A Mean Field Game of Optimal Portfolio Liquidation

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Outline

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Portfolio Liquidation

- almost all trading nowadays takes place in limit order markets
  - limit order book: list of prices and available liquidity
  - limited liquidity available at each price level
- large orders tend to adversely affect prices
- there is a substantial literature on optimal portfolio liquidation
- mostly single player models are considered in the literature

Single player benchmark model

Consider an order to sell $X > 0$ shares by time $T > 0$:

- ξ_t: *rate* of trading (control)
- $X_t = X - \int_0^t \xi_s \, ds$: remaining position (controlled state)
- $\widetilde{S}_t = \sigma W_t - \kappa \int_0^t \xi_s \, ds - \eta \xi_t$: transaction price
- $X_T = 0$: liquidation constraint

Single player benchmark model

The *liquidation costs* are then defined as

$$\begin{aligned}\mathcal{C} &= \text{book value} - \text{revenue} \\ &= S_0X - \int_0^T \widetilde{S}_t\xi_t\,dt = S_0X - \int_0^T S_t\,dX_t + \int_0^T \eta\xi_t^2\,dt\end{aligned}$$

Taking expectations and doing partial integration,

$$\mathbb{E}[\mathcal{C}] = E\left[\int_0^T \kappa X_t\xi_t + \eta\xi_t^2\,dt\right]$$

Single player benchmark model

Adding a risk term λX^2, the optimization problem is to minimize

$$J(\xi) = E\left[\int_0^T \kappa X_t \xi_t + \eta \xi_t^2 + \lambda X_t^2 \, dt\right]$$

This is a LQ control problem with liquidation constraint $X_T = 0$.

- PDE or BS(P)DE with terminal state constraint
- we will obtain a FBSDE with with terminal state constraint

Multi-player models

A multi-player model

- N ex-ante identical investors
- an asset price process of the form:

$$dS_t = \underbrace{-\frac{\kappa_t}{N}\sum_{j=1}^{N}\xi_t^j\,dt}_{\text{mean field interaction}} + \sigma\,dW_t.$$

- all agents trade in the same market
- average trading adds an adverse price impact

A multi-player model

The cost (interaction, liquidation, risk) for each player $i = 1, .., N$ is

$$J^i(\xi) = E \int_0^T \left(\underbrace{X_t^i \frac{\kappa_t}{N} \sum_{j=1}^{N} \xi_t^j}_{\text{interaction cost}} + \underbrace{\eta_t^i (\xi_t^i)^2}_{\text{trading cost}} + \underbrace{\lambda_t^i (X_t^i)^2}_{\text{market risk}} \right) dt.$$

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- what is the precise informational environment?

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- what is the precise informational environment?
- what does it mean for the mathematics?

Outline

A multi-player benchmark model

Carmona-Lacker (2016) consider a price impact model:

$$dX^i = -\xi_t^i \, dt + \sigma \, dW_t^i$$

$$dS_t = -\frac{\kappa}{N} \sum_{j=1}^{N} \xi_t^j \, dt + \sigma^0 dW_t^0.$$

- martingale optimality approach ("weak formulation")
- customer flow ("regularizes control problem")
- compact action space (hence no liquidation constraint)

A multi-player benchmark model

The resulting MFG ("representative player") is:

$$\begin{cases} X_t = X - \int_0^t \xi_s \, ds + \sigma W_t \\ J(\xi;\mu) = E \int_0^T \left(\kappa \mu_t X_t + \eta \xi_t^2 + \lambda X_t^2 \right) dt, \\ \mu_t = E[\xi_t]. \end{cases}$$

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- should one use conditional expectation, given W^0?

Outline

What we change

We consider conditional expectations

$$\mu_t = E[\xi_t | \mathcal{F}_t^W]$$

and

- unbounded action spaces
- no external customer flow
- which makes a liquidation constraint possible

Liquidation constraint results in FBSDE with singular coefficients.

