IPAM MGA Tutorial: Geometry–Conscious Techniques in Signal Representation

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Outline

- Importance of Geometry/Edges in Image Representations
- ENO Wavelets (Chan & Zhou)
- Polyharmonic Local Trig. Transforms (Saito et al.)
- Multiscale Inpainting Transforms (A. Cohen et al.)
- Summary

Acknowledgment

- Tony Chan
- Albert Cohen
- Hao-Min Zhou
- IPAM/UCLA
- NSF & ONR

Geometry/Edges: Still an issue after all these years

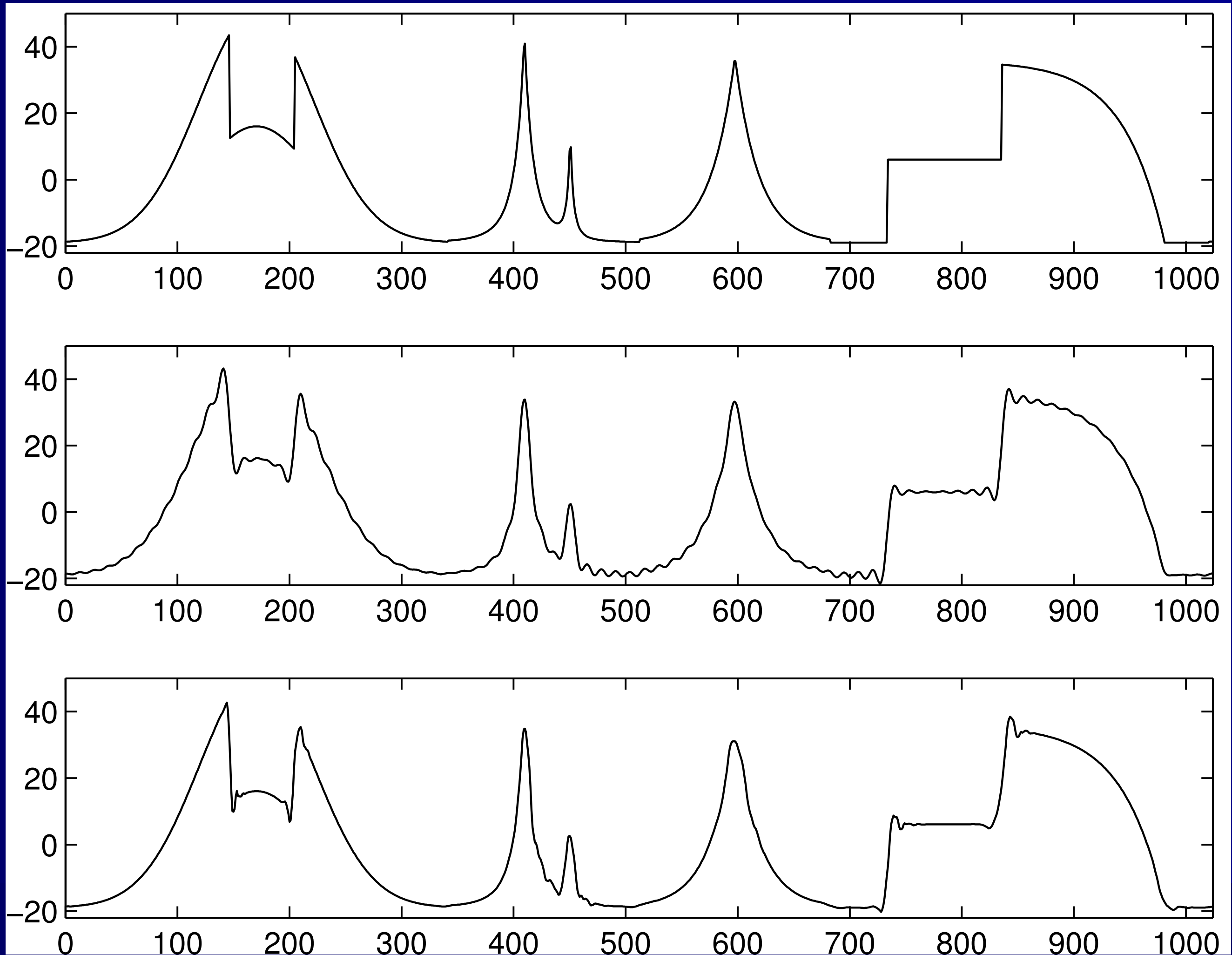
- Image representation, approximation, compression, denoising $\Longrightarrow$ geometry/edges make them tougher
- Object detection, object description, shape analysis $\Longrightarrow$ geometry/edges are indispensable

Purpose of This Tutorial

- Introduce a few recent methods that deals with edges and geometry proactively
- Not include wedgelets, ridgelets, curvelets (already covered by Donoho-Candès) bandlets (will be discussed by Mallat in WS 1), surflets (will be discussed by Baraniuk tomorrow?), and others

Gibbs Phenomenon around Discontinuities

Linear Approximation using 16% of the coefficients



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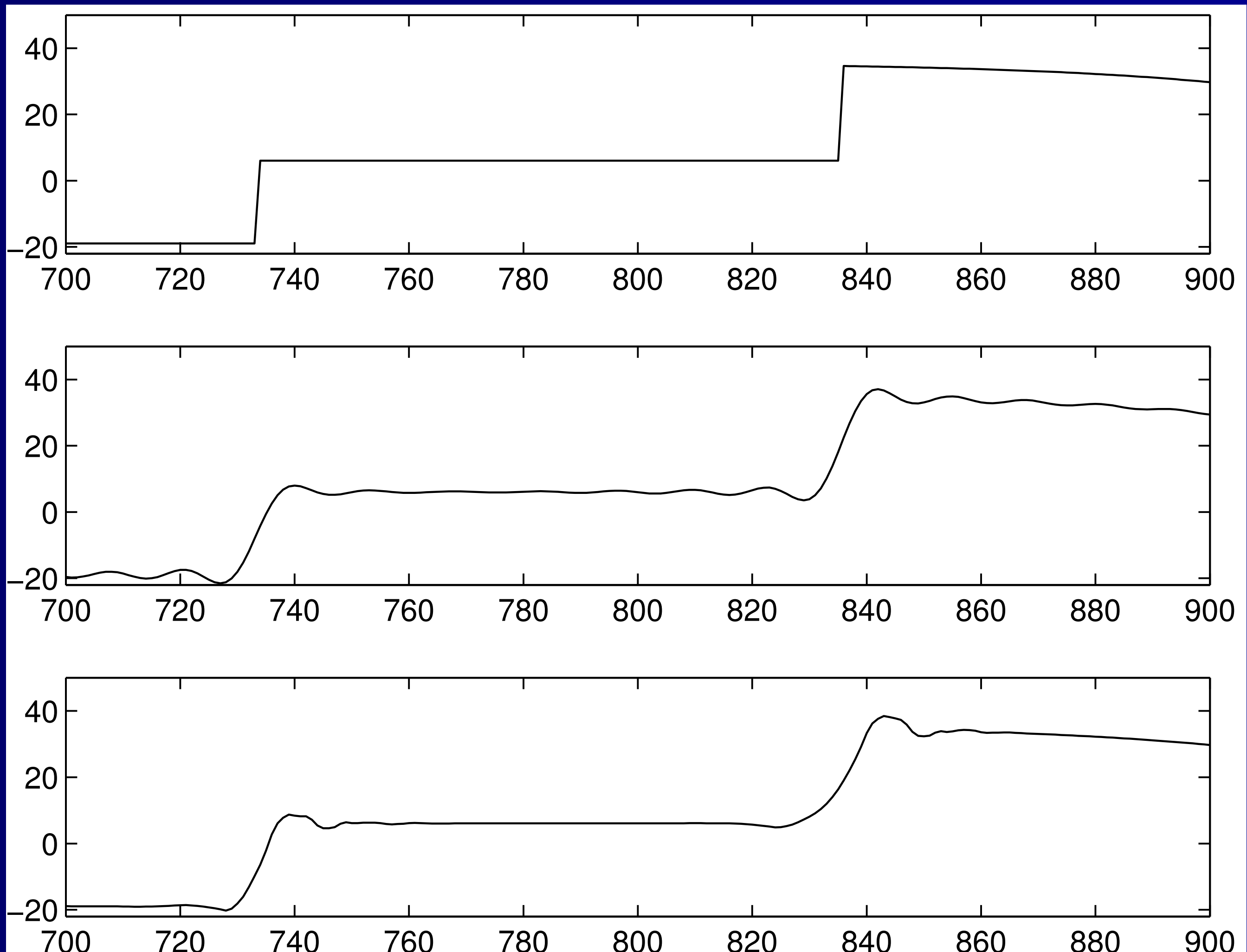


Essentially-Non-Oscillatory (ENO) Wavelets of Chan & Zhou

- Objective: Approximate discontinuous functions without oscillations near the discontinuities without too much computational burden
- Strategy: Avoid computing the wavelet coefficients using data from both sides of the discontinuities

ENO Wavelet Transform Algorithm

- Step 0: Set $j = J$ (the finest scale)
- Step 1: Apply the standard wavelet transform to compute the coefficients at scale $j-1$ using the low frequency coefficients at scale j
- Step 2: Detect discontinuities of the low frequency coefficients at scale j using high frequency coefficients scale $j-1$
- Step 3: Extrapolate the scale $j-1$ coefficients and modify them around the edges
- Step 4: Set $j \leftarrow j-1$; if $j = j_0$ (desired coarsest level), then stop; otherwise go to Step 1

ENO Wavelets: Setup

- Let $H = \{h_\ell\}_{\ell=0}^{L-1}$, $G = \{g_\ell\}_{\ell=0}^{L-1}$ be the pair of conjugate mirror of the standard Daubechies wavelet with $p = L/2$ vanishing moments.
- Let j be the scale (level) with $j = J > 0$ is the finest scale and $j = j_0 < J$ is the coarsest scale.
- Let $\alpha_{j,k}$, $\beta_{j,k}$ be the low and high frequency wavelet coefficients at scale j, respectively. Let

$$\alpha_{j,k} = \sum_{\ell=0}^{L-1} h_\ell \alpha_{j+1,2k+\ell}, \qquad \beta_{j,k} = \sum_{\ell=0}^{L-1} g_\ell \alpha_{j+1,2k+\ell},$$

with $j = J-1, J-2, \cdots, j_0$, and $\alpha_{J,k} = f(k/2^J)$.

