ESTIMATION OF RANDOM FIELDS

A.G. Ramm

LMA/CNRS, Marseille 13402, cedex 20, France
and Mathematics Department, Kansas State
University, Manhattan, KS 66506, USA
ramm@math.ksu.edu
www.math.ksu.edu/~ramm

$$\mathcal{U}(x) = s(x) + n(x), \quad x \in \mathbb{R}^r, \tag{1}$$

$$\overline{s(x)} = \overline{n(x)} = 0, \tag{2}$$

$$\overline{\mathcal{U}^*(x)\mathcal{U}(y)} := R(x,y), \quad \overline{\mathcal{U}^*(x)s(y)} := f(x,y), \tag{3}$$

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

$$LU := \int_D h(x,y)U(y)dy, \tag{4}$$

$$\epsilon := \overline{(LU - As)^2} = \min, \tag{5}$$

$$Rh := \int_D R(x,y)h(z,y)dy = f(x,z),$$

$$x, z \in \overline{D} := D \cup \Gamma, \tag{6}$$

$$Rh := \int_D R(x,y)h(y)dy = f(x), \quad x \in \overline{D}. \tag{7}$$

The questions are: in what functional space should one look for the solution? Is the solution unique? Does the solution to (7) provide the solution to the estimation problem (5)? Does the solution depend continuously on the data, e.g. on f and on $R(x,y)$? How does one compute the solution analytically and numerically? What are the properties of the solution, for example, what is the order of singularity of the solution? What is the singular support of the solution? What are the properties of the operator R as an operator in $L^2(D)$? How does the solution of singularly perturbed problem $\sigma^2 h + Rh = f$ in D

behaves when $\sigma \to 0$, where $\sigma^2 > 0$ is a parameter? This parameter has statistical meaning: it is a variance of the white-noise component in the noise.

Class $\mathcal{R}$ of kernels $R(x,y)$:

$$R(x,y) = \int_\Lambda P(\lambda)Q^{-1}(\lambda)\Phi(x,y,\lambda)d\rho(\lambda). \qquad (8)$$

Let $R(x,y) \in \mathcal{R}$, $\alpha := \frac{1}{2}s(q-p)$, $H^\ell(D)$ be the Sobolev spaces and $\dot{H}^{-\ell}(D)$ be its dual with respect to $H^0(D) = L^2(D)$. The space $\dot{H}^{-\ell}(D)$ consists of distributions in $H^{-\ell}(\mathbb{R}^r)$ with support in the closure of D.

Then the solution to equation (7) solves estimation problem (5) if and only if $h \in \dot{H}^{-\alpha}(D)$. The operator $R : \dot{H}^{-\alpha}(D) \to H^\alpha(D)$ is an isomorphism. The singular support of the solution $h \in \dot{H}^{-\alpha}(D)$ of equation (7) is $\Gamma = \partial D$. The analytic formula for h is of the form $h = Q(\mathcal{L})G$, where G is a solution to some interface elliptic boundary value problem and the differentiation is taken in the sense of distributions.

This theory can be used in many applications: in signal and image processing, underwater acoustics, geophysics, optics, etc, see [1]. In particular, the following question is answered by our theory: suppose a random field (1) is observed in a ball B and one wants to estimate $s(x_0)$, where x_0 is the center of B. What is the optimal size of the radius of B? If the radius is too small then the estimate is not accurate. If it is too large then the estimate is not better than the one obtained from the observations in a ball of smaller radius, so that the efforts are wasted.

II. FORMULATION OF BASIC RESULTS.

Let $\mathcal{L}$ be an elliptic selfadjoint in $H = L^2(\mathbb{R}^r)$ operator of order s. Let Λ, $\Phi(x,y,\lambda)$, $d\rho(\lambda)$ be the spectrum, spectral kernel and spectral measure of $\mathcal{L}$, respectively. A function $F(\mathcal{L})$ is defined as an operator on H with the kernel

$$F(\mathcal{L})(x,y) = \int_\Lambda F(\lambda)\Phi(x,y,\lambda)d\rho(\lambda) \qquad (1)$$

and

$$D(F(\mathcal{L})) = \{f : f \in H, \int_{-\infty}^{\infty} |F(\lambda)|^2 d(E_\lambda f, f) < \infty\},$$

where

$$(E_\lambda f, f) = \int_{-\infty}^{\lambda} \left\{ \int\int \Phi(x, y, \mu) f(y) \overline{f(y)} dx dy \right\} d\rho(\mu),$$
$$\int := \int_{\mathbb{R}^r}. \qquad (2)$$

$$R(x, y) = \int_\Lambda P(\lambda) Q^{-1}(\lambda) \Phi(x, y, \lambda) d\rho(\lambda), \qquad (3)$$

where $P(\lambda) > 0$ and $Q(\lambda) > 0$, $\forall \lambda \in \Lambda$, and Λ, Φ, $d\rho$ correspond to an elliptic selfadjoint operator $\mathcal{L}$ in $H = L^2(\mathbb{R}^r)$.

