Cones of measures

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Quantitative and Computational Aspects of Metric Geometry

Based on joint work with
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Motivating questions

To what extent does the *regularity* of a measure determine the geometry of its *support*?

- Given a Radon measure μ in $\mathbb{R}^n$ what information does the quantity $\frac{\mu(B(x,r))}{r^s}$ for $x \in \mathbb{R}^n$, $r > 0$ and $s > 0$ encode?
  In particular if for μ-a.e $x \in \mathbb{R}^n$ the s-density exists, i.e.

  $$0 < \theta^s(\mu, x) = \lim_{r \to 0} \frac{\mu(B(x,r))}{r^s} < \infty$$

  where $0 \leq s \leq n$, how regular is μ?

- To what extent can the structure of the boundary of a domain in $\mathbb{R}^n$ ($n \geq 3$) be understood in terms of the behavior of the harmonic measure?

The regular behavior of the measure of balls determines the rectifiability properties of its support

- Besicovitch (1928-1938): if $m = 2$ and $n = 1$ then μ is 1-rectifiable, i.e. the support of μ can be covered by countably many C^1 1-dimensional submanifolds ($+$ a set of μ measure 0).
- Marstrand (1955-1965): if the density exists on a set of μ positive measure then n is an integer.
- Preiss (1986) : μ is n-rectifiable.

Tools from Geometric Measure Theory

- Distances between Radon measures - Complete separable metric space of Radon measures in $\mathbb{R}^n$.
- Cones of measures
- Tangent measures
- Connectivity properties of the cone of tangent measures to a Radon measure.

Distances between Radon measures

- Let Φ and Ψ be Radon measures in $\mathbb{R}^n$. Let K be a compact set in $\mathbb{R}^n$ define $F_K(\Phi, \Psi)$, by

$$\sup\left\{\left|\int f d\Phi - \int f d\Psi\right| : \mathrm{spt}\, f \subset K, f \geq 0,\ f \text{ Lipschitz},\ \mathrm{Lip}\, f \leq 1\right\}.$$

- If $K = \overline{B}(0, r)$, $F_K = F_r$.

- $F_K(\Phi) = \int \mathrm{dist}\,(z, K^c)\, d\Phi(z)$.

A complete metric

- $\sum_{p=0}^{\infty} 2^{-p} \min\{1, F_p(\Phi, \Psi)\}$ defines a complete separable metric on the space of Radon measures on $\mathbb{R}^n$.
- Let $\mu_1, \mu_2, \dots$ and μ be Radon measures on $\mathbb{R}^n$. $\mu = \lim_{i\to\infty} \mu_i$ iff
  - $\lim_{i\to\infty} F_r(\mu_i, \mu) = 0 \quad \forall\, r > 0$.
  - $\mu = \lim_{i\to\infty} \mu_i$ iff $\mu_i \rightharpoonup \mu$.

Distances, dilations and translations

Let Φ be a Radon measure in $\mathbb{R}^n$. For $x \in \mathbb{R}^n$ and $r > 0$ define $T_{x,r} : \mathbb{R}^n \to \mathbb{R}^n$ by the formula $T_{x,r}(z) = (z - x)/r$.

- $T_{x,r}[\Phi](B(0,s)) := \Phi(T_{x,r}^{-1}(B(0,s))) = \Phi(B(x,sr))$ for $s > 0$.
- $\displaystyle\int f(z) dT_{x,r}[\Phi](z) = \int f\left(\frac{z-x}{r}\right) d\Phi(z)$
- $F_{B(x,r)}(\Phi) = rF_1(T_{x,r}[\Phi])$
- $F_{B(x,r)}(\Phi, \Psi) = rF_1(T_{x,r}[\Phi], T_{x,r}[\Psi])$