Informational environments

"Private value environment" (PVE):

- $\lambda^i \in \sigma(W, W^i)$
- time-inconsistent optimization problem

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"Private value environment" (PVE):

- $\lambda^i \in \sigma(W, W^i)$
- time-inconsistent optimization problem

"Common value environment" (CVE):

- $\lambda^i \in \sigma(W)$
- time-consistent optimization problem

Private values

Dynamics and information

Recall that

$$J(\xi;\mu) = E\int_0^T \left(\kappa_t\mu_t X_t + \eta_t\xi_t^2 + \lambda_t X_t^2\right)\, dt$$

$$dS_t = -\kappa_t\mu_t\, dt + \sigma\, dW_t$$

$$\mu_t = E[\xi_t|\mathcal{F}_t^W]$$

and assume that the cost coefficients satisfy (modulo regularity)

$$\eta \in \sigma(W), \quad \kappa \in \sigma(W), \quad \lambda \in \sigma(\widetilde{W})$$

where $\widetilde{W} = (W, W')$ for independent Brownian motions W, W', and that

$$\xi \in \sigma(\widetilde{W}).$$

The FBSDE

Solving the MFG = solving the conditional mean field FBSDE:

$$\begin{cases} dX_t &= -\dfrac{Y_t}{2\eta_t}\,dt \\ -dY_t &= \left(\kappa_t E\left[\dfrac{Y_t}{2\eta_t}\Big|\mathcal{F}_t^W\right] + 2\lambda_t X_t\right)\,dt - Z_t\,d\widetilde{W}_t \\ X_0 &= X \\ Y_T & \text{to be determined.} \end{cases}$$

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Remark

In the common value environment,

$$-dY_t = \left(\kappa_t \frac{Y_t}{2\eta_t} + 2\lambda_t X_t\right) dt - Z_t\, dW_t.$$

Affine ansatz

The ansatz $Y = AX + B$, yields a system of Riccati-type equations:

$$\begin{cases} -dA_t = \left(2\lambda_t - \frac{A_t^2}{2\eta_t}\right) dt - Z_t^A \, d\widetilde{W}_t, \\ \quad A_t = \mathcal{O}\left((T-t)^{-1}\right) \quad (t \to T); \\ -dB_t = \left(\frac{\kappa_t}{2\eta_t} E\left[\underbrace{A_t X_t + B_t}_{=Y_t} \middle| \mathcal{F}_t^W\right] - \frac{A_t B_t}{2\eta_t}\right) dt - Z_t^B \, d\widetilde{W}_t, \\ \quad B_T = 0 \end{cases}$$

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- benefit: known, yet still singular terminal conditions

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- benefit: the equation for A is decoupled

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- benefit: known, yet still singular terminal conditions
- benefit: the equation for A is decoupled
- drawback: the equation for B has singular coefficients

An equivalent FBSDE

We need to solve a mean-field FBSDE with singular coefficients:

$$\begin{cases} dX_t = -\dfrac{A_t X_t + B_t}{2\eta_t}\, dt \\ X_0 = x \\ -dB_t = \left(\dfrac{\kappa_t}{2\eta_t} E\left[A_t X_t + B_t \middle| \mathcal{F}_t^W \right] - \dfrac{A_t B_t}{2\eta_t} \right) dt - Z_t^B \, d\widetilde{W}_t, \\ B_T = 0. \end{cases}$$

Theorem (Fu, Graewe, H. (2017))

Under boundedness assumptions on the coefficients there exists a unique (global) solution to the mean-field FBSDE. The solution characterizes the solution to the MFG.

On the proof

- short-time solution: fixed-point or FBSDE methods
- global solution
  - standard continuation method fails
  - we use a modified continuation method

Short-time solution

$$\Phi_t(\mu) := E\left[x e^{-\int_0^t \frac{A_u}{2\eta_u} du} \frac{A_t}{2\eta_t} + \frac{B_t}{2\eta_t} \right.$$
$$\left. - \frac{A_t}{2\eta_t} \int_0^t \frac{B_r(\mu)}{2\eta_r} e^{-\int_r^t \frac{A_u}{2\eta_u} du} dr \middle| \mathcal{F}_t^W \right] = \mu_t$$

where

$$-dA_t = \left(2\lambda_t - \frac{A_t^2}{2\eta_t}\right) dt - Z_t^A \, d\widetilde{W}_t,$$
$$A_t = \mathcal{O}(\frac{1}{T-t}) \quad (t \to T)$$
$$-dB_t(\mu) = \left(\frac{\kappa_t}{2\eta_t}\mu_t - \frac{A_t B_t}{2\eta_t}\right) dt - Z_t^B \, d\widetilde{W}_t,$$
$$B_T = 0.$$

