Probabilistic approach to Mean-Field Games

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Based on joint works with R. Carmona, P. Cardaliaguet, D. Crisan, J.F. Chassagneux, D. Lacker, J.M. Lasry and P.L. Lions
Part I. Motivation
Part I. Motivation

a. General philosophy
Basic purpose

- Interacting particles / players
  - controlled players in mean-field interaction
  - particles have dynamical states $\leftrightarrow$ stochastic diff. equation
  - mean-field $\leftrightarrow$ symmetric interaction with whole population
    - no privileged interaction with some particles
- Associate cost functional with each player
  - find equilibria w.r.t. cost functionals
  - shape of the equilibria for a large population?
Basic purpose

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• Associate cost functional with each player
  ○ find equilibria w.r.t. cost functionals
  ○ shape of the equilibria for a large population?
• Different notions of equilibria
  ○ players decide on their own $\leadsto$ find a consensus inside the population $\Rightarrow$ notion of Nash equilibrium
  ○ players obey a common center of decision $\leadsto$ minimize the global cost to the collectivity
• Both cases $\leadsto$ asymptotic equilibria as the number of players $\uparrow \infty$?
Asymptotic formulation

- **Paradigm**
  - mean-field / symmetry ⇐⇒ propagation of chaos / LLN
  - reduce the asymptotic analysis to one typical player with interaction with a *theoretical* distribution of the population?
  - decrease the complexity to solve asymptotic formulation first
Asymptotic formulation

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• **Program**
  - **Existence** of asymptotic equilibria? **Uniqueness**? **Shape**?
  - Use asymptotic equilibria as quasi-equilibria in finite-game
  - Prove convergence of equilibria in finite-player-systems
Asymptotic formulation

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- Asymptotic formulation of Nash equilibria ⇔ Mean-field games!
  [Lasry-Lions (06), Huang-Caines-Malhamé (06), Cardaliaguet, Achdou, Gangbo, Gomes, Porretta (PDE), Bensoussan, Carmona, D., Kolokoltsov, Lacker, Yam (Probability)]

- Common center of decision ⇔ optimal control of McKean-Vlasov SDEs
Part I. Motivation

b. Equilibria within a finite system
General formulation

- Controlled system of $N$ interacting particles with mean-field interaction through the global state of the population
  - dynamics of particle number $i \in \{1, \ldots, N\}$
    \[
    dX^i_t = b(X^i_t, \text{global state of the collectivity}, \alpha^i_t)\,dt \\
    \in \mathbb{R}^d \\
    + \sigma(X^i_t, \text{global state}) \,dW^i_t \quad \text{idiosyncratic noises} \\
    + \sigma^0(X^i_t, \text{global state}) \,dB_t \quad \text{common/systemic noise}
    \]
General formulation

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    + \sigma(X_t^i, \text{global state}) dW_t^i \text{ idiosyncratic noises} \\
    + \sigma^0(X_t^i, \text{global state}) dB_t \text{ common/systemic noise}
    \]

- Rough description of the probabilistic set-up
  - $(B_t, W^1, \ldots, W^N)_{0 \leq t \leq T}$ independent B.M. with values in $\mathbb{R}^d$
  - $(\alpha_t^i)_{0 \leq t \leq T}$ progressively-measurable processes with values in $A$ (closed convex $\subset \mathbb{R}^k$)
  - i.i.d. initial conditions $\perp$ noises
Empirical measure

- Code the state of the population at time $t$ through

$$
\bar{\mu}_t^N = \frac{1}{N} \sum_{i=1}^{N} \delta_{X_t^i}
$$

\implies \text{probability measure on } \mathbb{R}^d

- $\mathcal{P}_2(\mathbb{R}^d) \implies \text{set of probabilities on } \mathbb{R}^d \text{ with finite 2nd moments}$
Empirical measure

- Code the state of the population at time $t$ through
  \[ \bar{\mu}_t^N = \frac{1}{N} \sum_{i=1}^{N} \delta_{X^i_t} \]
  \[ \mapsto \text{probability measure on } \mathbb{R}^d \]
  - $\mathcal{P}_2(\mathbb{R}^d) \mapsto$ set of probabilities on $\mathbb{R}^d$ with finite 2nd moments

- Express the coefficients as
  \[ b : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times A \to \mathbb{R}^d, \]
  \[ \sigma, \sigma^0 : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^{d \times d}, \]
  - examples: $b(x, \mu, \alpha) = b(x, \int_{\mathbb{R}^d} \varphi d\mu, \alpha)$, $\int_{\mathbb{R}^d} b(x, v, \alpha)d\mu(v)$
  - rewrite the dynamics of the particles

\[ dX^i_t = b(X^i_t, \bar{\mu}_t^N, \alpha^i_t)dt + \sigma(X^i_t, \bar{\mu}_t^N)dW^i_t + \sigma^0(X^i_t, \bar{\mu}_t^N)dB_t \]
Empirical measure

- Code the state of the population at time $t$ through:
  \[
  \bar{\mu}^N_t = \frac{1}{N} \sum_{i=1}^N \delta_{X^i_t}
  \]

  $\Rightarrow$ probability measure on $\mathbb{R}^d$

  - $\mathcal{P}_2(\mathbb{R}^d)$ $\Rightarrow$ set of probabilities on $\mathbb{R}^d$ with finite 2nd moments

- Express the coefficients as:
  \[b : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times A \to \mathbb{R}^d,\]
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  - examples: $b(x, \mu, \alpha) = b(x, \int_{\mathbb{R}^d} \varphi d\mu, \alpha), \quad \int_{\mathbb{R}^d} b(x, v, \alpha) d\mu(v)$

  - rewrite the dynamics of the particles

    \[dX^i_t = b(X^i_t, \bar{\mu}^N_t, \alpha^i_t) dt + \sigma(X^i_t, \bar{\mu}^N_t) dW^i_t + \sigma^0(X^i_t, \bar{\mu}^N_t) dB_t\]

- **Cost functional** to player $i \in \{1, \ldots, N\}$

  \[J^i(\alpha^1, \alpha^2, \ldots, \alpha^N) = \mathbb{E}\left[ g(X^i_T, \bar{\mu}^N_T) + \int_0^T f(X^i_t, \bar{\mu}^N_t, \alpha^i_t) dt \right]\]

  - same $(f, g)$ for all $i$ but $J^i$ depends on the others through $\bar{\mu}^N$
Nash equilibrium

• Each player is willing to minimize its own cost functional
  o need for a consensus $\sim$ Nash equilibrium
Nash equilibrium

- Each player is willing to minimize its own cost functional
  - need for a consensus $\leadsto$ Nash equilibrium
- Say that a $N$-tuple of strategies $(\alpha^1, \star, \ldots, \alpha^N, \star)$ is a consensus if
  - no interest for any player to leave the consensus
  - change $\alpha^i, \star \leadsto \alpha^i \implies J^i \nearrow$

$$J^i(\alpha^1, \star, \ldots, \alpha^i, \star, \ldots, \alpha^N, \star) \leq J^i(\alpha^1, \star, \ldots, \alpha^i, \ldots \alpha^N, \star)$$
Nash equilibrium

• Each player is willing to minimize its own cost functional
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  ◦ change $\alpha^i, \star \leadsto \alpha^i \Rightarrow J^i \nearrow$

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J^i(\alpha^1, \star, \ldots, \alpha^i, \star, \ldots, \alpha^N, \star) \leq J^i(\alpha^1, \star, \ldots, \alpha^i, \ldots, \alpha^N, \star)
\]

• Meaning of freezing $\alpha^1, \star, \ldots, \alpha^{i-1}, \star, \alpha^{i+1}, \star, \alpha^N, \star$
  ◦ freezing the processes $\leadsto$ Nash equilibrium in open loop
  ◦ $\alpha_t^i = \alpha^i(t, X^1_t, \ldots, X^N_t) \leadsto$ each function $\alpha^i$ is a Markov feedback
  $\leadsto$ Nash over of Markov loop
    ◦ leads to different equilibria! but expect that there is no difference in the asymptotic setting
Part I. Motivation

c. Example
**Exhaustible resources** [Guéant Lasry Lions]

- *N producers of oil* \( \sim X^i_t \) (estimated reserve) at time *t*
  
  \[
  dX^i_t = -\alpha^i_t \, dt + \sigma X^i_t \, dW^i_t
  \]
  
  - \( \alpha^i_t \) \( \sim \) instantaneous production rate
  - \( \sigma \) common volatility for the perception of the reserve
  - should be a constraint \( X^i_t \geq 0 \)

- Optimize the profit of a producer

  \[
  J_i(\alpha_1, \ldots, \alpha_N) = \mathbb{E} \int_0^\infty \exp(-rt)(P_t - c(\alpha^i_t)) \, dt
  \]
  
  - *P* \( t \) is selling price, *c* cost production
  - mean-field constraint

  \[
  P_t = P\left(\frac{1}{N} \sum_{i=1}^{N} \alpha^i_t\right)
  \]

  - slightly different! \( \rightsquigarrow \) interaction through the law of the control

  \[
  \text{[Gomes al., Carmona D., Cardaliaguet Lehalle]}\]
Exhaustible resources [Guéant Lasry Lions]

- $N$ producers of oil $\sim X^i_t$ (estimated reserve) at time $t$

\[ dX^i_t = -\alpha^i_t dt + \sigma X^i_t dW^i_t \]

\( \alpha^i_t \sim \) instantaneous production rate

\( \sigma \) common volatility for the perception of the reserve

\( \sigma \) should be a constraint $X^i_t \geq 0$

- Optimize the profit of a producer

\[ J^i(\alpha^1, \ldots, \alpha^N) = \mathbb{E} \int_0^\infty \exp(-rt)(\alpha^i_t P_t - c(\alpha^i_t))dt \]

\( P_t \) is selling price, $c$ cost production

- mean-field constraint $\sim$ selling price is a function of the mean-production

\[ P_t = P\left(\frac{1}{N} \sum_{i=1}^N \alpha^i_t \right) \]

\( \alpha \) slightly different! $\sim$ interaction through the law of the control

$\sim$ extended MFG [Gomes al., Carmona D., Cardaliaguet Lehalle]
Part II. From propagation of chaos to MFG
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a. McKean-Vlasov SDEs
General uncontrolled particle system

- Remove the control and the common noise!

\[ dX_t^i = b(X_t^i, \bar{\mu}_t^N)dt + \sigma(X_t^i, \bar{\mu}_t^N)dW_t^i \]

- \( X_0^1, \ldots, X_N^i \) i.i.d. (and \( \perp \) of noises), \( \bar{\mu}_t^N = \frac{1}{N} \sum_{i=1}^{N} \delta_{X_t^i} \)

- \( \exists! \) if the coefficients are Lipschitz in all the variables \( \leadsto \) need a suitable distance on space of measures
General uncontrolled particle system

- Remove the control and the common noise!

\[ dX_t^i = b(X_t^i, \bar{\mu}_t^N)dt + \sigma(X_t^i, \bar{\mu}_t^N)dW_t^i \]

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- \( \exists! \) if the coefficients are Lipschitz in all the variables \( \rightsquigarrow \) need a suitable distance on space of measures

- Use the Wasserstein distance on \( \mathcal{P}_2(\mathbb{R}^d) \)

\[ \mu, \nu \in \mathcal{P}_2(\mathbb{R}^d), \quad W_2(\mu, \nu) = \left( \inf_{\pi} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^2 d\pi(x, y) \right)^{1/2}, \]

where \( \pi \) has \( \mu \) and \( \nu \) as marginals on \( \mathbb{R}^d \times \mathbb{R}^d \)

- \( X \) and \( X' \) two r.v.'s \( \Rightarrow \) \( W_2(\mathcal{L}(X), \mathcal{L}(X')) \leq \mathbb{E}[|X - X'|^2]^{1/2} \)

- Example \( W_2\left( \frac{1}{N} \sum_{i=1}^{N} \delta_{x_i}, \frac{1}{N} \sum_{i=1}^{N} \delta_{x_i'} \right) \leq \left( \frac{1}{N} \sum_{i=1}^{N} |x_i - x_i'|^2 \right)^{1/2} \)
**McKean-Vlasov SDE**

- Expect some **decorrelation / averaging** in the system as $N \uparrow \infty$
  - replace the empirical measure by the theoretical law

$$dX_t = b(X_t, \mathcal{L}(X_t))dt + \sigma(X_t, \mathcal{L}(X_t))dW_t$$

- **Cauchy-Lipschitz theory**
  - assume $b$ and $\sigma$ Lipschitz continuous on $\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \Rightarrow$ unique solution for any given initial condition in $L^2$
  - proof works as in the standard case taking advantage of

$$\mathbb{E} \left[ \left| (b, \sigma)(X_t, \mathcal{L}(X_t)) - (b, \sigma)(X'_t, \mathcal{L}(X'_t)) \right|^2 \right] \leq C \mathbb{E} \left[ |X_t - X'_t|^2 \right]$$
**McKean-Vlasov SDE**

- Expect some **decorrelation / averaging** in the system as $N \uparrow \infty$
  
  - replace the empirical measure by the theoretical law
  
  \[
  dX_t = b(X_t, \mathcal{L}(X_t))dt + \sigma(X_t, \mathcal{L}(X_t))dW_t
  \]

- Cauchy-Lipschitz theory
  
  - assume $b$ and $\sigma$ Lipschitz continuous on $\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \Rightarrow$ unique solution for any given initial condition in $L^2$
  
