Probabilistic approach to Mean-Field Games

IPAM Conference (UCLA)

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August 28 2017

Based on joint works with R. Carmona, P. Cardaliaguet, D. Crisan, J.F. Chassagneux, D. Lacker, J.M. Lasry and P.L. Lions

Part I. Motivation

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a. General philosophy

Basic purpose

- Interacting particles / players
  - controlled players in mean-field interaction
  - particles have dynamical states $\leadsto$ stochastic diff. equation
  - mean-field $\leadsto$ symmetric interaction with whole population
no privileged interaction with some particles
- Associate cost functional with each player
  - find **equilibria** w.r.t. cost functionals
  - shape of the equilibria for a large population?

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  - shape of the equilibria for a large population?
- Different notions of equilibria

  ◦ players decide on their own $\rightsquigarrow$ find a consensus inside the population $\Rightarrow$ notion of Nash equilibrium

  ◦ players obey a common center of decision $\rightsquigarrow$ minimize the global cost to the collectivity

- Both cases $\rightsquigarrow$ asymptotic equilibria as the number of players $\uparrow \infty$?

Asymptotic formulation

- Paradigm
  - mean-field / symmetry $\rightsquigarrow$ propagation of chaos / LLN
  - reduce the asymptotic analysis to one typical player with interaction with a theoretical distribution of the population?
  - decrease the complexity to solve asymptotic formulation first

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  - Existence of asymptotic equilibria? Uniqueness? Shape?
  - Use asymptotic equilibria as quasi-equilibria in finite-game
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- Asymptotic formulation of Nash equilibria $\rightsquigarrow$ Mean-field games! [Lasry-Lions (06), Huang-Caines-Malhamé (06), Cardaliaguet, Achdou, Gangbo, Gomes, Porreta (PDE), Bensoussan, Carmona, D., Kolokoltsov, Lacker, Yam (Probability)]
- Common center of decision $\rightsquigarrow$ optimal control of McKean-Vlasov SDEs

Part I. Motivation

b. Equilibria within a finite system

General formulation

- Controlled system of N interacting particles with mean-field interaction through the global state of the population
  - dynamics of particle number $i \in \{1, \dots, N\}$

$$\underbrace{dX_t^i}_{\in \mathbb{R}^d} = b(X_t^i, \text{global state of the collectivity}, \alpha_t^i)dt$$

$$+ \sigma(X_t^i, \text{global state}) \quad \underbrace{dW_t^i}_{\text{idiosyncratic noises}}$$

$$+ \sigma^0(X_t^i, \text{global state}) \quad \underbrace{dB_t}_{\text{common/systemic noise}}$$

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• Rough description of the probabilistic set-up

∘ $(B_t, W^1, \dots, W^N)_{0 \le t \le T}$ independent B.M. with values in $\mathbb{R}^d$

∘ $(\alpha_t^i)_{0 \le t \le T}$ progressively-measurable processes with values in A (closed convex $\subset \mathbb{R}^k$)

∘ i.i.d. initial conditions $\perp\!\!\!\perp$ noises

Empirical measure

- Code the state of the population at time t through $\boxed{\bar{\mu}_t^N = \frac{1}{N}\sum_{i=1}^N \delta_{X_t^i}}$

$\rightsquigarrow$ probability measure on $\mathbb{R}^d$

  - $\mathcal{P}_2(\mathbb{R}^d) \rightsquigarrow$ set of probabilities on $\mathbb{R}^d$ with finite 2nd moments

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- Express the coefficients as $\begin{array}{l} b : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times A \to \mathbb{R}^d, \\ \sigma, \sigma^0 : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^{d\times d}, \end{array}$

  - examples: $b(x,\mu,\alpha) = b(x, \int_{\mathbb{R}^d} \varphi d\mu, \alpha), \quad \int_{\mathbb{R}^d} b(x,v,\alpha) d\mu(v)$
  - rewrite the dynamics of the particles

$$dX_t^i = b(X_t^i, \bar{\mu}_t^N, \alpha_t^i)dt + \sigma(X_t^i, \bar{\mu}_t^N)dW_t^i + \sigma^0(X_t^i, \bar{\mu}_t^N)dB_t$$

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- Cost functional to player $i \in \{1, \ldots, N\}$

$$J^i(\alpha^1, \alpha^2, \ldots, \alpha^N) = \mathbb{E}\left[g(X_T^i, \bar{\mu}_T^N) + \int_0^T f(X_t^i, \bar{\mu}_t^N, \alpha_t^i)dt\right]$$

  - same (f, g) for all i but J^i depends on the others through $\bar{\mu}^N$

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- Say that a N-tuple of strategies $(\alpha^{1,\star},\dots,\alpha^{N,\star})$ is a consensus if
  - no interest for any player to leave the consensus
  - change $\alpha^{i,\star} \rightsquigarrow \alpha^{i} \Rightarrow J^i \nearrow$

$$J^i(\alpha^{1,\star},\dots,\alpha^{i,\star},\dots,\alpha^{N,\star}) \le J^i(\alpha^{1,\star},\dots,\alpha^{i},\dots\alpha^{N,\star})$$

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$$J^i(\alpha^{1,\star}, \dots, \alpha^{i,\star}, \dots, \alpha^{N,\star}) \leq J^i(\alpha^{1,\star}, \dots, \alpha^i, \dots \alpha^{N,\star})$$

- Meaning of freezing $\alpha^{1,\star}, \dots, \alpha^{i-1,\star}, \alpha^{i+1,\star}, \alpha^{N,\star}$
  - freezing the processes $\rightsquigarrow$ Nash equilibrium in open loop
  - $\alpha_t^i = \alpha^i(t, X_t^1, \dots, X_t^N) \rightsquigarrow$ each function α^i is a Markov feedback

$\rightsquigarrow$ Nash over of Markov loop

  - leads to different equilibria! but expect that there is no

difference in the asymptotic setting

Part I. Motivation

c. Example

Exhaustible resources [Guéant Lasry Lions]

- N producers of oil $\rightsquigarrow X_t^i$ (estimated reserve) at time t

$$dX_t^i = -\alpha_t^i dt + \sigma X_t^i dW_t^i$$

  - $\alpha_t^i \rightsquigarrow$ instantaneous production rate
  - σ common volatility for the perception of the reserve
  - should be a constraint $X_t^i \geq 0$

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- Optimize the profit of a producer

$$J^i(\boldsymbol{\alpha}^1, \dots, \boldsymbol{\alpha}^N) = \mathbb{E} \int_0^\infty \exp(-rt)(\alpha^i_t P_t - c(\alpha^i_t))dt$$

  - P_t is selling price, c cost production
  - mean-field constraint $\rightsquigarrow$ selling price is a function of the mean-production

$$P_t = P\Big(\frac{1}{N}\sum_{i=1}^{N} \alpha^i_t\Big)$$

  - slightly different! $\rightsquigarrow$ interaction through the law of the control

$\rightsquigarrow$ extended MFG [Gomes al., Carmona D., Cardaliaguet Lehalle]

Part II. From propagation of chaos to MFG

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a. McKean-Vlasov SDEs

General uncontrolled particle system

- Remove the control and the common noise!

$$dX_t^i = b(X_t^i, \bar{\mu}_t^N)dt + \sigma(X_t^i, \bar{\mu}_t^N)dW_t^i$$

  - $X_0^1, \ldots, X_N^i$ i.i.d. (and $\perp\!\!\!\perp$ of noises), $\quad \bar{\mu}_t^N = \frac{1}{N}\sum_{i=1}^{N} \delta_{X_t^i}$

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- Use the Wasserstein distance on $\mathcal{P}_2(\mathbb{R}^d)$

$$\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d), \quad W_2(\mu,\nu) = \left(\inf_\pi \int_{\mathbb{R}^d \times \mathbb{R}^d} |x-y|^2 d\pi(x,y) \right)^{1/2},$$

where π has μ and ν as marginals on $\mathbb{R}^d \times \mathbb{R}^d$

  - X and X' two r.v.'s $\Rightarrow W_2(\mathcal{L}(X), \mathcal{L}(X')) \leq \mathbb{E}[|X - X'|^2]^{1/2}$

- Example $W_2\Big(\frac{1}{N}\sum_{i=1}^N \delta_{x_i}, \frac{1}{N}\sum_{i=1}^N \delta_{x'_i}\Big) \leq \Big(\frac{1}{N}\sum_{i=1}^N |x_i - x'_i|^2\Big)^{1/2}$

McKean-Vlasov SDE

- Expect some decorrelation / averaging in the system as $N \uparrow \infty$
  - replace the empirical measure by the theoretical law

$$dX_t = b(X_t, \mathcal{L}(X_t))dt + \sigma(X_t, \mathcal{L}(X_t))dW_t$$

- Cauchy-Lipschitz theory
  - assume b and σ Lipschitz continuous on $\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \Rightarrow$ unique solution for any given initial condition in L^2
  - proof works as in the standard case taking advantage of

$$\mathbb{E}\Big[\big|(b,\sigma)(X_t, \mathcal{L}(X_t)) - (b,\sigma)(X'_t, \mathcal{L}(X'_t))\big|^2\Big] \leq C\mathbb{E}[|X_t - X'_t|^2]$$

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- Propagation of chaos
  - each $(X_t^i)_{0 \leq t \leq T}$ converges in law to the solution of MKV SDE
  - particles get independent in the limit $\rightsquigarrow$ for k fixed:

$$(X_t^1, \ldots, X_t^k)_{0 \leq t \leq T} \underset{\mathcal{L}}{\longrightarrow} \mathcal{L}(\text{MKV})^{\otimes k} = \mathcal{L}((X_t)_{0 \leq t \leq T})^{\otimes k} \quad \text{as } N \nearrow \infty$$