ENO Wavelets: Discontinuity Detection

- Let $I_{j,k}$ be an interval at scale $j+1$ where $\alpha_{j,k}$ and $\beta_{j,k}$ are computed. Note $I_{j,k-1} \cap I_{j,k} \neq \emptyset$ if $L \geq 4$.
- In the smooth region, we have

$$|\beta_{j,k}| = (1 + O(\Delta x_j))|\beta_{j,k-1}|,$$

where $\Delta x_j = 2^{-j}$.

- If the filter stencil contains a discontinuity in $f^{(m)}(x)$ for some m in $[0, p)$, then

$$|\beta_{j,k}| = |f^{(m)}(x_0+) - f^{(m)}(x_0-)|O(\Delta x_j^m).$$

ENO Wavelets: Discontinuity Detection . . .

- Thus, the detection algorithm should be:
  - If $I_{j,k-1}$ does not contain any discontinuity and $|\beta_{j,k}| \le \tau_j |\beta_{j,k-1}|$ for some $\tau_j > 1$, then $I_{j,k}$ does not contain any discontinuity either.
  - Otherwise, $I_{j,k}$ contains discontinuities.
- Discontinuities cannot be too closely located; must be at least $(L+1)\Delta x_{j+1}$ apart.

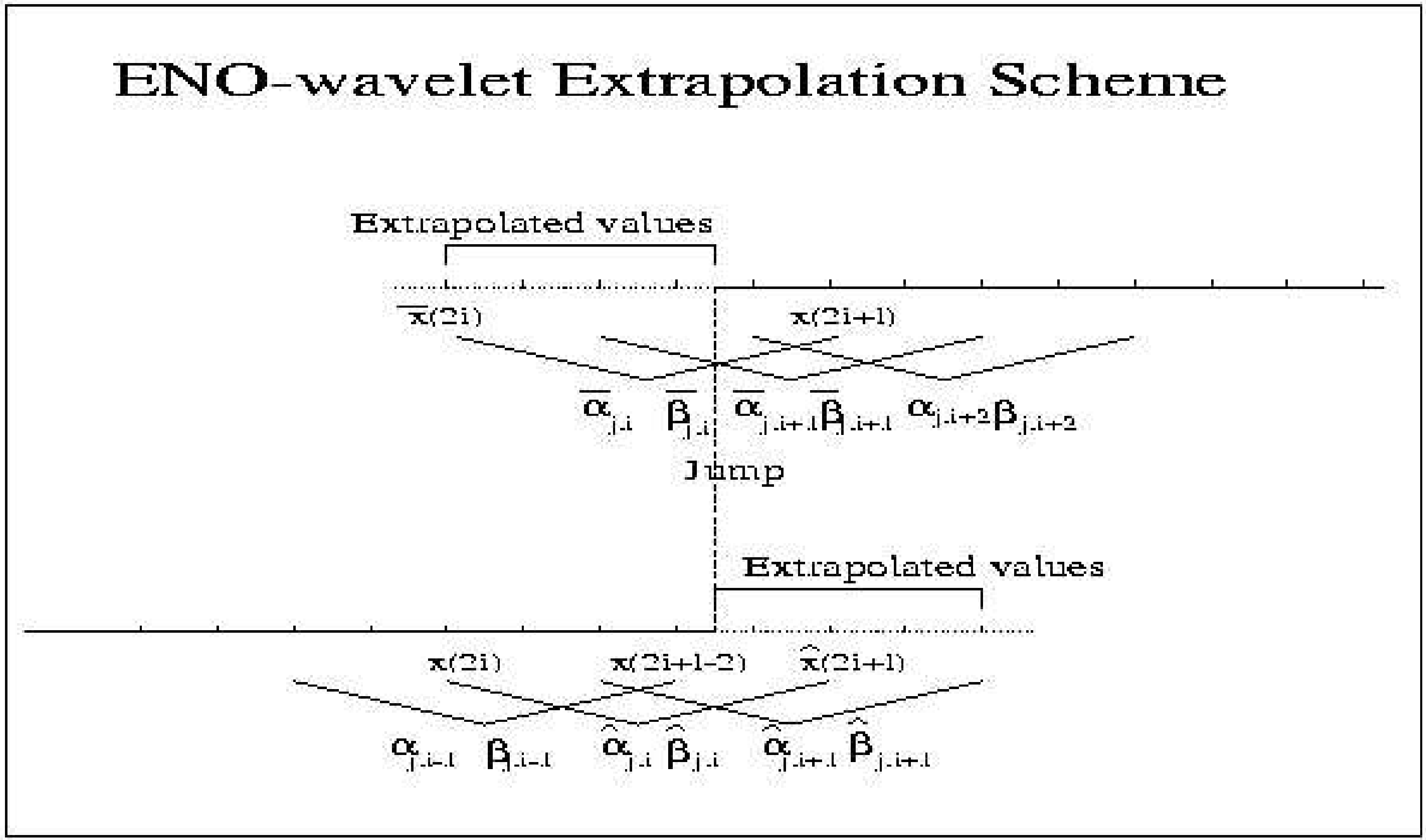
ENO Wavelets: Extrapolation of Coefficients

- At each discontinuity, need to extrapolate the coarse scale coefficients from left to right and from right to left.
- For the wavelet with p vanishing moments, use a polynomial degree $p-1$ to extrapolate those from one side of the discontinuity to the other side.
- Can use either the exact fit or the least square fit

Why Extrapolate Coarse Scale? ...

- Naïve approach: extrapolate the original function or finer scale low frequency coefficients directly. Suppose the discontinuity is located between α_{j+1,K_ℓ} and α_{j+1,K_r}, where $K_\ell = 2k + L - 3$, $K_r = 2k + L - 2$.
- Using the left side info $\{\alpha_{j+1,m}\}_{m=K_\ell-p+1}^{K_\ell}$, extrapolate into the right side to get $\{\hat{\alpha}_{j+1,m}\}_{m=K_r}^{K_r+L-2}$.
- Using the right side info $\{\alpha_{j+1,m}\}_{m=K_r}^{K_r+p-1}$, extrapolate into the left side to get $\{\bar{\alpha}_{j+1,m}\}_{m=K_\ell-L+2}^{K_\ell}$.
- Then we can compute $\{\hat{\alpha}_{j,m}, \hat{\beta}_{j,m}\}$ and $\{\bar{\alpha}_{j,m}, \bar{\beta}_{j,m}\}$ both for $k \le m \le k + p - 2$.

ENO Wavelets: Extrapolation of Coefficients . . .



Why Extrapolate Coarse Scale? ...

From left to right:

$$
\begin{bmatrix} \hat{\alpha}_{j,k} \\ \hat{\beta}_{j,k} \end{bmatrix} = \begin{bmatrix} \sum_{\ell=0}^{L-3} h_\ell \alpha_{j+1,2k+\ell} + h_{L-2}\hat{\alpha}_{j+1,2k+L-2} + h_{L-1}\hat{\alpha}_{j+1,2k+L-1} \\ \sum_{\ell=0}^{L-3} g_\ell \alpha_{j+1,2k+\ell} + g_{L-2}\hat{\alpha}_{j+1,2k+L-2} + g_{L-1}\hat{\alpha}_{j+1,2k+L-1} \end{bmatrix}
$$

$$
= \begin{bmatrix} \delta_{j,k} \\ \gamma_{j,k} \end{bmatrix} + A \begin{bmatrix} \hat{\alpha}_{j+1,2k+L-2} \\ \hat{\alpha}_{j+1,2k+L-1} \end{bmatrix}, \quad A = \begin{bmatrix} h_{L-2} & h_{L-1} \\ g_{L-2} & g_{L-1} \end{bmatrix}.
$$

Can do similarly from right to left.

ENO Wavelets: Coarse Scale Extrapolation

- Storage inefficiency (i.e., twice as many coefficients around the discontinuities)
- Solution: extrapolate the scale j coefficients and modify them within scale j
- Because A is singular for the standard Daubechies wavelets, $\hat{\alpha}_{j,k}$ ($\bar{\alpha}_{j,k}$) and $\hat{\beta}_{j,k}$ ($\bar{\beta}_{j,k}$) are not independent.
- In fact, we have:

$$\begin{bmatrix} \hat{\alpha}_{j,k} \\ \hat{\beta}_{j,k} \end{bmatrix} - \begin{bmatrix} \delta_{j,k} \\ \gamma_{j,k} \end{bmatrix} \in \mathcal{R}(A).$$

ENO Wavelets: Coarse Scale Extrapolation

- From this, we can determine $\hat{\beta}_{j,k}$ if $\hat{\alpha}_{j,k}$ is known, and vice versa.
- Thus, the following strategy is used:
  - L$\rightarrow$R: Extrapolate $\hat{\alpha}_{j,k}$, then determine $\hat{\beta}_{j,k}$.
  - R$\rightarrow$L: Set $\bar{\beta}_{j,k} = 0$, then determine $\bar{\alpha}_{j,k}$.
- No need to store $\hat{\alpha}_{j,k}$ and $\bar{\beta}_{j,k}$, which are always recoverable.
- An indicator (binary variable) for discontinuity locations needs to be stored for the inverse transform.