  - proof works as in the standard case taking advantage of
    
    \[
    \mathbb{E}\left[\left| (b, \sigma)(X_t, \mathcal{L}(X_t)) - (b, \sigma)(X'_t, \mathcal{L}(X'_t)) \right|^2 \right] \leq C \mathbb{E}[|X_t - X'_t|^2]
    \]

- Propagation of chaos
  
  - each $(X^i_t)_{0 \leq t \leq T}$ converges in law to the solution of MKV SDE
  
  - particles get **independent** in the limit $\sim$ for $k$ fixed:
    
    \[
    (X^1_t, \ldots, X^k_t)_{0 \leq t \leq T} \xrightarrow{\mathcal{L}} \mathcal{L}(\text{MKV})^\otimes k = \mathcal{L}((X_t)_{0 \leq t \leq T})^\otimes k \quad \text{as } N \uparrow \infty
    \]
  
  - $\lim_{N \uparrow \infty} \sup_{0 \leq t \leq T} \mathbb{E}[(W_2(\bar{\mu}^N_t, \mathcal{L}(X_t))^2)] = 0$
Part II. From propagation of chaos to MFG

b. Formulation of the asymptotic problems
Ansatz

- Go back to the finite game
- **Ansatz** \(\sim\) at equilibrium

\[
\alpha_i^* = \alpha^N(t, X_i^t, \bar{\mu}_t^N) \approx \alpha(t, X_i^t, \bar{\mu}_t^N)
\]

- particle system at equilibrium

\[
dX_i^t \approx b(X_i^t, \bar{\mu}_t^N, \alpha(t, X_i^t, \bar{\mu}_t^N))dt + \sigma(X_i^t, \alpha(t, X_i^t, \bar{\mu}_t^N))dW_i^t
\]

- particles should decorrelate as \(N \uparrow \infty\)
- \(\bar{\mu}_t^N\) should stabilize around some deterministic limit \(\mu_t\)
Ansatz

- Go back to the finite game

- Ansatz \( \sim \) at equilibrium

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\alpha^i\approx = \alpha^N(t, X_i^t, \bar{\mu}_t^N) \approx \alpha(t, X_i^t, \bar{\mu}_t^N)
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- particle system at equilibrium

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\]

- particles should decorrelate as \( N \uparrow \infty \)

- \( \bar{\mu}_t^N \) should stabilize around some deterministic limit \( \mu_t \)

- What about an intrinsic interpretation of \( \mu_t \)?

  - should describe the global state of the population in equilibrium

  - in the limit setting, any particle that leaves the equilibrium should not modify \( \mu_t \sim \) leaving the equilibrium means that the cost increases \( \sim \) any particle in the limit should solve an optimal control problem in the environment \((\mu_t)_{0\leq t\leq T}\)
Matching problem of MFG

- Define the asymptotic equilibrium state of the population as the solution of a fixed point problem.

\[ \text{Definition of asymptotic equilibrium state.} \]

\[ \begin{align*}
\text{Problem setup:} & \quad \text{Define a flow of probability measures} \\
& \quad \text{Solve a stochastic optimal control problem.} \\
& \quad \text{Find a fixed point.} \\
\end{align*} \]

\[ \text{Optimal control problem formulation.} \]

\[ \begin{align*}
\text{Problem statement:} & \quad \text{Define an optimal control problem} \\
& \quad \text{Solve for the optimal control.} \\
& \quad \text{Prove convergence.} \\
\end{align*} \]

\[ \text{Proof of convergence.} \]
Matching problem of MFG

- Define the asymptotic equilibrium state of the population as the solution of a **fixed point problem**

\[(1) \text{ fix a flow of probability measures } (\mu_t)_{0 \leq t \leq T} \text{ (with values in } \mathcal{P}_2(\mathbb{R}^d)) \]
Matching problem of MFG

• Define the asymptotic equilibrium state of the population as the solution of a fixed point problem

\[(1) \text{ fix a flow of probability measures } (\mu_t)_{0 \leq t \leq T} \text{ (with values in } \mathcal{P}_2(\mathbb{R}^d))\]

\[(2) \text{ solve the stochastic optimal control problem in the environment } (\mu_t)_{0 \leq t \leq T}\]

\[dX_t = b(X_t, \mu_t, \alpha_t)dt + \sigma(X_t, \mu_t)dW_t\]

○ with \(X_0 = \xi\) being fixed on some set-up \((\Omega, \mathcal{F}, \mathbb{P})\) with a \(d\)-dimensional B.M.

○ with \(J(\alpha) = \mathbb{E}\left[ g(X_T, \mu_T) + \int_0^T f(X_t, \mu_t, \alpha_t)dt \right] \)
Matching problem of MFG

- Define the asymptotic equilibrium state of the population as the solution of a [fixed point problem]

  (1) **fix a flow of probability measures** \((\mu_t)_{0 \leq t \leq T}\) (with values in \(\mathcal{P}_2(\mathbb{R}^d)\))

  (2) solve the **stochastic optimal control problem** in the environment 

  \[dX_t = b(X_t, \mu_t, \alpha_t)dt + \sigma(X_t, \mu_t)dW_t\]

  ○ with \(X_0 = \xi\) being fixed on some set-up \((\Omega, \mathcal{F}, \mathbb{P})\) with a \(d\)-dimensional B.M.

  ○ with [cost] \(J(\alpha) = \mathbb{E}\left[ g(X_T, \mu_T) + \int_0^T f(X_t, \mu_t, \alpha_t)dt \right]\)

  (3) let \((X_t^{*, \mu})_{0 \leq t \leq T}\) be the unique optimizer (under nice assumptions)

  ~ find \((\mu_t)_{0 \leq t \leq T}\) such that 

  \[\mu_t = \mathcal{L}(X_t^{*, \mu}), \quad t \in [0, T]\]

- Not a proof of convergence!
Part II. From propagation of chaos to MFG

c. Forward-backward systems
PDE point of view: HJB

• PDE characterization of the optimal control problem when $\sigma$ is the identity

• Value function in environment $(\mu_t)_{0 \leq t \leq T}$

$$u(t, x) = \inf_{\alpha \text{ processes}} \mathbb{E}[g(X_T, \mu_T) + \int_t^T f(X_s, \mu_s, \alpha_s) ds | X_t = x]$$
PDE point of view: HJB

- PDE characterization of the optimal control problem when $\sigma$ is the identity

- Value function in environment $(\mu_t)_{0 \leq t \leq T}$

  $$u(t, x) = \inf_{\alpha \text{ processes}} \mathbb{E}\left[ g(X_T, \mu_T) + \int_t^T f(X_s, \mu_s, \alpha_s) ds | X_t = x \right]$$

- $U$ solution Backward HJB

  $$\left( \partial_t u + \frac{\partial^2 u}{2} \right)(t, x) + \inf_{\alpha \text{ scalar}} \left[ b(x, \mu_t, \alpha) \partial_x u(t, x) + f(x, \mu_t, \alpha) \right] = 0$$

  standard Hamiltonian in HJB

- $H(x, \mu, \alpha, z) = b(x, \mu, \alpha) \cdot z + f(x, \mu, \alpha)$

  $\circ \alpha^*(x, \mu, z) = \arg\min_{\alpha \in A} H(x, \mu, \alpha, z) \leadsto \alpha^* = \alpha^*(x, \mu_t, \partial_x u(t, x))$

- Terminal boundary condition: $u(T, \cdot) = g(\cdot, \mu_T)$

- Pay attention that $u$ depends on $(\mu_t)_t$!
Fokker-Planck

- Need for a PDE characterization of \((\mathcal{L}(X_t^{\star,\mu}))_t\)

- Dynamics of \(X_t^{\star,\mu}\) at equilibrium
  \[
  dX_t^{\star,\mu} = b(X_t^{\star,\mu}, \mu_t, \alpha^*(X_t^{\star,\mu}, \mu_t, \partial_x u(t, X_t^{\star,\mu})))dt + dW_t
  \]

- Law \((X_t^{\star,\mu})_{0 \leq t \leq T}\) satisfies Fokker-Planck (FP) equation
  \[
  \partial_t \mu_t = -\text{div}(b(x, \mu_t, \alpha^*(x, \mu_t, \partial_x u(t, x))\mu_t) + \frac{1}{2} \partial_{xx} \mu_t
  
  b^*(t, x)
  \]
Fokker-Planck

- Need for a PDE characterization of $(\mathcal{L}(X_t^{*,\mu}))_t$

- Dynamics of $X^{*,\mu}$ at equilibrium

$$dX_t^{*,\mu} = b(X_t^{*,\mu}, \mu_t, \alpha^*(X_t^{*,\mu}, \mu_t, \partial_x u(t, X_t^{*,\mu})))dt + dW_t$$

- Law $(X_t^{*,\mu})_{0 \leq t \leq T}$ satisfies Fokker-Planck (FP) equation

$$\partial_t \mu_t = -\text{div}(b(x, \mu_t, \alpha^*(x, \mu_t, \partial_x u(t, x)))\mu_t) + \frac{1}{2} \partial_{xx} \mu_t + b^*(t, x)$$

- MFG equilibrium described by forward-backward in $\infty$ dimension

  Fokker-Planck (forward)
  HJB (backward)

- $\infty$ dimensional analogue of

$$\dot{x}_t = b(x_t, y_t)dt, \quad x_0 = x^0$$
$$\dot{y}_t = -f(x_t, y_t)dt, \quad y_T = g(x_T)$$
Optimal control and FBSDEs

- Environment \((\mu_t)_{0 \leq t \leq T}\) is fixed and cost functional of the type

\[
J(\alpha) = \mathbb{E}\left[ g(X_T, \mu_T) + \int_0^T f(X_t, \mu_t, \alpha_t) dt \right]
\]

○ assume \(f\) and \(g\) continuous and at most of quadratic growth

◦ \(\sigma\) invertible, \(H\) strict convex in \(\alpha\) and coeff. bounded in \(x\) ⇒ \(G, F = (\partial_x g, \partial_x H)\) \(\sigma\) independent of \(x\) ⇒ represent gradient value function!

◦ choose \((\mu_t)_{0 \leq t \leq T}\) as the law of optimal path! ⇒ characterized by FBSDE of McKean-Vlasov type
Optimal control and FBSDEs

• Environment $(\mu_t)_{0 \leq t \leq T}$ is fixed and cost functional of the type

$$J(\alpha) = \mathbb{E}\left[g(X_T, \mu_T) + \int_0^T f(X_t, \mu_t, \alpha_t)dt\right]$$

○ assume $f$ and $g$ continuous and at most of quadratic growth

• Interpret optimal paths as the forward component of an FBSDE $\rightsquigarrow$

On $(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathcal{F}$ generated by $(\xi, (W_t)_{0 \leq t \leq T})$

$$X_t = X_0 + \int_0^t b(X_s, \mu_s, Y_s, Z_s) \, ds + \int_0^t \sigma(X_s, \mu_s) \, dW_s$$

$$Y_t = G(X_T, \mu_T) + \int_t^T F(X_s, \mu_s, Y_s, Z_s) \, ds - \int_t^T Z_s \, dW_s$$
Optimal control and FBSDEs

- Environment $(\mu_t)_{0 \leq t \leq T}$ is fixed and cost functional of the type

$$J(\alpha) = \mathbb{E}\left[ g(X_T, \mu_T) + \int_0^T f(X_t, \mu_t, \alpha_t)dt \right]$$

  - assume $f$ and $g$ continuous and at most of quadratic growth

- Interpret optimal paths as the forward component of an FBSDE $\leadsto$ On $(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathcal{F}$ generated by $(\xi, (W_t)_{0 \leq t \leq T})$

  $X_t = X_0 + \int_0^t b(X_s, \mu_s, Y_s, Z_s) \, ds + \int_0^t \sigma(X_s, \mu_s) \, dW_s$

  $Y_t = G(X_T, \mu_T) + \int_t^T F(X_s, \mu_s, Y_s, Z_s) \, ds - \int_t^T Z_s \, dW_s$

  - $\sigma$ invertible, $H$ strict convex in $\alpha$ and coeff. bounded in $x \Rightarrow ((G, F) = (g, f)) \Rightarrow$ represent value function!
Optimal control and FBSDEs

• Environment \((\mu_t)_{0 \leq t \leq T}\) is fixed and cost functional of the type

\[
J(\alpha) = \mathbb{E}\left[ g(X_T, \mu_T) + \int_0^T f(X_t, \mu_t, \alpha_t) dt \right]
\]

○ assume \(f\) and \(g\) continuous and at most of quadratic growth

• Interpret optimal paths as the forward component of an FBSDE \(\Rightarrow\) On \((\Omega, \mathbb{F}, \mathbb{P})\) with \(\mathbb{F}\) generated by \((\xi, (W_t)_{0 \leq t \leq T})\)

\[
X_t = X_0 + \int_0^t b\left(X_s, \mu_s, Y_s, Z_s\right) ds + \int_0^t \sigma(X_s, \mu_s) dW_s
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Y_t = G(X_T, \mu_T) + \int_t^T F\left(X_s, \mu_s, Y_s, Z_s\right) ds - \int_t^T Z_s dW_s
\]

○ \(\sigma\) invertible, \(H\) strict convex in \(\alpha\) and coeff. bounded in \(x\) \(\Rightarrow\)

\(((G, F) = (g, f)) \Rightarrow \) represent value function!

