Scaling Limits for Large Stochastic Networks

Amarjit Budhiraja
Department of Statistics & Operations Research
University of North Carolina at Chapel Hill

IPAM Workshop on Mean Field Games, August 2017

Results.

- *Rate Control under Heavy Traffic with Strategic Servers* (Bayraktar, B. and Cohen (2016)).
  - N-player game for single server queues.
  - Each server has a cost function it seeks to minimize.
  - Objective: Compute (near) Nash equilibria.
  - Asymptotic Model: Mean Field Game for reflected diffusions.

Results.

- *Rate Control under Heavy Traffic with Strategic Servers* (Bayraktar, B. and Cohen (2016)).
    - N-player game for single server queues.
    - Each server has a cost function it seeks to minimize.
    - Objective: Compute (near) Nash equilibria.
    - Asymptotic Model: Mean Field Game for reflected diffusions.
- *Controlled Weakly Interacting Large Finite State Systems with Simultaneous Jumps* (B. and Friedlander (2016)).
    - Rate Control for large finite state jump Markov processes.
    - Central Controller.
    - Objective: Optimize system performance.
    - Asymptotic Model: Drift Control for Degenerate Time - Inhomogeneous Diffusions.

Results.

- *Coding and Load Balancing Mechanisms in Cloud Storage Systems* (B. and Friedlander (2017)).
  - Large number of file stored "in pieces" over a large number of servers.
  - Each file stored in equally sized pieces across L servers s.t. any k pieces recover the full file.
  - Objective: Model Simplification (LLN and CLT for fluctuations).
  - Asymptotic Model: SDE in ℓ_2 driven by cylindrical Brownian motion.
- *Power of d Schemes on Erdős-Rényi Graphs* (B., Mukherjee and Wu (2017)).
  - Each server has an associated queue in infinite capacity buffer.
  - An Erdős-Rényi graph (possibly time varying) describes the neighborhood of any server.
  - An arriving job chooses a server at random which then queries $d-1$ neighbors at random and sends the job to shortest queue.
  - Objective: LLN (Annealed and Quenched).
  - Asymptotic Model: Same infinite system of ODE as the 'fully connected' system. $(np_n \to \infty)$

Rate Control with Strategic Servers.

- Sequence of d single server queues. (arrival rate λ^n, service rate μ^n)
- Critically Loaded: $n^{-1/2}(\lambda^n - \mu^n) \to c$.
- Limit (under usual scaling) given by a reflected BM.
- I.e. if $Q_i^n(t)$ is queue length of i-th queue then $\tilde{Q}_i^n \doteq Q_i^n/\sqrt{n}$ converges to a BM with drift c, reflected at 0.
- Here consider a setting where each server can exercise control of arrival/service rates. [Arise in service networks, cloud computing, limit order books...]

Rate Control with Strategic Servers.

- control can depend on 'everything' up to current time.
- ...also rates can depend on queue state and the empirical measure.
- Each server aims to minimize its individual cost.
- Interested in (near) Nash equilibria.
- For large d (even with diffusion approximations) computing Nash eqilibria is computationally intractable.
- Approach: Heavy traffic + large d asymptotic regime.

Problem Setting.

- Fix T (time horizon) and L (buffer size). Control set: U a compact set.
- Controlled rates:

$$\lambda^{N,i}(t) = aN + \lambda(t, \tilde{\nu}^N(t), \tilde{Q}_i^N(t), \alpha_i^N(t))\sqrt{N} + o(\sqrt{N}),$$
$$\mu^{N,i}(t) = aN + \mu(t, \tilde{\nu}^N(t), \tilde{Q}_i^N(t), \alpha_i^N(t))\sqrt{N} + o(\sqrt{N}).$$

- $\tilde{Q}^N = Q^N/\sqrt{N}$, $\tilde{\nu}^N(t) = \frac{1}{N}\sum_{i=1}^N \delta_{\tilde{Q}_i^N(t)}$.
- $V^{N,i}, Z^{N,i}$ unit rate independent Poisson processes.
- State equation:

$$Q^{N,i}(t) = Q^{N,i}(0) + V^{N,i}\left(\int_0^t 1_{\{\tilde{Q}^{N,i}(s)<L\}}\lambda^{N,i}(s)ds\right) - Z^{N,i}\left(\int_0^t 1_{\{\tilde{Q}^{N,i}(s)>0\}}\mu^{N,i}(s)ds\right).$$

Problem Setting.