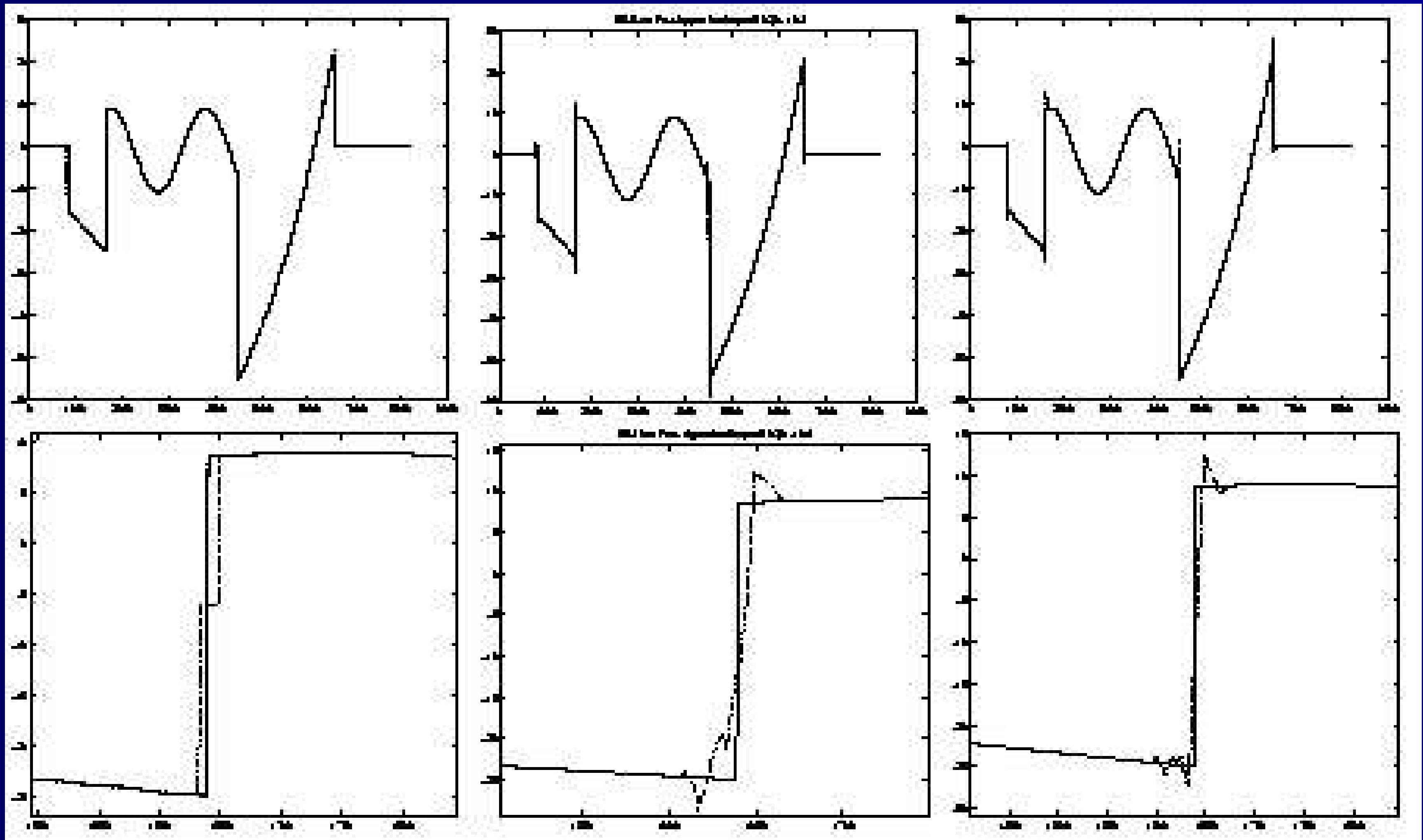
ENO Wavelets: Approximation Theorem

Theorem: Let $f(x)$ be a piecewise continuous function over Ω with bounded pth derivatives in each piece of smooth region, and let $f_j(x)$ be its jth level ENO-wavelet projection. If the minimum distance between any two consecutive discontinuities in f larger than $(L+1)\Delta x_{j+1}$, then

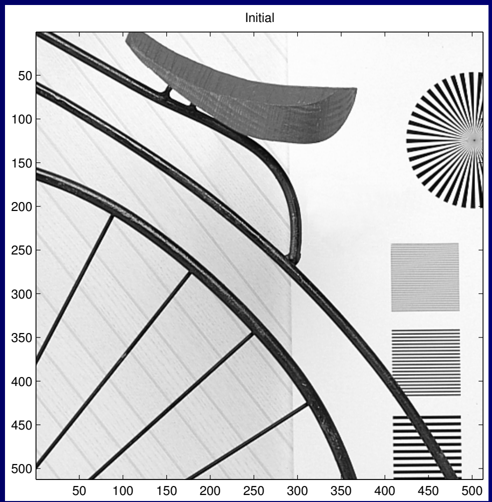
$$\|f - f_j\|_{L^q(\Omega)} \leq C(\Delta x_j)^p \|f^{(p)}\|_{L^q(\Omega \setminus K)},$$

where K is set of points of discontinuities in f, and $q = 2, \infty$.

ENO Wavelets: 1D Case



ENO Wavelets: 2D Case (Linear)



(a) Original

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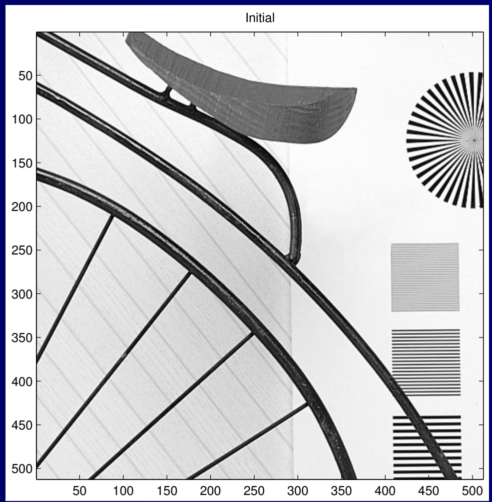

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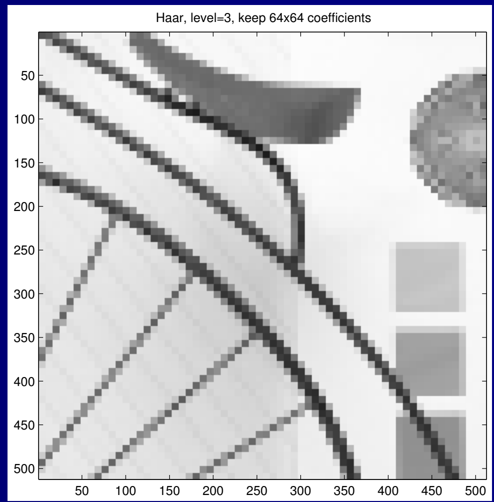

(b) Haar

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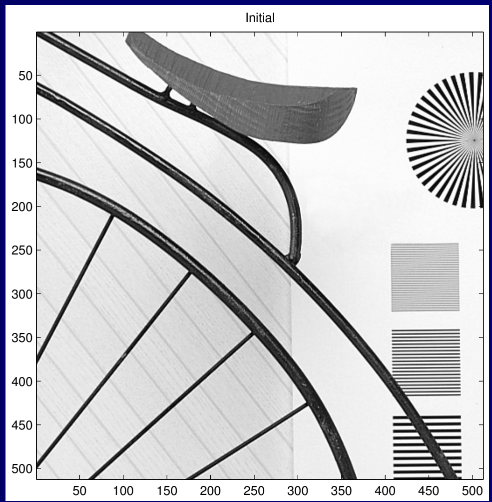

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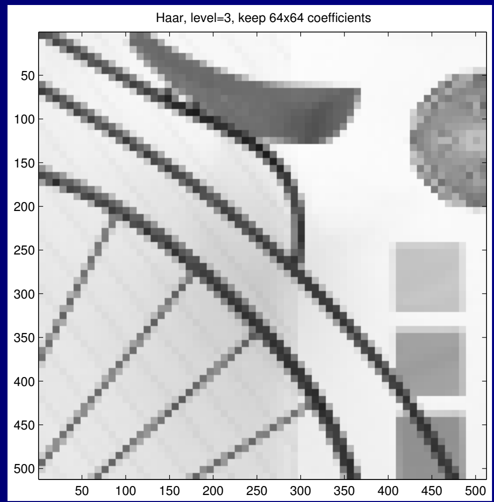

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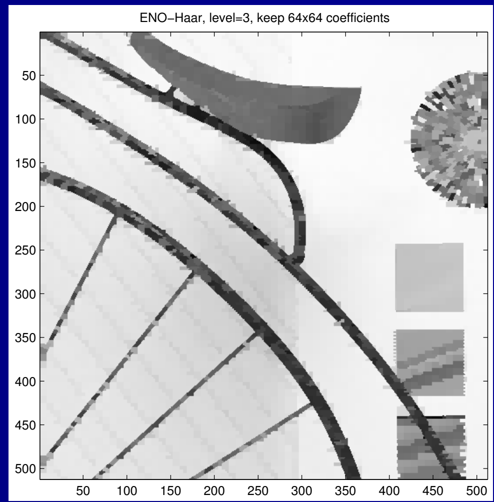
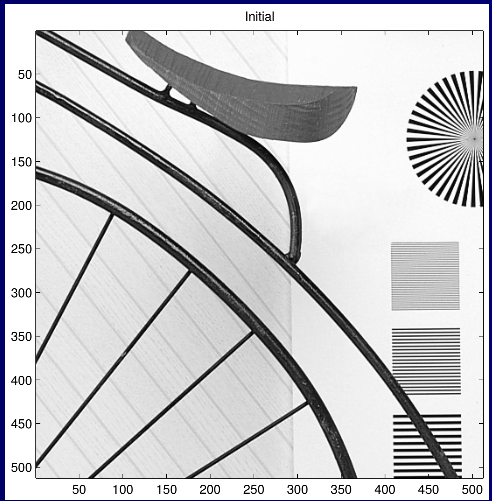


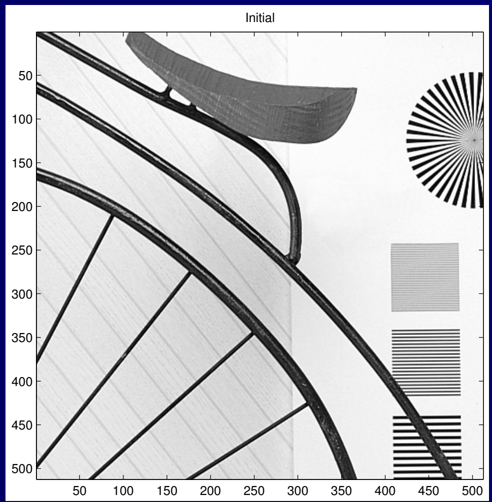
(c) ENO Haar

ENO Wavelets: 2D Case (Nonlinear)