○ \(H\) strict convex in \((x, \alpha) \Rightarrow\) Pontryagin! \(((G, F) = (\partial_x g, \partial_x H))\) \(\sigma\)

indep. of \(x\) \(\Rightarrow\) represent gradient value function!
Optimal control and FBSDEs

• Environment \((\mu_t)_{0 \leq t \leq T}\) is fixed and cost functional of the type

\[
J(\alpha) = \mathbb{E}\left[ g(X_T, \mu_T) + \int_0^T f(X_t, \mu_t, \alpha_t) dt \right]
\]

○ assume \(f\) and \(g\) continuous and at most of quadratic growth

• Interpret optimal paths as the forward component of an FBSDE \(\rightsquigarrow\)
On \((\Omega, \mathcal{F}, \mathbb{P})\) with \(\mathcal{F}\) generated by \((\xi, (W_t)_{0 \leq t \leq T})\)

\[
X_t = X_0 + \int_0^t b(X_s, \mu_s, Y_s, Z_s) \, ds + \int_0^t \sigma(X_s, \mu_s) \, dW_s
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Y_t = G(X_T, \mu_T) + \int_t^T F(X_s, \mu_s, Y_s, Z_s) \, ds - \int_t^T Z_s \, dW_s
\]

○ \(\sigma\) invertible, \(H\) strict convex in \(\alpha\) and coeff. bounded in \(x\) \(\Rightarrow\) \(((G, F) = (g, f)) \Rightarrow\) represent value function!

○ \(H\) strict convex in \((x, \alpha)\) \(\Rightarrow\) Pontryagin! \(((G, F) = (\partial_x g, \partial_x H))\) (\(\sigma\) indep. of \(x\)) \(\Rightarrow\) represent gradient value function!

○ choose \((\mu_t)_{0 \leq t \leq T}\) as the law of optimal path! \(\Rightarrow\) characterize by FBSDE of McKean-Vlasov type
MKV FBSDE for the value function

- Consider, on \((\Omega, \mathcal{F}, \mathbb{P})\), the MKV FBSDE

\[
X_t = \xi + \int_0^t b(X_s, \mathcal{L}(X_s), \alpha^*(X_s, \mathcal{L}(X_s), Z_s\sigma^{-1}(X_s, \mathcal{L}(X_s)))) \, ds \\
+ \int_0^t \sigma(X_s, \mathcal{L}(X_s)) dW_s \\
Y_t = g(X_T, \mathcal{L}(X_T)) \\
+ \int_t^T f(X_s, \mathcal{L}(X_s), \alpha^*(X_s, \mathcal{L}(X_s), Z_s\sigma^{-1}(X_s, \mathcal{L}(X_s)))) \, ds - \int_t^T Z_s dW_s
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\]

- Connection with PDE formulation

\[
Y_s = u(s, X_s), \quad Z_s = \partial_x u(s, X_s)\sigma(X_s, \mu_s)
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MKV FBSDE for the value function

- Consider, on \((\Omega, F, P)\), the MKV FBSDE

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- Connection with PDE formulation

\[
Y_s = u(s, X_s), \quad Z_s = \partial_x u(s, X_s)\sigma(X_s, \mu_s)
\]

- Unique minimizer for each \((\mu_t)_{0 \leq t \leq T}\) if
  - \(b, f, g, \sigma, \sigma^{-1}\) bounded in \((x, \mu)\), Lipschitz in \(x\)
  - \(b\) linear in \(\alpha\) and \(f\) strictly convex and loc. Lip in \(\alpha\), with \(\text{Lip}(f)\) at most of linear growth in \(\alpha\)
MKV FBSDE for the Pontryagin principle

- Consider, on \((\Omega, \mathcal{F}, \mathbb{P})\), the MKV FBSDE

\[
X_t = \xi + \int_0^t b(X_s, \mathcal{L}(X_s), \alpha^*(X_s, \mathcal{L}(X_s), Y_s)) \, ds + \int_0^t \sigma(\mathcal{L}(X_s))dW_s
\]

\[
Y_t = \partial_x g(X_T, \mathcal{L}(X_T)) + \int_t^T \partial_x H(X_s, \mathcal{L}(X_s), \alpha^*(X_s, \mathcal{L}(X_s), Y_s), Y_s) \, ds - \int_t^T Z_s dW_s
\]
MKV FBSDE for the Pontryagin principle

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X_t = \xi + \int_0^t b(X_s, \mathcal{L}(X_s), \alpha^*(X_s, \mathcal{L}(X_s), Y_s)) \, ds + \int_0^t \sigma(\mathcal{L}(X_s)) \, dW_s
\]

\[
Y_t = \partial_x g(X_T, \mathcal{L}(X_T))
\]

\[
+ \int_t^T \partial_x H(X_s, \mathcal{L}(X_s), \alpha^*(X_s, \mathcal{L}(X_s), Y_s), Y_s) \, ds - \int_t^T Z_s dW_s
\]

• Connection with PDE formulation

\[
Y_s = \partial_x u(s, X_s), \quad Z_s = \partial_x^2 u(s, X_s) \sigma(\mu_s)
\]
MKV FBSDE for the Pontryagin principle

- Consider, on \((\Omega, \mathcal{F}, \mathbb{P})\), the MKV FBSDE

\[
X_t = \xi + \int_0^t b(X_s, \mathcal{L}(X_s), \alpha^*(X_s, \mathcal{L}(X_s), Y_s)) \, ds + \int_0^t \sigma(\mathcal{L}(X_s)) \, dW_s
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- Connection with PDE formulation

\[
Y_s = \partial_x u(s, X_s), \quad Z_s = \partial_x^2 u(s, X_s) \sigma(\mu_s)
\]

- Unique minimizer for each \((\mu_t)_{0 \leq t \leq T}\) if
  
  - \(\sigma\) indep. of \(x\) and \(b(x, \mu, \alpha) = b_0(\mu) + b_1 x + b_2 \alpha\)
  
  - \(\partial_x f, \partial_\alpha f, \partial_x g\) \(L\)-Lipschitz in \((x, \alpha)\)
  
  - \(g\) and \(f\) convex in \((x, \alpha)\) with \(f\) strict convex in \(\alpha\)
Seeking a solution

- Any way $\leadsto$ two-point-boundary-problem $\Rightarrow$
  - Cauchy-Lipschitz theory in small time only
  - if Lipschitz coefficients (including the direction of the measure)
    $\leadsto$ existence and uniqueness in short time (see later on)
    $\leadsto$ existence and uniqueness of MFG equilibria in small time
Seeking a solution

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    $\leadsto$ existence and uniqueness of MFG equilibria in small time

- What about arbitrary time?
  - existence $\leadsto$ fixed point over the measure argument by means of compactness arguments
    Schauder’s theorem
  - uniqueness $\leadsto$ require additional assumption
Seeking a solution

• Any way $\leadsto$ two-point-boundary-problem $\Rightarrow$
  
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  $\leadsto$ existence and uniqueness of MFG equilibria in small time

• What about arbitrary time?
  
  ◦ existence $\leadsto$ fixed point over the measure argument by means of compactness arguments

    Schauder’s theorem

  ◦ uniqueness $\leadsto$ require additional assumption

• Other question $\leadsto$ connection with social optimization?
  
  ◦ potential games $\leadsto$ MFG solution is also a social optimizer (but for other coefficients)
Part III. Solving MFG

a. Schauder fixed point theorem without common noise
Statement of the Schauder fixed point theorem

• Generalisation of Brouwer’s theorem from finite to infinite dimension

• Let \((V, \| \cdot \|)\) be a normed vector space
  - \(\emptyset \neq E \subset V\) with \(E\) closed and convex
  - \(\phi : E \to E\) continuous such that \(\phi(E)\) is relatively compact
  - \(\Rightarrow\) existence of a fixed point to \(\phi\)
Statement of the Schauder fixed point theorem

- Generalisation of Brouwer’s theorem from finite to infinite dimension
- Let \((V, \| \cdot \|)\) be a normed vector space
  - \(\emptyset \neq E \subset V\) with \(E\) closed and convex
  - \(\phi : E \to E\) continuous such that \(\phi(E)\) is relatively compact
  - \(\Rightarrow\) existence of a fixed point to \(\phi\)
- In MFG \(\sim\) what is \(V\), what is \(E\), what is \(\phi\)?
  - recall that MFG equilibrium is a flow of measures \((\mu_t)_{0 \leq t \leq T}\)
    \[ E \subset C([0, T], \mathcal{P}_2(\mathbb{R}^d)) \]
  - need to embed into a linear structure
    \[ C([0, T], \mathcal{P}_2(\mathbb{R}^d)) \subset C([0, T], \mathcal{M}_1(\mathbb{R}^d)) \]
  - \(\mathcal{M}_1(\mathbb{R}^d)\) set of signed measures \(\nu\) with \(\int_{\mathbb{R}^d} |x| d|\nu|(x) < \infty\)
Compactness on the space of probability measures

- Equip $\mathcal{M}_1(\mathbb{R}^d)$ with a norm $\| \cdot \|$ and restrict to $\mathcal{P}_1(\mathbb{R}^d)$ such that
  - convergence of $(\nu_n)_{n \geq 1}$ in $\mathcal{P}_1(\mathbb{R}^d)$ implies weak convergence
    \[
    \forall h \in C_b(\mathbb{R}^d, \mathbb{R}), \quad \lim_{n \to \infty} \int_{\mathbb{R}^d} h\nu_n = \int_{\mathbb{R}^d} h\nu
    \]
  - if $(\nu_n)_{n \geq 1}$ has uniformly bounded moments of order $p > 2$
    \[
    \text{Unif. square integrability } \Rightarrow W_2(\nu_n, \nu) \to 0
    \]
  - says that the input in the coefficients varies continuously!
    \[
    b(x, \nu_n, y, z), \quad \sigma(x, \nu_n), \quad F(x, \nu_n, y, z), \quad G(x, \nu_n)
    \]
Compactness on the space of probability measures

• Equip $\mathcal{M}_1(\mathbb{R}^d)$ with a norm $\| \cdot \|$ and restrict to $\mathcal{P}_1(\mathbb{R}^d)$ such that
  ○ convergence of $(\nu_n)_{n \geq 1}$ in $\mathcal{P}_1(\mathbb{R}^d)$ implies weak convergence
    $\forall h \in C_b(\mathbb{R}^d, \mathbb{R}), \quad \lim_{n \to \infty} \int_{\mathbb{R}^d} h d\nu_n = \int_{\mathbb{R}^d} h d\nu$
  ○ if $(\nu_n)_{n \geq 1}$ has uniformly bounded moments of order $p > 2$
    Unif. square integrability $\Rightarrow W_2(\nu_n, \nu) \to 0$
  ○ says that the input in the coefficients varies continuously!
    $b(x, \nu_n, y, z), \sigma(x, \nu_n), F(x, \nu_n, y, z), G(x, \nu_n)$

• Compactness $\implies$ if $(\nu_n)_{n \geq 1}$ has bounded moments of order $p > 2$
  ○ $(\nu_n)_{n \geq 1}$ admits a weakly convergent subsequence
  ○ then convergence for $W_2$ by unif. integrability and for $\| \cdot \|$ also
Application to MKV FBSDE

- Choose $E$ as continuous $(\mu_t)_{0 \leq t \leq T}$ from $[0, T]$ to $\mathcal{P}_2(\mathbb{R}^d)$

$$\sup_{0 \leq t \leq T} \int_{\mathbb{R}^d} |x|^4 d\mu_t(x) \leq K \quad \text{for some } K$$
Application to MKV FBSDE

- Choose $E$ as continuous $(\mu_t)_{0 \leq t \leq T}$ from $[0, T]$ to $\mathcal{P}_2(\mathbb{R}^d)$

\[
\sup_{0 \leq t \leq T} \int_{\mathbb{R}^d} |x|^4 d\mu_t(x) \leq K \quad \text{for some } K
\]

- Construct $\phi \sim \text{fix} (\mu_t)_{0 \leq t \leq T}$ in $E$ and solve

\[
X_t = \xi + \int_0^t b(X_s, \mu_s, Y_s, Z_s) + \int_0^t \sigma(X_s, \mu_s) dW_s
\]

\[
Y_t = G(X_T, \mu_T) + \int_t^T F(X_s, \mu_s, Y_s, Z_s) \, ds - \int_t^T Z_s dW_s
\]

\[
\circ \text{ let } \phi(\mu = (\mu_t)_{0 \leq t \leq T}) = (\mathcal{L}(X^\mu_t))_{0 \leq t \leq T}
\]
Application to MKV FBSDE

- Choose $E$ as continuous $(\mu_t)_{0 \leq t \leq T}$ from $[0, T]$ to $\mathcal{P}_2(\mathbb{R}^d)$

\[
\sup_{0 \leq t \leq T} \int_{\mathbb{R}^d} |x|^4 \, d\mu_t(x) \leq K \quad \text{for some } K
\]

- Construct $\phi \leadsto$ fix $(\mu_t)_{0 \leq t \leq T}$ in $E$ and solve

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X_t = \xi + \int_0^t b(X_s, \mu_s, Y_s, Z_s) + \int_0^t \sigma(X_s, \mu_s) \, dW_s
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Y_t = G(X_T, \mu_T) + \int_t^T F(X_s, \mu_s, Y_s, Z_s) \, ds - \int_t^T Z_s \, dW_s
\]

- Assume bounded coefficients and $\mathbb{E}[|\xi|^4] < \infty$

  \[
  \text{choose } K \text{ such that } \mathbb{E}[|X_t^\mu|^4] \leq K
  \]

\[
\Rightarrow \text{E stable by } \phi
\]

- \[
W_2(\mathcal{L}(X_t^\mu), \mathcal{L}(X_s^\mu)) \leq C \mathbb{E}[|X_t^\mu - X_s^\mu|^2]^{1/2} \leq C|t - s|^{1/2}
\]
Conclusion

