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Monotoniciy methods in mean-field games

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Mean-field games

- Mean-field game (MFG) theory is the study of strategic decision making in large populations of interacting agents.
- MFGs were introduced in 2006/07 in the Engineering community by P. Caines and his co-workers and in the Mathematics community by J. M. Lasry and P. L. Lions.

The measure-potential framework

Many problems with large populations, including MFG, fit the following framework:

- A probability density m for the population distribution;
- A "potential", "pressure", or "value function" u that encodes the effects of the population in the environment;
- A PDE for u that depends on m (typically, a nonlinear elliptic or parabolic equation)
- An evolution PDE for m driven by the potential u.

Applications

- Crowd and population models;
- Chemotaxis, herding, and flocking;
- Economic growth and socio-economic models;
- Price formation and price impact;
- Traffic flow;
- Energy systems.

Mean-field models

A canonical MFG comprises:

- a Hamilton-Jacobi (HJ) equation
- a transport of Fokker-Planck (FP) equation
- The HJ and the FP equations are fully coupled and the FP equation is the adjoint of the linearization of the HJ equation.

Monotonicity

- Monotonicity properties for MFGs encode crowd aversion behavior;
- As a consequence of crowd aversion, agents then to spread in a uniform way.

Deterministic optimal control

Fix a Lagrangian, $L(x, v) : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$, and a terminal cost, $\psi : \mathbb{T}^d \to \mathbb{R}$, and set

$$u(x, t) = \inf_{\mathbf{x}} \int_t^T L(\mathbf{x}, \dot{\mathbf{x}}) ds + \psi(\mathbf{x}(T)),$$

where the infimum is taken over all Lipschitz trajectories with $\mathbf{x}(t) = x$.

Fundamental questions

Under natural assumptions on L (e.g. strict or uniform convexity in v) and ψ (e.g. boundedness from below), can we:

- Prove the existence of optimal trajectories, $\mathbf{x}$;
- Characterize the value function, u;
- Compute optimal trajectories.

Existence of optimal trajectories

Existence of optimal trajectories can be proven by semicontinuity arguments - the direct method in the calculus of variations.
Furthermore, optimal trajectories solve the Euler-Lagrange equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{x}}} - \frac{\partial L}{\partial \mathbf{x}} = 0.$$

The value function solves a PDE - the Hamilton-Jacobi equation.

Hamilton-Jacobi equation

The Hamiltonian is

$$H(p,x)=\sup_{v\in\mathbb{R}^d} -v\cdot p - L(x,v).$$

If the value function u is smooth then it solves the Hamilton-Jacobi equation

$$-u_t + H(x, D_x u) = 0$$

together with the terminal condition

$$u(x,T)=\psi(x).$$

In general, the value function is only a viscosity solution.

Verification theorem

Verification Theorem

If L is convex and u is a smooth solution to the Hamilton-Jacobi equation then

$$\dot{\mathbf{x}} = -D_p H(\mathbf{x}, D_x u(\mathbf{x}, t))$$

is an optimal trajectory.

ODE flows

Consider the ODE

$$\dot{\mathbf{x}} = b(\mathbf{x}, t)$$

and the corresponding flow $\phi_t(x)$

$$\mathbf{x}(t) = \phi_t(\mathbf{x}(0)).$$

Push-forward

The push-forward, $\phi_t \sharp m_0$, of a probability measure $m_0(x)$ through ϕ_t is the probability measure defined by

$$\int \psi(x) dm(x,t) = \int \psi(\phi_t(x)) dm_0(x).$$

Transport equation

Then, $m(x, t)$ is a weak solution of the transport equation

$$m_t + \mathrm{div}(b(x, t)m) = 0.$$

In particular, if $b = -D_pH(D_xu(x, t), x)$,

$$m_t - \mathrm{div}(D_pHm) = 0.$$

Mean-field games

In a mean-field game, L depends on m. The value function is

$$u(x,t) = \inf_{\mathbf{v}} E \int_t^T L(\mathbf{x}, \dot{\mathbf{x}}, m(\cdot, s))ds + \psi(\mathbf{x}(T)),$$

where the evolution of m is determined by the optimal trajectories.

Initial-terminal value problem

u solves the Hamilton-Jacobi equation

$$-u_t + H(D_x u, x, m) = 0,$$

and agents follow the optimal dynamics. Thus, m solves

$$m_t - \operatorname{div}(D_p H m) = 0.$$

Additionally,

$$u(x, T) = \psi(x) \qquad m(x, 0) = m_0(x).$$

Stationary problem

Find $u, m : \mathbb{T}^d \to \mathbb{R}$, m probability, and $\overline{H} \in \mathbb{R}$ solving

$$\begin{cases} H(x, Du, m) = \overline{H} \\ -\operatorname{div}(D_p H m) = 0. \end{cases}$$

Then, $\tilde{u}(x,t) = u(x) + \overline{H}t$ and $\tilde{m}(x,t) = m(x)$ solves the time-dependent problem.

Key questions

- Can we characterize uniqueness for a wide class of mean-field games?
- Can we solve explicit examples?
- Can we prove existence?
- Can we develop effective numerical methods?

Model Hamiltonians and Lagrangians

An agents in state x and moving at speed v incurs a cost, L, by unit of time

$$L(x, v, m) = m^\alpha \frac{|v|^{\gamma'}}{\gamma'} - V(x) + g(m),$$

V periodic, smooth, $\alpha \geq 0$, $\gamma > 1$, $\frac{1}{\gamma} + \frac{1}{\gamma'} = 1$.
The corresponding Hamiltonian is

$$H(x, p, m) = \frac{m^\alpha}{\gamma} \left| \frac{p}{m^\alpha} \right|^\gamma + V(x) - g(m).$$

Model interpretation

- Cost of moving: $m^\alpha |v|^{\gamma'}$, $\frac{1}{\gamma}+\frac{1}{\gamma'}=1$.
  - $\alpha>0$ – congestion
  - large γ' (small γ) – higher cost of moving at high speed
- $-V(x)$ accounts for spatial preferences
- $g(m)$ encodes interactions:
  - Non-local: $g(m)=G(\eta * m)$, η smooth mollifier;
  - Power-like: $g(m)=m^\alpha$;
  - Logarithm: $g(m)=\ln m$;
  - Inverse power $g(m)=-\frac{1}{m^\alpha}$.

Parabolic MFGs

By adding noise to the agent's dynamics, we obtain parabolic (or elliptic, for stationary problems) MFGs:

$$\begin{cases} -u_t + H(x, Du, m) = \Delta u \\ m_t - \operatorname{div}(D_p H m) = \Delta m. \end{cases}$$

Constraints

- Agents can feel the effect of other agents either directly, e.g. by looking at agents's density in a neighborhood, or indirectly, through constraints.
- Density constraints that may be imposed through location surcharges.
- Integral constraints, e.g. supply or demand constraints, can be imposed through a price.