(a) Original

ENO Wavelets: 2D Case (Nonlinear)



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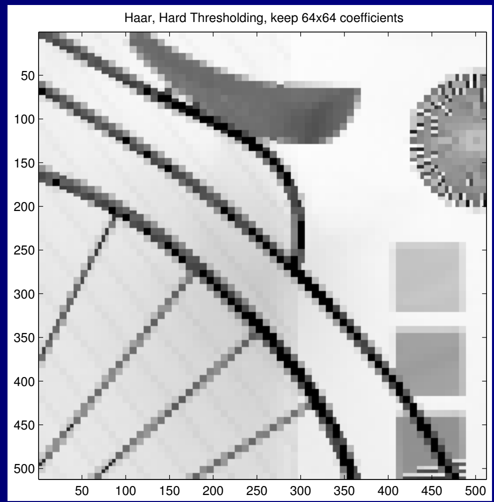

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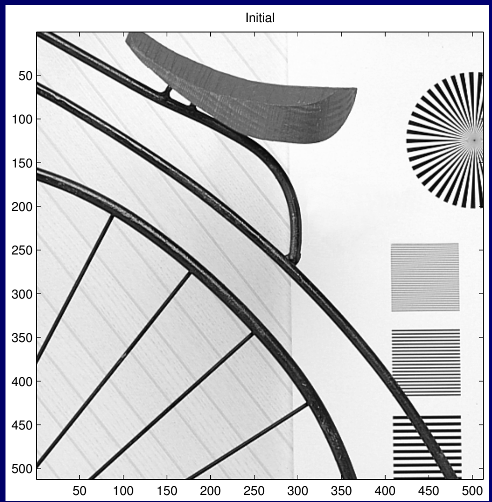

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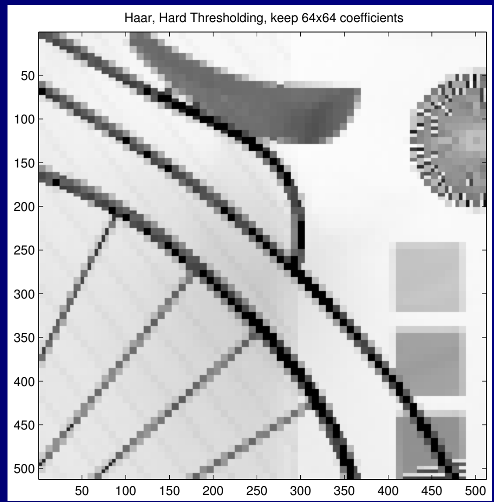

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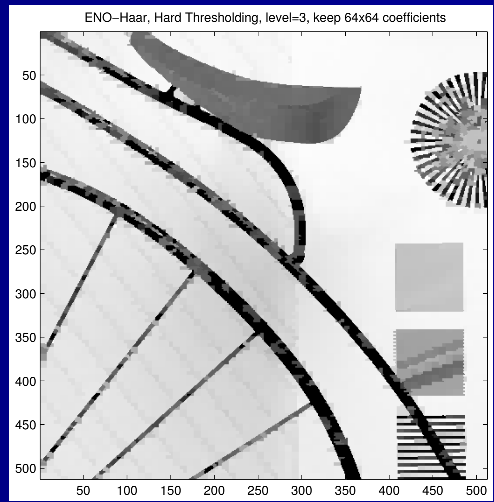

(c) ENO Haar

ENO Wavelets: Remarks

- Difficulty of detecting edges reliably in noisy signals
- Dealing 2D edges in tensor product of 1D
- Averbuch & Coifman had a similar idea: image enhancement via wavelet zoom-in, filling the missing fine scale coefficients via estimation from coarser scale coefficients.

ENO Wavelets: Remarks . . .

- A. Gelb, E. Tadmor, J. Tanner: discontinuity detection, approximation of piecewise smooth function using filters & molifiers on the Fourier domain
- G. Beylkin, NS, X. Shen, H. Xiao: Use of prolate spheroidal wave functions for piecewise BL functions
- Suggestion: organize a half day workshop on this issue with Greg Beylkin, Jared Tanner, Hong Xiao, and others as speakers.

Polyharmonic Local Sine Transform (PHLST)

- Want to do Fourier analysis of data locally
- Want to get rid of the edge effect and Gibbs phenomenon
- Do not want any overlaps among blocks (not like local cosines)
- Want to efficiently represent regions of more general shapes other than rectangular blocks

Basic Ideas of PHLST

- Split the domain Ω into a set of subregions (often rectangular blocks) $\{\Omega_j\}_j$ and cut the data/function f into pieces using the characteristic functions, $f_j = \chi_{\Omega_j} f$

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- Sines and cosines are eigenfunctions of the Laplacian on a rectangular box

1D Case

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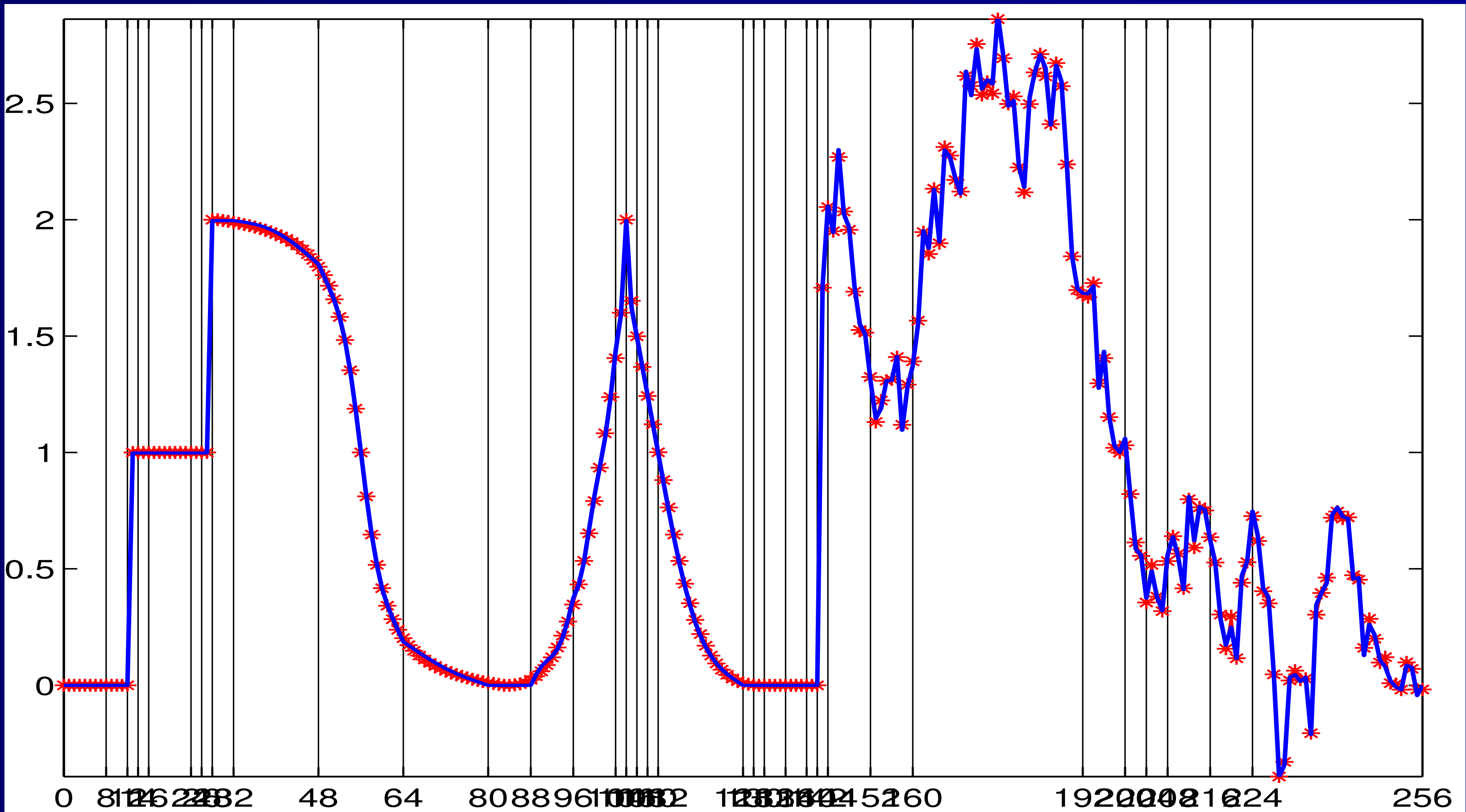
- v_j is a trigonometric polynomial
- These two compensate the shortcomings of each other:
- High order algebraic polynomial $\Longrightarrow$ Runge phenomenon
- Trigonometric polynomial on an interval $\Longrightarrow$ Gibbs phenomenon

1D Example: Compression Ratio ≈ 6



Polyharmonic Global Sine Transform on a Rectangle

- Consider a function $f \in C^{2m}(\overline{\Omega})$, where $m = 1, 2, \ldots$, and $\Omega \subset \mathbb{R}^n$ (e.g., $\Omega = [0,1]^n$), but not periodic.

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$$f(\boldsymbol{x}) = u(\boldsymbol{x}) + v(\boldsymbol{x}),$$

Polyharmonic Global Sine Transform on a Rectangle

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- Decompose this function into the following two components:

$$f(\boldsymbol{x}) = u(\boldsymbol{x}) + v(\boldsymbol{x}),$$

- $u(\boldsymbol{x})$ satisfies the following polyharmonic equation:

$$\Delta^m u = 0 \quad \text{in } \Omega.$$

PHGST on a Rectangle . . .