- Consider continuous $\mu = (\mu_t)_{0 \leq t \leq T}$ from $[0, T]$ to $\mathcal{P}_2(\mathbb{R}^d)$
  - for any $t \mapsto (\phi(\mu))_t$ in a compact subset of $\mathcal{P}_2(\mathbb{R}^d)$
  - $[0, T] \ni t \mapsto (\phi(\mu))_t$ is uniformly continuous in $\mu$
  - by Arzelà-Ascoli $\Rightarrow$ output lives in a compact subset of $E \subset C([0, T], \mathcal{P}_2(\mathbb{R}^d))$ (and thus of $C([0, T], M_1(\mathbb{R}^d))$

- Continuity of $\phi$ on $E \mapsto$ stability of the solution of FBSDEs with respect to a continuous perturbation of the environment
Conclusion

• Consider continuous $\mu = (\mu_t)_{0\leq t\leq T}$ from $[0, T]$ to $\mathcal{P}_2(\mathbb{R}^d)$
  
  ○ for any $t \mapsto (\phi(\mu))_t$ in a compact subset of $\mathcal{P}_2(\mathbb{R}^d)$
  
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• Continuity of $\phi$ on $E \mapsto$ stability of the solution of FBSDEs with respect to a continuous perturbation of the environment

• Refinements to allow for unbounded coefficients

  ○ for the Value-Function FBSDE $\mapsto b$ linear in $\alpha$, $f$ strictly convex in $\alpha$, with derivatives in $\alpha$ at most of linear growth in $\alpha$

  ○ Pontryagin principle $\mapsto b$ linear in $(x, \alpha)$ and $f$ convex in $(x, \alpha)$ with derivatives at most of linear growth with weak-mean reverting conditions

  $\langle x, \partial_x f(0, \delta_x, 0) \rangle \geq -c(1 + |x|)$ and $\langle x, \partial_x g(0, \delta_x) \rangle \geq -c(1 + |x|)$
**Linear-quadratic in** $d = 1$

- **Apply previous results** with
  - $b(t, x, \mu, \alpha) = a_t x + a'_t \mathbb{E}(\mu) + b_t \alpha$
  - $g(x, \mu) = \frac{1}{2}[qx + q'\mathbb{E}(\mu)]^2 \iff (\text{mean-reverting}) \; qq' \geq 0$
  - $f(t, x, \mu, \alpha) = \frac{1}{2}[\alpha^2 + (m_t x + m'_t \mathbb{E}(\mu))^2] \iff (\text{mean-rev.}) \; m_t m'_t \geq 0$
Linear-quadratic in $d = 1$

- **Apply previous results** with
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- **Compare with direct method** $\leadsto$ Pontryagin

\[
\begin{align*}
  dX_t &= [a_t X_t + a'_t \mathbb{E}(X_t) - b_t^2 Y_t] dt + \sigma dW_t \\
  dY_t &= -[a_t Y_t + m_t (m_t X_t + m'_t \mathbb{E}(X_t))] dt + Z_t dW_t \\
  Y_T &= q[qX_T + q'\mathbb{E}(X_T)]
\end{align*}
\]

- take the mean

\[
\begin{align*}
  d\mathbb{E}(X_t) &= [(a_t + a'_t)\mathbb{E}(X_t) - b_t^2 \mathbb{E}(Y_t)] dt \\
  d\mathbb{E}(Y_t) &= -[a_t \mathbb{E}(Y_t) + m_t (m_t + m'_t) \mathbb{E}(X_t)] dt \\
  \mathbb{E}(Y_T) &= q(q + q')\mathbb{E}(X_T)
\end{align*}
\]
Linear-quadratic in $d = 1$

- **Apply previous results** with
  - $b(t, x, \mu, \alpha) = a_t x + a_t' \mathbb{E}(\mu) + b_t \alpha_t$
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  \mathbb{E}(Y_T) &= q(q + q') \mathbb{E}(X_T)
\end{align*}
\]

- existence and uniqueness if $q(q + q') \geq 0, m_t (m_t + m_t') \geq 0$
Part III. Solving MFG

b. Uniqueness criterion
A counter-example to uniqueness

- Consider the MKV FBSDE

\[
\begin{align*}
    dX_t &= b(\mathbb{E}(Y_t))dt + dW_t, \quad X_0 = x_0 \\
    dY_t &= -f(\mathbb{E}(X_t))dt + Z_t dW_t, \quad Y_T = g(\mathbb{E}(X_T))
\end{align*}
\]

- take bounded and Lipschitz coefficients \(\leadsto\) existence of a solution
- uniqueness may not hold!
- completely different of the system with \(b(Y_t), f(X_t)\) and \(g(X_T)\) for which uniqueness holds true!
A counter-example to uniqueness

- Consider the MKV FBSDE

\[ dX_t = b(\mathbb{E}(Y_t))dt + dW_t, \quad X_0 = x_0 \]
\[ dY_t = -f(\mathbb{E}(X_t))dt + Z_t dW_t, \quad Y_T = g(\mathbb{E}(X_T)) \]

- take bounded and Lipschitz coefficients \( \rightsquigarrow \) existence of a solution

- uniqueness may not hold!

- completely different of the system with \( b(Y_t), f(X_t) \) and \( g(X_T) \) for which uniqueness holds true!

- Proof \( \rightsquigarrow \) take the mean

\[ d\mathbb{E}(X_t) = b(\mathbb{E}(Y_t))dt, \quad \mathbb{E}(X_0) = x_0 \]
\[ d\mathbb{E}(Y_t) = -f(\mathbb{E}(X_t))dt, \quad \mathbb{E}(Y_T) = g(\mathbb{E}(X_T)) \]

- led back to counter-example for FBSDE \( \rightsquigarrow \) choose \( b, f \) and \( g \) equal to the identity on a compact subset
Lasry Lions monotonicity condition

- Recall following FBSDE result
  - ∃! may hold for the Pontryagin system if convex $g$ and $H$
  - Convexity $\iff$ monotonicity of $\partial_x g$ and $\partial_x H$
  - What is monotonicity condition in the direction of the measure?
Lasry Lions monotonicity condition

• Recall following FBSDE result
  ◦ $\exists!$ may hold for the Pontryagin system if convex $g$ and $H$
  ◦ convexity $\iff$ monotonicity of $\partial_x g$ and $\partial_x H$
  ◦ what is monotonicity condition in the direction of the measure?

• Lasry Lions monotonicity condition
  ◦ $b, \sigma$ do not depend on $\mu$
  ◦ $f(x, \mu, \alpha) = f_0(x, \mu) + f_1(x, \alpha)$ ($\mu$ and $\alpha$ are separated)
  ◦ monotonicity property for $f_0$ and $g$ w.r.t. $\mu$
    \[ \int_{\mathbb{R}^d} (f_0(x, \mu) - f_0(x, \mu')) d(\mu - \mu')(x) \geq 0 \]
    \[ \int_{\mathbb{R}^d} (g(x, \mu) - g(x, \mu')) d(\mu - \mu')(x) \geq 0 \]
Lasry Lions monotonicity condition

• Recall following FBSDE result
  ○ ∃! may hold for the Pontryagin system if convex $g$ and $H$
  ○ convexity $\iff$ monotonicity of $\partial_x g$ and $\partial_x H$
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• Lasry Lions monotonicity condition
  ○ $b, \sigma$ do not depend on $\mu$
  ○ $f(x, \mu, \alpha) = f_0(x, \mu) + f_1(x, \alpha)$ ($\mu$ and $\alpha$ are separated)
  ○ monotonicity property for $f_0$ and $g$ w.r.t. $\mu$
    
    $$
    \int_{\mathbb{R}^d} \left( f_0(x, \mu) - f_0(x, \mu') \right) d(\mu - \mu')(x) \geq 0
    $$
    
    $$
    \int_{\mathbb{R}^d} \left( g(x, \mu) - g(x, \mu') \right) d(\mu - \mu')(x) \geq 0
    $$

• Example: $h(x, \mu) = \int_{\mathbb{R}^d} L(z, \rho \ast \mu(z))\rho(x - z)dz$ where $L$ is $\nearrow$ in second variable and $\rho$ is even
Monotonicity restores uniqueness

- Assume that for any input $\mu = (\mu_t)_{0 \leq t \leq T}$ unique optimal control $\alpha^* \cdot \mu$
  - + existence of an MFG for a given initial condition
Monotonicity restores uniqueness

• Assume that for any input $\mu = (\mu_t)_{0 \leq t \leq T}$ unique optimal control $\alpha^\star \mu$
  ◦ + existence of an MFG for a given initial condition

• Lasry Lions $\Rightarrow$ uniqueness of MFG equilibrium!
Monotonicity restores uniqueness

- Assume that for any input $\mu = (\mu_t)_{0 \leq t \leq T}$ unique optimal control $\alpha^* \cdot \mu$
  - + existence of an MFG for a given initial condition
- Lasry Lions $\Rightarrow$ uniqueness of MFG equilibrium!
  - if two different equilibria $\mu$ and $\mu' \mapsto \alpha^* \cdot \mu \neq \alpha^* \cdot \mu'$

\[
J^\mu(\alpha^* \cdot \mu) < J^\mu(\alpha^* \cdot \mu') \quad \text{and} \quad J^{\mu'}(\alpha^* \cdot \mu') < J^{\mu'}(\alpha^* \cdot \mu)
\]
- cost under $\mu$
  - cost under $\mu'$

\[
\mathbb{E} \left[ g(X^\star, \mu_T, \mu'_T) - g(X^\star, \mu_T, \mu_T) \right] - \left[ g(X^\star, \mu'_T, \mu'_T) - g(X^\star, \mu_T, \mu_T) \right] > 0
\]
- same for $f_0$ $\Rightarrow$ LHS must be $\leq 0$
Monotonicity restores uniqueness

- Assume that for any input $\mu = (\mu_t)_{0 \leq t \leq T}$ unique optimal control $\alpha^{*\mu}$
  - + existence of an MFG for a given initial condition

- Lasry Lions $\Rightarrow$ uniqueness of MFG equilibrium!
  - if two different equilibria $\mu$ and $\mu' \sim \alpha^{*\mu} \neq \alpha^{*\mu'}$

$$J^{\mu}(\alpha^{*\mu}) < J^{\mu}(\alpha^{*\mu'}) \quad \text{and} \quad J^{\mu'}(\alpha^{*\mu'}) < J^{\mu'}(\alpha^{*\mu})$$

so that

$$J^{\mu'}(\alpha^{*\mu}) - J^{\mu'}(\alpha^{*\mu'}) + J^{\mu}(\alpha^{*\mu'}) - J^{\mu}(\alpha^{*\mu}) > 0$$

$$J^{\mu'}(\alpha^{*\mu}) - J^{\mu}(\alpha^{*\mu}) - [J^{\mu'}(\alpha^{*\mu'}) - J^{\mu}(\alpha^{*\mu'})] > 0$$
Monotonicity restores uniqueness

- Assume that for any input $\mu = (\mu_t)_{0 \leq t \leq T}$ unique optimal control $\alpha^* \cdot \mu$
  - + existence of an MFG for a given initial condition
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\]

so that

\[
J^{\mu'}(\alpha^* \cdot \mu) - J^{\mu'}(\alpha^* \cdot \mu') + J^{\mu}(\alpha^* \cdot \mu') - J^{\mu}(\alpha^* \cdot \mu) > 0 \\
J^{\mu'}(\alpha^* \cdot \mu) - J^{\mu}(\alpha^* \cdot \mu) - [J^{\mu'}(\alpha^* \cdot \mu') - J^{\mu}(\alpha^* \cdot \mu')] > 0
\]

\[
E \left[ g(X_T^*, \mu'_T) - g(X_T^*, \mu_T) - (g(X_T^*, \mu'_T) - g(X_T^*, \mu_T)) + \ldots \right] > 0
\]

\[
\int_{\mathbb{R}^d} (g(x, \mu'_T) - g(x, \mu_T))d\mu_T(x) \quad \int_{\mathbb{R}^d} (g(x, \mu'_T) - g(x, \mu_T))d\mu'_T(x)
\]

- same for $f_0 \Rightarrow$ LHS must be $\leq 0$
Part IV. Solving MFG with a Common Noise

a. Formulation
MFG with a common noise

- Mean field game with common noise $B$
  - asymptotic formulation for a finite player game with
    \[
    dX^i_t = b(X^i_t, \bar{\mu}^N_t, \alpha^i_t)dt + \sigma(X^i_t, \bar{\mu}^N_t)dW^i_t + \sigma^0(X^i_t, \bar{\mu}^N_t)dB_t
    \]
  - uncontrolled version $\sim$ asymptotic SDE with $\bar{\mu}^N_t$ replaced by $\mathcal{L}(X_t|(B_s)_{0\leq s\leq T}) = \mathcal{L}(X_t|(B_s)_{0\leq s\leq t})$
  - particles become independent conditional on $B$ and converge to the solution
    \[
    dX_t = b(X_t, \mathcal{L}(X|B))dt + \sigma(X_t, \mathcal{L}(X|B))dW_t + \sigma^0(X_t, \mathcal{L}(X|B))dB_t
    \]
MFG with a common noise