  - $\lim_{N \nearrow \infty} \sup_{0 \leq t \leq T} \mathbb{E}[(W_2(\bar{\mu}_t^N, \mathcal{L}(X_t))^2] = 0$

Part II. From propagation of chaos to MFG

b. Formulation of the asymptotic problems

Ansatz

- Go back to the finite game
- Ansatz $\rightsquigarrow$ at equilibrium

$$\boxed{\alpha_t^{i,\star} = \alpha^N(t, X_t^i, \bar{\mu}_t^N) \approx \alpha(t, X_t^i, \bar{\mu}_t^N)}$$

  - particle system at equilibrium

$$dX_t^i \approx b\Big(X_t^i, \bar{\mu}_t^N, \alpha(t, X_t^i, \bar{\mu}_t^N)\Big)dt + \sigma\Big(X_t^i, \alpha(t, X_t^i, \bar{\mu}_t^N)\Big)dW_t^i$$

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- What about an $\boxed{\text{intrinsic interpretation of } \mu_t}$?
  - should describe the global state of the population in equilibrium
  - in the limit setting, any particle that leaves the equilibrium should not modify μ_t $\rightsquigarrow$ leaving the equilibrium means that the cost increases $\rightsquigarrow$ any particle in the limit should solve an optimal control problem in the environment $(\mu_t)_{0 \le t \le T}$

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$$dX_t = b(X_t, \mu_t, \alpha_t)dt + \sigma(X_t, \mu_t)dW_t$$

∘ with $X_0 = \xi$ being fixed on some set-up $(\Omega, \mathbb{F}, \mathbb{P})$ with a d-dimensional B.M.

∘ with cost $J(\alpha) = \mathbb{E}\Big[g(X_T, \mu_T) + \int_0^T f(X_t, \mu_t, \alpha_t)dt\Big]$

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(3) let $(X_t^{\star,\mu})_{0\le t\le T}$ be the unique optimizer (under nice assumptions)
$\rightsquigarrow$ find $(\mu_t)_{0\le t\le T}$ such that

$$\mu_t = \mathcal{L}(X_t^{\star,\mu}), \quad t\in[0,T]$$

- Not a proof of convergence!

Part II. From propagation of chaos to MFG

c. Forward-backward systems

PDE point of view: HJB

- PDE characterization of the optimal control problem when σ is the identity
- Value function in environment $(\mu_t)_{0 \leq t \leq T}$

$$u(t,x) = \inf_{\alpha \text{ processes}} \mathbb{E}\Big[g(X_T, \mu_T) + \int_t^T f(X_s, \mu_s, \alpha_s) ds \,|\, X_t = x \Big]$$

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- U solution Backward HJB

$$\Big(\partial_t u + \frac{\partial^2_{xx} u}{2}\Big)(t,x) + \underbrace{\inf_{\alpha \text{ scalar}} [b(x,\mu_t,\alpha)\partial_x u(t,x) + f(x,\mu_t,\alpha)]}_{\text{standard Hamiltonian in HJB}} = 0$$

- $H(x,\mu,\alpha,z) = b(x,\mu,\alpha)\cdot z + f(x,\mu,\alpha)$
  - $\alpha^\star(x,\mu,z) = \text{argmin}_{\alpha \in A} H(x,\mu,\alpha,z) \rightsquigarrow \alpha^\star = \alpha^\star(x,\mu_t,\partial_x u(t,x))$
- Terminal boundary condition: $u(T,\cdot) = g(\cdot,\mu_T)$
- Pay attention that u depends on $(\mu_t)_t$!

Fokker-Planck

- Need for a PDE characterization of $(\mathcal{L}(X_t^{\star,\mu}))_t$
- Dynamics of $X^{\star,\mu}$ at equilibrium

$$dX_t^{\star,\mu} = b(X_t^{\star,\mu}, \mu_t, \alpha^\star(X_t^{\star,\mu}, \mu_t, \partial_x u(t, X_t^{\star,\mu})))dt + dW_t$$

- Law $(X_t^{\star,\mu})_{0 \le t \le T}$ satisfies Fokker-Planck (FP) equation

$$\partial_t \mu_t = -\mathrm{div}(\underbrace{b(x, \mu_t, \alpha^\star(x, \mu_t, \partial_x u(t, x))}_{b^\star(t,x)}\mu_t) + \frac{1}{2}\partial_{xx}^2 \mu_t$$

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- MFG equilibrium described by forward-backward in ∞ dimension

Fokker-Planck (forward)
HJB (backward)

  - ∞ dimensional analogue of

$$\dot{x}_t = b(x_t, y_t)dt, \quad x_0 = x^0$$
$$\dot{y}_t = -f(x_t, y_t)dt, \quad y_T = g(x_T)$$

Optimal control and FBSDEs

- Environment $(\mu_t)_{0\le t\le T}$ is fixed and cost functional of the type

$$J(\alpha) = \mathbb{E}\Big[g(X_T, \mu_T) + \int_0^T f(X_t, \mu_t, \alpha_t)dt\Big]$$

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- Interpret optimal paths as the forward component of an FBSDE $\rightsquigarrow$

On $(\Omega,\mathbb{F},\mathbb{P})$ with $\mathbb{F}$ generated by $(\xi,(W_t)_{0\le t\le T})$

$$X_t = X_0 + \int_0^t b\left(X_s,\mu_s,Y_s,Z_s\right) ds + \int_0^t \sigma(X_s,\mu_s)dW_s$$

$$Y_t = G(X_T,\mu_T) + \int_t^T F\left(X_s,\mu_s,Y_s,Z_s\right) ds - \int_t^T Z_s dW_s$$

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  - H strict convex in $(x,\alpha) \Rightarrow$ Pontryagin! $((G,F) = (\partial_x g,\partial_x H))$ (σ

indep. of x) $\Rightarrow$ represent gradient value function!

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On $(\Omega,\mathbb{F},\mathbb{P})$ with $\mathbb{F}$ generated by $(\xi,(W_t)_{0\le t\le T})$

$$X_t = X_0 + \int_0^t b\left(X_s,\mu_s,Y_s,Z_s\right) ds + \int_0^t \sigma(X_s,\mu_s)dW_s$$

$$Y_t = G(X_T,\mu_T) + \int_t^T F\left(X_s,\mu_s,Y_s,Z_s\right) ds - \int_t^T Z_s dW_s$$

∘ σ invertible, H strict convex in α and coeff. bounded in $x \Rightarrow$ $((G,F)=(g,f)) \Rightarrow$ represent value function!

∘ H strict convex in $(x,\alpha) \Rightarrow$ Pontryagin! $((G,F)=(\partial_x g,\partial_x H))$ (σ indep. of x) $\Rightarrow$ represent gradient value function!

∘ choose $(\mu_t)_{0\le t\le T}$ as the law of optimal path! $\Rightarrow$ characterize by FBSDE of McKean-Vlasov type

MKV FBSDE for the value function

- Consider, on $(\Omega, \mathbb{F}, \mathbb{P})$, the MKV FBSDE

$$
\begin{aligned}
X_t &= \xi + \int_0^t b\Big(X_s, \mathcal{L}(X_s), \alpha^\star(X_s, \mathcal{L}(X_s), Z_s\sigma^{-1}(X_s, \mathcal{L}(X_s)))\Big)\, ds \\
&\quad + \int_0^t \sigma(X_s, \mathcal{L}(X_s))dW_s \\
Y_t &= g(X_T, \mathcal{L}(X_T)) \\
&\quad + \int_t^T f\Big(X_s, \mathcal{L}(X_s), \alpha^\star(X_s, \mathcal{L}(X_s), Z_s\sigma^{-1}(X_s, \mathcal{L}(X_s)))\Big)\, ds - \int_t^T Z_s dW_s
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\end{aligned}
$$

- Connection with PDE formulation

$$
\boxed{Y_s = u(s, X_s), \qquad Z_s = \partial_x u(s, X_s)\sigma(X_s, \mu_s)}
$$

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\boxed{Y_s = u(s, X_s), \quad Z_s = \partial_x u(s, X_s)\sigma(X_s, \mu_s)}
$$

- Unique minimizer for each $(\mu_t)_{0 \le t \le T}$ if
  - $b, f, g, \sigma, \sigma^{-1}$ bounded in (x, μ), Lipschitz in x
  - b linear in α and f strictly convex and loc. Lip in α, with Lip(f) at most of linear growth in α

MKV FBSDE for the Pontryagin principle

- Consider, on $(\Omega, \mathbb{F}, \mathbb{P})$, the MKV FBSDE

$$
\begin{aligned}
X_t &= \xi + \int_0^t b\Big(X_s, \mathcal{L}(X_s), \alpha^\star(X_s, \mathcal{L}(X_s), Y_s)\Big)\, ds + \int_0^t \sigma(\mathcal{L}(X_s)) dW_s \\
Y_t &= \partial_x g(X_T, \mathcal{L}(X_T)) \\
&\quad + \int_t^T \partial_x H\Big(X_s, \mathcal{L}(X_s), \alpha^\star(X_s, \mathcal{L}(X_s), Y_s), Y_s\Big)\, ds - \int_t^T Z_s dW_s
\end{aligned}
$$

MKV FBSDE for the Pontryagin principle

- Consider, on $(\Omega, \mathbb{F}, \mathbb{P})$, the MKV FBSDE

$$X_t = \xi + \int_0^t b\Big(X_s, \mathcal{L}(X_s), \alpha^\star(X_s, \mathcal{L}(X_s), Y_s)\Big)\, ds + \int_0^t \sigma(\mathcal{L}(X_s))dW_s$$

$$Y_t = \partial_x g(X_T, \mathcal{L}(X_T))$$

$$+ \int_t^T \partial_x H\Big(X_s, \mathcal{L}(X_s), \alpha^\star(X_s, \mathcal{L}(X_s), Y_s), Y_s\Big)\, ds - \int_t^T Z_s dW_s$$