- The boundary condition for u is:

$$\frac{\partial^{p_\ell} u}{\partial\, \nu^{p_\ell}} = \frac{\partial^{p_\ell} f}{\partial\, \nu^{p_\ell}} \quad \text{on } \Gamma = \partial\Omega, \qquad \ell = 0, \ldots, m-1,$$

where p_ℓ is the order of the normal derivatives to be specified ($p_0 \equiv 0 \Leftrightarrow u = f$ on Γ).

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  where p_ℓ is the order of the normal derivatives to be specified ($p_0 \equiv 0 \Leftrightarrow u = f$ on Γ).

- Now set $v(\boldsymbol{x}) = f(\boldsymbol{x}) - u(\boldsymbol{x})$, which we will call the residual component with

$$\frac{\partial^{p_\ell} v}{\partial\, \nu^{p_\ell}} = 0 \quad \text{on } \Gamma, \qquad \ell = 0, \ldots, m-1.$$

A Specific Example: $m = 1$ (Laplace) Case

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = f & \text{on } \Gamma. \end{cases}$$

Variational formulation $\Longrightarrow$ minimum gradient interpolation:

$$\min_{u \in H^1(\Omega)} \int_\Omega |\nabla u|^2 \, \mathrm{d}\boldsymbol{x} \quad \text{subject to the above boundary condition.}$$

Note that in 1D, this is simply a line.

A Specific Example: $m = 2$ (Biharmonic) Case

$$\begin{cases} \Delta^2 u = 0 & \text{in } \Omega, \\ u = f, \; \dfrac{\partial^2 u}{\partial \nu^2} = \dfrac{\partial^2 f}{\partial \nu^2} & \text{on } \Gamma. \end{cases}$$

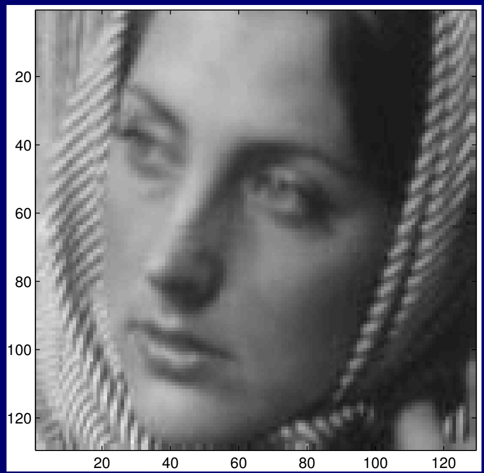
Variational formulation $\Longrightarrow$ minimum curvature interpolation:

$$\min_{u \in H^2(\Omega)} \int_\Omega \left(\Delta u + 2 \sum_{j \neq k} \partial_j \partial_k u \right)^2 \mathrm{d}\boldsymbol{x},$$

subject to the above boundary condition.

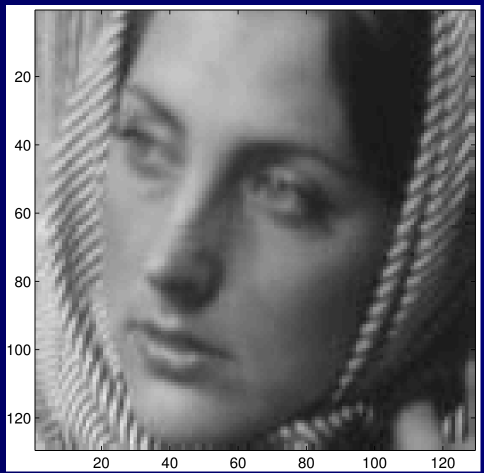
Note that in 1D, this is simply a cubic polynomial.

PHGST on a Rectangle . . .



(a) Original

PHGST on a Rectangle . . .



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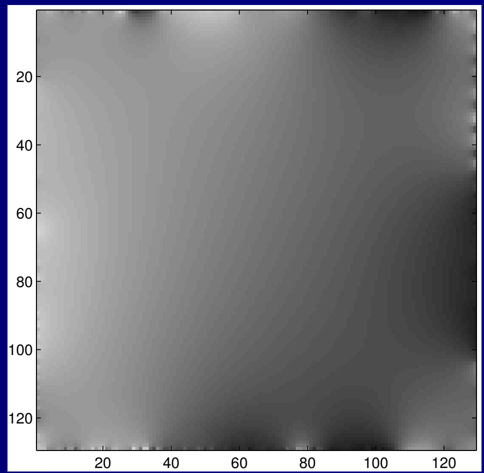
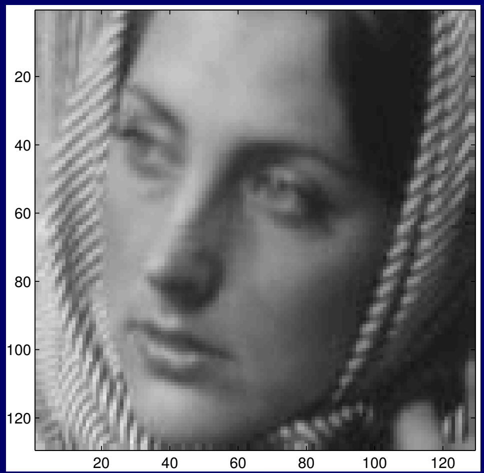


(b) u component

PHGST on a Rectangle ...



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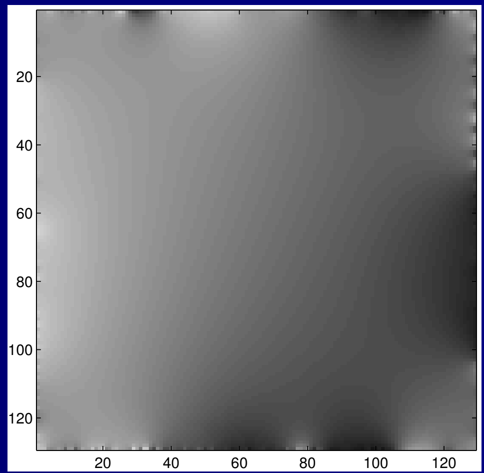

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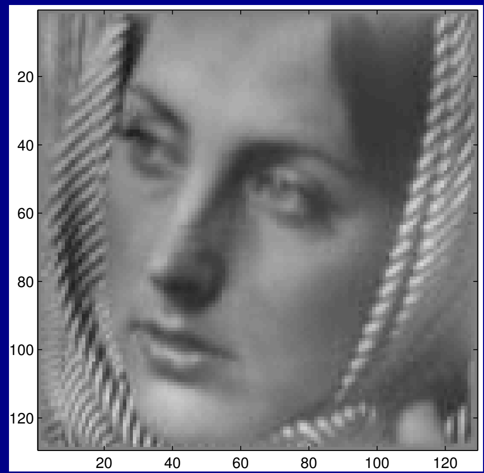
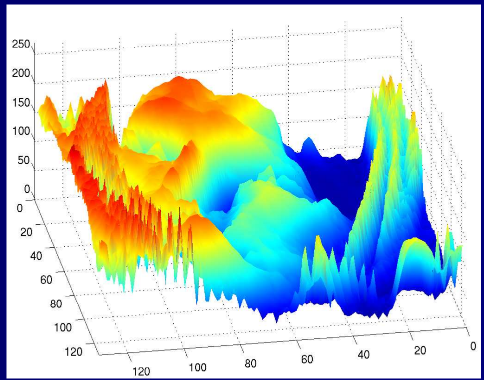


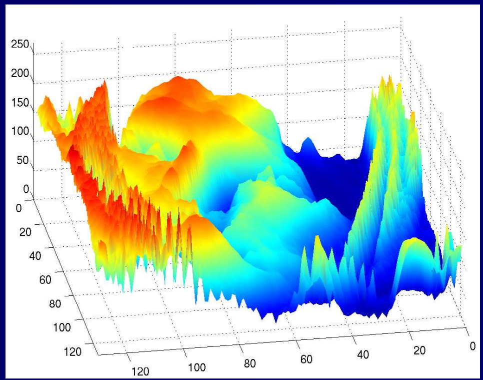
(c) v component

PHGST on a Rectangle . . .



(a) Original

PHGST on a Rectangle . . .



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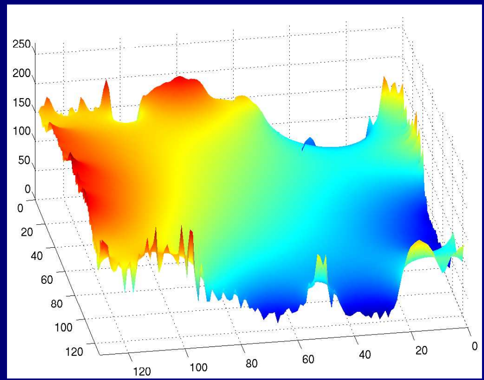
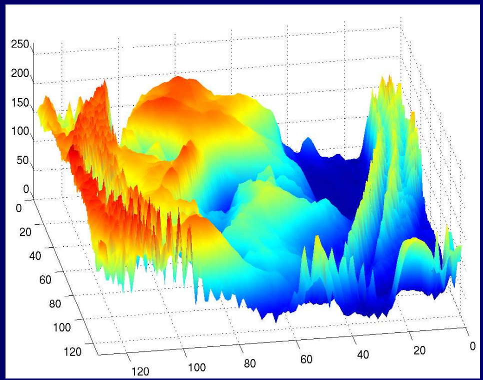


(b) u component

PHGST on a Rectangle . . .



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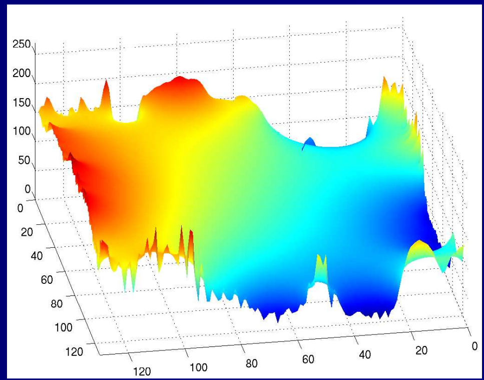
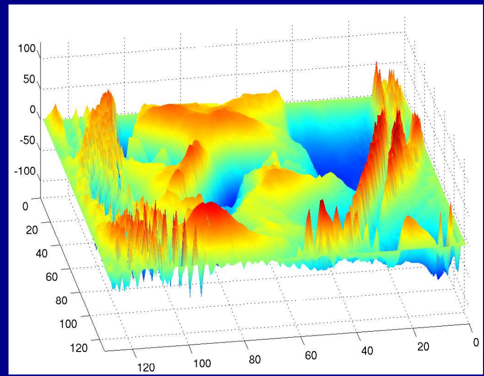


(b) u component



(c) v component

PHGST on a Rectangle . . .