Pointwise constraints

We look at a MFG with pointwise constraints

$$\beta(m) = 0,$$

where β is a convex function. A way to enforce the constraint is to add Lagrange multiplier, $\lambda(x)$. An example is the stationary model

$$\begin{cases} H(x, Du, m) = \overline{H} + \lambda(x)\beta'(m) \\ -\operatorname{div}(D_p Hm) = 0 \\ \beta(m) = 0. \end{cases}$$

Density constraints

For $\beta(m) = \max(m - \varphi, 0)$, the pointwise constraint $\beta(m) = 0$ becomes

$$m \leq \varphi.$$

Formally, we obtain the MFG

$$\begin{cases} H(x, Du, m) = \overline{H} + \lambda(x) 1_{m-\varphi=0} \\ -\operatorname{div}(D_p H m) = 0. \end{cases}$$

$\lambda(x)$ is a location surcharge at the places where the constraint is saturated.

Integral constraints

- The actions agents can take can be constrained by global production constraints;
- These global constraints determine a price agents need to pay.

Example

Consider the Lagrangian

$$L(x, v, \varpi) = \frac{v^2}{2} + \varpi v - V(x)$$

where

- The quadratic term represents non-linear costs faced by agents;
- ϖv is the price paid by moving at speed v.

Given ϖ the corresponding MFG (which is uncoupled) is

$$\begin{cases} -u_t + \frac{(\varpi + u_x)^2}{2} + V(x) = 0 \\ m_t - (m(\varpi + u_x))_x = 0. \end{cases}$$

Each agent moves at speed

$$v = -(\varpi + u_x).$$

Global production constrains

We require an average production constraint

$$\int (\varpi + u_x) m = \zeta(t)$$

where $\zeta(t)$ corresponds to an aggregate production constraint.

A MFG with integral constrains

Hence, the MFG with integral contraint problem is: find u, m, and ϖ satisfying

$$\begin{cases} -u_t + \frac{(\varpi + u_x)^2}{2} + V(x) = 0 \\ m_t - (m(\varpi + u_x))_x = 0 \\ \int (\varpi + u_x) m = \zeta(t), \end{cases}$$

with initial-terminal conditions for u and m.

Monotone operators

Let H be a Hilbert space. $A : D \subset H \to H$ is a monotone operator if

$$(A(w) - A(z), w - z) \geq 0, \qquad \forall w, z \in D.$$

A variational inequality is the problem: find $u \in D$ such that

$$(A(w), z - w) \geq 0, \qquad \forall z \in D.$$

Examples of monotone operators

- For $H = \mathbb{R}$, monotone operators are increasing functions
- Gradients of convex functions are monotone operators

Variational inequalities

If $A : H \to H$ is monotone, then $A(w) = 0$ if and only if w satisfies

$$(A(w), z - w) \geq 0, \qquad \forall z \in H.$$

Monotone building blocks - I

The operator

$$A\begin{bmatrix} m \\ u \end{bmatrix} = \begin{bmatrix} -u \\ m \end{bmatrix}$$

monotone in $L^2(\mathbb{T}^d) \times L^2(\mathbb{T}^d)$.
Proof:

$$\left(A\begin{bmatrix} m \\ u \end{bmatrix} - A\begin{bmatrix} \tilde{m} \\ \tilde{u} \end{bmatrix}, \begin{bmatrix} m \\ u \end{bmatrix} - \begin{bmatrix} \tilde{m} \\ \tilde{u} \end{bmatrix} \right)$$

$$= \int_{\mathbb{T}^d} -(u-\tilde{u})(m-\tilde{m}) + (m-\tilde{m})(u-\tilde{u}) = 0.$$

Monotone building blocks - II

The operator

$$A\begin{bmatrix} m \\ u \end{bmatrix} = \begin{bmatrix} u_t \\ m_t \end{bmatrix}$$

monotone for $(u, m) \in L^2([0, T]) \times L^2([0, T])$ with $m(0) = m_0$ and $u(T) = u_T$.

Proof:

$$\left(A\begin{bmatrix} m \\ u \end{bmatrix} - A\begin{bmatrix} \tilde{m} \\ \tilde{u} \end{bmatrix}, \begin{bmatrix} m \\ u \end{bmatrix} - \begin{bmatrix} \tilde{m} \\ \tilde{u} \end{bmatrix} \right)$$

$$= \int_0^T (u - \tilde{u})_t(m - \tilde{m}) + (m - \tilde{m})_t(u - \tilde{u})$$

$$= \int_0^T \frac{d}{dt}(u - \tilde{u})(m - \tilde{m})dt = 0,$$

due to the boundary conditions.

Monotone building blocks - III

If g is increasing, the operator

$$A \begin{bmatrix} m \\ u \end{bmatrix} = \begin{bmatrix} g(m) \\ 0 \end{bmatrix}$$

monotone in $D \subset L^2(\mathbb{T}^d) \times L^2(\mathbb{T}^d)$.
Proof:

$$\left(A \begin{bmatrix} m \\ u \end{bmatrix} - A \begin{bmatrix} \tilde{m} \\ \tilde{u} \end{bmatrix}, \begin{bmatrix} m \\ u \end{bmatrix} - \begin{bmatrix} \tilde{m} \\ \tilde{u} \end{bmatrix} \right)$$
$$= \int_{\mathbb{T}^d} (g(m) - g(\tilde{m}))(m - \tilde{m}) \geq 0,$$

and the inequality is strict if g is strictly increasing and $m \neq \tilde{m}$.

Monotone building blocks - IV

If $H(x,p)$ is convex in p, the operator

$$A\begin{bmatrix} m \\ u \end{bmatrix} = \begin{bmatrix} -H(x,Du) \\ -\operatorname{div}(mD_pH) \end{bmatrix}$$

monotone in its domain $D \subset L^2(\mathbb{T}^d) \times L^2(\mathbb{T}^d)$ for $m > 0$.
Proof: integration by parts...

$$
\begin{aligned}
&\int_{\mathbb{T}} \left(-\frac{u_x^2}{2} + \frac{\tilde{u}_x^2}{2} \right) (m - \tilde{m}) + \left(-(m u_x)_x + (\tilde{m}\tilde{u}_x)_x \right) (u - \tilde{u}) \\
&= \int_{\mathbb{T}} m \left(-\frac{u_x^2}{2} + \frac{\tilde{u}_x^2}{2} - u_x(u_x - \tilde{u}_x \right) \\
&\quad + \int_{\mathbb{T}} \tilde{m} \left(-\frac{\tilde{u}_x^2}{2} + \frac{u_x^2}{2} - \tilde{u}_x(\tilde{u}_x - u_x) \right) \\
&= \int_{\mathbb{T}} \frac{m + \tilde{m}}{2} (u_x - \tilde{u}_x)^2 \geq 0.
\end{aligned}
$$

Assembling the blocks - stationary

Then, if $H(x,p)$ is convex in p and g is increasing, the operator

$$A \begin{bmatrix} m \\ u \end{bmatrix} = \begin{bmatrix} -u - H(x,Du) + g(m) \\ m - \operatorname{div}(D_p H m) - 1 \end{bmatrix}$$

monotone in its domain $D \subset L^2 \times L^2$.