$$p = \deg P(\lambda), \quad q = \deg Q(\lambda), \quad s = \operatorname{ord} \mathcal{L}, \qquad (4)$$

$$\mathcal{L}u := \sum_{|j|\le s} a_j(x)\partial^j u, \tag{5}$$

where $j = (j_1, j_2 \ldots, j_r)$ is a multiindex, $\partial^j u = \partial^{j_1}_{x_1}\partial^{j_2}_{x_2}\ldots\partial^{j_r}_{x_r} u$ $|j| = j_1 + j_2 + \ldots j_r$, $j_m \ge 0$ are integers. The expression (5) is called elliptic if, for any real vector $t \in \mathbb{R}^r$, the equation

$$\sum_{|j|=s} a_j(x)t^j = 0$$

implies that $t = 0$. The expression

$$\mathcal{L}^+ u := \sum_{|j|\le s} (-1)^{|j|}\partial^j(a^*_j(x)u) \tag{6}$$

is called the formal adjoint with respect to $\mathcal{L}$. The star in (6) stands for complex conjugate.

$$\hat{\mathcal{U}} := L\mathcal{U} := \int_D h(x,y)\mathcal{U}(y)dy, \tag{7}$$

$$\epsilon := \overline{|\hat{\mathcal{U}} - As|^2} = \min. \tag{8}$$

The kernel $h(x,y)$ in (7) is a distribution, so that, by L. Schwartz's theorem about kernels, estimate (7) is the most general linear estimate. The operator A in (8) is assumed to be known. It is an arbitrary operator not necessarily linear. In the case when $A\mathcal{U} = \mathcal{U}$, that is $A = I$, where I is the identity operator, the estimation problem (8) is called the filtering problem. From (8) and (7) one obtains

$$\begin{aligned}
\epsilon = & \overline{\int_{\mathcal{D}} h(x,y)\mathcal{U}(y)dy \int_{\mathcal{D}} h^*(x,z)\mathcal{U}^*(z)dz} \\
& - 2Re\overline{\int h(x,z)\mathcal{U}(z)dz(As)^*(x)} + \overline{|As(x)|^2} \\
= & \int_{\mathcal{D}}\int_{\mathcal{D}} h(x,y)h^*(x,z)R(z,y)dzdy \\
& - 2Re\int h^*(x,z)f(z,x)dz + \overline{|As(x)|^2} = \min. \qquad (9)
\end{aligned}$$

Here

$$f(y,x) := \overline{\mathcal{U}^*(y)(As(x))} = f^*(x,y), \qquad (10)$$

the bar stands for the mean value and the star stands for complex conjugate. By the standard procedure one finds that a necessary condition for the minimum in (9) is:

$$\int_D R(z,y)h(x,y)dy = f(z,x), \quad x,z \in \overline{D} := D \cup \Gamma. \tag{11}$$

$$Rh := \int_D R(x,y)h(y)dy = f(x), \quad x \in \overline{D} \tag{12}$$

is basic for estimation theory.

We have suppressed the dependence on x in (11) and have written x in place of z in (12).

Let us show that the class $\mathcal{R}$ of kernels, that is the class of random fields that we introduced, is a natural one. To see this, recall that in the one-dimensional case, studied analytically in the literature, the covariance functions are of the form $R(x,y) = R(x-y)$, $x,y \in \mathbb{R}^1$,

$$\tilde{R}(\lambda) := \int_{-\infty}^{\infty} R(x)\exp(-i\lambda x)dx = P(\lambda)Q^{-1}(\lambda),$$

where $P(\lambda)$ and $Q(\lambda)$ are positive polynomials. This case is a very particular case of the kernels in the class $\mathcal{R}$. Indeed, take $r = 1$, $\mathcal{L} = -i\frac{d}{dx}$, $\Lambda = (-\infty, \infty)$, $d\rho(\lambda) = d\lambda$, $\Phi(x, y, \lambda) = (2\pi)^{-1}\exp\{i\lambda(x-y)\}$. Then formula (3) gives the above class of convolution covariance functions with rational Fourier transforms. If $p = q$, where p and q are defined in (4), then the basic equation (12) can be written as

$$Rh := \sigma^2 h(x) + \int_D R_1(x,y)h(y)dy = f(x), \quad x \in \overline{D}, \quad \sigma^2 > 0, \tag{13}$$

where

$$P(\lambda)Q^{-1}(\lambda) = \sigma^2 + P_1(\lambda)Q^{-1}(\lambda), \; p_1 := \deg P_1 < q, \tag{14}$$

and $\sigma^2 > 0$ is interpreted as the variance of the white noise component of the observed signal $\mathcal{U}(x)$.