- Connection with PDE formulation

$$\boxed{Y_s = \partial_x u(s, X_s), \quad Z_s = \partial_x^2 u(s, X_s)\sigma(\mu_s)}$$

MKV FBSDE for the Pontryagin principle

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$$X_t = \xi + \int_0^t b\Big(X_s, \mathcal{L}(X_s), \alpha^\star(X_s, \mathcal{L}(X_s), Y_s)\Big)\, ds + \int_0^t \sigma(\mathcal{L}(X_s)) dW_s$$

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- Unique minimizer for each $(\mu_t)_{0 \le t \le T}$ if
  - σ indep. of x and $b(x, \mu, \alpha) = b_0(\mu) + b_1 x + b_2 \alpha$
  - $\partial_x f, \partial_\alpha f, \partial_x g$ L-Lipschitz in (x, α)
  - g and f convex in (x, α) with f strict convex in α

Seeking a solution

- Any way $\rightsquigarrow$ two-point-boundary-problem $\Rightarrow$
  - Cauchy-Lipschitz theory in small time only
  - if Lipschitz coefficients (including the direction of the measure)

$\rightsquigarrow$ existence and uniqueness in short time (see later on)

$\rightsquigarrow$ existence and uniqueness of MFG equilibria in small time

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  $\rightsquigarrow$ existence and uniqueness of MFG equilibria in small time

- What about arbitrary time?

  ∘ existence $\rightsquigarrow$ fixed point over the measure argument by means of compactness arguments

  Schauder's theorem

  ∘ uniqueness $\rightsquigarrow$ require additional assumption

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- What about arbitrary time?

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  Schauder's theorem

  ◦ uniqueness $\rightsquigarrow$ require additional assumption

- Other question $\rightsquigarrow$ connection with social optimization?

  ◦ potential games $\rightsquigarrow$ MFG solution is also a social optimizer (but for other coefficients)

Part III. Solving MFG

a. Schauder fixed point theorem without common noise

Statement of the Schauder fixed point theorem

- Generalisation of Brouwer's theorem from finite to infinite dimension
- Let $(V, \|\cdot\|)$ be a normed vector space
  - $\emptyset \neq E \subset V$ with E closed and convex
  - $\phi : E \to E$ continuous such that $\phi(E)$ is relatively compact
  - $\Rightarrow$ existence of a fixed point to ϕ

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  - $\Rightarrow$ existence of a fixed point to ϕ

- In MFG $\rightsquigarrow$ $\boxed{\text{what is } V\text{, what is } E\text{, what is } \phi\text{?}}$
  - recall that MFG equilibrium is a flow of measures $(\mu_t)_{0 \leq t \leq T}$
  $$E \subset \mathcal{C}([0,T], \mathcal{P}_2(\mathbb{R}^d))$$
  - need to embed into a linear structure
  $$\mathcal{C}([0,T], \mathcal{P}_2(\mathbb{R}^d)) \subset \mathcal{C}([0,T], \mathcal{M}_1(\mathbb{R}^d))$$
  - $\mathcal{M}_1(\mathbb{R}^d)$ set of signed measures ν with $\int_{\mathbb{R}^d} |x| d|\nu|(x) < \infty$

Compactness on the space of probability measures

- Equip $\mathcal{M}_1(\mathbb{R}^d)$ with a norm $\|\cdot\|$ and restrict to $\mathcal{P}_1(\mathbb{R}^d)$ such that
  - convergence of $(\nu_n)_{n\geq 1}$ in $\mathcal{P}_1(\mathbb{R}^d)$ implies weak convergence

$$\forall h \in C_b(\mathbb{R}^d, \mathbb{R}), \quad \lim_{n\to\infty} \int_{\mathbb{R}^d} h d\nu_n = \int_{\mathbb{R}^d} h d\nu$$

  - if $(\nu_n)_{n\geq 1}$ has uniformly bounded moments of order $p > 2$

$$\text{Unif. square integrability} \Rightarrow W_2(\nu_n, \nu) \to 0$$

  - says that the input in the coefficients varies continuously!

$$b(x, \nu_n, y, z),\ \sigma(x, \nu_n),\ F(x, \nu_n, y, z),\ G(x, \nu_n)$$

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$$b(x, \nu_n, y, z),\ \sigma(x, \nu_n),\ F(x, \nu_n, y, z),\ G(x, \nu_n)$$

- Compactness $\rightsquigarrow$ if $(\nu_n)_{n\geq 1}$ has bounded moments of order $p > 2$
  - $(\nu_n)_{n\geq 1}$ admits a weakly convergent subsequence
  - then convergence for W_2 by unif. integrability and for $\|\cdot\|$ also

Application to MKV FBSDE

- Choose E as continuous $(\mu_t)_{0\leq t\leq T}$ from $[0,T]$ to $\mathcal{P}_2(\mathbb{R}^d)$

$$\sup_{0\leq t\leq T}\int_{\mathbb{R}^d}|x|^4 d\mu_t(x)\leq K \qquad \text{for some } K$$

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$$\sup_{0\le t\le T}\int_{\mathbb{R}^d}|x|^4 d\mu_t(x)\le K \qquad \text{for some } K$$

- Construct $\phi \rightsquigarrow$ fix $(\mu_t)_{0\le t\le T}$ in E and solve

$$X_t = \xi + \int_0^t b\left(X_s,\mu_s,Y_s,Z_s\right) + \int_0^t \sigma(X_s,\mu_s)dW_s$$

$$Y_t = G(X_T,\mu_T) + \int_t^T F(X_s,\mu_s,Y_s,Z_s)\,ds - \int_t^T Z_s dW_s$$

  - let $\boxed{\phi\left(\boldsymbol{\mu} = (\mu_t)_{0\le t\le T}\right) = (\mathcal{L}(X_t^{\boldsymbol{\mu}}))_{0\le t\le T}}$

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  - let $\boxed{\phi\big(\boldsymbol{\mu} = (\mu_t)_{0\le t\le T}\big) = (\mathcal{L}(X_t^{\boldsymbol{\mu}}))_{0\le t\le T}}$

- Assume bounded coefficients and $\mathbb{E}[|\xi|^4] < \infty$
  - choose K such that $\mathbb{E}[|X_t^{\boldsymbol{\mu}}|^4] \le K$

$$\Rightarrow \boxed{E \text{ stable by } \phi}$$

  - $W_2(\mathcal{L}(X_t^{\boldsymbol{\mu}}),\mathcal{L}(X_s^{\boldsymbol{\mu}})) \le C\mathbb{E}[|X_t^{\boldsymbol{\mu}} - X_s^{\boldsymbol{\mu}}|^2]^{1/2} \le C|t-s|^{1/2}$

Conclusion

- Consider continuous $\boldsymbol{\mu} = (\mu_t)_{0\leq t\leq T}$ from $[0,T]$ to $\mathcal{P}_2(\mathbb{R}^d)$
  - for any $t \rightsquigarrow (\phi(\boldsymbol{\mu}))_t$ in a compact subset of $\mathcal{P}_2(\mathbb{R}^d)$
  - $[0,T] \ni t \mapsto (\phi(\boldsymbol{\mu}))_t$ is uniformly continuous in $\boldsymbol{\mu}$
  - by Arzelà-Ascoli $\Rightarrow$ output lives in a compact subset of $E \subset \mathcal{C}([0,T],\mathcal{P}_2(\mathbb{R}^d))$ (and thus of $\mathcal{C}([0,T],\mathcal{M}_1(\mathbb{R}^d))$)
- Continuity of ϕ on $E \rightsquigarrow$ stability of the solution of FBSDEs with respect to a continuous perturbation of the environment

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- Continuity of ϕ on $E \rightsquigarrow$ stability of the solution of FBSDEs with respect to a continuous perturbation of the environment
- Refinements to allow for unbounded coefficients
  - for the Value-Function FBSDE $\rightsquigarrow b$ linear in α, f strictly convex in α, with derivatives in α at most of linear growth in α
  - Pontryagin principle

    $\rightsquigarrow b$ linear in (x, α) and f convex in (x, α) with derivatives at most of linear growth with weak-mean reverting conditions

$$\langle x, \partial_x f(0, \delta_x, 0) \rangle \ge -c(1 + |x|) \quad \text{and} \quad \langle x, \partial_x g(0, \delta_x) \rangle \ge -c(1 + |x|)$$

Linear-quadratic in $d = 1$

- Apply previous results with
  - $b(t, x, \mu, \alpha) = a_t x + a'_t \mathbb{E}(\mu) + b_t \alpha_t$
  - $g(x, \mu) = \frac{1}{2}[qx + q' \mathbb{E}(\mu)]^2 \leftrightsquigarrow$ (mean-reverting) $qq' \geq 0$
  - $f(t, x, \mu, \alpha) = \frac{1}{2}[\alpha^2 + (m_t x + m'_t \mathbb{E}(\mu))^2] \leftrightsquigarrow$ (mean-rev.) $m_t m'_t \geq 0$

Linear-quadratic in $d = 1$

- Apply previous results with
  - $b(t,x,\mu,\alpha) = a_t x + a_t' \mathbb{E}(\mu) + b_t \alpha_t$
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- Compare with direct method $\rightsquigarrow$ Pontryagin

$$
\begin{aligned}
dX_t &= [a_t X_t + a_t' \mathbb{E}(X_t) - b_t^2 Y_t]dt + \sigma dW_t \\
dY_t &= -[a_t Y_t + m_t(m_t X_t + m_t' \mathbb{E}(X_t))]dt + Z_t dW_t \\
Y_T &= q[qX_T + q' \mathbb{E}(X_T)]
\end{aligned}
$$