- Skorohod map:

  For $\psi \in D([0,T]:\mathbb{R})$ with $\psi(0) \in [0,L]$, say $(\varphi, \zeta_1, \zeta_2) \in \mathcal{D}([0,T]:\mathbb{R}^3)$ solves the Skorohod problem for ψ if:

  - For every $t \in [0,T]$, $\varphi(t) = \psi(t) + \zeta_1(t) - \zeta_2(t) \in [0,L]$.
  - ζ_i are nonnegative and nondecreasing, $\zeta_1(0) = \zeta_2(0) = 0$, and

$$\int_{[0,T]} 1_{(0,L]}(\varphi(s))d\zeta_1(s) = \int_{[0,T]} 1_{[0,L)}(\varphi(s))d\zeta_2(s) = 0.$$

  Write $\Gamma(\psi) = (\varphi, \zeta_1, \zeta_2)$ and refer to Γ as the Skorohod map.

Problem Setting.

- State evolution using the Skorokhod map:

$$(\tilde{Q}_i^N, \tilde{Y}_i^N, \tilde{R}_i^N)(t)$$
$$= \Gamma\left(\tilde{Q}_i^N(0) + \int_0^{\cdot} \tilde{b}_i^N(s)ds + \tilde{A}_i^N(\cdot) - \tilde{D}_i^N(\cdot) + o(1)\right)(t), \quad t \in [0, T].$$

- $\tilde{b}_i^N(t) \doteq b(t, \tilde{\nu}^N(t), \tilde{Q}_i^N(t), \alpha_i^N(t)),\ b \doteq \lambda - \mu,$

$$\tilde{Y}_i^N(t) \doteq \frac{1}{\sqrt{N}} \int_0^t 1_{\{\tilde{Q}_i^N(s)=0\}} \mu_i^N(s)ds,\ \tilde{R}_i^N(t) \doteq \frac{1}{\sqrt{N}} \int_0^t 1_{\{\tilde{Q}_i^N(s))=L\}} \lambda_i^N(s)ds.$$

$$\langle \tilde{A}_i^N, \tilde{A}_j^N \rangle(t) = \delta_{ij} \frac{1}{N} \int_0^t 1_{\{\tilde{Q}_i^N(s)<L\}} \lambda_i^n(s)ds,\ \langle \tilde{D}_i^N, \tilde{D}_j^N \rangle(t) = \dots$$

Control Problem.

- $\mathcal{U}^N$ is the class of all admissible controls $\alpha^N = (\alpha^{N,1}, \ldots, \alpha^{N,N})$.
- Cost for initial condition $\tilde{Q}^N(0)$ and control α^N:

$$J^{N,i}(\tilde{Q}^N(0); \alpha^N) \doteq E\Big[\int_0^T f(t, \tilde{\nu}^N(t), \tilde{Q}^{N,i}(t), \alpha^{N,i}(t))dt + g(\tilde{\nu}^N(T), \tilde{Q}^{N,i}(T)) - \int_0^T y(t, \tilde{\nu}^N(t))d\tilde{Y}^{N,i}(t) + \int_0^T r(t, \tilde{\nu}^N(t))d\tilde{R}^{N,i}(t)\Big].$$

- Asymptotic Nash Equilibrium: Sequence of admissible controls $\{\tilde{\alpha}^{N,i} : 1 \le i \le N\}_{N\in\mathbb{N}}$ is an asymptotic Nash equilibrium if for every j, and every sequence of admissible controls $\{\beta^N\}_{N=1}^{\infty}$ for the j-th player,

$$\limsup_{N\to\infty} J^{N,j}(\tilde{Q}^N(0); \tilde{\alpha}^{N,1}, \ldots, \tilde{\alpha}^{N,N}) \le \liminf_{N\to\infty} J^{N,j}(\tilde{Q}^N(0); \tilde{\alpha}^{N,1}, \ldots, \tilde{\alpha}^{N,j-1}, \beta^N, \tilde{\alpha}^{N,j+1}, \ldots, \tilde{\alpha}^{N,N}).$$

Mean Field Game.