If $p < q$, then the noise in $\mathcal{U}(x)$ is colored, it does not contain a white noise component.

§2. Formulation of the results.

1. Basic results.

Theorem 1. *If* $R(x,y) \in \mathcal{R}$ *then the operator* R *in (1.12) is an isomorphism between the spaces* $\dot{H}^{-\alpha}$ *and* H^{α}. *The solution to (1.12) of minimal order of singularity,* $\operatorname{ord} h \le \alpha$, *can be calculated by the formula:*

$$h(x) = Q(\mathcal{L})G, \tag{1}$$

where

$$G(x) = \begin{cases} g(x) + v(x) & \text{in } D, \\ u(x) & \text{in } \Omega := \mathbb{R}^r \setminus D, \end{cases} \tag{2}$$

$g(x) \in H^{s(p+q)/2}$ *is an arbitrary fixed solution to the equation*

$$P(\mathcal{L})g = f \quad \text{in} \quad D, \tag{3}$$

and the functions $u(x)$ *and* $v(x)$ *are the unique solution to the following transmission problem (4)-(6):*

$$Q(\mathcal{L})u = 0 \quad in \quad \Omega, \quad u(\infty) = 0, \tag{4}$$

$$P(\mathcal{L})v = 0 \quad in \quad D, \tag{5}$$

$$\partial_N^j u = \partial_N^j (v + g) \quad on \quad \Gamma, \quad 0 \leq j \leq \frac{s(p+q)}{2} - 1. \tag{6}$$

By $u(\infty) = 0$ *we mean* $\lim u(x) = 0$ *as* $|x| \to \infty$ *.*

Corollary 1. *If* f *is smooth, then*

$$sing\ supp h = \Gamma. \tag{7}$$

Corollary 2. *If* $P(\lambda) = 1$ *, then the transmission problem (4)-(6) reduces to the Dirichlet problem in* Ω *:*

$$Q(\mathcal{L})u = 0 \quad in \quad \Omega, \quad u(\infty) = 0, \tag{8}$$

$$\partial_N^j u = \partial_N^j f \quad on \quad \Gamma, \quad 0 \leq j \leq \frac{sq}{2} - 1, \tag{9}$$

and (1) takes the form

$$h = Q(\mathcal{L})F, \quad F = \begin{cases} f & \textit{in } D, \\ u & \textit{in } \Omega. \end{cases} \tag{10}$$

Corollary 1 follows immediately from formulas (1) and (2) since $g(x) + v(x)$ and $u(x)$ are smooth inside D and Ω respectively. Corollary 2 follows immediately from Theorem 1: if $P(\lambda) = 1$ then $g = f$, $v = 0$, and $p = 0$.

Let $\omega(\lambda) \geq 0$, $\omega(\lambda) \in C(R^1)$, $\omega(\infty) = 0$,

$$\omega := \max_{\lambda \in \Lambda} \omega(\lambda), \tag{11}$$

$$R(x,y) = \int_\Lambda \omega(\lambda)\Phi(x,y,\lambda)d\rho(\lambda), \tag{12}$$

$\lambda_j = \lambda_j(D)$ be the eigenvalues of the operator $R : L^2(D) \to L^2(D)$ with kernel (12), arranged so that

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots > 0. \tag{13}$$

Theorem 2. *If* $D \subset D'$ *then* $\lambda_j \leq \lambda'_j$ *, where* $\lambda'_j = \lambda_j(D')$. *If*

$$\sup_{x\in\mathbb{R}^r} \int |R(x,y)|dy := A < \infty, \qquad (14)$$

then

$$\lambda_{1\infty} = \omega, \qquad (15)$$

where

$$\lim_{D\to\mathbb{R}^r} \lambda_1(D) := \lambda_{1\infty}, \qquad (16)$$

and $\omega := \max_{\lambda\in\Lambda} \omega(\lambda)$.