Cones of measures

- A set $\mathcal{M}$ of non-zero Radon measures in $\mathbb{R}^n$ is a cone if $c\Psi \in \mathcal{M}$ whenever $\Psi \in \mathcal{M}$ and $c > 0$.
- A cone $\mathcal{M}$ is a d-cone if $T_{0,r}[\Psi] \in \mathcal{M}$ whenever $\Psi \in \mathcal{M}$ and $r > 0$.
- Let $\mathcal{M}$ be a d-cone, and let Φ a Radon measure in $\mathbb{R}^n$ such that for $s > 0$, $0 < F_s(\Phi) < \infty$ then the *distance* between Φ and $\mathcal{M}$ is

$$d_s(\Phi, \mathcal{M}) = \inf \left\{ F_s \left(\frac{\Phi}{F_s(\Phi)}, \Psi \right) : \Psi \in \mathcal{M} \text{ and } F_s(\Psi) = 1 \right\},$$

$$d_s(\Phi, \mathcal{M}) = 1 \text{ if } F_s(\Phi) = 0 \text{ or } F_s(\Phi) = +\infty.$$

- If $\mu = \lim_{i\to\infty} \mu_i$ and $F_s(\mu) > 0$ then $d_s(\mu, \mathcal{M}) = \lim_{i\to\infty} d_s(\mu_i, \mathcal{M})$.

Basis of cones

- The basis of a d-cone $\mathcal{M}$ of Radon measures is the set $\{\Psi \in \mathcal{M} : F_1(\Psi) = 1\}$.
- $\mathcal{M}$ has a closed (resp. compact) basis, if its basis is closed (resp. compact) in the topology induced by the metric

$$\sum_{p=0}^{\infty} 2^{-p} \min\{1, F_p(\cdot, \cdot)\}.$$

Tangent measures

- Let η be a Radon measure in $\mathbb{R}^n$, $x \in \mathbb{R}^n$, a non-zero Radon measure ν in $\mathbb{R}^n$ is said to be a tangent measure of η at x, i.e. $\nu \in \mathrm{Tan}\,(\eta, x)$, if there are sequences $r_k \searrow 0$ and $c_k > 0$ such that

$$\nu = \lim_{k \to \infty} c_k T_{x,r_k}[\eta].$$

- (Preiss) For a.e. $a \in \mathbb{R}^n$, if $\Psi \in \mathrm{Tan}\,(\eta, a)$ then
  - $T_{x,\rho}[\Psi] \in \mathrm{Tan}\,(\eta, a)$ for all $x \in \mathrm{spt}\,\Psi$ and all $\rho > 0$
  - $\mathrm{Tan}\,(\Psi, x) \subset \mathrm{Tan}\,(\eta, a)$ for all $x \in \mathrm{spt}\,\Psi$.

Cones of measures are connected (Preiss, KPT)

Main Theorem: Let $\mathcal{F}$ and $\mathcal{M}$ be d-cones such that $\mathcal{F} \subset \mathcal{M}$ is relatively closed. Suppose that $\mathcal{M}$ has a compact basis and

$$\begin{cases} \exists\, \epsilon_0 > 0 \text{ such that } \forall\, \epsilon \in (0, \epsilon_0) \text{ there exists no } \mu \in \mathcal{M} \text{ satisfying} \\ d_{r_0}(\mu, \mathcal{F}) = \epsilon \text{ and } d_r(\mu, \mathcal{F}) \leq \epsilon \quad \forall r \geq r_0 > 0. \end{cases} \tag{P}$$

Then for a Radon measure η and $x \in \operatorname{spt} \eta$ if

$$\operatorname{Tan}(\eta, x) \subset \mathcal{M} \text{ and } \operatorname{Tan}(\eta, x) \cap \mathcal{F} \neq \emptyset \quad \text{then } \operatorname{Tan}(\eta, x) \subset \mathcal{F}.$$

Corollary: Let $\mathcal{F}$ and $\mathcal{M}$ be d-cones such that $\mathcal{F} \subset \mathcal{M}$ is relatively closed. Suppose that $\mathcal{M}$ has a compact basis and

$$\exists\, \epsilon_0 > 0 \text{ such that if } d_r(\mu, \mathcal{F}) < \epsilon_0 \text{ for all } r \geq r_0 > 0 \text{ then } \mu \in \mathcal{F} \qquad \text{(P')}$$

Then for a Radon measure η and $x \in \operatorname{spt} \eta$ if

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An application to Harmonic Analysis: Hausdorff dimension of the harmonic measure for domains in $\mathbb{R}^n$ with $n \geq 3$.