- (Lasry and Lions (2006), Huang, Malhamé and Caines (2006), Carmona and Delarue (2013), Carmona and Lacker (2015)...).
- ... a fixed point problem on $\mathcal{P}_{T,L} = \mathcal{P}(C([0,T] : [0,L]))$.
- For fixed $x \in [0,L]$ and $\nu \in \mathcal{P}_{T,L}$ consider a stochastic control problem:
- Filtered probability space: $\Xi = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P, B)$.
- An admissible pair on Ξ: Stochastic processes (α, Z), such that
  - $\alpha = \{\alpha(s)\}_{0 \le s \le T}$ is a U-valued $\mathcal{F}_s$-progressively measurable process,
  - $Z = \{Z(s)\}_{0 \le s \le T}$ is a $[0,L] \times \mathbb{R}_+ \times \mathbb{R}_+$ valued $\mathcal{F}_s$-adapted continuous process.

such that (α, Z) satisfy...

Mean Field Game.

$$Z(t) = (X, Y, R)(t) = \Gamma \left(x + \int_0^{\cdot} \bar{b}(s) ds + \sigma B(\cdot) \right)(t), \quad t \in [0, T],$$

where

$$\bar{b}(s) \doteq b(s, \nu(s), X(s), \alpha(s)), \; s \in [0, T],$$

and $\nu(s)$ is the marginal of ν at time instant s and $\sigma = \sqrt{2a}$.

- Denote by $\mathcal{A}(\Xi, x, \nu)$ the collection of all admissible pairs (α, Z).

Mean Field Game.

- Cost function in the MFG. Given $\nu \in \mathcal{P}_{T,L}$, $x \in [0, L]$ and a system Ξ, let $(\alpha, Z) \in \mathcal{A}(\Xi, x, \nu)$. Define

$$J_\nu(x, \alpha, Z) \doteq E\Big[\int_0^T f(s, \nu(s), X(s), \alpha(s))ds + g(\nu(T), X(T)) - \int_0^T y(s, \nu(s))dY_s + \int_0^T r(s, \nu(s))dR_s \Big].$$

- Value function:

$$V_\nu(x) = \inf_{\Xi} \inf_{(\alpha,Z)\in\mathcal{A}(\Xi,x,\nu)} J_\nu(x, \alpha, Z).$$

- Denote by $V_\nu(t, x)$ the value function for the control problem over $[t, T]$.

Mean Field Game.

- A **solution to the MFG** with initial condition $x \in [0, L]$ is defined to be a $\nu \in \mathcal{P}_{T,L}$ such that there exist a system Ξ and an $(\alpha, Z) \in \mathcal{A}(\Xi, x, \nu)$ such that $Z = (X, Y, R)$ satisfies
  - $V_\nu(0, x) = J_\nu(0, x, \alpha, Z)$.
  - $P \circ X^{-1} = \nu$
- If there exists a unique such ν, we refer to $V_\nu(0, x)$ as the **value of the MFG** with initial condition x.

Solving the Mean Field Game.

- Condition A: Functions b, f, g, y, r are Lipschitz. For every $(t, \eta, x, p) \in [0, T] \times \mathcal{P}([0, L]) \times [0, L] \times \mathbb{R}$, there is a unique $\hat{\alpha}(t, \eta, x, p) \in U$ such that

$$\hat{\alpha}(t, \eta, x, p) = \arg\min_{u \in U} h(t, \eta, x, u, p).$$

$$h(t, \eta, x, u, p) = f(t, \eta, x, u) + b(t, \eta, x, u)p.$$

- For $c \in (0, \infty)$, let $\mathcal{M}_c$ be the collection of all $\nu \in \mathcal{P}_{T,L}$ such that

$$\sup_{0 \leq s < t \leq T} \frac{W_1(\nu(t), \nu(s))}{(t-s)^{1/2}} \leq c$$

and let

$$\mathcal{M}_0 = \cup_{c>0} \mathcal{M}_c.$$

Solving the Mean Field Game.