Assembling the blocks - time-dependent

Then, if $H(x,p)$ is convex in p and g is increasing, the operator

$$A\begin{bmatrix} m \\ u \end{bmatrix} = \begin{bmatrix} u_t - H(x,Du) + g(m) \\ m_t - \mathrm{div}(D_pHm) - 1 \end{bmatrix}$$

monotone in its domain $D \subset L^2 \times L^2$.

Other model monotone MFGs

Stationary

$$\begin{cases} \Delta u - H(x, Du) + g(m) + \overline{H} = 0 \\ -\Delta m - \operatorname{div}(D_p H m) = 0. \end{cases}$$

Stationary congestion problem

$$\begin{cases} \Delta u - m^\alpha H(x, \frac{Du}{m^\alpha}) + \overline{H} = 0 \\ -\Delta m - \operatorname{div}(D_p H m) = 0. \end{cases}$$

Here, $g(m)$ is increasing (e.g. m^γ or $\ln m$), or non-local monotone, $H(x, p)$ convex in p, $0 < \alpha < 2$.

Pointwise constraints

For a convex function, β, and $\lambda(x) \geq 0$, the operator

$$A \begin{bmatrix} m \\ u \\ \lambda \end{bmatrix} = \begin{bmatrix} \lambda\beta'(m) \\ 0 \\ \beta(m) \end{bmatrix}$$

is monotone in $D \subset L^2 \times L^2 \times L^2$.

Price models

For $H(x,p)$ convex in p, the operator

$$A\begin{bmatrix} m \\ u \\ \varpi \end{bmatrix} = \begin{bmatrix} -H(x,\varpi+u_x) \\ -\operatorname{div}(mD_pH(x,\varpi+u_x)) \\ \int D_pH(x,\varpi+u_x)m \end{bmatrix}$$

is monotone in $D \subset L^2 \times L^2 \times \mathbb{R}$.
Proof: ...

Uniqueness

Often, monotonicity gives uniqueness. Given two solutions (m, u) and $(\tilde{m}, \tilde{u})$, we have

$$0 = \left(A \begin{bmatrix} m \\ u \end{bmatrix} - A \begin{bmatrix} \tilde{m} \\ \tilde{u} \end{bmatrix}, \begin{bmatrix} m \\ u \end{bmatrix} - \begin{bmatrix} \tilde{m} \\ \tilde{u} \end{bmatrix} \right) \geq 0.$$

Example

For

$$A\begin{bmatrix} m \\ u \end{bmatrix} = \begin{bmatrix} u_t - \frac{u_x^2}{2} + m \\ m_t - (mu_x)_x \end{bmatrix},$$

we get

$$0 = \int_0^T \int (m+\tilde{m})\frac{(u_x - \tilde{u}_x)^2}{2} + (m-\tilde{m})^2 \geq 0.$$

This implies $m = \tilde{m}$ and then, uniqueness of solution of

$$-u_t + \frac{u_x^2}{2} = m$$

gives $u = \tilde{u}$.

Separated MFGs without congestion

Solve

$$\begin{cases} H(x, Du) = g(m) + \overline{H} \\ -\operatorname{div}(mD_pH(x, Du)) = 0 \end{cases}$$

for $u, m : \mathbb{T}^d \to \mathbb{R}$, $m \geq 0$, and $\overline{H} \in \mathbb{R}$.
Let $G : \mathbb{R}_0^+ \to \mathbb{R}$, convex, and $g = G'$.

Constrained minimization

Minimize

$$(m, v) \mapsto \int_{\mathbb{T}^d} m(x)L(x, v(x)) + G(m)$$

under the constraint

$$-\operatorname{div}(m(x)v(x)) = 0,$$

$m \geq 0$ and $\int_{\mathbb{T}^d} m dx = 1$.

Dual minimization

Minimize

$$u \mapsto \int_{\mathbb{T}^d} G^*(H(x, Du))dx.$$

Joint minimization

Minimize (or look for critical points)

$$(m,u) \mapsto J[m,u] = \int_{\mathbb{T}^d} -m(x)H(x, D_x u) + G(m(x))dx,$$

$m \geq 0$ and $\int_{\mathbb{T}^d} m dx = 1$.

MFG as a (pseudo) gradient

The preceding functional is convex in m and concave in u. Hence

$$\begin{bmatrix} m \\ u \end{bmatrix} \mapsto \begin{bmatrix} \frac{\delta J}{\delta m} \\ -\frac{\delta J}{\delta u} \end{bmatrix}$$

is a monotone map.

A functional for congestion problems

Let

$$J[u,m]=\int_{\mathbb{T}^d}\left(\frac{|P+Du|^\gamma}{\gamma(\alpha-1)m^{\alpha-1}}-Vm+G(m)\right)dx.$$

Then, if (u,m), with $u,m:\mathbb{T}^d\to\mathbb{R}$ and $m>0$, is a smooth enough minimizer of J under the constraint

$$\int_{\mathbb{T}^d} m\,dx=1,$$

then (u,m) solves

$$\begin{cases}\frac{|P+Du|^\gamma}{\gamma m^\alpha}+V(x)=g(m)+\overline{H}\\ -\operatorname{div}(m^{1-\alpha}|P+Du|^{\gamma-2}(P+Du))=0.\end{cases}$$

MFGs as gradients

For $1 < \alpha \leq \gamma$, the preceding functional is jointly convex in m and u. Hence

$$\begin{bmatrix} m \\ u \end{bmatrix} \mapsto \begin{bmatrix} \frac{\delta J}{\delta m} \\ \frac{\delta J}{\delta u} \end{bmatrix}$$

is a monotone map.