• Mean field game with common noise $B$
  
  o asymptotic formulation for a finite player game with $A = \mathbb{R}^k$ and
    \[ dX_t^i = \left( b(X_t^i, \bar{\mu}_t^N) + \alpha_t^i \right) dt + \sigma dW_t^i + \eta dB_t \]

  o uncontrolled version $\sim \bar{\mu}_t^N$ replaced by $\mathcal{L}(X_t|B)$

• Equilibrium as a fixed point $\sim$ time $[0, T]$, state in $\mathbb{R}^d$
  
  o candidate $\sim (\mu_t)_{t \in [0, T]} \mathbb{F}^B$ prog-meas with values in space of probability measures with a finite second moment $\mathcal{P}_2(\mathbb{R}^d)$

  o representative player with control $\alpha$
    \[ dX_t = (b(X_t, \mu_t) + \alpha_t) dt + \sigma dW_t + \eta dB_t \]
    \[ \sim X_0 \sim \mu_0, \sigma, \eta \in \{0, 1\}, W \text{ and } B \mathbb{R}^d\text{-valued \perp B.M.} \]

  o cost functional $J(\alpha) = \mathbb{E} \left[ g(X_T, \mu_T) + \int_0^T (f(X_t, \mu_t) + \frac{1}{2}|\alpha_t|^2) dt \right]$

  o find $(\mu_t)_{t \in [0, T]}$ such that $\mu_t = \mathcal{L}(X_t^{\text{optimal}} | (B_s)_{0 \leq s \leq T})$
MFG with a common noise

- Mean field game with common noise $B$
  - asymptotic formulation for a finite player game with
    \[ dX^i_t = \left( b(X^i_t, \bar{\mu}^N_t) + \alpha^i_t \right) dt + \sigma dW^i_t + \eta dB_t \]
  - uncontrolled version $\sim \bar{\mu}^N_t$ replaced by $\mathcal{L}(X_t|B)$
- Equilibrium as a fixed point $\sim$ time $[0, T]$, state in $\mathbb{R}^d$
  - candidate $\sim (\mu_t)_{t \in [0,T]} \mathbb{F}^B$ prog-meas with values in space of probability measures with a finite second moment $\mathcal{P}_2(\mathbb{R}^d)$
  - representative player with control $\alpha$
    \[ dX^i_t = (b(X^i_t, \mu_t) + \alpha^i_t) dt + \sigma dW_t + \eta dB_t \]
    $\sim \Rightarrow X_0 \sim \mu_0$, $\sigma, \eta \in \{0, 1\}$, $W$ and $B$ $\mathbb{R}^d$-valued $\perp$ B.M.
  - cost functional $J(\alpha) = \mathbb{E}\left[ g(X_T, \mu_T) + \int_0^T \left( f(X_t, \mu_t) + \frac{1}{2}|\alpha_t|^2 \right) dt \right]$
  - find $(\mu_t)_{t \in [0,T]}$ such that $\mu_t = \mathcal{L}(X_t^{\text{optimal}}| (B_s)_{0 \leq s \leq t})$
Forward-backward formulation

- Forward-backward formulation must account for \((\mu_t)_{0 \leq t \leq T}\) random
  - systems of two forward-backward SPDEs [Carmona D, Cardaliaguet D Lasry Lions]
Forward-backward formulation

- Forward-backward formulation must account for $(\mu_t)_{0 \leq t \leq T}$ random
  - systems of two forward-backward SPDEs

  $\sim \Rightarrow$ one backward stochastic HJB equation [Peng]

  $d_t u(t, x) + \left( b(x, \mu_t) \cdot D_x u(t, x) + \frac{\sigma^2 + \eta^2}{2} \Delta_x u(t, x) + f(x, \mu_t) - \frac{1}{2} |D_x u(t, x)|^2 \right)$

  Laplace generator  standard Hamiltonian in HJB

  $+ \eta \text{div}[v(t, x)] dt - \eta v(t, x) \cdot dB_t = 0$

  Ito Wentzell cross term  backward term

  with boundary condition: $u(T, \cdot) = g(\cdot, \mu_T)$

  $\sim \Rightarrow$ one forward stochastic Fokker-Planck equation

  $d_t \mu_t = \left( -\text{div}(\mu_t[b(x, \mu_t) - D_x u(t, x)]) dt + \frac{\sigma^2 + \eta^2}{2} \text{trace}(\partial^2_{xx} \mu_t) \right) dt$

  $- \eta \text{div}(\mu_t dB_t)$
Forward-backward formulation

- Forward-backward formulation must account for \((\mu_t)_{0\leq t\leq T}\) random
  - systems of two forward-backward SPDEs
  - systems of two forward-backward McKV SDEs [Carmona D, Buckdahn (al.), Lacker]
Forward-backward formulation

- Forward-backward formulation must account for \((\mu_t)_{0 \leq t \leq T}\) random
  - systems of two forward-backward SPDEs
  - systems of two forward-backward McKV SDEs

\(\leadsto\) two ways: represent the value function or optimal control

- **Representation of the value function** \(\sigma = 1\)

\[
\begin{align*}
  dX_t &= b(X_t, \mathcal{L}(X_t|B))dt - Z_t dt + dW_t + \eta dB_t \\
  dY_t &= -f(X_t, \mathcal{L}(X_t|B))dt - \frac{1}{2}|Z_t|^2 dt + Z_t dW_t + \zeta_t dB_t \\
  Y_T &= g(X_T, \mathcal{L}(X_T|B))
\end{align*}
\]

- **Representation of the optimal control (Pontryagin)**

\[
\begin{align*}
  dX_t &= b(X_t, \mathcal{L}(X_t|B))dt - Y_t dt + \sigma dW_t + \eta dB_t \\
  dY_t &= -\partial_x H(X_t, \mathcal{L}(X_t|B), Y_t) dt + Z_t dW_t + \zeta_t dB_t \\
  Y_T &= \partial_x g(X_T, \mathcal{L}(X_T|B))
\end{align*}
\]
Forward-backward formulation

- Forward-backward formulation must account for \((\mu_t)_{0 \leq t \leq T}\) random
  - systems of two forward-backward SPDEs
  - systems of two forward-backward McKV SDEs

  \[\sim\] two ways: represent the value function or optimal control

- **Representation of the value function** \(\sigma = 1\)
  \[
  \begin{align*}
  dX_t &= b(X_t, \mathcal{L}(X_t|B))dt - Z_t dt + dW_t + \eta dB_t \\
  dY_t &= -f(X_t, \mathcal{L}(X_t|B))dt - \frac{1}{2}|Z_t|^2 dt + Z_t dW_t + \zeta_t dB_t \\
  Y_T &= g(X_T, \mathcal{L}(X_T|B))
  \end{align*}
  \]

- **Representation of the optimal control** (Pontryagin)
  \[
  \begin{align*}
  dX_t &= b(X_t, \mathcal{L}(X_t|B))dt - Y_t dt + \sigma dW_t + \eta dB_t \\
  dY_t &= -\partial_x H(X_t, \mathcal{L}(X_t|B), Y_t)dt + Z_t dW_t + \zeta_t dB_t \\
  Y_T &= \partial_x g(X_T, \mathcal{L}(X_T|B))
  \end{align*}
  \]

- Analysis of these equations?
Part IV. Solving MFG with a Common Noise

b. Strong solutions
Implementing Picard theorem

- Easiest way to construct solutions is to implement Picard theorem
  - shall see next how to make use of Schauder’s theorem
- Forward-backward system of McKean-Vlasov type

\[ dX_t = \left( b(X_t, \mathcal{L}(X_t|B)) - Z_t \right) dt + dW_t + \eta dB_t \]

\[ dY_t = -\left( f(X_t, \mathcal{L}(X_t|B)) + \frac{1}{2}|Z_t|^2 \right) dt + Z_t dW_t + \zeta_t dB_t \]

\[ Y_T = g(X_T, \mathcal{L}(X_T|B)) \]

- \( Z_t \) should be \( \partial_x u(t, X_t) \leadsto \) bounded and \( x \)-Lipschitz coefficients
  \( \Rightarrow \) \( L^\infty \) bound

\( \leadsto \) replace quadratic term by general bounded \( f \)
Implementing Picard theorem

• Easiest way to construct solutions is to implement Picard theorem
  ◦ shall see next how to make use of Schauder’s theorem

• Forward-backward system of McKean-Vlasov type

\[
\begin{align*}
  dX_t &= \left( b(X_t, \mathcal{L}(X_t|B)) - Z_t \right) dt + dW_t + \eta dB_t \\
  dY_t &= -f(X_t, \mathcal{L}(X_t|B), Z_t) dt + Z_t dW_t + \zeta t dB_t \\
  Y_T &= g(X_T, \mathcal{L}(X_T|B))
\end{align*}
\]

  ◦ Cauchy-Lipschitz theory in small time only!

• **Theorem** If $K$-Lipschitz coefficients $\Rightarrow \exists!$ for $T \leq c(K)$
  ◦ for any initial condition $X_0 \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; \mathbb{R}^d)$

• **Question** How to go further?
Decoupling field \((T \leq c(K))\)

• Recall non MKV case \(\exists U : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}\) such that

\[
Y_t = U(t, X_t) \iff U(t, x) = Y_t^{t,x} \quad \text{(with } X_t^{t,x} = x) \]

○ keep fact for extending solutions is to bound \(\text{Lip}_x(U)\)
Decoupling field \((T \leq c(K))\)

- Recall non MKV case \(\exists U : [0, T] \times \mathbb{R}^d \to \mathbb{R}\) such that
  \[ Y_t = U(t, X_t) \iff U(t, x) = Y_{t,x}^t \text{ (with } X_{t,x}^t = x) \]

- MKV setting \(\leadsto\) state variable is in \(\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)\)

  \(\leadsto\) need to construct \(U(t, x, \mu)\) \(t \in [0, T], x \in \mathbb{R}^d, \mu \in \mathcal{P}_2(\mathbb{R}^d)\)

- Two-step procedure [Crisan Chassagneux D, Buckdahn (al.)]
  - 1st step \(\leadsto\) MKV FBSDE with \(X_t \sim \mu, X_t \perp \perp (W, B)\)
    
    \[ dX_s = \left( b(X_s, \mathcal{L}(X_s|B)) - Z_s \right) ds + dW_s + \eta dB_s \]
    
    \[ dY_s = -f(X_s, \mathcal{L}(X_s|B), Z_s) ds + Z_s dW_s + \zeta_s dB_s, \quad Y_T = g(X_T, \mathcal{L}(X_T|B)) \]

    \(\leadsto\) \((\mathcal{L}(X_s|B))_{t \leq s \leq T}\) only depends on \(X_t\) through \(\mu\)
Decoupling field \((T \leq c(K))\)

- Recall non MKV case \(\exists U : [0, T] \times \mathbb{R}^d \to \mathbb{R}\) such that
  \[
  Y_t = U(t, X_t) \quad \Leftrightarrow \quad U(t, x) = Y_t^{t,x} \text{ (with } X_t^{t,x} = x)\]

- MKV setting \(\mapsto\) state variable is in \(\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)\)
  \(\mapsto\) need to construct \(U(t, x, \mu)\) \(t \in [0, T], x \in \mathbb{R}^d, \mu \in \mathcal{P}_2(\mathbb{R}^d)\)

- Two-step procedure [Crisan Chassagneux D, Buckdahn (al.)]
  - 1st step \(\mapsto\) MKV FBSDE with \(X_t \sim \mu, X_t \perp \perp (W, B)\)
    \[
    dX_s = \left( b(X_s, \mathcal{L}(X_s|B)) - Z_s \right)ds + dW_s + \eta dB_s
    
    dY_s = -f(X_s, \mathcal{L}(X_s|B), Z_s)ds + Z_s dW_s + \zeta_s dB_s, \quad Y_T = g(X_T, \mathcal{L}(X_T|B))
    
    - 2nd step \(\mapsto\) non-MKV FBSDE with \(x_t = x\) and 1st step input
    \[
    dx_s = \left( b(x_s, \mathcal{L}(X_s|B)) - z_s \right)ds + dW_s + \eta dB_s
    
    dy_s = -f(x_s, \mathcal{L}(X_s|B), z_s)dt + z_s dW_s + \varsigma_s dB_s, \quad y_T = g(x_T, \mathcal{L}(X_T|B))
    
    - let \(U(t, x, \mu) = y_t\) \(\Rightarrow Y_t = U(t, X_t, \mu) = U(t, X_t, \mathcal{L}(X_t|B))\)
Controlling the Lipschitz constant

- **Non-MKV setting** ⇒ may control the Lipschitz constant by monotonicity or ellipticity conditions
  
  ~⇒ start with **monotonicity** ⇒ $B$ has no role ⇒ simplify $\eta = 0$

- **Come back to cost structure** ⇒ **monotonicity** of $f$ (same with $g$)

  \[
  \int_{\mathbb{R}^d} [f(x, \mu) - f(x, \mu')]d(\mu - \mu')(x) \geq 0 \quad \text{[Lions]}
  \]

- **Theorem** [L, C C D, Cardaliaguet (al.)] If $b \equiv 0$, $f$ and $g$ bounded, monotone and Lipschitz ⇒ **bound on** $\text{Lip}_\mu U$ and $\exists!$ on any $[0, T]$
Controlling the Lipschitz constant

- Non-MKV setting \( \leadsto \) may control the Lipschitz constant by monotonicity or ellipticity conditions

\( \leadsto \) start with monotonicity \( \leadsto \) \( B \) has no role \( \Rightarrow \) simplify \( \eta = 0 \)

- Come back to cost structure \( \leadsto \) monotonicity of \( f \) (same with \( g \))

\[ \int_{\mathbb{R}^d} [f(x, \mu) - f(x, \mu')] d(\mu - \mu')(x) \geq 0 \quad \text{[Lions]} \]

