# Proper isometric actions of wreath products

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13 January 2009

This is a report on joint work with Y. de CORNULIER and Y. STALDER. We address the following question: how to embed a finitely generated group $G$ equivariantly into $L^p$? (where $G$ acts on $L^p$ by some affine isometric action).

## 1 Links with Assaf's lecture

Let $G$ be a finitely generated group, with word length $|.|_S$ w.r.t. a finite generating set $S \subset G$. Define the $L^1$-equivariant compression of $G$, denoted by $\alpha_1^\sharp(G)$, as the supremum of the set of $\alpha \in [0,1]$ for which there exists a $G$-equivariant map $f : G \to L^1$ such that $C.|g|_S^\alpha \leq \|f(g)\|_1$ for some constant $C > 0$ and every $g \in G$.

Recall that, if $G, H$ are two groups, the *wreath product* $H \wr G$ is defined as the semi-direct product $H \wr G = W \rtimes G$, where $W =: H^{(G)}$ is the group of finitely supported maps $G \to H$ and $G$ acts on $W$ by left multiplication at the source.

**Theorem 1** *Let $H$ be any finitely generated group and let $\mathbb{F}_k$ be the free group on $k$ generators ($1 \leq k < \infty$). Then $\alpha_1^\sharp(H \wr \mathbb{F}_k) = \alpha_1^\sharp(H)$.*

The proof of this result uses, on the one hand, the easy solution of the Traveling Salesman Problem on a tree (a finite tree on $n$ vertices is covered by a walk of length $2(n-1)$); on the other hand measured walls structures, which we now define.

## 2 Measured walls structures

A *measured walls structure* is a pair $(X, \mu)$ where $X$ is a set and $\mu$ is a Borel measure on the power set $2^X$ (viewed as a compact space), such that for every

$x, y \in X : d_\mu(x,y) =: \mu(\mathcal{A}_x \triangle \mathcal{A}_y) < +\infty$, where $\mathcal{A}_x =: \{A \subset X : x \in A\}$ (a clopen subset of $2^X$).

Observe that $d_\mu$ satisfies the triangle inequality, hence defines a pseudo-distance on $X$.

**Example 1** *Let $X$ be the vertex set of a tree. Define a half-tree as one class of the partition of $X$ obtained by removing some edge. Define a measure $\mu$ on $2^X$ as counting measure supported on the set of half-trees, i.e. $\mu(B)$ is the number of half-trees belonging to $B$, for $B \subset 2^X$. Then $d_\mu(x,y) = 2.dist(x,y)$, the tree distance from $x$ to $y$.*

Say that a kernel $k : X \times X \to \mathbb{R}^+$ is $L^p$-embeddable, if there exists $f : X \to L^p$ such that $k(x,y) = \|f(x) - f(y)\|_p$. Note that

- $L^2$-embeddable implies $L^1$-embeddable;
- $k$ is $L^2$-embeddable if and only if $k^2$ is conditionally negative definite.

**Proposition 1 (Robertson-Steger 1998)** *Let $X$ be a countable set, $k$ a kernel on $X$. TFAE:*

*i) $k$ is measure-definite, i.e. there exists a measure space $(Y, \mathcal{B}, \nu)$ and a map $F : X \to \mathcal{B}$ such that $k(x,y) = \nu(F(x) \triangle F(y))$;*

*ii) $k$ is $L^1$-embeddable;*

*iii) $k = d_\mu$ for some measured walls structure $(X, \mu)$.* □

We shall need the equivariant version.

**Theorem 2 (Chatterji-Drutu-Haglund 2007)** *Let $X$ be a countable $G$-set, and let $k$ be a $G$-invariant kernel on $X$. Consider the following properties:*

*(i) $k$ is measure-definite;*

*(ii) $(X, k)$ is $G$-equivariantly embeddable into a $G$-metric space $Y$, which is isometrically embeddable into $L^1$;*

*(iii) $k = d_\mu$ for some $G$-invariant measured wall structure $(X, \mu)$;*

*(iv) $(X, k)$ is $G$-equivariantly embeddable into $L^1$.*

*Then: $(iii) \Leftrightarrow (iv) \Rightarrow (i) \Leftrightarrow (ii)$. Moreover:*

*a) If $G$ is amenable, then all four properties are equivalent;*

*b) If $k$ is a priori assumed to be $L^2$-embeddable, then it satisfies all four conditions.*

**Remark:** If $G = \mathbb{F}_2$, the implication $(i) \Rightarrow (iv)$ is an *open question.*

# 3 a-T-menability

## 3.1 Definitions and examples

**Definition 1** *A locally compact group $G$ is* **a-T-menable**, *or has the* **Haagerup property**, *if $G$ satisfies the following equivalent conditions:*

- *$G$ admits a proper equivariant embedding into a Hilbert space;*
- *there exists a proper function $G \to \mathbb{R}^+$ which is conditionally negative definite;*
- *$G$ admits a proper action on some measured walls structure.*

**Example 2**
- *Crystallographic groups (by a classical result of Bieberbach (1910), these are the only discrete groups which embed properly equivariantly into finite-dimensional Hilbert spaces, i.e. Euclidean spaces);*
- *free groups;*
- *amenable groups;*
- *closed subgroups of $SO(n, 1)$ and $SU(n, 1)$;*
- *Coxeter groups;*
- *countable subgroups of $GL_2$ of a field...*

On the other hand, non-compact groups with Kazhdan's property (T) (e.g. $SL_n(\mathbb{R}), SL_n(\mathbb{Z})$, $n \geq 3$) have the property that every equivariant map to a Hilbert space is bounded, hence cannot be proper.