Set

$$\bar{J}[u,m]=\int_{\mathbb{T}^d}\left[\bar{f}(\nabla u,m)-Vm+G(m)\right]dx, \tag{1}$$

where, for $(p,m)\in\mathbb{R}^d\times\mathbb{R}_0^+$,

$$\bar{f}(p,m)=\begin{cases}\frac{|P+p|^\gamma}{\gamma(\alpha-1)m^{\alpha-1}} & \text{if } m\neq 0,\\ +\infty & \text{if } m=0 \text{ and } p\neq -P,\\ 0 & \text{if } m=0 \text{ and } p=-P.\end{cases} \tag{2}$$

Existence of minimizers

Assume that $V \in L^\infty(\mathbb{T}^d)$ and that there exist positive constants, θ and C, such that

$$G(z) \geq \frac{1}{C} z^{\theta+1} - C \text{ for all } z > 0,$$

then the preceding minimization problem has a solution $(u, m) \in W^{1,\gamma(1+\theta)/(\alpha+\theta)} \times L^{1+\theta}$ for all $1 < \alpha \leq \gamma$.

Proof: Convexity and Coercivity.

Proof - Convexity

Suppose that $1 < \alpha \leq \gamma$. Then, the function $\bar{f}$ is convex and lower semi-continuous in $\mathbb{R}^d \times \mathbb{R}_0^+$.

Proof - Coercivity

Suppose

$$\sup_{n\in\mathbb{N}} \left|\bar{J}[u_n, m_n]\right| \leq C.$$

Let

$$U_n = \{x \in \mathbb{T}^d : \nabla u_n \neq -P\}.$$

Then

$$\int_{\mathbb{T}^d} \bar{f}(\nabla u_n, m_n)\, dx = \int_{U_n} \frac{|P + \nabla u_n|^\gamma}{\gamma(\alpha - 1) m_n^{\alpha-1}}\, dx \leq C.$$

Proof - Coercivity

Because $q = \frac{\gamma}{\alpha}$, using the prior estimate and Young's inequality, we obtain

$$\begin{aligned}\int_{\mathbb{T}^d} |P + \nabla u_n|^q \, dx &= \int_{U_n} |P + \nabla u_n|^{\frac{\gamma}{\alpha}} \, dx = \int_{U_n} \frac{|P + \nabla u_n|^{\frac{\gamma}{\alpha}}}{m_n^{\frac{\alpha-1}{\alpha}}} m_n^{\frac{\alpha-1}{\alpha}} \, dx \\ &\leq \frac{1}{\alpha} \int_{U_n} \frac{|P + \nabla u_n|^{\gamma}}{m_n^{\alpha-1}} \, dx + \frac{\alpha-1}{\alpha} \int_{\mathbb{T}^d} m_n \, dx \\ &\leq C.\end{aligned}$$

First-order MFGs

- First-order mean-field games with logarithmic nonlinearities:

$$\begin{cases} H(Du,x) = \ln m + \overline{H} \\ -\operatorname{div}(D_pHm) = 0, \end{cases}$$

have smooth solutions (Evans).

- However, this is not the general situation. Even simple MFGs as

$$\begin{cases} \frac{u_x^2}{2} + \lambda V(x) = m + \overline{H}(\lambda), \\ -(mu_x)_x = 0, \end{cases}$$

may fail to have smooth solutions.

In this case, m is given by

$$m(x,\lambda) = \left(\lambda V(x) - \overline{H}(\lambda)\right)^+,$$

and

$$\frac{u_x^2}{2} = \left(\lambda V(x) - \overline{H}(\lambda)\right)^-.$$

If λ is small, the condition $\int_{\mathbb{T}} m = 1$ gives

$$\overline{H}(\lambda) = \lambda \int_{\mathbb{T}} V - 1;$$

that is,

$$m(x,\lambda) = 1 + \lambda \left(V(x) - \int_{\mathbb{T}} V \right).$$

For $|\lambda|$ large, the condition $m > 0$ fails.

Formation of regions with $m = 0$ as λ increases

$$V(x) = \lambda \sin(2\pi(x + 1/4))$$

Current formulation

Consider the MFG

$$\begin{cases} \frac{(u_x+p)^2}{2} + V(x) = g(m) + \overline{H}, \\ -(m(u_x+p))_x = 0. \end{cases} \tag{3}$$

From the preceding equation, $j = m(u_x + p)$ is constant. Next, we solve for u_x and replace it in the first equation. in the first equation.

Monotonicity and solvability

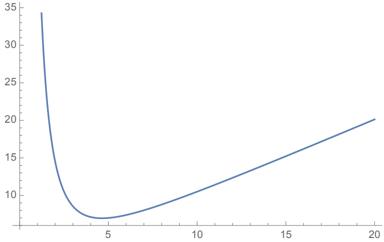
For $j \neq 0$, we have

$$\begin{cases} \frac{j^2}{2m^2} - g(m) = \overline{H} - V(x), \\ m > 0, \int\limits_{\mathbb{T}} m dx = 1, \\ \int\limits_{\mathbb{T}} \frac{1}{m} dx = \frac{p}{j}. \end{cases}$$

If g is monotone, first equation above has a unique solution m for each j, $\overline{H}$, and x. Moreover, $m > 0$ and $\overline{H}$ is determined by the normalization condition $\int m = 1$.

The non-monotone case, $g(m) = -m$

$$\begin{cases} \frac{j^2}{2m^2} + m = \overline{H} - V(x); \\ m > 0, \ \int\limits_{\mathbb{T}} m dx = 1; \\ \int\limits_{\mathbb{T}} \frac{1}{m} dx = \frac{p}{j}. \end{cases} \tag{4}$$



$$F_j(m) = \frac{j^2}{2m^2} + m, \ \min F_j = \frac{3j^{2/3}}{2}$$

Explicit solutions for $g(m) = -m,\ j > 0$

A lower bound for $\overline{H}$ is

$$\overline{H} \geq \overline{H}_j^{cr} = \max_{\mathbb{T}} V + \frac{3j^{2/3}}{2}. \tag{5}$$

For any $\overline{H}$ satisfying (5), let $m_{\overline{H}}^-$ and $m_{\overline{H}}^+$ be the solutions of

$$\frac{j^2}{2(m_{\overline{H}}^{\pm}(x))^2} + m_{\overline{H}}^{\pm}(x) = \overline{H} - V(x),$$

with $0 \leq m_{\overline{H}}^-(x) \leq m_{\overline{H}}^+(x)$.

Explicit solutions for $g(m) = -m, \; j > 0$

The function m is agrees almost everywhere with $m^{\pm}$. However, because

$$p + u_x = \frac{j}{m},$$

the viscosity condition implies that m can only have upwards jumps.