- **Theorem** [L, C C D, Cardaliaguet (al.)] If \( b \equiv 0, f \) and \( g \) bounded, monotone and Lipschitz \( \Rightarrow \) bound on \( \text{Lip}_\mu U \) and \( \exists! \) on any \([0, T]\)

- **Strategy** Investigate derivative of the flow in \( L^2 \)

\( \leadsto \) for \( \xi, \chi \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; \mathbb{R}^d) \)

\[ (\partial_\chi X_\xi^s, \partial_\chi Y_\xi^s, \partial_\chi Z_\xi^s) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left( \underbrace{X_\xi^s + \varepsilon \chi - X_\xi^s}, \underbrace{Y_\xi^s + \varepsilon \chi - Y_\xi^s}, \underbrace{Z_\xi^s + \varepsilon \chi - Z_\xi^s} \right) \]

in \( \mathbb{E} \left[ \sup_{0 \leq s \leq T} |s| \right] \quad \text{in} \quad \mathbb{E} \left( \int_0^T |s| \, ds \right) \]

- provide a bound for \( (\partial_\chi X_\xi^s, \partial_\chi Y_\xi^s, \partial_\chi Z_\xi^s) \)
Derivative on the Wasserstein space

- Differentiation on $\mathcal{P}_2(\mathbb{R}^d)$ taken from Lions

- Consider $U : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$

**Lifted-version of $U$**

\[ \hat{U} : L^2(\Omega, \mathbb{P}; \mathbb{R}^d) \ni X \mapsto U(\text{Law}(X)) \]

- $U$ differentiable if $\hat{U}$ Fréchet differentiable
Derivative on the Wasserstein space

- Differentiation on $\mathcal{P}_2(\mathbb{R}^d)$ taken from Lions
- Consider $U : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$
- Lifted-version of $U$

$$
\hat{U} : L^2(\Omega, \mathbb{P}; \mathbb{R}^d) \ni X \mapsto U(\text{Law}(X))
$$

- $U$ differentiable if $\hat{U}$ Fréchet differentiable

- Differential of $U$

  - Fréchet derivative of $\hat{U}$ [see also Zhang (al.)]

$$
D\hat{U}(X) = \partial_\mu U(\mu)(X), \quad \partial_\mu U(\mu) : \mathbb{R}^d \ni v \mapsto \partial_\mu U(\mu)(v) \quad \mu = \mathcal{L}(X)
$$

- derivative of $U$ at $\mu \sim \partial_\mu U(\mu) \in L^2(\mathbb{R}^d, \mu; \mathbb{R}^d)$
Derivative on the Wasserstein space

- Differentiation on $\mathcal{P}_2(\mathbb{R}^d)$ taken from Lions
- Consider $U : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$
- Lifted-version of $U$

\[ \hat{U} : L^2(\Omega, \mathbb{P}; \mathbb{R}^d) \ni X \mapsto U(\text{Law}(X)) \]

○ $U$ differentiable if $\hat{U}$ Fréchet differentiable

- Differential of $U$

  ○ Fréchet derivative of $\hat{U}$ [see also Zhang (al.)]

\[ D\hat{U}(X) = \partial_\mu U(\mu)(X), \quad \partial_\mu U(\mu) : \mathbb{R}^d \ni v \mapsto \partial_\mu U(\mu)(v) \quad \mu = \mathcal{L}(X) \]

○ derivative of $U$ at $\mu \leadsto \partial_\mu U(\mu) \in L^2(\mathbb{R}^d, \mu; \mathbb{R}^d)$

- Finite dimensional projection

\[ \partial_{x_i} \left[ U\left(\frac{1}{N} \sum_{j=1}^{N} \delta_{x_j}\right)\right] = \frac{1}{N} \partial_\mu U\left(\frac{1}{N} \sum_{j=1}^{N} \delta_{x_j}\right)(x_i), \quad x_1, \ldots, x_N \in \mathbb{R}^d \]
Application to the coupled case \((b \equiv 0)\)

- Return to coupled case \(\leadsto\) estimate \(\partial_{\chi} Y_{0}^{\xi}\)

\[
\partial_{\chi} Y_{0}^{\xi} = \partial_{x} U(0, \xi, L(\xi)) \cdot \chi + \tilde{\Omega} \left[ \partial_{\mu} U(0, \xi, L(\xi))(\tilde{\xi}) \cdot \tilde{\chi} \right]
\]

\(\tilde{\Omega} = \text{copy space}\)

- \(\text{Lip}_{\mu}\) estimate on \(U \Leftrightarrow\) bound of \(\mathbb{E}[|\partial_{\mu} U(0, \xi, L(\xi))(\xi)|^{2}]^{1/2}\)
Application to the coupled case \( (b \equiv 0) \)

- Return to coupled case \( \rightsquigarrow \) estimate \( \partial Y^\xi_0 \)

\[
\partial Y^\xi_0 = \partial X U(0, \xi, \mathcal{L}(\xi)) \cdot \chi + \mathbb{E}[\partial U(0, \xi, \mathcal{L}(\xi))(\tilde{\xi}) \cdot \tilde{\chi}]
\]

\( \tilde{\Omega} = \text{copy space} \)

- **Lip** \( \mu \) estimate on \( U \Leftrightarrow \) bound of \( \mathbb{E}[|\partial U(0, \xi, \mathcal{L}(\xi))|^2]^{1/2} \)

- Estimate \( (\partial X_t)_t \) first \( \rightsquigarrow \) dynamics of \( (X_t)_t \) and \( (\partial X_t)_t \)

\[
dX_t = -\partial X U(t, X_t, \mathcal{L}(X_t))dt + dW_t
\]

\[
d\partial X_t = -\left(\partial^2 X U(t, X_t, \mathcal{L}(X_t))\right)\partial X_t
\]

\[
+ \mathbb{E}[\partial U(t, X_t, \mathcal{L}(X_t))(\tilde{X}_t) \partial \tilde{X}_t]dt
\]

- \( \partial^2 X U \) already estimated! (thanks to Laplace)
Application to the coupled case \((b \equiv 0)\)

- Return to \textbf{coupled case} \(\leadsto\) estimate \(\partial_{\chi} Y_0^\xi\)

\[
\partial_{\chi} Y_0^\xi = \partial_{\chi} U(0, \xi, L(\xi)) \cdot \chi + \mathbb{E}[\partial_\mu U(0, \xi, L(\xi))(\tilde{\xi}) \cdot \tilde{\chi}]
\]

\(\tilde{\Omega} = \text{copy space}\)

- \textbf{Lip}_\mu \text{ estimate} on \(U \Leftrightarrow\) bound of \(\mathbb{E}[|\partial_\mu U(0, \xi, L(\xi))(\xi)|^2]^{1/2}\)

- \textbf{Estimate} \((\partial_{\chi} X_t)_t\) first \(\leadsto\) dynamics of \((X_t)_t\) and \((\partial_{\chi} X_t)_t\)

\[
d\mathbb{E}[|\partial_{\chi} X_t|^2] = -2\mathbb{E}\left[\partial_{\chi} X_t \cdot (\partial_{xx} U(X_t, L(X_t))\partial_{\chi} X_t)\right]dt
\]

\[
- 2\mathbb{E}\mathbb{E}\left[\partial_{\chi} X_t \cdot (\partial_\mu (\partial_{x} U)(X_t, L(X_t))(\tilde{X}_t)\tilde{\partial}_{\chi} X_t)\right]dt
\]

- \(\partial_{xx} U\) already estimated! (thanks to Laplace)
Application to the coupled case \((b \equiv 0)\)

- Return to coupled case \(\sim\) estimate \(\partial_x Y_0^\xi\)

\[
\partial_x Y_0^\xi = \partial_x U(0, \xi, \mathcal{L}(\xi)) \cdot \chi + \mathbb{E}[\partial_\mu U(0, \xi, \mathcal{L}(\xi))(\tilde{\xi} \cdot \tilde{\chi})]
\]

\(\tilde{\Omega} = \text{copy space}\)

- \(\text{Lip}_\mu\) estimate on \(U \iff\) bound of \(\mathbb{E}[|\partial_\mu U(0, \xi, \mathcal{L}(\xi))(\xi)|^2]^{1/2}\)

- Estimate \((\partial_x X_t)_t\) first \(\sim\) dynamics of \((X_t)_t\) and \((\partial_x X_t)_t\)

\[
\begin{align*}
d\mathbb{E}[|\partial_x X_t|^2] &= -2\mathbb{E}\left[\partial_x X_t \cdot (\partial_{xx}^2 U(X_t, \mathcal{L}(X_t))\partial_x X_t)\right]dt \\
&- 2\mathbb{E}\tilde{\mathbb{E}}\left[\partial_x X_t \cdot (\partial_\mu (\partial_x U)(X_t, \mathcal{L}(X_t))(\tilde{X}_t)\overline{\partial_x X_t})\right]dt
\end{align*}
\]

- \(\partial_{xx}^2 U\) already estimated! (thanks to Laplace)

- Propagation of monotonicity

\[
\mathbb{E}\tilde{\mathbb{E}}\left[\partial_x X_t \cdot (\partial_x (\partial_\mu U)(t, X_t, \mathcal{L}(X_t))(\tilde{X}_t)\overline{\partial_x X_t})\right] \geq 0 \Rightarrow \mathbb{E}[|\partial_x X_T|^2] \leq C\mathbb{E}[|\chi|^2]
\]

- insert into the backward equation
Part IV. Solving MFG with a Common Noise

c. Weak solutions
Fixed point without uniqueness

• Solution by compactness argument (without monotonicity)
  ○ use of **Schauder’s fixed point theorem**

• Disentangle sources of noise \(\sim\) product probability space
  \[\Omega = \Omega^0 \times \Omega^1, \quad \mathcal{F} = \mathcal{F}^0 \otimes \mathcal{F}^1, \quad \mathcal{P} = \mathcal{P}^0 \otimes \mathcal{P}^1\]
  ○ \((\Omega^0, \mathcal{F}^0, \mathcal{P}^0) \sim \text{common noise } B; (\Omega^1, \mathcal{F}^1, \mathcal{P}^1) \sim \text{noise } W\)

• Fixed point \((\mu_t)_{0 \leq t \leq T}\) as \(\mathcal{F}^0\) prog. meas. process
  ○ \(\mathcal{F}^0 = \mathcal{F}^B\) and \(\mathcal{F}^1 = \mathcal{F}^W \Rightarrow \text{optimal path under } (\mu_t)_{0 \leq t \leq T}\) given by
  \[dX_t = \left( b(X_t, \mu_t) - Z_t \right) dt + dW_t + \eta dB_t\]
  \[dY_t = -\left( f(X_t, \mu_t) + \frac{1}{2}|Z_t|^2 \right) dt + Z_t dW_t + \zeta_t dB_t, \quad Y_T = g(X_T, \mu_T)\]

• Solve \(\mu_t(\omega^0) = \mathcal{L}(X^\text{optimal}_t | \mathcal{F}^0_T)(\omega^0)\) for \(t \in [0, T]\) and \(\omega^0 \in \Omega^0\)

  \(\sim\) fixed point in \(\left(C([0, T], \mathcal{P}_2(\mathbb{R}^d))\right)^{\Omega^0}\)
  ○ much too big space for tractable compactness \(\sim\) strategy is to discretize common noise
Discretization method [Carmona D Lacker]

- General principle \(\leadsto\) discretization of the fixed point
  - choice of the conditioning \(\leadsto\) canonical space for \((B_t)_{0\leq t\leq T}\)
  \(\leadsto\) \(\mathcal{L}(X_t | \mathcal{F}_T^0) = \mathcal{L}(X_t | (B_s)_{0\leq s\leq T})\)
    - \(\mathcal{L}(X_t | (B_s)_{0\leq s\leq T}) \leadsto \mathcal{L}(X_t | \text{process with finite support})\)
Discretization method [Carmona D Lacker]

- General principle \(\Rightarrow\) discretization of the fixed point
  - choice of the conditioning \(\Rightarrow\) canonical space for \((B_t)_{0 \leq t \leq T}\)
    \[\mathcal{L}(X_t | \mathcal{F}_t^0) = \mathcal{L}(X_t | (B_s)_{0 \leq s \leq T})\]
  - \(\mathcal{L}(X_t | (B_s)_{0 \leq s \leq T}) \Rightarrow \mathcal{L}(X_t | \text{process with finite support})\)

- Choice of the process with finite support
  - \(\Pi\) projection on spatial grid \(\{x_1, \ldots, x_P\} \subset \mathbb{R}^d\)
  - \(t_1, \ldots, t_N\) time mesh \(\subset [0, T]\)
  - \(\hat{B}_{t_i} = \Pi(B_{t_i})\)

- Conditioning
  - fixed point condition on \(\mathcal{L}(X_t | \hat{B}_{t_1}, \ldots, \hat{B}_{t_i})\) for \(t \in [t_i, t_{i+1}]\)
  - input \(\Rightarrow\) sequence of processes on each \([t_i, t_{i+1}]\) with values in \(\mathcal{P}_2(\mathbb{R}^d)\) and only depending on the realizations of \((\hat{B}_{t_1}, \ldots, \hat{B}_{t_i})\)

fixed point in \(\prod_{i=1}^N C([t_i, t_{i+1}]; \mathcal{P}_2(\mathbb{R}^d))^{iP}\)
Solution under discrete conditioning