  - take the mean

$$
\begin{aligned}
d\mathbb{E}(X_t) &= [(a_t + a_t')\mathbb{E}(X_t) - b_t^2 \mathbb{E}(Y_t)]dt \\
d\mathbb{E}(Y_t) &= -[a_t \mathbb{E}(Y_t) + m_t(m_t + m_t')\mathbb{E}(X_t)]dt \\
\mathbb{E}(Y_T) &= q(q + q')\mathbb{E}(X_T)
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- Apply previous results with
  - $b(t,x,\mu,\alpha) = a_t x + a_t'\mathbb{E}(\mu) + b_t\alpha_t$
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\mathbb{E}(Y_T) &= q(q + q')\mathbb{E}(X_T)
\end{aligned}
$$

- existence and uniqueness if $q(q + q') \geq 0$, $m_t(m_t + m_t') \geq 0$

Part III. Solving MFG

b. Uniqueness criterion

A counter-example to uniqueness

- Consider the MKV FBSDE

$$
\begin{aligned}
dX_t &= b(\mathbb{E}(Y_t))dt + dW_t, \quad X_0 = x_0 \\
dY_t &= -f(\mathbb{E}(X_t))dt + Z_t dW_t, \quad Y_T = g(\mathbb{E}(X_T))
\end{aligned}
$$

◦ take bounded and Lipschitz coefficients $\rightsquigarrow$ existence of a solution

◦ uniqueness may not hold!

◦ completely different of the system with $b(Y_t), f(X_t)$ and $g(X_T)$ for which uniqueness holds true!

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• Proof $\rightsquigarrow$ take the mean

$$
\begin{aligned}
d\mathbb{E}(X_t) &= b(\mathbb{E}(Y_t))dt, \quad \mathbb{E}(X_0) = x_0 \\
d\mathbb{E}(Y_t) &= -f(\mathbb{E}(X_t))dt, \quad \mathbb{E}(Y_T) = g(\mathbb{E}(X_T))
\end{aligned}
$$

∘ led back to counter-example for FBSDE $\rightsquigarrow$ choose b, f and g equal to the identity on a compact subset

Lasry Lions monotonicity condition

- Recall following FBSDE result
  - $\exists!$ may hold for the Pontryagin system if convex g and H
  - convexity $\leadsto$ monotonicity of $\partial_x g$ and $\partial_x H$
  - what is monotonicity condition in the direction of the measure?

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  - what is monotonicity condition in the direction of the measure?
- Lasry Lions monotonicity condition
  - b, σ do not depend on μ
  - $f(x,\mu,\alpha) = f_0(x,\mu) + f_1(x,\alpha)$ (μ and α are separated)
  - monotonicity property for f_0 and g w.r.t. μ

$$\int_{\mathbb{R}^d} (f_0(x,\mu) - f_0(x,\mu'))d(\mu - \mu')(x) \geq 0$$

$$\int_{\mathbb{R}^d} (g(x,\mu) - g(x,\mu'))d(\mu - \mu')(x) \geq 0$$

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$$\int_{\mathbb{R}^d} (g(x,\mu) - g(x,\mu'))d(\mu - \mu')(x) \geq 0$$

- Example: $h(x,\mu) = \displaystyle\int_{\mathbb{R}^d} L(z, \rho \star \mu(z))\rho(x-z)dz$ where L is $\nearrow$ in second variable and ρ is even

Monotonicity restores uniqueness

- Assume that for any input $\boldsymbol{\mu} = (\mu_t)_{0 \leq t \leq T}$ unique optimal control $\boldsymbol{\alpha}^{\star,\boldsymbol{\mu}}$
  - + existence of an MFG for a given initial condition

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- Lasry Lions $\Rightarrow$ uniqueness of MFG equilibrium!
  - if two different equilibria $\boldsymbol{\mu}$ and $\boldsymbol{\mu}'$ $\rightsquigarrow \alpha^{\star,\boldsymbol{\mu}} \neq \alpha^{\star,\boldsymbol{\mu}'}$

$$\underbrace{J^{\boldsymbol{\mu}}(\alpha^{\star,\boldsymbol{\mu}})}_{\text{cost under } \boldsymbol{\mu}} < J^{\boldsymbol{\mu}}(\alpha^{\star,\boldsymbol{\mu}'}) \quad \text{and} \quad \underbrace{J^{\boldsymbol{\mu}'}(\alpha^{\star,\boldsymbol{\mu}'})}_{\text{cost under } \boldsymbol{\mu}'} < J^{\boldsymbol{\mu}'}(\alpha^{\star,\boldsymbol{\mu}})$$

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$$\underbrace{J^{\boldsymbol{\mu}}(\alpha^{\star,\boldsymbol{\mu}})}_{\text{cost under } \boldsymbol{\mu}} < J^{\boldsymbol{\mu}}(\alpha^{\star,\boldsymbol{\mu}'}) \quad \text{and} \quad \underbrace{J^{\boldsymbol{\mu}'}(\alpha^{\star,\boldsymbol{\mu}'})}_{\text{cost under } \boldsymbol{\mu}'} < J^{\boldsymbol{\mu}'}(\alpha^{\star,\boldsymbol{\mu}})$$

so that

$$J^{\boldsymbol{\mu}'}(\alpha^{\star,\boldsymbol{\mu}}) - J^{\boldsymbol{\mu}'}(\alpha^{\star,\boldsymbol{\mu}'}) + J^{\boldsymbol{\mu}}(\alpha^{\star,\boldsymbol{\mu}'}) - J^{\boldsymbol{\mu}}(\alpha^{\star,\boldsymbol{\mu}}) > 0$$
$$J^{\boldsymbol{\mu}'}(\alpha^{\star,\boldsymbol{\mu}}) - J^{\boldsymbol{\mu}}(\alpha^{\star,\boldsymbol{\mu}}) - [J^{\boldsymbol{\mu}'}(\alpha^{\star,\boldsymbol{\mu}'}) - J^{\boldsymbol{\mu}}(\alpha^{\star,\boldsymbol{\mu}'})] > 0$$

Monotonicity restores uniqueness

- Assume that for any input $\boldsymbol{\mu} = (\mu_t)_{0 \le t \le T}$ unique optimal control $\alpha^{\star,\boldsymbol{\mu}}$
    - + existence of an MFG for a given initial condition
- Lasry Lions $\Rightarrow$ uniqueness of MFG equilibrium!
    - if two different equilibria $\boldsymbol{\mu}$ and $\boldsymbol{\mu}' \rightsquigarrow \alpha^{\star,\boldsymbol{\mu}} \neq \alpha^{\star,\boldsymbol{\mu}'}$

$$\underbrace{J^{\boldsymbol{\mu}}(\alpha^{\star,\boldsymbol{\mu}})}_{\text{cost under } \boldsymbol{\mu}} < J^{\boldsymbol{\mu}}(\alpha^{\star,\boldsymbol{\mu}'}) \quad \text{and} \quad \underbrace{J^{\boldsymbol{\mu}'}(\alpha^{\star,\boldsymbol{\mu}'})}_{\text{cost under } \boldsymbol{\mu}'} < J^{\boldsymbol{\mu}'}(\alpha^{\star,\boldsymbol{\mu}})$$

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$$J^{\boldsymbol{\mu}'}(\alpha^{\star,\boldsymbol{\mu}}) - J^{\boldsymbol{\mu}}(\alpha^{\star,\boldsymbol{\mu}}) - [J^{\boldsymbol{\mu}'}(\alpha^{\star,\boldsymbol{\mu}'}) - J^{\boldsymbol{\mu}}(\alpha^{\star,\boldsymbol{\mu}'})] > 0$$

$$\mathbb{E}\Big[\underbrace{g(X_T^{\star,\boldsymbol{\mu}}, \mu_T') - g(X_T^{\star,\boldsymbol{\mu}}, \mu_T)}_{\int_{\mathbb{R}^d} (g(x,\mu_T') - g(x,\mu_T)) d\mu_T(x)} - \underbrace{\big(g(X_T^{\star,\boldsymbol{\mu}'}, \mu_T') - g(X_T^{\star,\boldsymbol{\mu}'}, \mu_T)\big)}_{\int_{\mathbb{R}^d} (g(x,\mu_T') - g(x,\mu_T)) d\mu_T'(x)} + \dots \Big] > 0$$

- same for $f_0 \Rightarrow$ LHS must be ≤ 0

Part IV. Solving MFG with a Common Noise

a. Formulation

MFG with a common noise

- Mean field game with common noise B
  - asymptotic formulation for a finite player game with

$$dX_t^i = b(X_t^i, \bar{\mu}_t^N, \alpha_t^i)dt + \sigma(X_t^i, \bar{\mu}_t^N)dW_t^i + \sigma^0(X_t^i, \bar{\mu}_t^N)dB_t$$

  - uncontrolled version $\rightsquigarrow$ asymptotic SDE with $\bar{\mu}_t^N$ replaced by $\mathcal{L}(X_t|(B_s)_{0\le s\le T}) = \mathcal{L}(X_t|(B_s)_{0\le s\le t})$
  - particles become independent conditional on B and converge to the solution

$$dX_t = b(X_t, \mathcal{L}(X|B))dt + \sigma(X_t, \mathcal{L}(X|B))dW_t + \sigma^0(X_t, \mathcal{L}(X|B))dB_t$$

MFG with a common noise

- Mean field game with common noise B
  - asymptotic formulation for a finite player game with $A = \mathbb{R}^k$ and

$$dX_t^i = \left(b(X_t^i, \bar{\mu}_t^N) + \alpha_t^i\right)dt + \sigma dW_t^i + \eta dB_t$$

  - uncontrolled version $\rightsquigarrow$ $\bar{\mu}_t^N$ replaced by $\mathcal{L}(X_t|B)$
- Equilibrium as a fixed point $\rightsquigarrow$ time $[0, T]$, state in $\mathbb{R}^d$
  - candidate $\rightsquigarrow$ $(\mu_t)_{t\in[0,T]}$ $\mathbb{F}^B$ prog-meas with values in space of probability measures with a finite second moment $\mathcal{P}_2(\mathbb{R}^d)$
  - representative player with control α

$$dX_t = (b(X_t, \mu_t) + \alpha_t)dt + \sigma dW_t + \eta dB_t$$

$\rightsquigarrow$ $X_0 \sim \mu_0$, $\sigma, \eta \in \{0, 1\}$, W and B $\mathbb{R}^d$-valued $\perp\!\!\!\perp$ B.M.