- Under Condition A, for $\nu \in \mathcal{M}_0$, V_ν is the unique $H_{2+\frac{1}{2}}$ solution of:

$$-D_t\phi - H(t,\nu(t),x,D\phi) - \frac{1}{2}\sigma^2 D^2\phi = 0, \qquad (t,x) \in [0,T] \times [0,L],$$

with BC: $\phi(T,x) = g(\nu(T),x)$,

$$D\phi(t,0) = y(t,\nu(t)), \text{ and } D\phi(t,L) = r(t,\nu(t)), \ t \in [0,T],$$

where $H(t,\eta,x,p) = \inf_{u\in U} h(t,\eta,x,u,p)$.

- $\alpha(u,\omega) \doteq \hat{\alpha}(u,\nu(u),X(u,\omega),DV_\nu(u,X(u,\omega)))$ is the (essentially unique) optimal feedback control.

Solving the Mean Field Game.

- Fix ν and denote the state process under the optimal feedback control for the associated stochastic control problem by X^{ν}.

- Define Φ on $\mathcal{M}_0$ as $\Phi(\nu) \doteq P \circ (X^{\nu})^{-1}$.

- Theorem. Under Condition A Φ has a fixed point $\bar{\nu}$. This fixed point is a solution of the MFG.
  - Proof idea: Schauder's fixed point theorem.
  - Main technical step: Proving the continuity of Φ.

- Under an additional monotonicity condition the solution is unique.

Asymptotic Nash Equilibrium.

- Given a solution $\bar{\nu}$ of MFG, define 'feedback controls' for the N-player game as

$$\tilde{\alpha}^{N,i}(t) \doteq \hat{\alpha}(t, \bar{\nu}(t), \tilde{Q}^{N,i}(t), DV_{\bar{\nu}}(t, \tilde{Q}^{N,i}(t))).$$

- Condition B. The drift $b(t, \eta, x, \alpha) \equiv b(t, x, \alpha)$. Initial conditions are exchangeable and $\tilde{Q}^{N,i}(0) \to x$ for some $x \in [0, L]$.

- For a control β^N, let

$$\tilde{\alpha}^N_{-1}(t) \doteq (\beta^n, \tilde{\alpha}^{N,2}(t), \ldots, \tilde{\alpha}^{N,N}(t)).$$

- Theorem. Under Conditions A and B, $\tilde{\alpha}^N$ is an asymptotic Nash equilibrium:

$$\limsup_{N\to\infty} J^{N,1}(\tilde{Q}^N(0), \tilde{\alpha}^N) = V_{\bar{\nu}}(0, x) \leq \liminf_{N\to\infty} J^{N,1}(\tilde{Q}^N(0), \tilde{\alpha}^N_{-1}).$$

Proof Sketch.

- Step 1 As $N \to \infty$, $\tilde{\nu}_{-1}^N \to \bar{\nu}$, where $\tilde{\nu}_{-1}^N = \frac{1}{N-1}\sum_{i=2}^{N-1} \delta_{\tilde{Q}^{N,i}}$.
- Step 2 When players uses $\tilde{\alpha}^N$, $(\tilde{Q}^{N,i}, \tilde{Y}^{N,i}, \tilde{R}^{N,i})$ converge to

$$(\tilde{Q}^i, \tilde{Y}^i, \tilde{R}^i) = \Gamma(x + \int_0^{\cdot} b(s, \tilde{Q}^i(s), \hat{\alpha}(s, \bar{\nu}(s), \tilde{Q}^i(s), DV_{\bar{\nu}}(s, \tilde{Q}^i(s))))ds + \sigma B^i).$$

  The costs converge as well.

- This proves: $\limsup_{N\to\infty} J^{N,1}(\tilde{Q}^N(0), \tilde{\alpha}^N) = V_{\bar{\nu}}(0, x)$.
- Step 3 When for every $i \neq 1$, the i-th player uses $\tilde{\alpha}^{N,i}$ the limit points of the cost (for the first player) are costs for the (relaxed version of) stochastic control problem for $\bar{\nu}$ and so are bounded below by $V_{\bar{\nu}}(0, x)$.
- This proves: $V_{\bar{\nu}}(0, x) \le \liminf_{N\to\infty} J^{N,1}(\tilde{Q}^N(0), \tilde{\alpha}_{-1}^N)$.