Theorem 3. *If* $\omega(\lambda) = |\lambda|^{-a}(1 + o(1))$ *as* $|\lambda| \to \infty$ *, and* $a > 0$ *, then the asymptotics of the eigenvalues of the operator* R *with kernel (12) is given by the formula:*

$$\lambda_j \sim cj^{-as/r} \quad as \quad j \to \infty, \quad c = const > 0, \quad (17)$$

where $c = \gamma^{as/r}$ *and*

$$\gamma := (2\pi)^{-r} \int_D \eta(x)dx, \qquad (18)$$

with

$$\eta(x) := meas\{t : t \in \mathbb{R}^r, \sum_{|\alpha|=|\beta|=s/2} a_{\alpha\beta}(x)t^{\alpha+\beta} \leq 1\}. \qquad (19)$$

Here the form $a_{\alpha\beta}(x)$ *generates the principal part of the selfadjoint elliptic operator* $\mathcal{L}$ *:*

$$\mathcal{L}u = \sum_{|\alpha|=|\beta|=s/2} \partial^\alpha(a_{\alpha\beta}(x))\partial^\beta u + \mathcal{L}_1, \quad ord\mathcal{L}_1 < s.$$

Corollary 3. *If* $\omega(\lambda) = P(\lambda)Q^{-1}(\lambda)$ *then* $a = q - p$, *where* $q = \deg Q$, $p = \deg P$, *and* $\lambda_n \sim cn^{-(q-p)s/r}$, *where* λ_n *are the eigenvalues of the operator in equation (1.12).*

Theorem 4. *Let A and Q be linear compact operators in a Hilbert space H, $B := A(I+Q)$, $N(I+Q) = \{0\}$, $dim Ran A = \infty$. Then*

$$\lim_{n\to\infty} \frac{s_n(B)}{s_n(A)} = 1.$$

If $s_n(A) = cn^{-p}[1 + O(n^{-p_1})]$ *and*
$||Qf|| \leq ||Af||^a ||f||^{1-a}, \quad 0 < a < 1$, *then*

$$s_n(B) = cn^{-p}[1 + O(n^{-q})], \quad q = \min(p_1, \frac{pa}{1+pa}).$$

If $\frac{pa}{1+pa} \geq p_1$, *then* $q = p_1$.

2. Generalizations.

First, let us consider a generalization of the class $\mathcal{R}$ of kernels for the case when there are several commuting differential operators. Let $\mathcal{L}_1, \dots \mathcal{L}_m$ be a system of commuting selfadjoint differential operators in $L^2(\mathbb{R}^r)$. There exists a spectral measure $d\mu(\xi)$ and a spectral kernel $\Phi(x, y, \xi)$, $\xi =$

$(\xi_1, \dots, \xi_m)$ such that a function $F(\mathcal{L}_1, \dots \mathcal{L}_m)$ is given by the formula

$$F(\mathcal{L}_1, \dots \mathcal{L}_m) = \int_M F(\xi)\phi(\xi)d\mu(\xi) \qquad (20)$$

where $\Phi(\xi)$ is the operator with kernel $\Phi(x, y, \xi)$. The domain of definition of the operator $F(L_1, \dots L_m)$ is the set of all functions $u \in L^2(\mathbb{R}^r)$ for which $\int_M |F(\xi)|^2(\Phi(\xi)u, u)d\mu < \infty$, M is the support of the spectral measure $d\mu$, and the parentheses denote the inner product in $L^2(\mathbb{R}^r)$.

For example, let $m = r$, $\mathcal{L}_j = -i\frac{\partial}{\partial x_j}$. Then

$$\xi = (\xi_1, \dots \xi_r), \quad d\mu = d\xi_1 \dots d\xi_r,$$
$$\phi(x, y, \xi) = (2\pi)^{-r} \exp\{i\xi \cdot (x - y)\},$$

where the dot denotes the inner product in $\mathbb{R}^r$.

If $F(\xi) = P(\xi)Q^{-1}(\xi)$, where $P(\xi)$ and $Q(\xi)$ are positive polynomials, and the operators $P(\mathcal{L}) := P(\mathcal{L}_1, \dots \mathcal{L}_r)$ and $Q(\mathcal{L}) := Q(\mathcal{L}_1, \dots \mathcal{L}_r)$ are elliptic of orders m and n respectively, $m < n$, then theorems analogous to Theorems 1-2 in section 1 hold with $sp = m$ and $sq = n$. Theorem 3 has also an analogue in which $as = n - m$ in formula (17).