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- The polyharmonic component u is smooth inside the domain Ω, and its values (and possibly its normal derivatives) matches those of data.
- The u component can be represented only by the boundary values $f|_\Gamma$. $\Longrightarrow$ No need to store the whole u.
- The residual component v becomes 0 at the boundary Γ.

PHGST on a Rectangle . . .

- Therefore, the v component is suitable for Fourier analysis. In fact, if $\Omega = [0, 1]^n$ and $p_\ell = 2\ell$, $\ell = 0, \ldots, m-1$, then the Fourier sine analysis should be used to get the matching normal derivatives up to order $2m-1$ by odd reflection at the boundaries.

PHGST on a Rectangle . . .

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- The frequency content (in particular, mid to high frequency range) of the original is retained in the residual $\Longrightarrow$ textures remain in v; shading is captured by u.
- We can get the decay rate as follows:

Theorem

Let $\Omega = [0,1]^n$, and $f \in C^{2m}(\overline{\Omega})$, but non-periodic. Assume further that $\partial_i^{2m+1} f$, $i = 1, \ldots, n$, exist and are of bounded variation. Furthermore, let $f = u + v$ be the PHLST representation where the polyharmonic component u is the solution of the polyharmonic equation of order m with the boundary condition

$$\frac{\partial^{2\ell} u}{\partial\, \nu^{2\ell}} = \frac{\partial^{2\ell} f}{\partial\, \nu^{2\ell}} \quad \text{on } \Gamma, \quad \ell = 0, \ldots, m-1.$$

Then, the Fourier sine coefficient $b_{\boldsymbol{k}}$ of the residual v is of $O\left(\|\boldsymbol{k}\|^{-2m-1}\right)$ for all $\boldsymbol{k} \neq \boldsymbol{0}$, where $\boldsymbol{k} = (k_1, \ldots, k_n)$, and $\|\boldsymbol{k}\|$ is the usual Euclidean (i.e., ℓ^2) norm of $\boldsymbol{k}$.

Polyharmonic Local Sine Transform

Now, consider a decomposition of Ω into a disjoint set of subdomains $\{\Omega_j\}$, i.e., $\overline{\Omega} = \cup_{j=1}^{J} \overline{\Omega}_j$. A typical example is $\Omega = (0,1)^n$, and Ω_j is a dyadic subcube. Then, restrict f on Ω_j, i.e., for each j, we decompose f locally as follows:

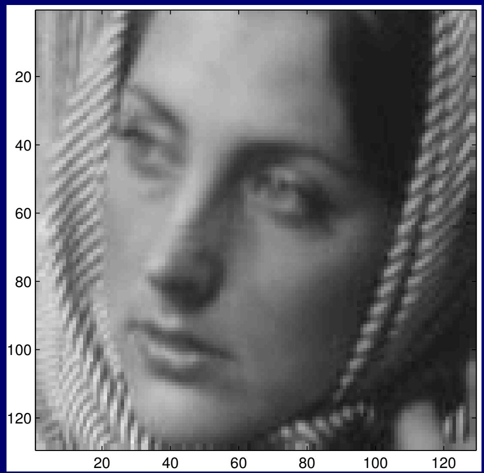
$$f \chi_{\Omega_j} = f_j = u_j + v_j,$$

where we follow the same recipe locally as in the global case. We call this decomposition of f into $\{u_j, v_j\}$ Polyharmonic Local Sine Transform (PHLST).

For $m = 1$: Laplace Local Sine Transform (LLST);

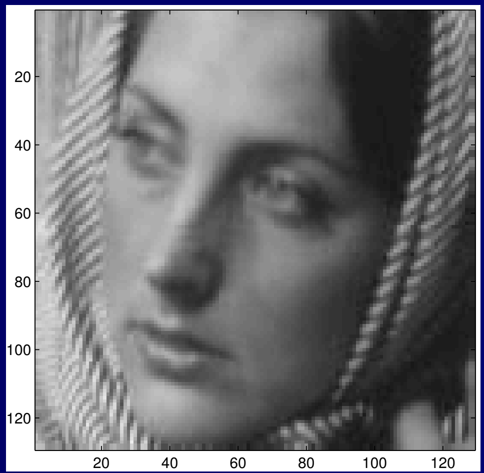
For $m = 2$: Biharmonic Local Sine Transform (BLST).

Polyharmonic Local Sine Transform ...

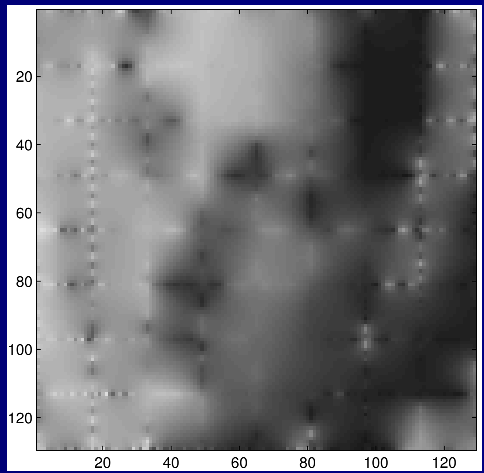


(a) Original

Polyharmonic Local Sine Transform . . .

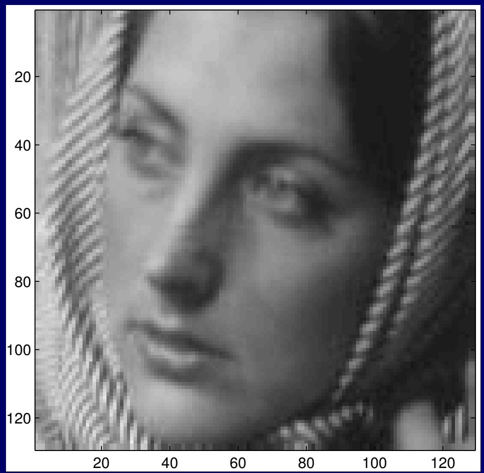


(a) Original

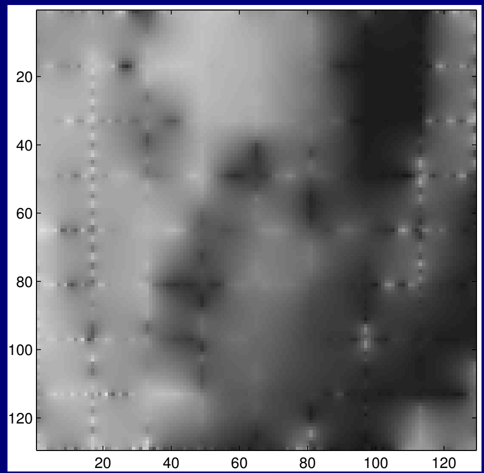


(b) $\cup_j u_j$

Polyharmonic Local Sine Transform . . .



(a) Original



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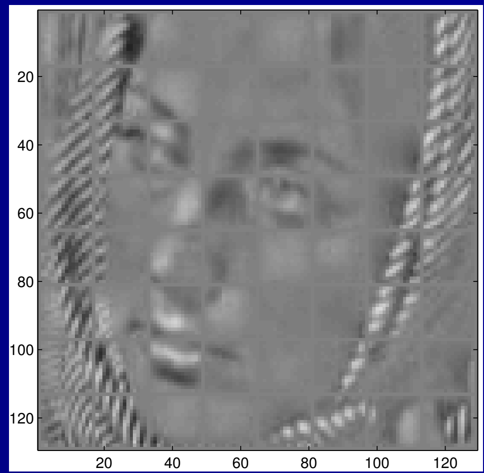

(c) $\cup_j v_j$

Polyharmonic Local Sine Transform . . .