Let $\omega^\pm$ denotes the harmonic measure of $\Omega^\pm \subset \mathbb{R}^n$, where $\Omega^+ = \Omega$ and $\Omega^- = \text{int}(\Omega^c)$ are NTA domains. If

$$\omega^+ \ll \omega^- \ll \omega^+ \text{ then } \mathcal{H} - \dim \omega^\pm \geq n - 1$$

where

$$\begin{aligned} \mathcal{H} - \dim \omega \quad = \quad & \inf \{k : \text{ there exists } E \subset \partial\Omega \text{ with } \mathcal{H}^k(E) = 0 \text{ and} \\ & \omega(E \cap K) = \omega(\partial\Omega \cap K) \text{ for all compact sets } K \subset \mathbb{R}^n\} \end{aligned}$$

Do there exist NTA domains $\Omega^\pm \subset \mathbb{R}^n$ such that $\omega^+ \ll \omega^- \ll \omega^+$ and $\mathcal{H} - \dim \omega^\pm > n - 1$?

Theorem: Let $\Omega^{\pm} \subset \mathbb{R}^n$ be NTA domains. Assume that ω^+ and ω^- are mutually absolutely continuous, then

$$\mathcal{H} - \dim \omega^+ = \mathcal{H} - \dim \omega^- = n - 1.$$

If furthermore $\mathcal{H}^{n-1} \llcorner \partial\Omega$ is a Radon measure then

$$\partial\Omega = G \cup N$$

where G is $(n-1)$-rectifiable and $\omega^{\pm}(N) = 0$.

Strategy

- Main goal show that at $\omega^{\pm}$-a.e. point in $\partial\Omega$ all blow-ups of $\partial\Omega$ are $(n-1)$-planes. This implies $\mathcal{H}-\dim \omega^{\pm} \leq n-1$.
- Show that for $\omega^{\pm}$-a.e. $Q \in \partial\Omega$, all tangent measures of $\omega^{\pm}$ at Q are $(n-1)$-flat, i.e. $\mathrm{Tan}(\omega^{\pm}, Q) \subset \mathcal{F}$ where

$$\mathcal{F} = \{c\mathcal{H}^{n-1} \llcorner V : c \in (0,\infty); V \in G(n, n-1)\}.$$

- Show that for $\omega^{\pm}$-a.e. $Q \in \partial\Omega$, $\mathcal{M} = \mathcal{F} \cup \mathrm{Tan}(\omega, Q)$ and $\mathcal{F}$ satisfy the hypothesis of the corollary, i.e. $\mathcal{M}$ has a compact basis, $\mathcal{F}$ is relatively closed, hypothesis (P') and $\mathcal{F} \cap \mathrm{Tan}(\omega, Q) \neq \emptyset$.

Blow-ups

Let $\Omega^{\pm} \subset \mathbb{R}^n$, $Q \in \partial\Omega$, $\{r_j\}_{j\geq 1}$ be such that $\lim_{j\to\infty} r_j = 0$. Consider the domains

$$\Omega_j^{\pm} = \frac{1}{r_j}(\Omega^{\pm} - Q)$$

the functions

$$u_j^{\pm}(X) = \frac{u^{\pm}(r_j X + Q)}{\omega^{\pm}(B(Q, r_j))} r_j^{n-2}$$

and the measures

$$\omega_j^{\pm}(E) = \frac{\omega^{\pm}(r_j E + Q)}{\omega^{\pm}(B(Q, r_j))} \text{ for } E \subset \mathbb{R}^n \text{ a Borel set.}$$

Here $u^{\pm}$ are the Green's functions corresponding to $\omega^{\pm}$ (w.l.o.g. $u^{\pm} = 0$ on $\Omega^{\mp}$).