  - cost functional $J(\alpha) = \mathbb{E}\left[g(X_T, \mu_T) + \int_0^T \left(f(X_t, \mu_t) + \frac{1}{2}|\alpha_t|^2\right)dt\right]$
  - find $(\mu_t)_{t\in[0,T]}$ such that $\mu_t = \mathcal{L}(X_t^{\text{optimal}}|(B_s)_{0\le s\le T})$

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Forward-backward formulation

- Forward-backward formulation must account for $(\mu_t)_{0 \leq t \leq T}$ random

    ◦ systems of two forward-backward SPDEs [Carmona D, Cardaliaguet D Lasry Lions]

Forward-backward formulation

- Forward-backward formulation must account for $(\mu_t)_{0\le t\le T}$ random
  - systems of two forward-backward SPDEs

$\rightsquigarrow$ one backward stochastic HJB equation [Peng]

$$d_t u(t,x) + \Big(\underbrace{b(x,\mu_t)\cdot D_x u(t,x) + \frac{\sigma^2+\eta^2}{2}\Delta_x u(t,x)}_{\text{Laplace generator}} + \underbrace{f(x,\mu_t) - \frac{1}{2}|D_x u(t,x)|^2}_{\text{standard Hamiltonian in HJB}}$$

$$+ \underbrace{\eta \mathrm{div}[v(t,x)]}_{\text{Ito Wentzell cross term}} \Big)dt - \underbrace{\eta v(t,x)\cdot dB_t}_{\text{backward term}} = 0$$

with boundary condition: $u(T,\cdot) = g(\cdot,\mu_T)$

$\rightsquigarrow$ one forward stochastic Fokker-Planck equation

$$d_t\mu_t = \Big(-\mathrm{div}(\mu_t[b(x,\mu_t) - D_x u(t,x)])dt + \frac{\sigma^2+\eta^2}{2}\mathrm{trace}(\partial^2_{xx}\mu_t)\Big)dt$$

$$- \eta \mathrm{div}(\mu_t dB_t)$$

Forward-backward formulation

- Forward-backward formulation must account for $(\mu_t)_{0 \le t \le T}$ random
  - systems of two forward-backward SPDEs
  - systems of two forward-backward McKV SDEs [Carmona D, Buckdahn (al.), Lacker]

Forward-backward formulation

- Forward-backward formulation must account for $(\mu_t)_{0\le t\le T}$ random
  - systems of two forward-backward SPDEs
  - systems of two forward-backward McKV SDEs

    $\rightsquigarrow$ two ways: represent the value function or optimal control

- Representation of the value function $\sigma = 1$

$$
\begin{aligned}
dX_t &= b(X_t, \mathcal{L}(X_t|B))dt - Z_t dt + dW_t + \eta dB_t \\
dY_t &= -f(X_t, \mathcal{L}(X_t|B))dt - \tfrac{1}{2}|Z_t|^2 dt + Z_t dW_t + \zeta_t dB_t \\
Y_T &= g(X_T, \mathcal{L}(X_T|B))
\end{aligned}
$$

- Representation of the optimal control (Pontryagin)

$$
\begin{aligned}
dX_t &= b(X_t, \mathcal{L}(X_t|B))dt - Y_t dt + \sigma dW_t + \eta dB_t \\
dY_t &= -\underbrace{\partial_x H(X_t, \mathcal{L}(X_t|B), Y_t)}_{H(x,\mu,y)=b(x,\mu)\cdot y + f(x,\mu,y)} dt + Z_t dW_t + \zeta_t dB_t \\
Y_T &= \partial_x g(X_T, \mathcal{L}(X_T|B))
\end{aligned}
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Forward-backward formulation

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Y_T &= \partial_x g(X_T, \mathcal{L}(X_T|B))
\end{aligned}
$$

- Analysis of these equations?

Part IV. Solving MFG with a Common Noise

b. Strong solutions

Implementing Picard theorem

- Easiest way to construct solutions is to implement Picard theorem
  - shall see next how to make use of Schauder's theorem
- Forward-backward system of McKean-Vlasov type

$$
\begin{aligned}
dX_t &= \Big(b(X_t, \mathcal{L}(X_t|B)) - Z_t\Big)dt + dW_t + \eta dB_t \\
dY_t &= -\Big(f(X_t, \mathcal{L}(X_t|B)) + \tfrac{1}{2}|Z_t|^2\Big)dt + Z_t dW_t + \zeta_t dB_t \\
Y_T &= g(X_T, \mathcal{L}(X_T|B))
\end{aligned}
$$

  - Z_t should be $\partial_x u(t, X_t)$ $\rightsquigarrow$ bounded and x-Lipschitz coefficients

$\Rightarrow L^\infty$ bound

$\rightsquigarrow$ replace quadratic term by general bounded f

Implementing Picard theorem

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\end{aligned}
$$

  - Cauchy-Lipschitz theory in small time only!
- Theorem If K-Lipschitz coefficients $\Rightarrow$ $\exists!$ for $T \le c(K)$
  - for any initial condition $X_0 \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; \mathbb{R}^d)$
- Question How to go further?

Decoupling field ($T \leq c(K)$)

- Recall non MKV case $\rightsquigarrow \exists U : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ such that

$$Y_t = U(t, X_t) \quad \Leftrightarrow \quad U(t,x) = Y_t^{t,x} \text{ (with } X_t^{t,x} = x)$$

  - keep fact for extending solutions is to bound $\mathrm{Lip}_x(U)$

Decoupling field ($T \leq c(K)$)

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- MKV setting $\rightsquigarrow$ state variable is in $\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$

$\rightsquigarrow$ need to construct $\boxed{U(t,x,\mu)}$ $\quad t \in [0,T], x \in \mathbb{R}^d, \mu \in \mathcal{P}_2(\mathbb{R}^d)$

- Two-step procedure [Crisan Chassagneux D, Buckdahn (al.)]

  ◦ 1st step $\rightsquigarrow$ $\boxed{\text{MKV FBSDE}}$ with $X_t \sim \mu$, $X_t \perp\!\!\!\perp (W,B)$

$$dX_s = \Big(b(X_s, \mathcal{L}(X_s|B)) - Z_s\Big)ds + dW_s + \eta dB_s$$

$$dY_s = -f(X_s, \mathcal{L}(X_s|B), Z_s)ds + Z_s dW_s + \zeta_s dB_s, \quad Y_T = g(X_T, \mathcal{L}(X_T|B))$$

$\rightsquigarrow (\mathcal{L}(X_s|B))_{t \leq s \leq T}$ only depends on X_t through μ

Decoupling field ($T \leq c(K)$)

- Recall non MKV case $\rightsquigarrow \exists U : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ such that

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$$dY_s = -f(X_s, \mathcal{L}(X_s|B), Z_s)ds + Z_s dW_s + \zeta_s dB_s, \quad Y_T = g(X_T, \mathcal{L}(X_T|B))$$

  - 2nd step $\rightsquigarrow$ $\boxed{\text{non-MKV FBSDE}}$ with $x_t = x$ and 1st step input

$$dx_s = \big(b(x_s, \mathcal{L}(X_s|B)) - z_s\big)ds + dW_s + \eta dB_s$$

$$dy_s = -f(x_s, \mathcal{L}(X_s|B), z_s)dt + z_s dW_s + \varsigma_s dB_s, \quad y_T = g(x_T, \mathcal{L}(X_T|B))$$

  - let $\boxed{U(t,x,\mu) = y_t}$ $\Rightarrow Y_t = U(t, X_t, \mu) = U(t, X_t, \mathcal{L}(X_t|B))$

Controlling the Lipschitz constant

• Non-MKV setting $\rightsquigarrow$ may control the Lipschitz constant by monotonicity or ellipticity conditions

$\leadsto$ start with monotonicity $\rightsquigarrow$ B has no role $\Rightarrow$ simplify $\eta = 0$

• Come back to cost structure $\rightsquigarrow$ monotonicity of f (same with g)

$$\int_{\mathbb{R}^d} [f(x,\mu) - f(x,\mu')] d(\mu - \mu')(x) \geq 0 \quad \text{[Lions]}$$

• Theorem [L, C C D, Cardaliaguet (al.)] If $b \equiv 0$, f and g bounded, monotone and Lipschitz $\Rightarrow$ bound on $\mathrm{Lip}_\mu U$ and $\exists!$ on any $[0,T]$

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- Strategy Investigate derivative of the flow in L^2

  $\leadsto$ for $\xi,\ \chi \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; \mathbb{R}^d)$

$$(\partial_\chi X_s^\xi, \partial_\chi Y_s^\xi, \partial_\chi Z_s^\xi) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \Big(\underbrace{X_s^{\xi+\varepsilon\chi} - X_s^\xi, Y_s^{\xi+\varepsilon\chi} - Y_s^\xi}_{\text{in } \mathbb{E}[\sup_{0\leq s\leq T} |\cdot_s|^2]}, \underbrace{Z_s^{\xi+\varepsilon\chi} - Z_s^\xi}_{\text{in } \mathbb{E}\int_0^T |\cdot_s|^2 ds} \Big)$$