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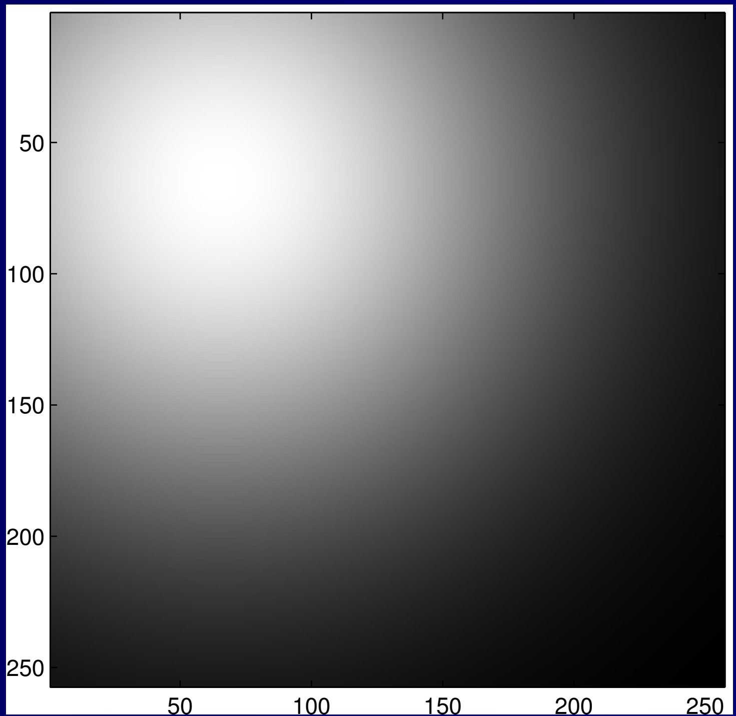
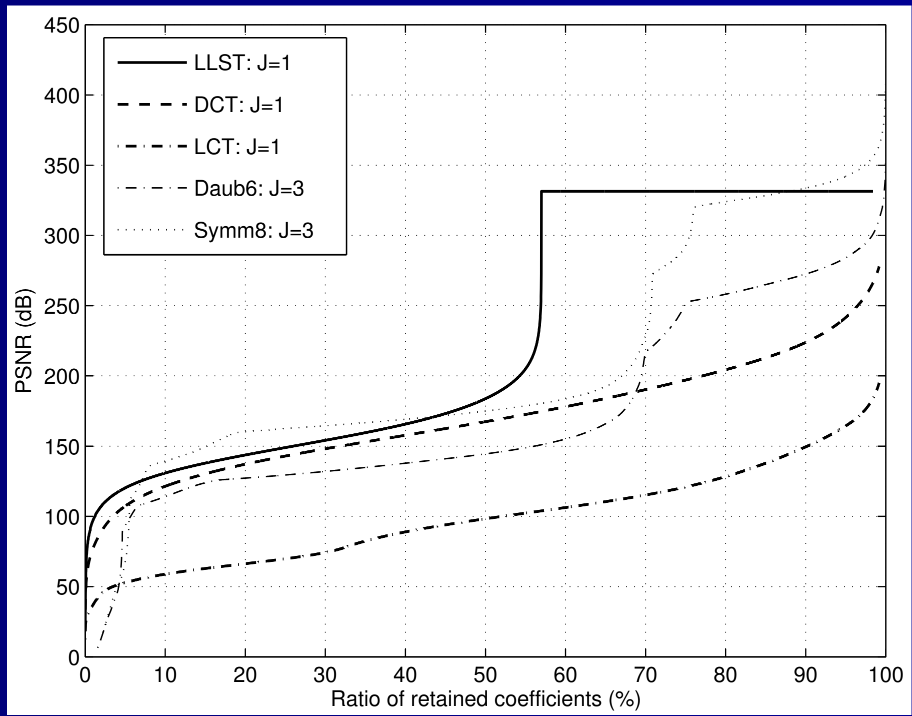
Polyharmonic Local Sine Transform . . .

- No spatial overlaps
- Decay of the Fourier sine coefficients are fast if Ω_j does not contain any singularity
- Can distinguish intrinsic singularities from the artificial discontinuities at Γ_j imposed by χ_{Ω_j}
- A fast and accurate (no finite difference approximation of the Laplace operator) algorithm (for both 2D and 3D) exists via FFT if Ω and Ω_j are dyadic cubes (Averbuch, Braverman, Israeli, and Vozovoi, 1998)