• Solve FBSDE

\[
\begin{align*}
    dX_t &= \left( b(X_t, \mathcal{L}(X_t | \hat{B}_{t_1}, \ldots, \hat{B}_{t_i})) - Z_t \right) dt + dW_t + \eta dB_t \\
    dY_t &= -\left( f(X_t, \mathcal{L}(X_t | \hat{B}_{t_1}, \ldots, \hat{B}_{t_i})) + \frac{1}{2} |Z_t|^2 \right) dt + Z_t dW_t + \zeta_t dB_t \\
    Y_T &= g(X_T, \mathcal{L}(X_T | \hat{B}_{t_1}, \ldots, \hat{B}_{t_N}))
\end{align*}
\]

• Strategy for the fixed point

○ input \( \mu = (\mu^1, \ldots, \mu^N) \) with

\[
\mu^i \in C([t_i, t_{i+1}]; \mathcal{P}_2(\mathbb{R}^d))^{x_1, \ldots, x_P}_i
\]

○ \( \mu_t = \mu^i_t(\hat{B}_{t_1}, \ldots, \hat{B}_{t_i}) \)

○ output given by

\[
\{x_1, \ldots, x_P\}_i \ni (a_1, \ldots, a_i) \mapsto \mathcal{L}(X_t | \hat{B}_{t_1} = a_1, \ldots, \hat{B}_{t_i} = a_i)
\]

• Stability for FBSDEs \( \leadsto \) continuity w.r.t input + compactness for laws \( \Rightarrow \) Schauder
Passing to the limit

- **Convergent subsequence as** $N, P \to \infty$?
  - use Pontryagin’s principle to describe optimal paths

$$dX_t = b(X_t, \mathcal{L}(X_t | \hat{B}_{t_1}, \ldots, \hat{B}_{t_i}))dt - Z_t dt + dW_t + \eta dB_t$$

$$dZ_t = -\partial_x H(X_t, \mathcal{L}(X_t | \hat{B}_{t_1}, \ldots, \hat{B}_{t_i}), Z_t)dt + dM_t$$

$$Z_T = \partial_x g(X_T, \mathcal{L}(X_T | \hat{B}_{t_1}, \ldots, \hat{B}_{t_N}))$$

$\leadsto$ $(M_t)_t$ martingale, $\mu_t = \mathcal{L}(X_t | \hat{B}_{t_1}, \ldots, \hat{B}_{t_i})$
Passing to the limit

- **Convergent subsequence as** $N, P \to \infty$?
  - use Pontryagin’s principle to describe optimal paths
    
    $dX_t = b(X_t, \mathcal{L}(X_t | \hat{B}_{t_1}, \ldots, \hat{B}_{t_i}))dt - Z_t dt + dW_t + \eta dB_t$
    
    $dZ_t = -\partial_x H(X_t, \mathcal{L}(X_t | \hat{B}_{t_1}, \ldots, \hat{B}_{t_i}), Z_t)dt + dM_t$
    
    $Z_T = \partial_x g(X_T, \mathcal{L}(X_T | \hat{B}_{t_1}, \ldots, \hat{B}_{t_N}))$

    $\leadsto (M_t)_t$ martingale, $\mu_t = \mathcal{L}(X_t | \hat{B}_{t_1}, \ldots, \hat{B}_{t_i})$

- **Tightness** of the laws of $(X_{t}^{N,P}, \mu_{t}^{N,P}, Z_{t}^{N,P}, M_{t}^{N,P}, B_{t}, W_{t})_{0 \leq t \leq T}$
  - tightness of $(X_{t}^{N,P})_{0 \leq t \leq T}$ in $C([0, T]; \mathbb{R}^d)$ by Kolmogorov
  - tightness of $(\mu_{t}^{N,P})_{0 \leq t \leq T}$ in $C([0, T]; \mathcal{P}_2(\mathbb{R}^d))$ since
    
    $\int_{\mathbb{R}^d} |x|^q d\mu_{t}^{N,P}(x) = \mathbb{E}[|X_{t}^{N,P}|^q | \mathcal{F}_T^0], \quad W_2(\mu_{t}^{N,P}, \mu_{s}^{N,P})^2 \leq \mathbb{E}[|X_{t}^{N,P} - X_{s}^{N,P}|^2 | \mathcal{F}_T^0]$
  
  - tightness $(Z_{t}^{N,P}, M_{t}^{N,P})_{0 \leq t \leq T}$ in $\mathcal{D}([0, T]; \mathbb{R}^d)$ with Meyer-Zheng
    
    $\leadsto (z_{t}^n)_{0 \leq t \leq T} \to (z_{t})_{0 \leq t \leq T}$ in $dt$-measure [Pardoux] for use in BSDE
Passing to the limit

- **Convergent subsequence as** $N, P \to \infty$?

  - use Pontryagin’s principle to describe optimal paths

\[
\begin{align*}
    dX_t &= b(X_t, \mathcal{L}(X_t | \hat{B}_{t_1}, \ldots, \hat{B}_{t_i})) dt - Z_t dt + dW_t + \eta dB_t \\
    dZ_t &= -\partial_x H(X_t, \mathcal{L}(X_t | \hat{B}_{t_1}, \ldots, \hat{B}_{t_i}), Z_t) dt + dM_t \\
    Z_T &= \partial_x g(X_T, \mathcal{L}(X_T | \hat{B}_{t_1}, \ldots, \hat{B}_{t_N}))
\end{align*}
\]

$\leadsto (M_t)_t$ martingale, $\mu_t = \mathcal{L}(X_t | \hat{B}_{t_1}, \ldots, \hat{B}_{t_i})$.

- **Tightness** of the laws of $(X_t^N, P, \mu_t^N, Z_t^N, M_t^N, B_t, W_t)_{0 \leq t \leq T}$

- **Limit process** $(X_t^\infty, \mu_t^\infty, Z_t^\infty, M_t^\infty, B_t^\infty, W_t^\infty)_{0 \leq t \leq T}$

  - identify $\leadsto \mu_t^\infty$ as conditional law of $X_t^\infty$ given information?

  $\leadsto$ pass to the limit in $\mu_t^N, P = \mathcal{L}(X_t^N, P | \hat{B}_{t_1}^N, \ldots, \hat{B}_{t_i}^N)$

  - solve optimization problem in environment $(\mu_t^\infty)_{0 \leq t \leq T}$?

  $\leadsto$ main difficulty $\leadsto$ loss of measurability of $\mu_t^\infty$ w.r.t $(B_s^\infty)_{0 \leq s \leq t} \Rightarrow$ weak solution only!
Strong vs. weak solutions

- Limiting FBSDE formulation

\[ dX_t^\infty = \left( b(X_t^\infty, \mu_t^\infty) - Z_t^\infty \right) dt + dW_t^\infty + \eta dB_t^\infty \]
\[ dZ_t^\infty = -\partial_x H(X_t^\infty, \mu_t^\infty, Z_t^\infty) dt + dM_t^\infty, \quad Z_T^\infty = \partial_x g(X_T^\infty, \mu_T^\infty) \]

\[ \leadsto \text{necessary condition for optimality only, but not a limitation} \]
\[ \leadsto \text{may pass to the limit in the optimality condition} \]

- \[ \text{cost } J(-Z^\infty) = \mathbb{E} \left[ g(X_T^\infty, \mu_T^\infty) + \int_0^T \left( f(X_t^\infty, \mu_t^\infty) + \frac{1}{2}|Z_t^\infty|^2 \right) dt \right] \]
Strong vs. weak solutions

- Limiting FBSDE formulation

\[
\begin{align*}
    dX_t^\infty &= \left( b(X_t^\infty, \mu_t^\infty) - Z_t^\infty \right) dt + dW_t^\infty + \eta dB_t^\infty \\
    dZ_t^\infty &= -\partial_x H(X_t^\infty, \mu_t^\infty, Z_t^\infty) dt + dM_t^\infty, \quad Z_T^\infty = \partial_x g(X_T^\infty, \mu_T^\infty)
\end{align*}
\]

\[\mapsto\text{necessary condition for optimality only, but not a limitation}\]
\[\mapsto\text{may pass to the limit in the optimality condition}\]

- Main question: What is the common information?

  - whole information \[\mapsto\mathcal{F}^\infty\text{ generated by } (X^\infty, \mu^\infty, B^\infty, W^\infty)\]

  - common environment \[\mapsto\text{ expect } (\mu^\infty, B^\infty)?\text{ should satisfy}\]

    \[\mapsto\text{fixed point } \mu_t^\infty = \mathcal{L}(X_t^\infty | \mu^\infty, B^\infty)\text{ (true)}\]

    \[\mapsto(\mu^\infty, B^\infty)X_0^\infty\text{ and }W^\infty \perp\perp\text{ (true) } (X_0^\infty, W^\infty) \mapsto\text{ proper noise}\]

    \[\mapsto\text{fair extra observation } \mapsto \sigma(X_0^\infty, \mu_s^\infty, B_s^\infty, W_s^\infty, s \leq T)\text{ and } \mathcal{F}_t^\infty\text{ conditional } \perp\perp\text{ on } \sigma(X_0^\infty, \mu_s^\infty, B_s^\infty, W_s^\infty, s \leq t)\text{ (???)}\]

    \[\mapsto\text{observation of private state has no bias on future of the environment (???)}\]
Strong vs. weak solutions

- Limiting FBSDE formulation

\[ dX_t^\infty = \left( b(X_t^\infty, \mu_t^\infty) - Z_t^\infty \right) dt + dW_t^\infty + \eta dB_t^\infty \]
\[ dZ_t^\infty = -\partial_x H(X_t^\infty, \mu_t^\infty, Z_t^\infty) dt + dM_t^\infty, \quad Z_T^\infty = \partial_x g(X_T^\infty, \mu_T^\infty) \]

⇒ necessary condition for optimality only, but not a limitation
⇒ may pass to the limit in the optimality condition

- Main question: What is the common information?
  - whole information ⇒ \( \mathbb{F}^\infty \) generated by \((X^\infty, \mu^\infty, B^\infty, W^\infty)\)
  - common environment ⇒ expect \((\mu^\infty, B^\infty)\)? should satisfy
    - fixed point \( \mu_t^\infty = \mathcal{L}(X_t^\infty | \mu^\infty, B^\infty) \) (true)
    - \((\mu^\infty, B^\infty) X_0^\infty \) and \( W^\infty \perp\perp \) (true) \((X_0^\infty, W^\infty) \) ⇒ proper noise
    - fair extra observation ⇒ \( \sigma(X_0^\infty, \mu_s^\infty, B_s^\infty, W_s^\infty, s \leq T) \) and \( \mathcal{F}_t^\infty \) conditional \( \perp\perp \) on \( \sigma(X_0^\infty, \mu_s^\infty, B_s^\infty, W_s^\infty, s \leq t) \) (???)
    - notion of compatibility [Jacod, Mémin, Kurtz] and [Buckdahn (al.)] for BSDEs
Strong vs. weak solutions

- Limiting FBSDE formulation
  \[ \rightsquigarrow \text{necessary condition for optimality only, but not a limitation} \Rightarrow \text{may pass to the limit in the optimality condition} \]

- Main question: What is the common information?
  - whole information \[ \rightsquigarrow \mathcal{F}_t^\infty \text{ generated by } (X_t^\infty, \mu_t^\infty, B_t^\infty, W_t^\infty) \]
  - common environment \[ \rightsquigarrow \text{expect } (\mu_t^\infty, B_t^\infty)? \text{ should satisfy} \]
    - fixed point \[ \mu_t^\infty = \mathcal{L}(X_t^\infty | \mu_t^\infty, B_t^\infty) \text{ (true)} \]
    - \[ (\mu_t^\infty, B_t^\infty) X_0^\infty \text{ and } W_t^\infty \perp \perp \text{ (true) } (X_0^\infty, W_t^\infty) \rightsquigarrow \text{proper noise} \]
    - fair extra observation \[ \rightsquigarrow \sigma(X_0^\infty, \mu_s^\infty, B_s^\infty, W_s^\infty, s \leq T) \text{ and} \mathcal{F}_t^\infty \text{ conditional } \perp \perp \text{ on } \sigma(X_0^\infty, \mu_s^\infty, B_s^\infty, W_s^\infty, s \leq t) \text{ (???)} \]

  - notion of compatibility [Jacod, Mémin, Kurtz] and [Buckdahn (al.)] for BSDEs

  \[ \rightsquigarrow \text{difficult to pass to the limit on compatibility } \Rightarrow \text{need to enlarge environment} \]
Strong vs. weak solutions

- Limiting FBSDE formulation $\implies$ necessary condition for optimality only, but not a limitation $\implies$ may pass to the limit in the optimality condition

- Main question: What is the common information?
  - whole information $\implies \mathbb{F}^\infty$ generated by $(X^\infty, \mu^\infty, B^\infty, W^\infty)$
  - common environment $\implies$ replace by $(\mathcal{M}^\infty, B^\infty)$
    $\implies \mathcal{M}_t^\infty$ limit in law of $\mathcal{L}(X_{\wedge t}^{N, P}, W_{\wedge t}^{N, P} | B^\infty)$
    $\implies$ fixed point $\mathcal{M}_t^\infty = \mathcal{L}(X_{\wedge t}^\infty, W_{\wedge t}^\infty | \mathcal{M}^\infty, B^\infty)$
    $\implies$ fixed point $\implies$ compatibility

- **Yamada-Watanabe**: strong ! for compatible solutions $\implies$ weak solutions are strong
  - strong solutions $\implies$ environment is adapted to $B^\infty$
  - example if monotonicity $\implies$ close the loop!
Part V. Master Equation

a. Derivation of equation
Setting

- **Assume** \( \exists \) for value function MKV FBSDE (\( \sigma = 1 \))

\[
\begin{align*}
    dX_s &= \left( b(X_s, \mathcal{L}(X_s|B)) - Z_s \right) ds + dW_s + \eta dB_s \\
    dY_s &= -f(X_s, \mathcal{L}(X_s|B), Z_s) ds + Z_s dW_s + \xi_s dB_s, \quad Y_T = g(X_T, \mathcal{L}(X_T|B))
\end{align*}
\]

\( \circ \) \( Y_t = U(t, X_t, \mu) = U(t, X_t, \mathcal{L}(X_t|B)) \)

- **Goal**: Expand the right-hand side to identify PDE for \( U \)!!!
Setting

- Assume $\exists!$ for value function MKV FBSDE ($\sigma = 1$)

\[ dX_s = \left( b(X_s, \mathcal{L}(X_s|B)) - Z_s \right) ds + dW_s + \eta dB_s \]

\[ dY_s = -f(X_s, \mathcal{L}(X_s|B), Z_s) ds + Z_s dW_s + \zeta_s dB_s, \quad Y_T = g(X_T, \mathcal{L}(X_T|B)) \]

- **Goal**: Expand the right-hand side to identify PDE for $U$!!