  - provide a bound for $(\partial_\chi X^\xi, \partial_\chi Y^\xi, \partial_\chi Z^\xi)$

Derivative on the Wasserstein space

- Differentiation on $\mathcal{P}_2(\mathbb{R}^d)$ taken from Lions
- Consider $U : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$
- Lifted-version of U

$$\hat{U} : L^2(\Omega, \mathbb{P}; \mathbb{R}^d) \ni X \mapsto U(\mathrm{Law}(X))$$

  - U differentiable if $\hat{U}$ Fréchet differentiable

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  - ∘ U differentiable if $\hat{U}$ Fréchet differentiable
- Differential of U
  - ∘ Fréchet derivative of $\hat{U}$ [see also Zhang (al.)]

$$D\hat{U}(X) = \partial_\mu U(\mu)(X), \quad \partial_\mu U(\mu) : \mathbb{R}^d \ni v \mapsto \partial_\mu U(\mu)(v) \quad \mu = \mathcal{L}(X)$$

  - ∘ derivative of U at $\mu \rightsquigarrow \partial_\mu U(\mu) \in L^2(\mathbb{R}^d, \mu; \mathbb{R}^d)$

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  - derivative of U at $\mu \rightsquigarrow \partial_\mu U(\mu) \in L^2(\mathbb{R}^d, \mu; \mathbb{R}^d)$

- Finite dimensional projection

$$\partial_{x_i}\Big[U\Big(\frac{1}{N}\sum_{j=1}^N \delta_{x_j}\Big)\Big] = \frac{1}{N}\partial_\mu U\Big(\frac{1}{N}\sum_{j=1}^N \delta_{x_j}\Big)(x_i), \quad x_1, \dots, x_N \in \mathbb{R}^d$$

Application to the coupled case $(b \equiv 0)$

- Return to coupled case $\rightsquigarrow$ estimate $\partial_\chi Y_0^\xi$

$$\partial_\chi Y_0^\xi = \partial_x U(0,\xi,\mathcal{L}(\xi)) \cdot \chi + \underbrace{\tilde{\mathbb{E}}[\partial_\mu U(0,\xi,\mathcal{L}(\xi))(\tilde{\xi}) \cdot \tilde{\chi}]}_{\tilde{\Omega} = \text{copy space}}$$

  - $\circ$ Lip_μ estimate on $U \Leftrightarrow$ bound of $\mathbb{E}[|\partial_\mu U(0,\xi,\mathcal{L}(\xi))(\xi)|^2]^{1/2}$

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  - Lip$_\mu$ estimate on $U \Leftrightarrow$ bound of $\mathbb{E}[|\partial_\mu U(0,\xi,\mathcal{L}(\xi))(\xi)|^2]^{1/2}$

- Estimate $(\partial_\chi X_t)_t$ first $\rightsquigarrow$ dynamics of $(X_t)_t$ and $(\partial_\chi X_t)_t$

$$\begin{aligned} dX_t &= -\partial_x U(t,X_t,\mathcal{L}(X_t))dt + dW_t \\ d\partial_\chi X_t &= -\Big(\partial^2_{xx} U(t,X_t,\mathcal{L}(X_t))\partial_\chi X_t \\ &\qquad + \tilde{\mathbb{E}}[\partial_\mu(\partial_x U)(t,X_t,\mathcal{L}(X_t))(\tilde{X}_t)\partial_\chi \tilde{X}_t]\Big)dt \end{aligned}$$

  - $\partial^2_{xx} U$ already estimated! (thanks to Laplace)

Application to the coupled case $(b \equiv 0)$

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$$\partial_\chi Y_0^\xi = \partial_x U(0,\xi,\mathcal{L}(\xi)) \cdot \chi + \underbrace{\tilde{\mathbb{E}}[\partial_\mu U(0,\xi,\mathcal{L}(\xi))(\tilde{\xi}) \cdot \tilde{\chi}]}_{\tilde{\Omega} = \text{copy space}}$$

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- Estimate $(\partial_\chi X_t)_t$ first $\rightsquigarrow$ dynamics of $(X_t)_t$ and $(\partial_\chi X_t)_t$

$$d\mathbb{E}[|\partial_\chi X_t|^2] = -2\mathbb{E}\Big[\partial_\chi X_t \cdot (\partial^2_{xx} U(X_t,\mathcal{L}(X_t))\partial_\chi X_t)\Big]dt - 2\mathbb{E}\tilde{\mathbb{E}}\Big[\partial_\chi X_t \cdot \Big(\partial_\mu(\partial_x U)(X_t,\mathcal{L}(X_t))(\tilde{X}_t)\widetilde{\partial_\chi X_t}\Big)\Big]dt$$

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Application to the coupled case ($b \equiv 0$)

- Return to coupled case $\rightsquigarrow$ estimate $\partial_\chi Y_0^\xi$

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- Estimate $(\partial_\chi X_t)_t$ first $\rightsquigarrow$ dynamics of $(X_t)_t$ and $(\partial_\chi X_t)_t$

$$\begin{aligned} d\mathbb{E}[|\partial_\chi X_t|^2] &= -2\mathbb{E}\Big[\partial_\chi X_t \cdot (\partial^2_{xx} U(X_t,\mathcal{L}(X_t))\partial_\chi X_t)\Big]dt \\ &\quad - 2\mathbb{E}\tilde{\mathbb{E}}\Big[\partial_\chi X_t \cdot \Big(\partial_\mu(\partial_x U)(X_t,\mathcal{L}(X_t))(\tilde{X}_t)\widetilde{\partial_\chi X_t}\Big)\Big]dt \end{aligned}$$

  - $\partial^2_{xx} U$ already estimated! (thanks to Laplace)

- Propagation of monotonicity

$$\mathbb{E}\tilde{\mathbb{E}}\Big[\partial_\chi X_t \cdot \Big(\partial_x(\partial_\mu U)(t,X_t,\mathcal{L}(X_t))(\tilde{X}_t)\widetilde{\partial_\chi X_t}\Big)\Big] \geq 0 \Rightarrow \boxed{\mathrm{E}\,[|\partial_\chi X_T|^2] \leq C\mathbb{E}[|\chi|^2]}$$

  - insert into the backward equation

Part IV. Solving MFG with a Common Noise

c. Weak solutions

Fixed point without uniqueness

- Solution by compactness argument (without monotonicity)
  - use of Schauder's fixed point theorem
- Disentangle sources of noise $\rightsquigarrow$ product probability space

$$\Omega = \Omega^0 \times \Omega^1, \quad \mathbb{F} = \mathbb{F}^0 \otimes \mathbb{F}^1, \quad \mathbb{P} = \mathbb{P}^0 \otimes \mathbb{P}^1$$

  - $(\Omega^0, \mathbb{F}^0, \mathbb{P}^0) \rightsquigarrow$ common noise B ; $(\Omega^1, \mathbb{F}^1, \mathbb{P}^1) \rightsquigarrow$ noise W
- Fixed point $(\mu_t)_{0 \le t \le T}$ as $\mathbb{F}^0$ prog. meas. process
  - $\mathbb{F}^0 = \mathbb{F}^B$ and $\mathbb{F}^1 = \mathbb{F}^W \Rightarrow$ optimal path under $(\mu_t)_{0 \le t \le T}$ given by

$$dX_t = \big(b(X_t, \mu_t) - Z_t\big)dt + dW_t + \eta dB_t$$

$$dY_t = -\Big(f(X_t, \mu_t) + \frac{1}{2}|Z_t|^2\Big)dt + Z_t dW_t + \zeta_t dB_t, \quad Y_T = g(X_T, \mu_T)$$

- Solve $\boxed{\mu_t(\omega^0) = \mathcal{L}(X_t^{\text{optimal}} \,|\, \mathcal{F}_T^0)(\omega^0)}$ for $t \in [0, T]$ and $\omega^0 \in \Omega^0$

  $\rightsquigarrow$ fixed point in $\Big(C([0, T], \mathcal{P}_2(\mathbb{R}^d))\Big)^{\Omega^0}$

  - much too big space for tractable compactness $\rightsquigarrow$ strategy is to discretize common noise

Discretization method [Carmona D Lacker]

- General principle $\leadsto$ discretization of the fixed point
  - choice of the conditioning $\leadsto \Omega^0$ canonical space for $(B_t)_{0\le t\le T}$

$\leadsto \mathcal{L}(X_t \,|\, \mathcal{F}_T^0) = \mathcal{L}(X_t \,|\, (B_s)_{0\le s\le T})$

  - $\mathcal{L}(X_t \,|\, (B_s)_{0\le s\le T}) \leadsto \mathcal{L}(X_t|\text{process with finite support})$

Discretization method [Carmona D Lacker]

- General principle $\rightsquigarrow$ discretization of the fixed point

  ◦ choice of the conditioning $\rightsquigarrow$ Ω^0 canonical space for $(B_t)_{0\le t\le T}$
$\rightsquigarrow \mathcal{L}(X_t \,|\, \mathcal{F}^0_T) = \mathcal{L}(X_t \,|\, (B_s)_{0\le s\le T})$

  ◦ $\mathcal{L}(X_t \,|\, (B_s)_{0\le s\le T}) \rightsquigarrow \mathcal{L}(X_t|$process with finite support$)$

- Choice of the process with finite support

  ◦ Π projection on spatial grid $\{x_1, \dots, x_P\} \subset \mathbb{R}^d$

  ◦ $t_1, \dots, t_N$ time mesh $\subset [0, T]$

  ◦ $\hat{B}_{t_i} = \Pi(B_{t_i})$

- Conditioning

  ◦ fixed point condition on $\mathcal{L}(X_t \,|\, \hat{B}_{t_1}, \dots, \hat{B}_{t_i})$ for $t \in [t_i, t_{i+1}]$