Another generalization of the class $\mathcal{R}$ of kernels is the following one. Since the kernel R is a covariance function, it must be a non-negative-definite kernel, that is, the quadratic form (Rh, h) must be non-negative. Here $(\cdot, \cdot)$ is the extension of L^2- inner product to the pairing between H^α and its dual space $\dot{H}^{-\alpha}$. Let $Q(x, \partial)$ and $P(x, \partial)$ be elliptic differential operators and

$$QR = P\delta(x - y) \quad \text{in} \quad \mathbb{R}^r. \tag{21}$$

Note that the kernels $R \in \mathcal{R}$ satisfy equation (21) with $Q = Q(\mathcal{L})$, $P = P(\mathcal{L})$, so that if $R \in \mathcal{R}$, then Q and P are selfadjoint commuting elliptic operators. We wish to generalize the class of kernels R so that nonselfadjoint and non-commuting elliptic operators can be considered also. Let $\text{ord}Q = n$, $\text{ord}P = m$, $n > m$. Under suitable assumptions either on the quadratic form (Rh, h) or on the formal differential elliptic operators Q and P, one can establish an analog of Theorem 1. For example, if there exist positive constants c_1 and c_2 such that

$$c_1||h||^2_{-\alpha} \leq (Rh, h) \leq c_2||h||^2_{-\alpha} \quad \forall h \in C_0^\infty(\mathbb{R}^r), \tag{22}$$

then the operator with the non-negative definite kernel R, satisfying (21) is an isomorphism of $\dot{H}^{-\alpha}(D)$ onto $H^\alpha(D)$.

Theorem 5. *Inequality (22) holds if (21) and the following two inequalities hold* $\forall h \in C_0^\infty(\mathbb{R}^r)$:

$$c_3||h||_{-\alpha+n} \leq ||Q^*h|| \leq c_4||h||_{-\alpha+n}, \tag{23}$$

$$c_5||h||_{-\alpha+n} \leq (PQ^*h, h) \leq c_6||h||_{-\alpha+n}. \tag{24}$$

III. EXAMPLES.

Some examples of one-dimensional equations of estimation theory are also included.

1. If $r = 1$, $\mathcal{L} = -i\partial$, $\partial = d/dx$, $\Phi(x, y, \lambda) = (2\pi)^{-1}\exp\{i\lambda(x-y)\}$, $d\rho = d\lambda$, then $R(x, y) \in \mathcal{R}$ if

$$R(x,y) = (2\pi)^{-1}\int_{-\infty}^{\infty}\tilde{R}(\lambda)\exp\{i\lambda(x-y)\}d\lambda, \quad (1)$$

where

$$\tilde{R}(\lambda) = P(\lambda)Q^{-1}(\lambda), \quad (2)$$

and $P(\lambda)$, $Q(\lambda)$ are positive polynomials.

2. If $r > 1$, $\mathcal{L} = (\mathcal{L}_1, \dots \mathcal{L}_r)$, $\mathcal{L}_r = -i\partial_r$, $\partial_r = \partial/\partial x_r$, $\Phi(x, y, \lambda) = (2\pi)^{-r}\exp\{i\lambda\cdot(x-y)\}$, $\lambda = (\lambda_1, \dots \lambda_r)$, $d\rho(\lambda) = d\lambda = d\lambda_1 \dots d\lambda_r$, then

$$R(x,y) = (2\pi)^{-r}\int_{\mathbb{R}^r}\tilde{R}(\lambda)\exp\{i\lambda\cdot(x-y)\}d\lambda, \quad (3)$$

where $\tilde{R}(\lambda)$ is given by (2), and

$$P(\lambda) = P(\lambda_1, \dots \lambda_r) > 0, \; Q(\lambda) = Q(\lambda_1 \dots \lambda_r) > 0, \quad (4)$$

are polynomials. For the operators $P(\mathcal{L})$ and $Q(\mathcal{L})$ to be elliptic of orders p and q respectively, one has to assume that, for any $|\lambda| > 0$, one has:

$$0 < c_1 \leq P(\lambda)|\lambda|^{-p} \leq c_2,$$
$$0 < c_3 \leq Q(\lambda)|\lambda|^{-q} \leq c_4, \quad \forall \lambda \in \mathbb{R}^r, \tag{5}$$

where $|\lambda| = (\lambda_1^2 + \cdots + \lambda_r^2)^{1/2}$ and c_j, $1 \leq j \leq 4$, are positive constants, independent of λ in the region $|\lambda| > c > 0$, where $c > 0$ is an arbitrary fixed constant.

3. If $r = 1$, $\mathcal{L} = -\frac{d^2}{dx^2}$, $D(\mathcal{L}) = \{u : u \in H^2(0,\infty)$, $u'(0) = 0\}$, $D(\mathcal{L}) =$ domain of $\mathcal{L}$, then

$$R(x,y) = \frac{1}{2}[A(|x+y|) + A(|x-y|)], \quad x, y \geq 0, \tag{6}$$

where

$$A(x) = \pi^{-1} \int_0^\infty P(\lambda) Q^{-1}(\lambda) \cos(\sqrt{\lambda} x) \lambda^{-1/2} d\lambda, \tag{7}$$

and $P(\lambda) > 0$, $Q(\lambda) > 0$ are polynomials.