Stationary solutions for different current levels

$V(x) = \sin(2\pi(x+1/4))$, each frame is a stationary solution for some $j > 0$.

Zero current unhappiness traps and non-uniqueness

$V(x) = \sin(2\pi(x + 1/4))$, each frame is a solution for $j = 0$.

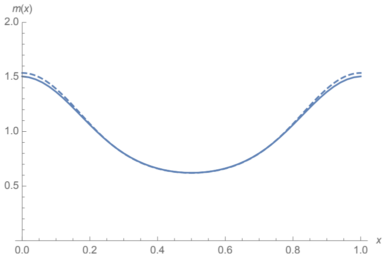
Lack of uniqueness of solutions

$V(x) = \sin(4\pi(x + 1/8))$, each frame is a stationary solution for $j = 0$.

"Unhappiness traps"

- Our solutions suggest that when $g(m) = -m$ agents prefer to stick together, rather than be at a better place (unhappiness traps).
- These results are consistent with the intuition that g models the crowd seeking preference of the agents.

Vanishing viscosity instability



Stability (monotone) vs instability (anti-monotone) - no viscosity dashed.

Stationary, elliptic MFGs

Theorem (G., Patrizi, Voskanyan)

Suppose H has quadratic growth, $g(m) = \ln m$ or $g(m) = m^\alpha$, $0 < \alpha < \alpha_d$. Then, the MFG

$$\begin{cases} -\Delta u + H(x, Du) = g(m) + \overline{H} \\ -\Delta m - \operatorname{div}(D_p H(x, Du) m) = 0, \end{cases}$$

has a unique classical solution $u, m \in C^\infty(\mathbb{T}^d)$, and $\overline{H} \in \mathbb{R}$.

This problem was examined first studied by G. and H. Sanchez-Morgado. Later, Voskanyan and Pimentel who considered a general growth Hamiltonian, and, afterwards, M. Cirant developed improved value α_d. Recently, Boccardo, Orsina and Porretta examined stationary problems in the light of weak solutions.

Stationary, elliptic MFGs

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This problem was examined first studied by G. and H. Sanchez-Morgado. Later, Voskanyan and Pimentel who considered a general growth Hamiltonian, and, afterwards, M. Cirant developed improved value α_d. Recently, Boccardo, Orsina and Porretta examined stationary problems in the light of weak solutions.

Congestion

Theorem (G., Mitake)

For $0 < \alpha < 1$, there exists a classical solution (u, m) to the congestion MFG

$$\begin{cases} u + V(x) + \frac{|Du|^2}{2m^\alpha} = \Delta u + \overline{H} \\ m - \operatorname{div}(m^{1-\alpha} Du) = \Delta m + 1. \end{cases}$$

Recently, G. and Evangelista proved the existence of solutions for congestion with general subquadratic Hamiltonians

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Time-dependent problems

- Initial-terminal problem

$$\begin{cases} -u_t + H(x, Du) = \Delta u + g(m) \\ m_t - \operatorname{div}(D_pH(x, Du)m) = \Delta m, \end{cases}$$

  $m(x,0)$, $u(x,T)$ given, $g(m) = \ln m$ or $g(m) = m^\alpha$, H sub or superquadratic.
  Weak sol.: Porretta; Smooth sol.: G., Pimentel, and Morgado; First-order: Cardaliaguet, Graber; Cardaliaguet, Porretta and Tonon. Log: G., Pimentel;
- Short-time congestion:

$$\begin{cases} -u_t + m^\alpha H\left(x, \frac{Du}{m^\alpha}\right) = \Delta u \\ m_t - \operatorname{div}(D_pHm) = \Delta m \end{cases}$$

  with $m(x,0)$, $u(x,T)$ given, $0 < \alpha < 1$, T small.
  Weak sol.: Graber; Smooth sol.: G. and Voskanyan

Additional models

- Logistic problems. Fokker-Planck equation becomes
$$-\Delta m - \operatorname{div}(D_pHm) = m(1-m) + 1.$$
A key difficulty is the lack of monotonicity.
G. and Ribeiro
- Obstacle-type problems:
$$\begin{cases} \beta_\epsilon(u) + H(Du, x) = g(m) \\ \beta'(u)m - \operatorname{div}(D_pHm) = 1. \end{cases}$$
Here, β_ϵ is an increasing function such that
$$\lim_{\epsilon\to 0} \beta_\epsilon(x) = \begin{cases} +\infty \text{ if } x > 0 \\ 0, \text{ otherwise.} \end{cases}$$
G. and Patrizi - only first order. Similar techniques apply to weakly coupled systems

Continuation method strategy

- Introduce a parameter, λ. For $\lambda = 0$, the model has an explicit solution; $\lambda = 1$ corresponds to the desired problem.
- Prove a-priori estimates uniformly in λ.
- Show that the implicit function applies.

Monotonicity and a priori estimates

Energy estimates can be regarded as a consequence of the monotonicity. For a solution (m, u) of

$$A\begin{bmatrix} m \\ u \end{bmatrix} = 0$$

and any test function $(\bar{m}, \bar{u})$

$$\begin{aligned}
&\left(-A\begin{bmatrix} \bar{m} \\ \bar{u} \end{bmatrix}, \begin{bmatrix} m \\ u \end{bmatrix} - \begin{bmatrix} \bar{m} \\ \bar{u} \end{bmatrix}\right) \\
&\quad = \left(A\begin{bmatrix} m \\ u \end{bmatrix} - A\begin{bmatrix} \bar{m} \\ \bar{u} \end{bmatrix}, \begin{bmatrix} m \\ u \end{bmatrix} - \begin{bmatrix} \bar{m} \\ \bar{u} \end{bmatrix}\right) \geq 0.
\end{aligned}$$

Example

Consider the MFG

$$\begin{cases} -u_t + \frac{u_x^2}{2} - m = 0 \\ m_t - (mu_x)_x = 0, \end{cases}$$

with initial-terminal conditions $m(x,0) = m_0$ and $u(x,T) = u_T(x)$.
The corresponding operator is

$$A \begin{bmatrix} m \\ u \end{bmatrix} = \begin{bmatrix} u_t - \frac{u_x^2}{2} + m \\ m_t - (mu_x)_x \end{bmatrix}.$$

First,

$$\left(A\begin{bmatrix} m \\ u \end{bmatrix} - A\begin{bmatrix} m_0 \\ u_0 \end{bmatrix}, \begin{bmatrix} m \\ u \end{bmatrix} - \begin{bmatrix} \bar{m} \\ \bar{u} \end{bmatrix}\right)$$

$$= \int_0^T \int_{\mathbb{T}} (m_0 + m)\frac{u_x^2}{2} + (m - m_0)(m - m_0).$$

Second

$$\left(-A\begin{bmatrix} m_0 \\ u_T \end{bmatrix}, \begin{bmatrix} m \\ u \end{bmatrix} - \begin{bmatrix} m_0 \\ u_T \end{bmatrix}\right)$$

is **linear** on (m, u).