**Theorem 3 (Higson-Kasparov 1996)** *A-T-menable groups satisfy the strongest form of the Baum-Connes conjecture* □

## 3.2 Permanence properties

- If $G_1, G_2$ have the Haagerup property, so has $G_1 \times G_2$.
- Let $N$ be a closed normal subgroup of $G$; if $N$ is a-T-menable and $G/N$ is amenable, then $G$ is a-T-menable.

**Bad news!** A-T-menability is not stable under arbitrary semi-direct products. Indeed, let $SL_2(\mathbb{Z})$ act linearly on $\mathbb{Z}^2$. Although both groups are a-T-menable, the semi-direct product $G = \mathbb{Z}^2 \rtimes SL_2(\mathbb{Z})$ is *not*: this is due to the *relative property (T)*: every $G$-equivariant from $G$ to a Hilbert space, is bounded on $\mathbb{Z}^2$, hence cannot be proper[1].

**Open question:** For which semi-direct products is the Haagerup property stable?

**Theorem 4** *A-T-menability is stable under wreath products.*

# 4 Gauges

Let $W$ be a group on which $G$ acts by automorphisms. Denote by $2^{(G)}$ the set of finite subsets of $G$.

**Definition 2** *A $G$-gauge on $W$ is a map $\psi : W \to 2^{(G)}$ such that, for every $w, w' \in W, g \in G$:*

- $\psi(1) = \emptyset$*;*
- *Symmetry:* $\psi(w) = \psi(w^{-1})$*;*
- *Sub-additivity:* $\psi(ww') \subset \psi(w) \cup \psi(w')$*;*
- *Equivariance:* $\psi(g(w)) = g.\psi(w)$*.*

**Example 3**
- *Set $W = Sym_0(G)$, the group of finitely supported permutations of $G$; then $\psi(w) = supp(w)$ defines a $G$-gauge.*
- *Set $W = H^{(G)}$, as in section 1; then $\psi(w) = supp(w)$ defines a $G$-gauge.*

For $F \in 2^{(G)}$, set $W_F =: \{w \in W : \psi(w) \subset F\}$; observe that $W_F$ is a subgroup of $W$.

**Theorem 5** *Assume that there exists a $G$-gauge $\psi$ and a $G$-invariant function $u : W \to \mathbb{R}^+$, conditionally of negative type, such that $u|_{W_F}$ is proper for every $F \in 2^{(G)}$. If $G$ is Haagerup, so is $W \rtimes G$.*

**Example 4** *(see Example 3)*

[1]The representation-theoretic form of the relative property (T) for $\mathbb{Z}^2 \rtimes SL_2(\mathbb{Z})$ was used by Margulis in 1973 to give the first explicit construction of expander graphs!

- *For $W = Sym_0(G)$, we have $W_F = Sym(F)$, a finite group; so we may take $u \equiv 0$.*

- *Take $W = H^{(G)}$, so that $W_F = H^F$; assume that $H$ is Haagerup; let $u : H \to \mathbb{R}^+$ be a proper function, conditionally of negative type. Extend $u$ to $W$ by $u(w) =: \sum_{g \in G} u(w_g)$. The assumptions of Theorem 5 are then verified, and we see that Theorem 5 implies Theorem 4.*

In the proof of Theorem 5, gauges are used to "lift" measured walls structures from $G$ to $W \rtimes G$.

If $A, F$ are subsets of $X$, we say that $A$ *cuts* $F$ if $A$ meets both $F$ and $X \backslash F$.

Assume that $(G, \mu)$ is an invariant measured walls structure on $G$; for $A \subset W \times G$, define

$$d_A(wg, w'g') = \begin{cases} 1 & \text{if } A \text{ cuts } \psi(w^{-1}w') \cup \{g, g'\} \\ 0 & \text{otherwise} \end{cases}$$

**Lemma 1** *$d_A$ is a cut-metric on $W \rtimes G$ (i.e. a $\{0,1\}$-valued pseudo-metric).* □

We can now sketch the proof of Theorem 5. Let $(B_i)_{i \in I}$ be the partition of $W \times G$ defined by "being at $d_A$-distance 0"; set $\nu_A = \frac{1}{2} \sum_{i \in I} \delta_{B_i}$, a measured walls structure on $W \times G$. Set then $\tilde{\mu} = \int_{2^G} \nu_A \, d\mu(A)$: a Borel measure on $2^{W \rtimes G}$, defining a measured walls structure with

$$d_{\tilde{\mu}}(wg, w'g') = \mu\{A \subset G : A \text{ cuts } \psi(w^{-1}w') \cup \{g, g'\}\}.$$

$\tilde{\mu}$ is $W \rtimes G$-invariant; moreover a subset of $W \times G$ is $d_{\tilde{\mu}}$-bounded if and only if it is contained in a product $W_B \times B'$, for some $d_\mu$-bounded sets $B, B' \subset G$. So if $G$ acts properly on $(G, \mu)$ then the action of $W \rtimes G$ on $(W \times G, \tilde{\mu})$ is proper, except for what happens on $W_B$: this is controlled by the function $u$. □