Indeed, one has for $\mathcal{L}$

$$\Phi(x, y, \lambda) d\rho(\lambda)$$
$$= \begin{cases} \pi^{-1} \cos(\sqrt{\lambda} x) \cos(\sqrt{\lambda} y) \lambda^{-1/2} d\lambda, & \lambda \geq 0, \\ 0, & \lambda < 0, \end{cases}$$

$0 \leq x$, $y < \infty$. Since

$$\cos(kx)\cos(ky) = \frac{1}{2}[\cos(kx - ky) + \cos(kx + ky)],$$
$$k = \sqrt{\lambda},$$

one obtains (6) and (7).

If one puts $\sqrt{\lambda} = k$ in (7), one gets

$$A(x) = \frac{2}{\pi} \int_0^\infty P(k^2) Q^{-1}(k^2) \cos(kx) dk, \qquad (8)$$

which is a cosine transform of a positive rational function of k. The eigenfunctions of $\mathcal{L}$, normalized in $L^2(0, \infty)$, are $\left(\frac{2}{\pi}\right)^{1/2} \cos(kx)$ and $d\rho = dk$

in the variable k. If $\mathcal{L} = -\frac{d^2}{dx^2}$ is determined in $L^2(0,\infty)$ by the boundary condition $u(0)=0$, then

$$R(x,y) = \frac{1}{2}[A(|x-y|) - A(x+y)], \quad x,y \geq 0, \quad (9)$$

where $A(x)$ is given by (8), the eigenfunctions of $\mathcal{L}$ with the Dirichlet boundary condition $u(0)=0$ are $\sqrt{\frac{2}{\pi}}\sin(kx)$, $d\rho = dk$ in the variable k, and $\Phi(x,y,k)d\rho(k) = \frac{2}{\pi}\sin(kx)\sin(ky)dk$.

For the Neumann boundary condition $u'(0)=0$, one gets: $\Phi(x,y,k)d\rho(k) =$ $\frac{2}{\pi}\cos(kx)\cos(ky)dk$.

4. If $\mathcal{L} = -\frac{d^2}{dx^2} + (\nu^2 - \frac{1}{4})x^{-2}$, $\nu \geq 0$, $x \geq 0$, then

$$\Phi(x,y,\lambda)d\rho(\lambda) = \begin{cases} \sqrt{x\lambda}J_\nu(x\lambda)\sqrt{y\lambda}J_\nu(y\lambda)d\lambda, & \text{if } \lambda \geq 0, \\ 0, & \text{if } \lambda < 0, \end{cases} \quad (10)$$

so that

$$R(x,y) = \sqrt{xy}\int_0^\infty P(\lambda)Q^{-1}(\lambda)J_\nu(\lambda x)J_\nu(\lambda y)\lambda d\lambda, \quad (11)$$

where $P(\lambda)$ and $Q(\lambda)$ are positive polynomials on the semiaxis $\lambda \geq 0$.

5. Let $R(x,y) = \exp(-a|x-y|)(4\pi|x-y|)^{-1}$, $x, y \in \mathbb{R}^3$, $a = \text{const} > 0$. Note that $(-\Delta + a^2)R = \delta(x-y)$ in $\mathbb{R}^3$. The kernel $R(x,y) \in \mathcal{R}$. One has $\mathcal{L} = (\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3)$, $\mathcal{L}_j = -i\partial_j$, $P(\lambda) = 1$, $Q(\lambda) = \lambda^2 + a^2$, $\lambda^2 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$, $\Phi d\rho = (2\pi)^{-3} \exp\{i\lambda \cdot (x-y)\} d\lambda$,

$$R(x,y) = (2\pi)^{-3} \int_{\mathbb{R}^3} \frac{\exp\{i\lambda \cdot (x-y)\}}{\lambda^2 + a^2} d\lambda. \qquad (12)$$

6. Let $R(x,y) = R(xy)$. Put $x = \exp(\xi)$, $y = \exp(-\eta)$. Then $R(xy) = R(\exp(\xi - y)) := R_1(\xi - y)$. If $R_1 \in \mathcal{R}$ with $\mathcal{L} = -i\partial$, then one can solve the equation

$$\int_a^b R(xy)h(y)dy = f(x), \quad a \leq x \leq b, \qquad (13)$$

analytically.