Then, reorganizing the terms

$$\int_0^T \int_{\mathbb{T}} (m_0 + m)\frac{u_x^2}{2} + m^2 \leq C(m_0, u_T).$$

Monotonicity and the implicit function theorem

- To use the implicit function theorem, we need the linearized operator to be an isomorphism and in particular injective.
- Formally, the monotonicity of the original operator gives monotonicity of the linearized operator and hence injectivity.

Monotonicity of the linearized operator

Let A be a monotone operator.

$$(A(w+\epsilon z) - A(w), z) \geq 0)$$

Divide the preceding inequality by $\epsilon > 0$ and we get

$$(Lz, z) \geq 0$$

where L is the linearization of A around w. Often, this inequality is strict for $z \neq 0$ and combined with $Lz = 0$ implies $z = 0$.

Example

Let

$$A\begin{bmatrix} m \\ u \end{bmatrix} = \begin{bmatrix} -\frac{u_x^2}{2} + m \\ -(mu_x)_x \end{bmatrix}.$$

The linearized operator is

$$L\begin{bmatrix} \eta \\ \varphi \end{bmatrix} = \begin{bmatrix} -u_x\varphi_x + \eta \\ -(m\varphi_x)_x - (\eta u_x)_x \end{bmatrix},$$

and

$$\left(L\begin{bmatrix} \eta \\ \varphi \end{bmatrix}, \begin{bmatrix} \eta \\ \varphi \end{bmatrix}\right) = \int \eta^2 + m\varphi_x^2 \geq 0.$$

Weak solutions to variational inequalities

w is a weak solution of the variational inequality if

$$(A(z), z - w) \geq 0$$

for all $z \in D$.
Solutions of the variational inequality are weak solutions.
Under continuity assumptions and if D is large enough, weak solutions are solutions.

Weak solutions - an example

If $H = \mathbb{R}$, a montone operator, A, is an increasing function. If A is continuous,

$$A(0) = 0$$

if and only if

$$A(z)(z - 0) = A(z)z \geq 0$$

for all z.

MFGs and variational inequalities

Consider the MFG

$$\begin{cases} u - \Delta u + H(x, Du) = g(m) \\ m - \Delta m - \operatorname{div}(D_p H m) = 1. \end{cases}$$

Then, if $H(x,p)$ is convex in p and g is increasing, the operator

$$A \begin{bmatrix} m \\ u \end{bmatrix} = \begin{bmatrix} -u + \Delta u - H(x, Du) + g(m) \\ m - \Delta m - \operatorname{div}(D_p H m) - 1 \end{bmatrix}$$

is monotone in its domain $D \subset L^2 \times L^2$.

Weak solutions

A weak solution of the MFG is a pair (m, u), $m \geq 0$, such that

$$\left\langle \begin{bmatrix} \eta \\ v \end{bmatrix} - \begin{bmatrix} m \\ u \end{bmatrix}, A \begin{bmatrix} \eta \\ v \end{bmatrix} \right\rangle_{\mathcal{D}'(\mathbb{T}^d)\times\mathcal{D}'(\mathbb{T}^d), C^\infty(\mathbb{T}^d)\times C^\infty(\mathbb{T}^d)} \geq 0$$

for all $(\eta, v) \in \mathcal{C}^\infty(\mathbb{T}^d; \mathbb{R}^+) \times \mathcal{C}^\infty(\mathbb{T}^d)$.

Existence of weak solutions

Main Theorem (Ferreira, G.)

Under suitable but general Assumptions, there exists a weak solution, $(m,u)\in\mathcal{D}'(\mathbb{T}^d)\times\mathcal{D}'(\mathbb{T}^d)$, $m\geq 0$, to the MFG

$$A\begin{bmatrix} m\\ u\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}.$$

Moreover, $(m,u)\in\mathcal{M}_{ac}\times W^{1,\gamma}$ for some $\gamma>1$ and $\int_{\mathbb{T}^d} m\,dx=1$.

Scope

First-order, second-order, degenerate elliptic, and congestion problems satisfying monotonicity conditions.

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Scope

First-order, second-order, degenerate elliptic, and congestion problems satisfying monotonicity conditions.

Proof - Minty method

Suppose A_ϵ is a monotone approximation to A and w_ϵ a solution of

$$A_\epsilon(w_\epsilon) = 0.$$

Then

$$(A_\epsilon(z), z - w_\epsilon) \geq 0.$$

If $w_\epsilon \rightharpoonup w$ in H and $A_\epsilon(z) \to A(z)$ in H, then

$$(A(z), w - z) \geq 0.$$

Proof - regularization

To build solutions, we consider the regularized operator

$$A_\epsilon \begin{bmatrix} \eta \\ v \end{bmatrix} = A \begin{bmatrix} \eta \\ v \end{bmatrix} + \epsilon \begin{bmatrix} \eta + \Delta^{2p}\eta \\ v + \Delta^{2p} v \end{bmatrix} + \begin{bmatrix} \beta_\epsilon(\eta) \\ 0 \end{bmatrix},$$

where p is large enough, and $\beta_\epsilon(s) = 0$ if $s \geq \epsilon$ and $\beta_\epsilon(s) = -\frac{1}{s^q}$ if $0 < s \leq \frac{\epsilon}{2}$.

Proof - solutions to the regularized problem

Because p is large, mild growth conditions on H ensure the existence of a unique classical solution with $m_\epsilon > 0$ to

$$A_\epsilon \begin{bmatrix} m_\epsilon \\ u_\epsilon \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Proof - structural hypothesis on H

To prove convergence of the regularized problem, we need structural hypothesis on H. These are quite involved, but boil down to monotonicity together with the identity

$$0 = \int \left[m_\epsilon - \epsilon - 1 \;\; u_\epsilon \right] A_\epsilon \begin{bmatrix} m_\epsilon \\ u_\epsilon \end{bmatrix}$$

implying weak pre-compactness of (m_ϵ, u_ϵ) as $\epsilon \to 0$.

Example

Theorem (Ferreira, G.)