Comments on Step 1: $\tilde{\nu}^N_{-1} \to \bar{\nu}$.

- Follows Kotelenez and Kurtz(2010).
- Define $\tilde{G}^{N,i} \doteq (\tilde{Q}^{N,i}, \tilde{Y}^{N,i}, \tilde{R}^{N,i})$ and

$$\Xi^N \doteq \frac{1}{N-1} \sum_{i=2}^{N-1} \delta_{\tilde{G}^{N,i}}.$$

- $\{\tilde{G}^{N,i}\}_{N \in \mathbb{N}}$ is $\mathcal{C}$-tight for each i.
- If $\tilde{G}^N = (\tilde{G}^{N,i})_{i=2}^{\infty}$ converges along a subsequence to $\tilde{G}$, then

$$(\tilde{G}^N, \Xi^N) \to (\tilde{G}, \Xi), \quad \Xi = \lim_{m \to \infty} \frac{1}{m} \sum_{i=1}^{m} \delta_{\tilde{G}^i}.$$

Comments on Step 1: $\tilde{\nu}^N_{-1} \to \bar{\nu}$.

- Characterization. For $i \geq 2$, with $\tilde{G}^i = (\tilde{Q}^i, \tilde{Y}^i, \tilde{R}^i)$,

$$(\tilde{Q}^i, \tilde{Y}^i, \tilde{R}^i) = \Gamma(x + \int_0^{\cdot} b(s, \tilde{Q}^i(s), \hat{\alpha}(s, \bar{\nu}(s), \tilde{Q}^i(s), DV_{\bar{\nu}}(s, \tilde{Q}^i(s))))ds + \sigma B^i).$$

- This shows that $\tilde{Q}^i$ are i.i.d. $\bar{\nu}$. Thus

$$\lim_{N\to\infty} \tilde{\nu}^N_{-1} = \lim_{N\to\infty} \Xi^N_{(1)} = \Xi_{(1)} = \lim_{m\to\infty} \frac{1}{m} \sum_{i=1}^{m} \delta_{\tilde{Q}^i} = \bar{\nu}.$$

Numerical Approximations.

- Construction of an asymptotic Nash equilibrium requires the solution of the MFG.
- Tractable expressions for the MFG solution are not available in general.
- Suitable numerical approximations needed.
- We introduce a Markov chain approximation method to construct numerical solutions of the MFG.

Outline of the Scheme.

- Discretize state space: For $h > 0$, $\mathbb{S}^h \doteq \{-h, 0, h, \ldots, L + h\}$.

- Introduce a sequence of controlled Markov chains $\{X_n^k\}_{n\in\mathbb{N}_0}$, one for each value of the discretization parameter $h^k > 0$, where $h^k \to 0$ as $k \to \infty$.

- Theorem. Suppose Condition A holds and that the MFG has a unique solution. Then the law $\nu^k \in \mathcal{P}_{T,L}$ of the continuous time interpolation of X^k converges to $\bar{\nu}$ for small T.

Construction of X_n^k.

- The k-th chain X_n^k is obtained by solving an approximate fixed point problem.

- Fix $\nu \in \mathcal{P}_{T,L}$. Formulate a MDP with transition kernel and cost depending on ν. The cost is the discretized analog of the cost in the MFG.

- Denote the law of the optimal state process (with piecewise linear interpolation) as $\Phi_k(\nu)$.

- Approximate Contraction. For some $q \in (0,1)$

$$W_1^2(\Phi_k(\nu), \Phi_k(\nu')) \leq q(h_k^2 + W_1^2(\nu, \nu')).$$

Construction of X_n^k.

- Let $\nu^m \doteq (\Phi_k)^m(\nu)$. Then

$$W_1^2(\Phi_k(\nu^m), \nu^m) \leq \frac{q}{1-q} h_k^2 + q^{m-1} W_1^2(\nu^2, \nu^1).$$

- Let

$$m(k) \doteq \min \left\{ m : q^{m-1} W_1^2(\nu^2, \nu^1) \leq \frac{q}{1-q} h_k^2 \right\}.$$

- Then $\bar{\nu}^k \doteq (\Phi_k)^{m(k)}(\nu)$ converges to $\bar{\nu}$ as $k \to \infty$.