7. Let $K_0(a|x|)$ be the modified Bessel function which can be defined by the formula

$$K_0(a|x|) = (2\pi)^{-1} \int_{\mathbb{R}^2} \frac{\exp(i\lambda \cdot x)}{\lambda^2 + a^2} d\lambda, \quad a > 0, \tag{14}$$

where $\lambda \cdot x = \lambda_1 x_1 + \lambda_2 x_2$. Then the kernel $R(x,y) := K_0(a|x-y|) \in \mathcal{R}$, $\mathcal{L} = (-i\partial_1, -i\partial_2)$, $r = 2$, $P(\lambda) = 1$, $Q(\lambda) = \lambda^2 + a^2$, $\Phi(x,y,\lambda)d\rho(\lambda) = (2\pi)^{-1} \exp\{i\lambda \cdot (x-y)\}d\lambda$.

8. Consider the equation

$$\int_D \frac{\exp(-a|x-y|)}{4\pi|x-y|} h(y)dy = f(x), \quad x \in \overline{D} \subset \mathbb{R}^3, \quad a > 0, \tag{15}$$

with kernel (12). By Theorem 1, one obtains the unique solution to equation (15) in $\dot{H}^{-1}(D)$:

$$h(x) = (-\Delta + a^2)f + \left(\frac{\partial f}{\partial N} - \frac{\partial u}{\partial N}\right)\delta_\Gamma, \tag{16}$$

where u is the unique solution to the Dirichlet problem in the exterior domain $\Omega := \mathbb{R}^3 \setminus D$:

$$(-\Delta + a^2)u = 0 \quad \text{in} \quad \Omega, \quad u|_\Gamma = f|_\Gamma, \qquad (17)$$

$\Gamma = \partial D = \partial\Omega$ is the boundary of D, and δ_Γ is the delta function with support Γ.

Let us derive formula (16). For kernel (12) one has $p = 0$, $q = 2$, $P(\lambda) = 1$, $Q(\lambda) = \lambda^2 + a^2$, $s = 1$, $\alpha = \frac{sq}{2} = 1$, and

$$h(x) = (-\Delta + a^2)G, \qquad (18)$$

with

$$G = \begin{cases} f & \text{in } D, \\ u & \text{in } \Omega, \end{cases} \qquad (19)$$

and u is the solution to (17). Indeed, since $P(\lambda) = 1$, one has $v = 0$ and $g = f$. In order to compute h by formula (18) one uses the definition of the derivative in the sense of distributions.

For any $\phi \in C_0^\infty(\mathbb{R}^r)$ one has:

$$
\begin{aligned}
&((-\Delta + a^2)G, \phi) = (G, (-\Delta + a^2)\phi) \\
&= \int_D f(-\Delta + a^2)\phi dx + \int_\Omega u(-\Delta + a^2)\phi dx \\
&= \int_D (-\Delta + a^2) f\phi dx + \int_\Omega (-\Delta + a^2) u\phi dx \\
&- \int_\Gamma \left(f\frac{\partial \phi}{\partial N} - \frac{\partial f}{\partial N}\phi \right) ds + \int_\Gamma \left(u\frac{\partial \phi}{\partial N} - \phi\frac{\partial u}{\partial N} \right) ds \\
&= \int_D (-\Delta + a^2) f\phi dx + \int_\Gamma \left(\frac{\partial f}{\partial N} - \frac{\partial u}{\partial N} \right) \phi ds, \qquad (20)
\end{aligned}
$$

where the condition $u = f$ on Γ was used. Formula (20) is equivalent to (16).

9. Consider the equation

$$
2\pi \int_D K_0(a|x-y|)h(y)dy = f(x), \qquad x \in \overline{D} \subset \mathbb{R}^2, \quad a > 0, \qquad (21)
$$

where $D = \{x : x \in \mathbb{R}^2, |x| \leq b\}$, and $K_0(x)$ is given by formula (14). The solution to (21) in $\dot{H}^{-1}(D)$ can be calculated by formula (16) in which $u(x)$ can be calculated explicitly

$$u(x) = \sum_{n=-\infty}^{\infty} f_n \frac{\exp(in\phi)K_n(ar)}{K_n(ab)}, \qquad (22)$$

where $x = (r, \phi)$, (r, ϕ) are polar coordinates in $\mathbb{R}^2$,

$$f_n := (2\pi)^{-1} \int_0^{2\pi} f(b, \phi) \exp(-in\phi) d\phi, \qquad (23)$$

$K_n(r)$ is the modified Bessel function of order n, which decays as $r \to +\infty$. One can easily calculate $\frac{\partial u}{\partial N} \big|_\Gamma$ in formula (16):

$$\frac{\partial u}{\partial N} \Big|_\Gamma = \frac{\partial u}{\partial r} \Big|_{r=b} = \sum_{n=-\infty}^{\infty} a f_n \frac{\exp(in\phi)K_n'(ab)}{K_n(ab)}. \qquad (24)$$

Formulas (16) and (24) give an explicit analytical formula for the solution to equation (21) in $\dot{H}^{-1}(D)$.