Let κ be a standard mollifier, $\alpha > 0$. Then, there exists a weak solution $u \in H^1$, $m \in L^{\alpha+1}$, $m \geq 0$ to

$$\begin{cases} u + \frac{|Du|^2}{2} + V(x) = m^\alpha + \kappa * m \\ m - \operatorname{div}(mDu) = 1. \end{cases}$$

That is, for all $(\eta, v) \in C^\infty$, $\eta > 0$, we have

$$\int \left(v + \frac{|Dv|^2}{2} + V(x) - \eta^\alpha - \kappa * m\right)(\eta - m) + \int (\eta - \operatorname{div}(\eta Dv) - 1)(v - u) \geq 0.$$

Example - further properties

Theorem (Ferreira, G.)

There exists a weak solution (u, m) such that

$$\begin{cases} -u - \frac{|Du|^2}{2} + V(x) + m^\alpha + \kappa * m \geq 0, & \text{in } \mathcal{D}' \\ m - \operatorname{div}(mDu) - 1 = 0, \text{a.e.}. \end{cases}$$

Moreover, if $\alpha > \max\left(\frac{d-4}{2}, 0\right)$

$$\Big(- u - \frac{|Du|^2}{2} + V(x) + m^\alpha + \kappa * m \Big) m = 0$$

almost everywhere.

Weak-strong uniqueness

Recently, V. Voskanyan proved a weak-strong uniqueness result for monotone MFGs; that is, if a MFG admits a strong solution then weak solutions are unique and agree with the strong solution.

Singular stationary problems

A natural application of monotonicity methods is the study of obstacle-type problems such as the second-order problem

$$\begin{cases} \beta_\epsilon(u) - \Delta u + H(Du, x) = g(m) \\ \beta'(u)m - \Delta m - \operatorname{div}(D_pHm) = 1 \end{cases}$$

or systems of variational inequalities (such as the ones arising in switching problems).

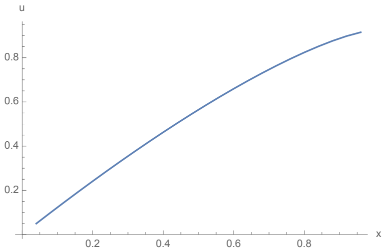
Stationary problems with boundary conditions

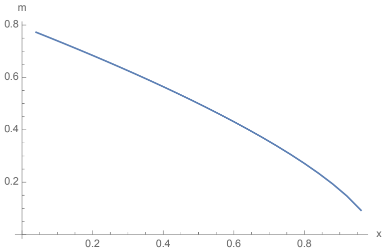
We examine the problem

$$\begin{cases} -u + m - \frac{{u_x}^2}{2} = 0 \quad \text{in } (0,1), \\ m - (mu_x)_x - 1 = 0 \quad \text{in } (0,1), \\ u(0) = 0, u(1) = u_1, \end{cases}$$

Given arbitrary u_1 do we always have a solution? or are the boundary conditions satisfied in viscosity sense?

Boundary conditions in viscosity sense





u (left) and m right with boundary condition (in the viscosity sense) $u(1) = 2$

Solutions that satisfy the boundary conditions

We introduce the regularization

$$\begin{cases} -u + m - \frac{{u_x}^2}{2} + \epsilon(m + m_{xxxx}) = 0 \quad \text{in } (0,1), \\ m - (m u_x)_x - \phi + \epsilon(u + u_{xxxx}) = 0 \quad \text{in } (0,1), \\ m_{xx}(0) = m_{xx}(1) = m_{xxx}(0) = m_{xxx}(1) = 0, \\ u(0) = 0, u(1) = u_1, u_{xx}(0) = u_{xx}(1) = 0. \end{cases}$$

Then, we have the following a-priori estimate:

Theorem (Ferreira, G., Tada)

There exits a positive constant, that depends only on the problem data such that

$$\int_0^1 [(m^\epsilon)^2 + \frac{m^\epsilon (u^\epsilon)^2}{2} + \frac{(u_x^\epsilon)^2}{2}]dx$$
$$+\epsilon \int_0^1 [(m^\epsilon)^2 + (m_{xx}^\epsilon)^2 + (u^\epsilon)^2 + (u_{xx}^\epsilon)^2]dx \leq C.$$

- From the preceding theorem, u is uniformly continuous in ϵ and converges to a function that satisfies the boundary conditions.
- This limit is a weak solution of the MFG.

Extensions: time-dependent problems

For time-dependent problems, we use an **elliptic** regularization

$$\begin{cases} -u_t + \frac{|Du|^2}{2} = \Delta u + \epsilon(\partial_t^2 + \Delta)^{2q} m + \beta_\epsilon(m) \\ m_t - \operatorname{div}(mDu) = \Delta m - \epsilon(\partial_t^2 + \Delta)^{2q} u \end{cases}$$

with suitable boundary conditions.

Long-time convergence for Hamilton-Jacobi equations

For convex Hamiltonians, the solutions of

$$-\epsilon u_t + \frac{|Du|^2}{2} + V(x) = 0$$

are known to converge as $\epsilon \to 0$ (long-time limit) to stationary solutions

$$\frac{|D\bar{u}|^2}{2} + V(x) = 0.$$

Monotonicity estimates

The original proof is due to Fathi. An approach based on ideas closely resembling MFGs was developed by Cagnetti, G., Mitake, and Tran. The key ingredient in the proof is the monotonicity estimate

$$\int_0^1 \int |Du - D\bar{u}|^2 m \to 0,$$

where m solves

$$m_t - \operatorname{div}(mDu) = 0.$$

Long-time convergence for MFGs

- For finite-state MFGs, monotonicity was used to establish the long-time convergence by G., Mohr, and Souza.
- For continuous time MFG, the convergence was established by Cardaliaguet, Lasry, Lions, and Porretta, also using monotonicity.

Long-time convergence

Consider the MFG

$$\begin{cases} -\epsilon u_t + \frac{|Du|^2}{2} = \Delta u + m \\ \epsilon m_t - \operatorname{div}(mDu) = \Delta m \end{cases}$$

with initial-terminal conditions

$$\begin{cases} u(x,1) = u_1(x) \\ m(x,0) = m_0(x). \end{cases}$$

Uniform bounds

We have the following a-priori bound

$$\int_0^1 \int_{\mathbb{T}^d} \frac{|Du^\epsilon|^2}{2}(m_0 + m^\epsilon) + (m^\epsilon)^2 \leq C,$$

where C is independent of ϵ.
Thus, through a subsequence $u^\epsilon \rightharpoonup u$ in $L^2([0,1], \dot{H}^1)$ and $m^\epsilon \rightharpoonup m$ in L^2.
A simple argument shows that for almost every $0 \leq t \leq 1$, $(u(x,t), m(x,t))$ is a weak solution of

$$\begin{cases} -\Delta u + \frac{|Du|^2}{2} = m + \overline{H}(t) \\ -\Delta m - \operatorname{div}(mDu) = 0. \end{cases}$$

Traveling waves

Consider the following non-monotonic MFG with congestion

$$\begin{cases} -u_t + \frac{u_x^2}{2m^\alpha} + Km^\alpha = 0 \\ m_t - (m^{1-\alpha} u_x)_x = 0. \end{cases}$$

Set $\alpha = 1$, $v = u_x$, and differentiate the first equation

$$\begin{cases} v_t + \left(v^2/(2m) + Km\right)_x = 0 \\ m_t - v_x = 0. \end{cases}$$

The following are traveling-wave solutions

$$m(t,x) = m_0(x + \sqrt{2K}t) \quad \text{and} \quad v(t,x) = v_0(x + \sqrt{2K}t)$$

for $v_0 = \sqrt{2K} m_0$.