- **Need for second-order derivatives**
  - $\partial_t U(t, x, \mu)$ and $\partial_x^2 U(t, x, \mu)$ bounded and Lipschitz in $(x, \mu)$
  - $\partial_\mu U(t, x, \mu)(v)$ is differentiable in $x$, $v$ and $\mu$
  - $\partial_x \partial_\mu U(t, x, \mu)(v)$, $\partial_v \partial_\mu U(x, \mu)(v)$ bounded and Lipschitz
  - $\partial^2_\mu U(t, x, \mu)(v, v')$ is bounded and Lipschitz
Setting

- Assume $\exists!$ for value function MKV FBSDE ($\sigma = 1$)

$$
\begin{align*}
    dX_s &= \left(b(X_s, \mathcal{L}(X_s|B)) - Z_s\right)ds + dW_s + \eta dB_s \\
    dY_s &= -f(X_s, \mathcal{L}(X_s|B), Z_s)ds + Z_s dW_s + \zeta_s dB_s, \quad Y_T = g(X_T, \mathcal{L}(X_T|B))
\end{align*}
$$

- $Y_t = U(t, X_t, \mu) = U(t, X_t, \mathcal{L}(X_t|B))$

- **Goal**: Expand the right-hand side to identify PDE for $U$!!!

- Need for **second-order derivatives**

  - $\partial_t U(t, x, \mu)$ and $\partial^2_x U(t, x, \mu)$ bounded and Lipschitz in $(x, \mu)$
  - $\partial_\mu U(t, x, \mu)(\nu)$ is differentiable in $x$, $\nu$ and $\mu$
  - $\partial_x \partial_\mu U(t, x, \mu)(\nu)$, $\partial_\nu \partial_\mu U(x, \mu)(\nu)$ bounded and Lipschitz
  - $\partial^2_\mu U(t, x, \mu)(\nu, \nu')$ is bounded and Lipschitz

- **Theorem**: [Gangbo Swiech, C D D, C D L L] If monotonicity and smooth coefficients, then $U$ is smooth
Itô’s formula on $\mathcal{P}_2(\mathbb{R}^d)$

- Process $dX_t = b_t dt + dW_t + dB_t \mathbb{E} \int_0^T |b_t|^2 dt < \infty$
  - disentangle sources of noise $\leadsto$ use product probability space
    \[ \Omega = \Omega^B \times \Omega^W, \quad \mathcal{F} = \mathcal{F}^B \otimes \mathcal{F}^W, \quad \mathbb{P} = \mathbb{P}^B \otimes \mathbb{P}^W \]
    - $\mathbb{P}^B \leadsto$ $B$, $\mathbb{P}^W \leadsto$ $W$, $\mathcal{L}(\cdot \mid \sigma(B)) = \mathcal{L}^W(\cdot)$
    - $\Omega = \Omega^B \times \Omega^W$, $\Omega^B$ carries $B$, $\Omega^W$ carries $W$
    - $\mu_t = \mathcal{L}(X_t)$: conditional law of $X_t$ given $B$
Itô’s formula on $\mathcal{P}_2(\mathbb{R}^d)$

- **Process** $dX_t = b_t dt + dW_t + dB_t$ $\mathbb{E} \int_0^T |b_t|^2 dt < \infty$
  
  - disentangle sources of noise $\leadsto$ use product probability space

  $\Omega = \Omega^B \times \Omega^W$, $\mathcal{F} = \mathcal{F}^B \otimes \mathcal{F}^W$, $\mathbb{P} = \mathbb{P}^B \otimes \mathbb{P}^W$

  - $(\Omega^B, \mathcal{F}^B, \mathbb{P}^B) \leadsto B$, $(\Omega^W, \mathcal{F}^W, \mathbb{P}^W) \leadsto W$, $\mathcal{L}(\cdot | \sigma(B)) = \mathcal{L}^W(\cdot)$

  - $\Omega = \Omega^B \times \Omega^W$, $\Omega^B$ carries $B$, $\Omega^W$ carries $W$

  - $\mu_t = \mathcal{L}(X_t)$: conditional law of $X_t$ given $B$

- $U$ Fréchet differentiable with $\mathbb{R}^d \ni v \mapsto \partial_\mu U(\mu, v)$ differentiable $(v, \mu)$
Itô’s formula on $\mathcal{P}_2(\mathbb{R}^d)$

- **Process** $dX_t = b_t dt + dW_t + dB_t$ $\mathbb{E} \int_0^T |b_t|^2 dt < \infty$
  - disentangle sources of noise $\rightsquigarrow$ use product probability space

\[ \Omega = \Omega^B \times \Omega^W, \quad \mathcal{F} = \mathcal{F}^B \otimes \mathcal{F}^W, \quad \mathbb{P} = \mathbb{P}^B \otimes \mathbb{P}^W \]

- $(\Omega^B, \mathcal{F}^B, \mathbb{P}^B) \rightsquigarrow B$, $(\Omega^W, \mathcal{F}^W, \mathbb{P}^W) \rightsquigarrow W$, $\mathcal{L}(\cdot | \sigma(B)) = \mathcal{L}^W(\cdot)$

- $\Omega = \Omega^B \times \Omega^W$, $\Omega^B$ carries $B$, $\Omega^W$ carries $W$

- $\mu_t = \mathcal{L}(X_t)$: conditional law of $X_t$ given $B$

- $U$ Fréchet differentiable with $\mathbb{R}^d \ni v \mapsto \partial_\mu U(\mu, v)$ differentiable $(v, \mu)$
  - Itô’s formula for $(U(\mu_t))_{t \geq 0}$?

\[
dU(\mu_t) = \mathbb{E}^W [b_t \cdot \partial_\mu U(\mu_t)(X_t)] + \mathbb{E}^W [\text{Trace}(\partial_v \partial_\mu U(\mu_t)(X_t))] dt
\]
\[
+ \frac{1}{2} \mathbb{E}^W \tilde{\mathbb{E}}^\tilde{W} [\text{Trace}(\partial^2_\mu U(\mu_t)(X_t, \tilde{X}_t))] dt + \mathbb{E}^W [\partial_\mu U(\mu_t)(X_t)] \cdot dB_t
\]

- $\tilde{\mathbb{E}}^{\tilde{W}}$ conditional expectation on a copy space $\Omega^B \times \tilde{\Omega}^W$
Identification of the master equation

- Identification of the $dt$ terms in the expansion of the identify:

$$Y_t = U(t, X_t, \mathcal{L}(X_t | B))$$
Identification of the master equation

- Identification of the $dt$ terms in the expansion of the identify:

$$Y_t = U(t, X_t, \mathcal{L}(X_t | B))$$

- Get the form of the full-fledged master equation

$$\begin{align*}
\partial_t U(t, x, \mu) &- \int_{\mathbb{R}^d} \partial_x U(t, v, \mu) \cdot \partial_\mu U(t, x, \mu)(v) d\mu(v) \\
+ f(x, \mu) &- \frac{1}{2} |\partial_x U(t, x, \mu)|^2 + \frac{1 + \eta^2}{2} \text{Trace}(\partial_x^2 U(t, x, \mu)) \\
+ \frac{1 + \eta^2}{2} &\int_{\mathbb{R}^d} \text{Trace}(\partial_v \partial_\mu U(t, x, \mu, v)) d\mu(v) \\
+ \eta^2 &\int_{\mathbb{R}^d} \text{Trace}(\partial_x \partial_\mu U(t, x, \mu, v)) d\mu(v) \\
+ \frac{\eta^2}{2} &\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \text{Trace}(\partial_\mu^2 U(t, x, \mu, v, v')) d\mu(v) d\mu(v') = 0
\end{align*}$$

- Not a HJB! (MFG ≠ optimization)
Part V. Master Equation

b. Application
Revisiting the $N$-player game

- Controlled dynamics

$$dX^i_t = \left( b(X^i_t, \bar{\mu}^N_t) + \alpha^i_t \right) dt + dW^i_t + \eta dB_t$$

- Cost functionals to player $i$

$$J^i(\alpha^1, \ldots, \alpha^N) = \mathbb{E}\left[ g(X^i_T, \bar{\mu}^N_T) + \int_0^T \left( f(X^i_s, \bar{\mu}^N_s) + \frac{1}{2}|\alpha^i_s|^2 \right) ds \right]$$

- Rigorous connection between $N$-player game and MFG?
Revisiting the \(N\)-player game

- Controlled dynamics
  \[
dX^i_t = \left( b(X^i_t, \bar{\mu}^N_t) + \alpha^i_t \right) dt + dW^i_t + \eta dB_t
\]

- Cost functionals to player \(i\)
  \[
  J^i(\alpha^1, \ldots, \alpha^N) = \mathbb{E}\left[ g(X^i_T, \bar{\mu}^N_T) + \int_0^T \left( f(X^i_s, \bar{\mu}^N_s) + \frac{1}{2} |\alpha^i_s|^2 \right) ds \right]
\]

- Rigorous connection between \(N\)-player game and MFG?

- Prove the convergence of the Nash equilibria as \(N\) tends to \(\infty\)
  
  - difficulty \(\sim\) no uniform smoothness on the optimal feedback function \(\alpha^{*,N}_{i,N}\) w.r.t to \(N\)
    \[
    \alpha^{*,i,N}_t = \alpha^{*,N}(X^i_t; X^1_t, \ldots, X^{i-1}_t, X^{i+1}_t, \ldots, X^N_t)
    \]
  - \(\sim\) no compactness on the feedback functions
  
  - weak compactness arguments on the control (notion of relaxed controls) for equilibria over open loop controls [Lacker, Fischer]
Revisiting the $N$-player game

- Controlled dynamics
  
  $$dX^i_t = \left( b(X^i_t, \bar{\mu}^N_t) + \alpha^i_t \right) dt + dW^i_t + \eta dB_t$$

- Cost functionals to player $i$
  
  $$J^i(\alpha^1, \ldots, \alpha^N) = \mathbb{E}\left[ g(X^i_T, \bar{\mu}^N_T) + \int_0^T \left( f(X^i_s, \bar{\mu}^N_s) + \frac{1}{2} |\alpha^i_s|^2 \right) ds \right]$$

- Rigorous connection between $N$-player game and MFG?

- Prove the convergence of the Nash equilibria as $N$ tends to $\infty$
  
  - difficulty $\leadsto$ no uniform smoothness on the optimal feedback function $\alpha^{*,N}$ w.r.t. to $N$
  
  $$\underbrace{\alpha^*_{t,i,N}}_{\text{optimal control to player } i} = \alpha^{*,N}(X^i_t; X^1, \ldots, X^{i-1}, X^{i+1}, \ldots, X^N)$$

  $\leadsto$ no compactness on the feedback functions

  - use the master equation [C D L L]:
    
    $$\text{expand } (U(t, X^i_t, \bar{\mu}^N_t))_{0 \leq t \leq T}$$

    and prove $\approx$ equilibrium cost to player $i$
Revisiting the $N$-player game

- Controlled dynamics

$$dX_t^i = \left( b(X_t^i, \bar{\mu}_t^N) + \alpha_t^i \right) dt + dW_t^i + \eta dB_t$$

- Cost functionals to player $i$

$$J_i^i(\alpha^1, \ldots, \alpha^N) = \mathbb{E}\left[ g(X_T^i, \bar{\mu}_T^N) + \int_0^T \left( f(X_s^i, \bar{\mu}_s^N) + \frac{1}{2}|\alpha_s^i|^2 \right) ds \right]$$

- Rigorous connection between $N$-player game and MFG?

- **Construct approximate Nash equilibria** (easier)

  - limit setting $\leadsto$ optimal control has the form

  $$\alpha_t^* = -\partial_x U(t, X_t, \underbrace{\mathcal{L}(X_t|B)}_{\text{population at equilibrium}})$$

  - in $N$-player game, use $\alpha_t^{i,N} = -\partial_x U(t, X_t^i, \bar{\mu}_t^N)$

  - almost Nash $\leadsto$ cost decreases at most of $\varepsilon_N$ under unilateral deviation where $\varepsilon_N \to 0$
Probabilistic Theory of Mean Field Games with Applications I
Mean Field FBSDEs, Control, and Games

Probabilistic Theory of Mean Field Games with Applications II
Mean Field Games with Common Noise and Master Equations