  ◦ input $\rightsquigarrow$ sequence of processes on each $[t_i, t_{i+1}]$ with values in $\mathcal{P}_2(\mathbb{R}^d)$ and only depending on the realizations of $(\hat{B}_{t_1}, \dots, \hat{B}_{t_i})$

fixed point in $\prod_{i=1}^N C([t_i, t_{i+1}]; \mathcal{P}_2(\mathbb{R}^d))^{iP}$

Solution under discrete conditioning

- Solve FBSDE

$$
\begin{aligned}
dX_t &= \Big(b(X_t, \mathcal{L}(X_t \,|\, \hat{B}_{t_1}, \dots, \hat{B}_{t_i})) - Z_t\Big)dt + dW_t + \eta dB_t \\
dY_t &= -\Big(f(X_t, \mathcal{L}(X_t \,|\, \hat{B}_{t_1}, \dots, \hat{B}_{t_i})) + \tfrac{1}{2}|Z_t|^2\Big)dt + Z_t dW_t + \zeta_t dB_t \\
Y_T &= g(X_T, \mathcal{L}(X_T \,|\, \hat{B}_{t_1}, \dots, \hat{B}_{t_N}))
\end{aligned}
$$

- Strategy for the fixed point
  - input $\mu = (\mu^1, \dots, \mu^N)$ with

$$
\mu^i \in \mathcal{C}([t_i, t_{i+1}]; \mathcal{P}_2(\mathbb{R}^d))^{\{x_1,\dots,x_P\}^i}
$$

  - $\mu_t = \mu_t^i(\hat{B}_{t_1}, \cdots, \hat{B}_{t_i})$
  - output given by

$$
\{x_1, \cdots, x_P\}^i \ni (a_1, \dots, a_i) \mapsto \mathcal{L}\Big(X_t \,|\, \hat{B}_{t_1} = a_1, \dots, \hat{B}_{t_i} = a_i\Big)
$$

- Stability for FBSDEs $\rightsquigarrow$ continuity w.r.t input + compactness for laws $\Rightarrow$ Schauder

Passing to the limit

- Convergent subsequence as $N, P \to \infty$?
  - use Pontryagin's principle to describe optimal paths

$$dX_t = b(X_t, \mathcal{L}(X_t \,|\, \hat{B}_{t_1}, \dots, \hat{B}_{t_i}))dt - Z_t dt + dW_t + \eta dB_t$$
$$dZ_t = -\partial_x H(X_t, \mathcal{L}(X_t \,|\, \hat{B}_{t_1}, \dots, \hat{B}_{t_i}), Z_t)dt + dM_t$$
$$Z_T = \partial_x g(X_T, \mathcal{L}(X_T \,|\, \hat{B}_{t_1}, \dots, \hat{B}_{t_N}))$$

$\rightsquigarrow (M_t)_t$ martingale, $\quad \mu_t = \mathcal{L}(X_t \,|\, \hat{B}_{t_1}, \dots, \hat{B}_{t_i})$

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$\rightsquigarrow$ $(M_t)_t$ martingale, $\quad \mu_t = \mathcal{L}(X_t \,|\, \hat{B}_{t_1}, \dots, \hat{B}_{t_i})$

- Tightness of the laws of $(X_t^{N,P}, \mu_t^{N,P}, Z_t^{N,P}, M^{N,P}, B_t, W_t)_{0 \le t \le T}$
  - tightness of $(X_t^{N,P})_{0 \le t \le T}$ in $\mathcal{C}([0,T]; \mathbb{R}^d)$ by Kolmogorov
  - tightness of $(\mu_t^{N,P})_{0 \le t \le T}$ in $\mathcal{C}([0,T]; \mathcal{P}_2(\mathbb{R}^d))$ since

$$\int_d |x|^q d\mu_t^{N,P}(x) = \mathbb{E}[|X_t^{N,P}|^q | \mathcal{F}_T^0], \quad W_2(\mu_t^{N,P}, \mu_s^{N,P})^2 \le \mathbb{E}[|X_t^{N,P} - X_s^{N,P}|^2 | \mathcal{F}_T^0]$$

  - tightness $(Z_t^{N,P}, M_t^{N,P})_{0 \le t \le T}$ in $\mathcal{D}([0,T]; \mathbb{R}^d)$ with Meyer-Zheng

$\rightsquigarrow$ $(z_t^n)_{0 \le t \le T} \to (z_t)_{0 \le t \le T}$ in dt-measure [Pardoux] for use in BSDE

Passing to the limit

- Convergent subsequence as $N, P \to \infty$?
  - use Pontryagin's principle to describe optimal paths

$$\begin{aligned} dX_t &= b(X_t, \mathcal{L}(X_t \,|\, \hat{B}_{t_1}, \dots, \hat{B}_{t_i}))dt - Z_t dt + dW_t + \eta dB_t \\ dZ_t &= -\partial_x H(X_t, \mathcal{L}(X_t \,|\, \hat{B}_{t_1}, \dots, \hat{B}_{t_i}), Z_t)dt + dM_t \\ Z_T &= \partial_x g(X_T, \mathcal{L}(X_T \,|\, \hat{B}_{t_1}, \dots, \hat{B}_{t_N})) \end{aligned}$$

  $\rightsquigarrow$ $(M_t)_t$ martingale, $\mu_t = \mathcal{L}(X_t \,|\, \hat{B}_{t_1}, \dots, \hat{B}_{t_i})$

- Tightness of the laws of $(X_t^{N,P}, \mu_t^{N,P}, Z_t^{N,P}, M^{N,P}, B_t, W_t)_{0 \le t \le T}$
- Limit process $(X_t^\infty, \mu_t^\infty, Z_t^\infty, M_t^\infty, B_t^\infty, W_t^\infty)_{0 \le t \le T}$
  - identify $\rightsquigarrow$ μ_t^∞ as conditional law of X_t^∞ given information?

    $\rightsquigarrow$ pass to the limit in $\mu_t^{N,P} = \mathcal{L}(X_t^{N,P} \,|\, \hat{B}_{t_1}^{N,P}, \dots, \hat{B}_{t_i}^{N,P})$
  - solve optimization problem in environment $(\mu_t^\infty)_{0 \le t \le T}$?

    $\rightsquigarrow$ main difficulty $\rightsquigarrow$ loss of measurability of μ_t^∞ w.r.t

$(B_s^\infty)_{0 \le s \le t} \Rightarrow$ **weak solution only!**

Strong vs. weak solutions

- Limiting FBSDE formulation

$$dX_t^\infty = \big(b(X_t^\infty, \mu_t^\infty) - Z_t^\infty\big)dt + dW_t^\infty + \eta dB_t^\infty$$
$$dZ_t^\infty = -\partial_x H(X_t^\infty, \mu_t^\infty, Z_t^\infty)dt + dM_t^\infty, \quad Z_T^\infty = \partial_x g(X_T^\infty, \mu_T^\infty)$$

$\rightsquigarrow$ necessary condition for optimality only, but not a limitation

$\rightsquigarrow$ may pass to the limit in the optimality condition

- ∘ cost $J(-Z^\infty) = \mathbb{E}\Big[g(X_T^\infty, \mu_T^\infty) + \int_0^T \big(f(X_t^\infty, \mu_t^\infty) + \tfrac{1}{2}|Z_t^\infty|^2\big)dt\Big]$

Strong vs. weak solutions

- Limiting FBSDE formulation

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⇝ necessary condition for optimality only, but not a limitation
⇝ may pass to the limit in the optimality condition

- Main question: What is the common information?
  - whole information ⇝ $\mathbb{F}^\infty$ generated by $(X^\infty, \mu^\infty, B^\infty, W^\infty)$
  - common environment ⇝ expect (μ^∞, B^∞)? should satisfy

⇝ fixed point $\mu_t^\infty = \mathcal{L}(X_t^\infty \,|\, \mu^\infty, B^\infty)$ (true)

⇝ (μ^∞, B^∞) X_0^∞ and W^∞ $\perp\!\!\!\perp$ (true) (X_0^∞, W^∞) ⇝ proper noise

⇝ fair extra observation ⇝ $\sigma(X_0^\infty, \mu_s^\infty, B_s^\infty, W_s^\infty, s \le T)$ and $\mathcal{F}_t^\infty$ conditional $\perp\!\!\!\perp$ on $\sigma(X_0^\infty, \mu_s^\infty, B_s^\infty, W_s^\infty, s \le t)$ (???)

⇝ observation of private state has no bias on future of the environment (???)