The contracting flow

If A is a monotone operator in a Hilbert space, then the flow

$$\dot{w} = -A(w)$$

is a contraction in H.

Monotone flow

We introduced the dynamic approximation

$$\begin{cases} \dot{m} = \frac{u_x^2}{2} + V(x) - \ln m \\ \dot{u} = (mu_x)_x. \end{cases}$$

If (u, m) and $(\tilde{u}, \tilde{m})$ are solutions of the previous flow, then

$$\frac{d}{dt} \int |m - \tilde{m}|^2 + |u - \tilde{u}|^2 \leq 0,$$

provided $m, \tilde{m} \geq 0$.

u and m evolution by monotone flow. $V(x) = \sin(2\pi x)$.

The congestion problem

$$\begin{cases} \dot{m} = -\frac{|u_x|^2}{m^{1/2}} - \sin(2\pi x) + \ln m \\ \dot{u} = \operatorname{div}(m^{1/2} u_x). \end{cases}$$

m error evolution.

A two-dimensional example

u and *m* error - monotone flow. $V(x,y) = \sin(2\pi x) + \sin(2\pi y)$.

Effective Hamiltonian

In homogenization theory, it is important to find the constant $\overline{H}(P)$ such that the Hamilton-Jacobi equation

$$H(P + D_x u, x) = \overline{H}(P)$$

has periodic solutions.
The number $\overline{H}$ is called the effective Hamiltonian.

Here, for illustration, we consider

$$\frac{(P+u_x)^2}{2}+V(x)=\overline{H}(P).$$

Modified flow

We introduced the flow

$$\begin{cases} \dot{m} = m\left(\frac{(P+u_x)^2}{2} + V(x) - \lambda(t)\right) \\ \dot{u} = (m(P+u_x))_x, \end{cases}$$

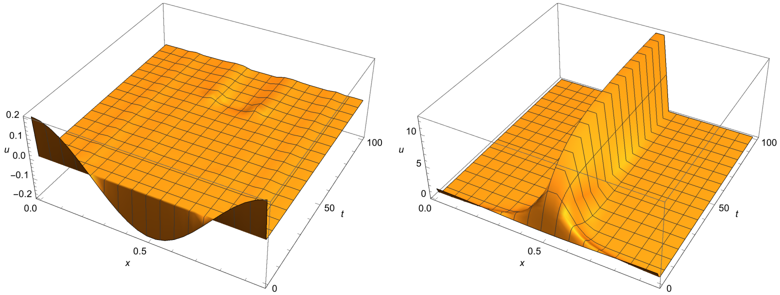
where

$$\lambda(t) = \frac{\int (\frac{(P+u_x)^2}{2} + V(x))m}{\int m}.$$

Formally, we have

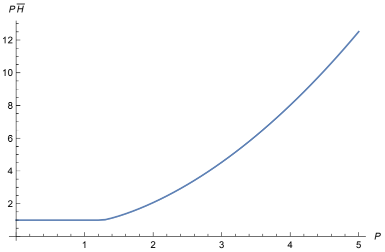
- Flow is a contraction
- $m(x, t)$ converges to a Mather measure
- the limit of $u(x, t)$ solves the stationary problem on $m > 0$
- $\lambda(t) \to \overline{H}$.

Evolution by the flow



Evolution of u (left) and m (right)

Computation of $\overline{H}$



Effective Hamiltonian

Flows with boundary conditions

In general, the monotone flow

$$\frac{d}{ds}\begin{bmatrix} m \\ u \end{bmatrix} = -A\begin{bmatrix} m \\ u \end{bmatrix}$$

may not preserve boundary conditions.

Conjugation of monotone operators

If A is a monotone operator and P is a self-adjoint projection then

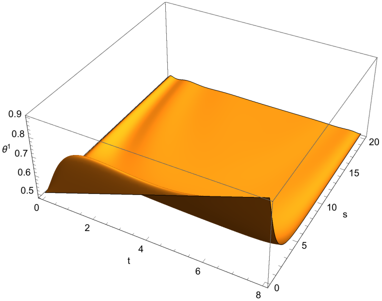
$$A_P = PAP$$

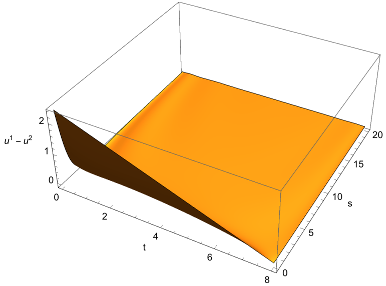
is a monotone operator.
For time-dependent MFGs, we can consider the H^1 projection on initia-terminal conditions and consider the modified flow

$$\frac{d}{ds}\begin{bmatrix} m \\ u \end{bmatrix} = -A_P \begin{bmatrix} m \\ u \end{bmatrix}.$$

This flow is still a contraction.

Initial-terminal conditions preserving flows





Of the density of distribution of players, $\theta(\cdot, s)$, and the difference of the value functions, $(u^1 - u^2)(\cdot, s)$, for a two-state MFG.

Collaborators and students

- Stationary MFGs - H. Morgado (UNAM), V. Voskanyan (KAUST), S. Patrizi (UT Austin), G. Pires (IST), R. Ribeiro (U. F. Londrina), M. Cirant (Padova).
- Time-dependent MFGs - E. Pimentel (PUC), H. Morgado (UNAM);
- Congestion - H. Mitake (Hiroshima), V. Voskanyan (KAUST), D. Evangelista (KAUST), L. Nurbekyan (KAUST), R. Ferreira (KAUST).
- Monotonicity - R. Ferreira (KAUST), T. Tada (KAUST).;
- Numerics - Noha AlMullah (Dammam U.), R. Ferreira (KAUST), J. Saude (IST and CMU);
- 1d MFGs - Nurbekyan (KAUST), M. Prazeres (KAUST), M. Sedjro (KAUST).