If $\omega^+ \ll \omega^- \ll \omega^+$ then the Radon-Nikodym derivative h exits and

$$h(Q) := \frac{d\omega^-}{d\omega^+}(Q) = \lim_{r\to 0} \frac{\omega^-(B(Q,r))}{\omega^+(B(Q,r))}.$$

Assume that $\log h \in C(\partial\Omega)$.

Theorem: There exists a sequence such that

$$\begin{aligned} \Omega_j^\pm &\to \Omega_\infty^\pm \text{ in the Hausdorff distance sense} \\ u_j^+ - u_j^- &\to u_\infty \text{ uniformly on compact sets,} \\ \lim_{j\to\infty} \omega_j^\pm &= \omega_\infty. \end{aligned}$$

Moreover $\Omega_\infty^\pm = \{u_\infty^\pm > 0\}$, $\omega_\infty^\pm$ is the harmonic measure of $\Omega_\infty^\pm$ with pole at infinity and u_∞ is a harmonic polynomial.

Harmonic measures corresponding to harmonic polynomials satisfy hypothesis (P')

Lemma: Let h be a harmonic polynomial in $\mathbb{R}^n$ such that $h(0) = 0$. Let ν be the corresponding harmonic measure. There exists $\epsilon_0 > 0$ such that if for some $r_0 > 0$

$$d_r(\nu, \mathcal{F}) < \epsilon_0 \;\text{ for } r \geq r_0, \;\text{ then } \nu \in \mathcal{F}.$$

Final ingredients of the proof

- Let h be a harmonic polynomial in $\mathbb{R}^n$, then $h^{-1}(0)$ decomposes into a disjoint union of the embedded C^1 submanifolds $\mathcal{R}$ together with a closed $(n-2)$-rectifiable set, $\mathcal{S}$.
- For ω-a.e. $Q \in \partial\Omega$ if $h = u_\infty$ and $Y \in \mathcal{R}$, $\mathrm{Tan}\,(\omega_\infty, Y) \subset \mathcal{F} \cap \mathrm{Tan}\,(\omega, Q)$.
- Let $\mathcal{M} = \mathcal{F} \cup \mathrm{Tan}\,(\omega, Q)$.
  - ω is a doubling measure then $\mathcal{M}$ is a d-cone with compact basis.
  - Hypothesis (P') holds.
  - For ω-a.e. $Q \in \partial\Omega$, $\mathcal{F} \cap \mathrm{Tan}\,(\omega, Q) \neq \emptyset$

  then the Corollary implies that $\mathrm{Tan}\,(\omega, Q) \subset \mathcal{F}$.
  Hence all blow ups of $\partial\Omega$ at Q converge in the Hausdorff distance sense to an $(n-1)$-plane, which implies that $\mathcal{H} - \dim \omega^{\pm} \leq n-1$.

Further applications

Let

$$\mathcal{F}_k = \{\omega : \omega \text{ harmonic measure corresponding to homogeneous polynomials of degree } k\}$$

Theorem: (Badger) Let $\Omega^{\pm} \subset \mathbb{R}^n$ be NTA domains. Assume that ω^+ and ω^- are mutually absolutely continuous, and that the Radon-Nikodym derivative of ω^- w.r.t. ω^+, h satisfies $\log h \in C(\partial\Omega)$. There exists $d \geq 1$ such that

$$\partial\Omega = \Gamma_1 \cap \cdots \Gamma_d$$

where for $Q \in \Gamma_k$ with $1 \leq k \leq d$

$$\mathrm{Tan}\,(\omega^{\pm}, Q) \subset \mathcal{F}_k.$$