERAL,

$DW_f(a,b,\theta)$ not of rapid decay off of $WF(f)$!

(6) DIRECTIONAL WAVELETS

φ radial oscillatory

$\tilde{\varphi}(x_1, x_2) = \varphi(10 x_1, x_2/10)$

directionally - sensitive

$\tilde{\varphi}_{ab\theta}(x_1, x_2) = \tilde{\varphi}(R_\theta(x-b)/a)/a$

$\uparrow$ DIRECTIONAL PARAMETER

$\widetilde{DW}_f(a,b,\theta) = \langle \tilde{\varphi}_{ab\theta}, f \rangle$

DIRECTIONAL WAVELET TRANSFORM

IN SPIRIT, SIMILAR TO EXISTING TRANSFORMS....

Example

$$\langle \nu, f \rangle = \int f(0, x_2)\, dx_2$$

$$\widetilde{DW}_\nu(a, 0, \theta) = \int \hat{\tilde{\varphi}}_{a,0,\theta}(\xi_1, 0)\, d\xi_1$$

$$= \int \hat{\tilde{\varphi}}(a\xi_1, 0-\theta)\, a\, d\xi_1$$

$$= \int \hat{\tilde{\varphi}}(u, 0-\theta)\, du = \Phi(\theta)$$

$\Phi(\theta)$ smooth (not sharp)

Not of rapid decay in directions $\theta \in$ ~~int~~ supp Φ.

CCT (WPT)

$$\Gamma_f(a,b,\theta) = \langle \gamma_{ab\theta}, f \rangle$$

$a < a_0$

$b \in \mathbb{R}^2$

$\theta \in [0, 2\pi)$

f a high pass function

$$f(x) = \int \Gamma_f(a,b,\theta)\, \gamma_{ab\theta}(x)\, \mu(da, db, d\theta)$$

$$\|f\|_2^2 = \int |\Gamma_f(a,b,\theta)|^2\, \mu(da\, db\, d\theta)$$

$$d\mu = \frac{da}{a^2}\, db\, d\theta$$

(7) How to get directional transform resolving WF ?

$W(r), \quad r \in (1/2, 2)$

$V(t), \quad t \in (-1/2, 1/2)$

ADMISSIBILITY

$$\int_0^\infty W(ar)^2 \frac{da}{a} = 1 \quad \forall r > 0$$

$$\int_{-1}^{1} V(u)^2 \, du = 1$$

Wave Packet

$$\hat{\gamma}_{a00}(r, \omega) = W(ar) \, V(\omega/\sqrt{a}) \, a^{3/4}$$

$$0 < a < a_0$$

$\gamma_{ab\theta}$ by translation, rotation

Notice scaling.

Thm.

$$\mathcal{R} = \{ x_0 : \ \Gamma_f \text{ decays rapidly near } (x_0, \theta_0) \text{ as } a \to 0 \}$$

$$WF(f) = \mathcal{R}^c$$