Strong vs. weak solutions

- Limiting FBSDE formulation

$$dX_t^\infty = \big(b(X_t^\infty, \mu_t^\infty) - Z_t^\infty\big)dt + dW_t^\infty + \eta dB_t^\infty$$
$$dZ_t^\infty = -\partial_x H(X_t^\infty, \mu_t^\infty, Z_t^\infty)dt + dM_t^\infty, \quad Z_T^\infty = \partial_x g(X_T^\infty, \mu_T^\infty)$$

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⇝ notion of compatibility [Jacod, Mémin, Kurtz] and [Buckdahn (al.)] for BSDEs

Strong vs. weak solutions

- Limiting FBSDE formulation

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$\rightsquigarrow$ notion of compatibility [Jacod, Mémin, Kurtz] and [Buckdahn (al.)] for BSDEs

$\rightsquigarrow$ difficult to pass to the limit on compatibility $\Rightarrow$ need to enlarge environment

Strong vs. weak solutions

- Limiting FBSDE formulation $\rightsquigarrow$ necessary condition for optimality only, but not a limitation $\leadsto$ may pass to the limit in the optimality condition
- Main question: What is the common information?
  - whole information $\leadsto$ $\mathbb{F}^\infty$ generated by $(X^\infty, \mu^\infty, B^\infty, W^\infty)$
  - common environment $\leadsto$ replace by $(\mathcal{M}^\infty, B^\infty)$

    $\rightsquigarrow$ $\mathcal{M}_t^\infty$ limit in law of $\mathcal{L}(X_{\cdot\wedge t}^{N,P}, W_{\cdot\wedge t}^{N,P} | B^\infty)$

    $\rightsquigarrow$ fixed point $\mathcal{M}_t^\infty = \mathcal{L}(X_{\cdot\wedge t}^\infty, W_{\cdot\wedge t}^\infty \,|\, \mathcal{M}^\infty, B^\infty)$

    $\rightsquigarrow$ fixed point $\Rightarrow$ compatibility
- Yamada-Watanabe: strong ! for compatible solutions $\Rightarrow$ weak solutions are strong
  - strong solutions $\leadsto$ environment is adapted to B^∞
  - example if monotonicity $\Rightarrow$ close the loop!

Part V. Master Equation

a. Derivation of equation

Setting

- Assume $\exists!$ for value function MKV FBSDE ($\sigma = 1$)

$$dX_s = \Big(b(X_s, \mathcal{L}(X_s|B)) - Z_s\Big)ds + dW_s + \eta dB_s$$

$$dY_s = -f(X_s, \mathcal{L}(X_s|B), Z_s)ds + Z_s dW_s + \zeta_s dB_s, \quad Y_T = g(X_T, \mathcal{L}(X_T|B))$$

  - $Y_t = U(t, X_t, \mu) = U(t, X_t, \mathcal{L}(X_t|B))$
- Goal: Expand the right-hand side to identify PDE for U!!!

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- Goal: Expand the right-hand side to identify PDE for U!!!
- Need for second-order derivatives
  - $\partial_t U(t, x, \mu)$ and $\partial_x^2 U(t, x, \mu)$ bounded and Lipschitz in (x, μ)
  - $\partial_\mu U(t, x, \mu)(v)$ is differentiable in x, v and μ
  - $\partial_x \partial_\mu U(t, x, \mu)(v)$, $\partial_v \partial_\mu U(x, \mu)(v)$ bounded and Lipschitz
  - $\partial_\mu^2 U(t, x, \mu)(v, v')$ is bounded and Lipschitz

Setting

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  - $\partial_\mu^2 U(t,x,\mu)(v,v')$ is bounded and Lipschitz
- Theorem: [Gangbo Swiech, C D D, C D L L] If monotonicity and smooth coefficients, then U is smooth

Itô's formula on $\mathcal{P}_2(\mathbb{R}^d)$

- Process $dX_t = b_t dt + dW_t + dB_t$ $\mathbb{E}\int_0^T |b_t|^2 dt < \infty$
  - disentangle sources of noise $\rightsquigarrow$ use product probability space

$$\Omega = \Omega^B \times \Omega^W, \quad \mathbb{F} = \mathbb{F}^B \otimes \mathbb{F}^W, \quad \mathbb{P} = \mathbb{P}^B \otimes \mathbb{P}^W$$

  - $(\Omega^B, \mathbb{F}^B, \mathbb{P}^B) \rightsquigarrow B, \quad (\Omega^W, \mathbb{F}^W, \mathbb{P}^W) \rightsquigarrow \mathrm{W}, \quad \mathcal{L}(\cdot|\sigma(B)) = \mathcal{L}^W(\cdot)$
  - $\Omega = \Omega^B \times \Omega^W$, Ω^B carries B, Ω^W carries W
  - $\mu_t = \mathcal{L}(X_t)$: conditional law of X_t given B

Itô's formula on $\mathcal{P}_2(\mathbb{R}^d)$

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- U Fréchet differentiable with $\mathbb{R}^d \ni v \mapsto \partial_\mu U(\mu, v)$ differentiable (v, μ)

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  - $\Omega = \Omega^B \times \Omega^W$, Ω^B carries B, Ω^W carries W
  - $\mu_t = \mathcal{L}(X_t)$: conditional law of X_t given B
- U Fréchet differentiable with $\mathbb{R}^d \ni v \mapsto \partial_\mu U(\mu, v)$ differentiable (v, μ)
  - Itô's formula for $(U(\mu_t))_{t \geq 0}$?

$$dU(\mu_t) = \mathbb{E}^W[b_t \cdot \partial_\mu U(\mu_t)(X_t)] + \mathbb{E}^W[\mathrm{Trace}(\partial_v \partial_\mu U(\mu_t)(X_t))]dt + \frac{1}{2}\mathbb{E}^W \tilde{\mathbb{E}}^{\tilde{W}}[\mathrm{Trace}(\partial_\mu^2 U(\mu_t)(X_t, \tilde{X}_t))]dt + \mathbb{E}^W[\partial_\mu U(\mu_t)(X_t)] \cdot dB_t$$

  - $\tilde{\mathbb{E}}^{\tilde{W}}$ conditional expectation on a copy space $\Omega^B \times \tilde{\Omega}^W$

Identification of the master equation

- Identification of the dt terms in the expansion of the identify:

$$Y_t = U\big(t, X_t, \mathcal{L}(X_t \,|\, B)\big)$$

Identification of the master equation

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$$Y_t = U(t, X_t, \mathcal{L}(X_t \,|\, B))$$

- Get the form of the full-fledged master equation

$$\begin{aligned}
&\partial_t U(t,x,\mu) - \int_{\mathbb{R}^d} \partial_x U(t,v,\mu) \cdot \partial_\mu U(t,x,\mu)(v) d\mu(v) \\
&\quad + f(x,\mu) - \frac{1}{2}|\partial_x U(t,x,\mu)|^2 + \frac{1+\eta^2}{2} \mathrm{Trace}\big(\partial_x^2 U(t,x,\mu)\big) \\
&\quad + \frac{1+\eta^2}{2} \int_{\mathbb{R}^d} \mathrm{Trace}\big(\partial_v \partial_\mu U(t,x,\mu,v)\big) d\mu(v) \\
&\quad + \eta^2 \int_{\mathbb{R}^d} \mathrm{Trace}\big(\partial_x \partial_\mu U(t,x,\mu,v)\big) d\mu(v) \\
&\quad + \frac{\eta^2}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathrm{Trace}\big(\partial_\mu^2 U(t,x,\mu,v,v')\big) d\mu(v) d\mu(v') = 0
\end{aligned}$$

  - Not a HJB! (MFG $\neq$ optimization)

Part V. Master Equation

b. Application

Revisiting the N-player game

- Controlled dynamics

$$dX_t^i = \Big(b(X_t^i, \bar{\mu}_t^N) + \alpha_t^i\Big)dt + dW_t^i + \eta dB_t$$

- Cost functionals to player i

$$J^i(\alpha^1, \ldots, \alpha^N) = \mathbb{E}\Big[g(X_T^i, \bar{\mu}_T^N) + \int_0^T \Big(f(X_s^i, \bar{\mu}_s^N) + \frac{1}{2}|\alpha_s^i|^2\Big)ds\Big]$$

- Rigorous connection between N-player game and MFG?

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- Rigorous connection between N-player game and MFG?
- Prove the convergence of the Nash equilibria as N tends to ∞

  ∘ difficulty $\rightsquigarrow$ no uniform smoothness on the optimal feedback function $\alpha^{\star,N}$ w.r.t to N

$$\underbrace{\alpha_t^{\star,i,N}}_{\text{optimal control to player } i} = \alpha^{\star,N}(X_t^i; \underbrace{X^1, \ldots, X^{i-1}, X^{i+1}, \ldots, X^N}_{\text{states of the others}})$$

  $\rightsquigarrow$ no compactness on the feedback functions

  ∘ weak compactness arguments on the control (notion of relaxed controls) for equilibria over open loop controls [Lacker, Fischer]

Revisiting the N-player game

- Controlled dynamics

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$$\underbrace{\alpha_t^{\star,i,N}}_{\text{optimal control to player } i} = \alpha^{\star,N}(X_t^i; \underbrace{X^1, \dots, X^{i-1}, X^{i+1}, \dots, X^N}_{\text{states of the others}})$$

  $\rightsquigarrow$ no compactness on the feedback functions

  ∘ use the master equation [C D L L]: expand $(U(t, X_t^i, \bar{\mu}_t^N))_{0 \le t \le T}$ and prove $\approx$ equilibrium cost to player i

Revisiting the N-player game

- Controlled dynamics

$$dX_t^i = \big(b(X_t^i, \bar{\mu}_t^N) + \alpha_t^i\big)dt + dW_t^i + \eta dB_t$$

- Cost functionals to player i

$$J^i(\alpha^1, \dots, \alpha^N) = \mathbb{E}\Big[g(X_T^i, \bar{\mu}_T^N) + \int_0^T \Big(f(X_s^i, \bar{\mu}_s^N) + \frac{1}{2}|\alpha_s^i|^2\Big)ds\Big]$$

- Rigorous connection between N-player game and MFG?
- Construct **approximate Nash equilibria** (easier)
  - limit setting $\rightsquigarrow$ optimal control has the form

$$\alpha_t^\star = -\partial_x U(t, X_t, \underbrace{\mathcal{L}(X_t|B)}_{\text{population at equilibrium}})$$

  - in N-player game, use $\alpha_t^{N,i} = -\partial_x U(t, X_t^i, \bar{\mu}_t^N)$
  - almost Nash $\rightsquigarrow$ cost decreases at most of ε_N under unilateral deviation where $\varepsilon_N \to 0$

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