IV. NUMERICAL SOLUTION OF THE BASIC INTEGRAL EQUATION.

We illustrate the ideas using the following simple equation:

$$Rh := \int_{-1}^{1} \exp(-|x-y|)h(y)dy = f(x), \quad -1 \le x \le 1. \qquad (1)$$

Here $r = 1$, $p = 0$, $q = 2$, $s = 1$, $\alpha = 1$, and the solution to (1) is:

$$h(x) = \frac{-f'' + f}{2} + \frac{\delta(x+1)\{-f'(-1) + f(-1)\}}{2} + \frac{\delta(x-1)\{f'(1) + f(1)\}}{2}. \qquad (2)$$

It is not possible to solve (1) by a regularization method: (1) does not have integrable solutions, in general.

Look for the solution of the form:

$$h_n = \sum_{j=-2}^{n} c_j \phi_j(x), \tag{3}$$

where

$$\phi_j(x) = \cos\left[j\frac{\pi}{2}(x+1)\right], \; 0 \le j < \infty, \;\; \phi_{-1} = \delta(x-1),$$

$$\phi_{-2} = \delta(x+1). \tag{4}$$

Find the coefficients from the condition:

$$\| Rh_n - f \|_1 = \min . \tag{5}$$

I have proved that (5) defines uniquely the coefficients, and

$$\| h_n - h \|_{-1} \to 0 \quad \text{as} \quad n \to \infty. \tag{6}$$

The approximate solution is stable in the following sense

$$\| h_n - h \|_{-1} \le c \, \| Rh_n - f \|_1, \tag{7}$$

where $c > 0$ is a constant which does not depend on n and $f \in H^1$, it depends on the kernel $R(x, y)$. An explicit bound on c is given in terms of the quantity:

$$\inf_{\lambda \in \mathbb{R}} \left\{(1 + \lambda^2)^{\alpha/s} P(\lambda) Q^{-1}(\lambda)\right\} := \gamma_1 > 0,$$

$$c = \| R^{-1} \|_{H^1 \to \dot{H}^{-1}} \leq \gamma_1^{-1}.$$

V. SINGULAR PERTURBATION.

$$Rh_\epsilon + \epsilon h_\epsilon = f,\ f \in H_+ := H^\alpha, \quad R: H_- \to H_+, \tag{1}$$

where $H_- := \dot{H}^{-\alpha}$. Let $Rh = f$.

Theorem 1. *One has $h_\epsilon \rightharpoonup h$ in H_- as $\epsilon \to 0$.*

Proof. One has

$$c||h_\epsilon||_-^2 \le (Rh_\epsilon, h_\epsilon) \le ||f||_+ ||h_\epsilon||_-.$$

Thus

$$||h_\epsilon||_- \lesssim c, \quad \textit{and} \quad h_\epsilon \rightharpoonup g \quad \textit{in} \quad H_-,$$

where $c>0$ stands for various estimation constants.

Claim 1: $g = h$.

Proof of Claim 1: $(Rh_\epsilon, p) + \epsilon(h_\epsilon, p) = (f, p)$ for all $p \in H_+$. Since $(Rh_\epsilon, p) = (h_\epsilon, Rp)$, and because $\epsilon(h_\epsilon, p) \to 0$ as $\epsilon \to 0$, one gets: $(Rg - f, p) = 0, \forall p \in H_+$, so $Rg = f$, and $g = h$ by the injectivity of R. □

Strong convergence

Theorem 2. $h_\varepsilon \xrightarrow[H_-]{} h$ as $\varepsilon \to 0$.

This follows from

Lemma. If $h_\varepsilon \rightharpoonup h$ and $\|h_\varepsilon\| \le \|h\|$ then $h_\varepsilon \to h$.

Proof of Lemma: $\|h_\varepsilon - h\|^2 = \|h_\varepsilon\|^2 + \|h\|^2 - 2\operatorname{Re}(h_\varepsilon, h)$
$\le 2\|h\|^2 - 2\operatorname{Re}(h_\varepsilon, h) \to 0$ as $\varepsilon \to 0$.

Proof of Thm 2. It is sufficient to prove $\|h_\varepsilon\|_- \le \|h\|_-$. One has

$$Rh_\varepsilon + \varepsilon h_\varepsilon = f = Rh.$$

Multiply by h_ε: $(Rh_\varepsilon, h_\varepsilon) \le (Rh, h_\varepsilon)$

$$\|h_\varepsilon\|_-^2 \le \|h\|_- \|h_\varepsilon\|_-.$$

So $\|h_\varepsilon\|_- \le \|h\|_-$. $\square$

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