Approximation Test: Smooth Function

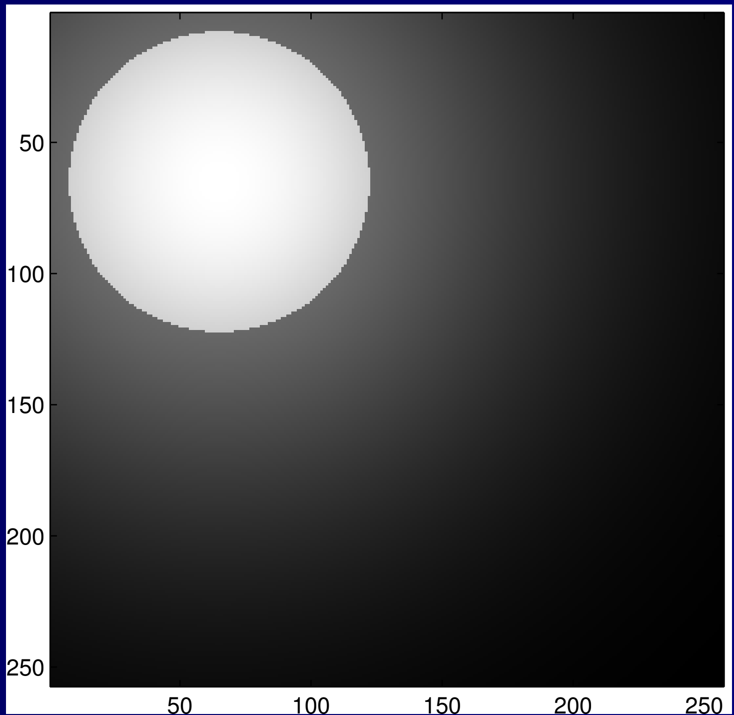


(a) Original

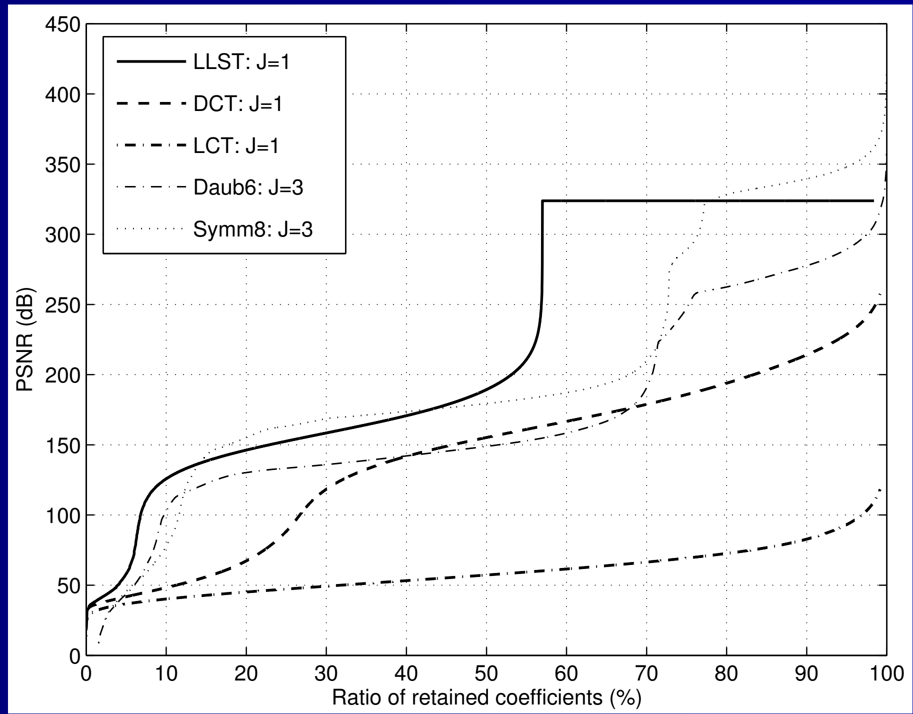


(b) PSNR

Approximation Test: Piecewise Smooth Function

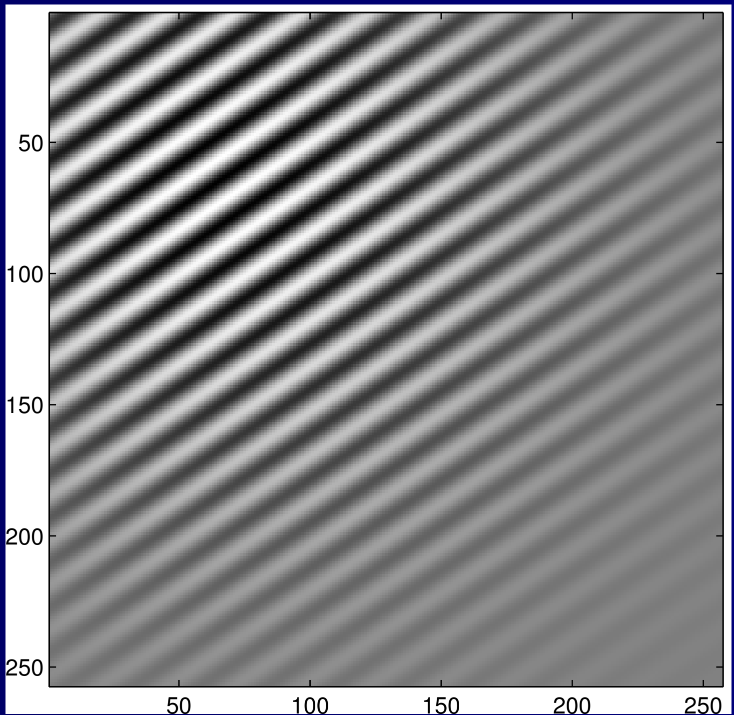


(a) Original

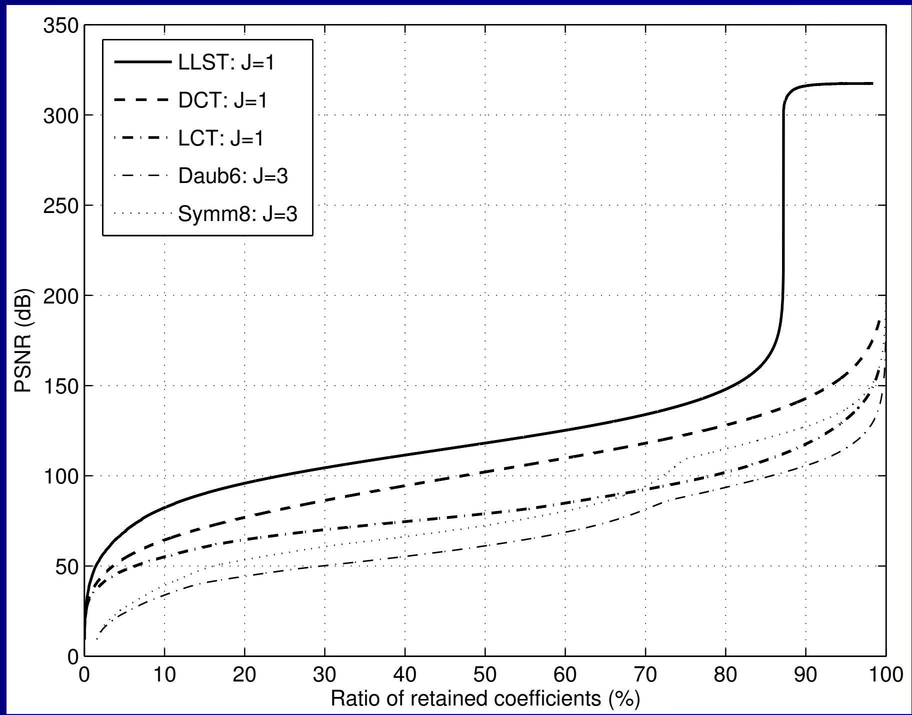


(b) PSNR

Approximation Test: Oscillatory Function

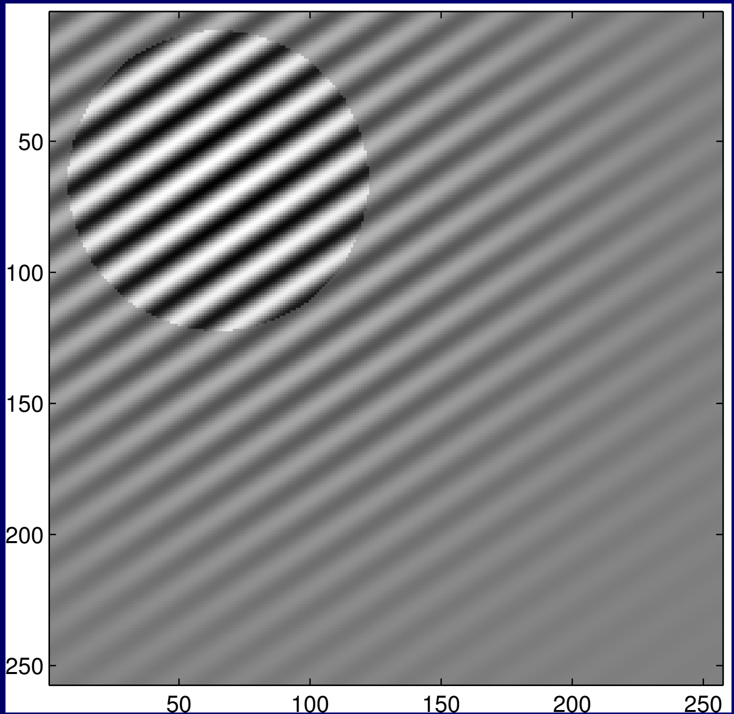


(a) Original

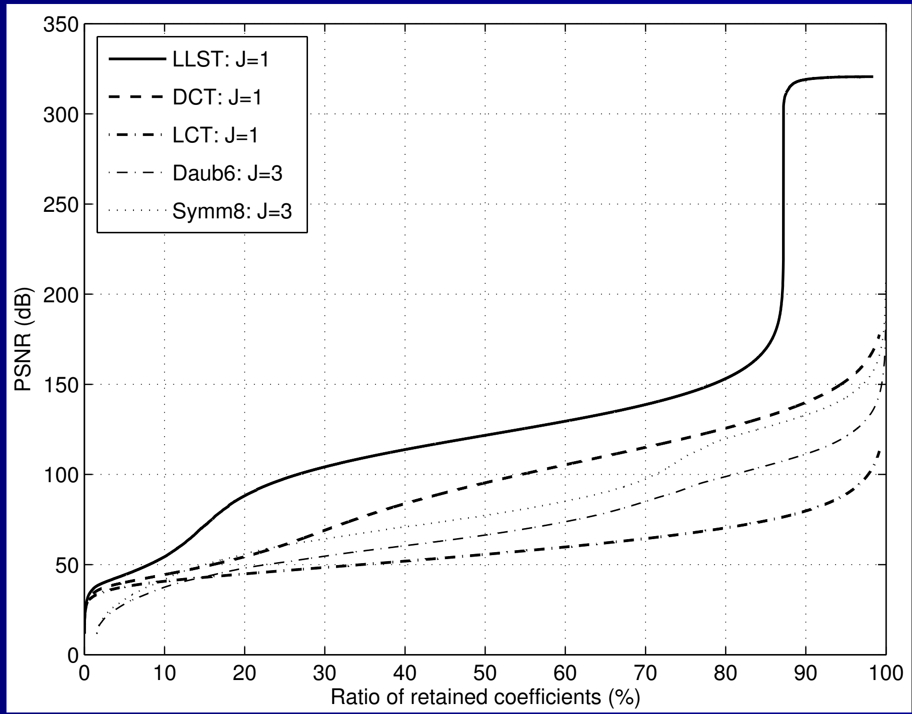


(b) PSNR

Approximation Test: Oscillatory Function with Discontinuity



(a) Original



(b) PSNR

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- Useful for interpolation and local feature computation (e.g., gradients, directional derivatives, etc.)

Multiscale Inpainting Transform (Cohen, Capricelli, Masnou, Romberg)

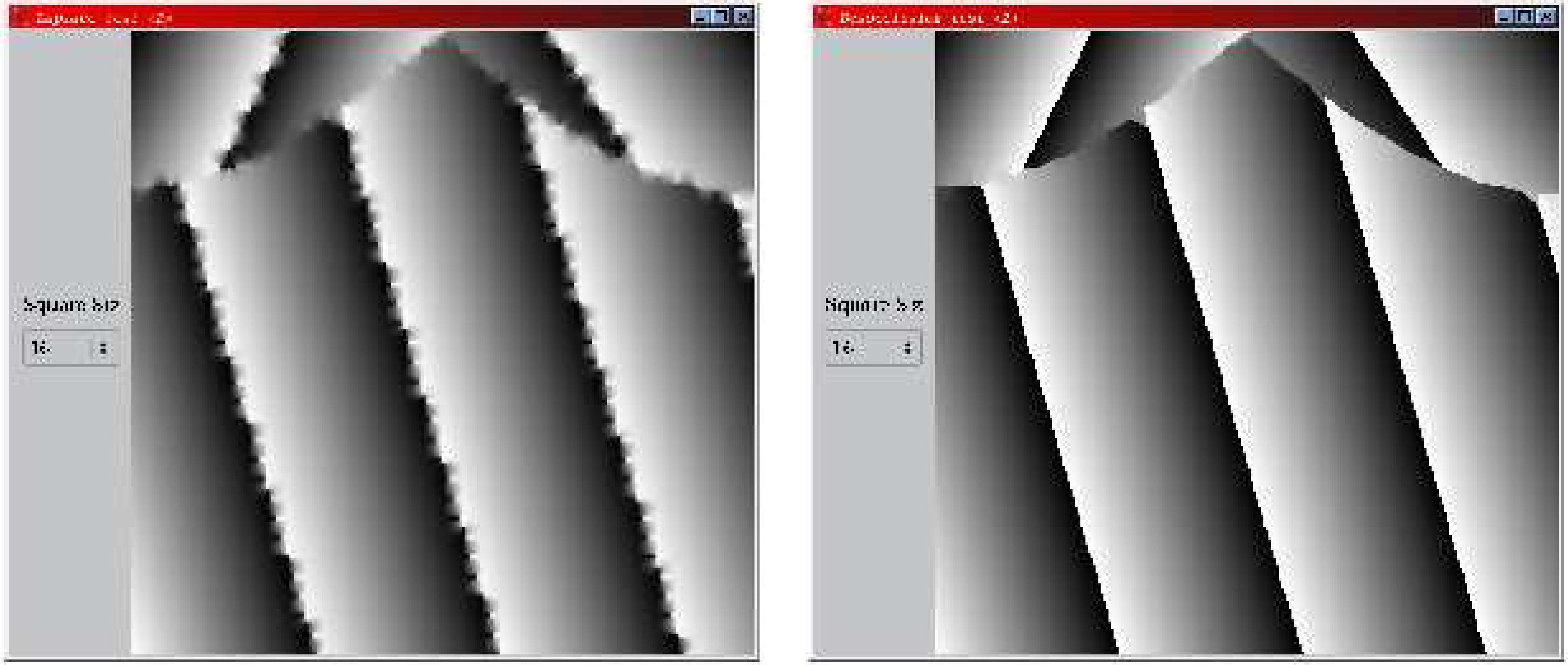
- In the case of LLST of Saito et al., the functional to be minimized is $J(u) = \int_\Omega |\nabla u|^2 \, \mathrm{d}x$.
- Instead, this approach uses the nonlinear inpainting algorithm using the functional $J(u) = \int_{-\infty}^{\infty} F(\partial E_t[u]) \, \mathrm{d}t$, where $E_t[u]$ is the level set of u at level t. Possible choices of F:
  - Level curve length: $F(\partial E_t[u]) = \int_{\partial E_t[u]} \mathrm{d}s$.
  - Level curve length + curvature: $F(\partial E_t[u]) = \int_{\partial E_t[u]} (1 + \mu|\kappa(s)|^p) \, \mathrm{d}s$. If $p = 1$, then the level curves of the solution are straight lines.
- Unfortunately, the solution is not unique and unstable.

Multiscale Inpainting Transforms . . .

- Use the nonlinear inpainting algorithm to compute u components at each scale (prediction) $\approx$ Ami Harten's discrete MRA.
- v components are the prediction errors, which are further processed by 1D multiscale transforms using linear lifting scheme.

Multiscale Inpainting Transforms . . .

Reconstruction of a geometric image from level $J - 4$



Linear inpainting　　　　Nonlinear inpainting

Multiscale Inpainting Transforms . . .

Reconstruction of a real image from level $J - 4$

Linear inpainting | Nonlinear inpainting

Summary

- Reviewed geometry/edge-conscious techniques
- Many approaches deal with edges proactively
- This is a "meeting ground" of image processing, harmonic analysis, PDEs, and shape optimization
- A large possibility for prediction operators from boundary
- Reliability of boundary information and noise is an issue

References

- T. F. Chan & H-M. Zhou: "ENO-wavelet transforms for piecewise smooth functions," *SIAM J. Num. Anal.*, vol.40, no.4, pp.1369–1404, 2001.
- A. Cohen's homepage: `http://www.ann.jussieu.fr/~cohen`
- N. Saito and J.-F. Remy: "A new local sine transform without overlaps: A combination of computational harmonic analysis and PDE," in *Wavelets: Applications in Signal and Image Processing X*, (M. A. Unser, A. Aldroubi, & A. F. Laine, eds.), Proc. SPIE 5207, pp.495–506, 2003.
- My homepage: `http://www.math.ucdavis.edu/~saito`