Global solution

We apply method of continuation to a FBSDE of the form:

$$\begin{cases} dX_t = \{-A_t X_t - \alpha B_t - b_t\}\, dt, & X_0 = 1, \\ -dB_t = \{-A_t B_t + \alpha \kappa_t \mathbb{E}[A_t X_t + B_t] + f_t\}\, dt - Z_t\, dW_t, & B_T = 0 \end{cases}$$

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- we are looking for X, B that belong to

$$\mathcal{H} := \left\{ Z : \operatorname{esssup}_{t,\omega} (T-t)^{-1} |Z_t(\omega)| < \infty \right\}$$

- this is why we only multiply the second term by α
- else $X, B = \mathcal{O}\left((T-t)^{\alpha}\right)$

Lemma

For $\alpha = 0$ there exists for every $b \in \mathcal{H}$ and $f \in L^\infty_\mathcal{F}(0, T; \mathbb{R})$ a unique solution (X, B, Z) in the right space to the equation above:

$$\begin{cases} X_t = xe^{-\int_0^t A_r\,dr} + \int_0^t b_s e^{-\int_s^t A_r\,dr}\,ds, \\ B_t = \mathbb{E}\left[\int_t^T f_s e^{-\int_t^s A_r\,dr}\,ds \,\middle|\, \mathcal{F}_t \right]. \end{cases}$$

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Lemma

If for some $\alpha \in [0,1]$ the above equation is solvable for every data $b \in \mathcal{H}$ and $f \in L^\infty_{\mathcal{F}}(0, T; \mathbb{R})$, then it is solvable for $\beta \in [\alpha, \alpha + \delta]$ for $\delta > 0$ small.

Common values

Common value environment

Assume that there is only common information:

$$\kappa \in \sigma(W), \qquad \lambda \in \sigma(W).$$

Then, because $\xi \in \sigma(W)$, the mean-field FBSDE reduces to

$$\begin{cases} dX_t &= -\frac{Y_t}{2\eta_t}\, dt \\ X_0 &= X \\ -dY_t &= \left(\kappa_t \frac{Y_t}{2\eta_t} + 2\lambda X_t\right) dt - Z_t\, dW_t \\ Y_T & \text{to be determined} \end{cases}$$

Common value environment

Ansatz $Y = AX$ reduces the FBSDE to the Riccati equation

$$-dA_t = \left(2\lambda_t + \frac{\kappa_t A_t}{2\eta_t} - \frac{A_t^2}{2\eta_t}\right) dt - Z_t dW_t, \quad A_T = \infty.$$

Theorem (F., Graewe, Horst (2017))

The above FBSDE has a unique global solution. It characterizes the solution to the MFG.

Example

If all coefficients are constants, the backward stochastic Riccati equation reduces to the ODE

$$-dA_t = \left(2\lambda + \frac{\kappa A_t}{2\eta} - \frac{A_t^2}{2\eta}\right), \quad A_T = \infty.$$

Its explicit solution is

$$A_t = 2\sqrt{\eta\lambda}\coth\left(\sqrt{\frac{\lambda}{\eta}}(T-t)\right) - \frac{2\eta\left(A_- e^{A_+ T} e^{A_- t} - A_+ e^{A_- T} e^{A_+ t}\right)}{e^{A_+ T} - e^{A_- T}}$$
$$- \frac{2\sqrt{\eta\lambda}\left(e^{A_+ T} e^{A_- t} - e^{A_- T} e^{A_+ t}\right)\coth\left(\sqrt{\frac{\lambda}{\eta}}(T-t)\right)}{e^{A_+ T} - e^{A_- T}}$$

Example

The optimal position and trading rate are

$$X^*(t) = \frac{e^{A_+ T} e^{A_- t} - e^{A_- T} e^{A_+ t}}{e^{A_+ T} - e^{A_- T}} X,$$

$$\xi^*_t = \frac{A_+ e^{A_- T} e^{A_+ t} - A_- e^{A_+ T} e^{A_- t}}{e^{A_+ T} - e^{A_- T}} X,$$

where

$$A_+ = \frac{\kappa + \sqrt{\kappa^2 + 16\eta\lambda}}{4\eta}, \quad A_- = \frac{\kappa - \sqrt{\kappa^2 + 16\eta\lambda}}{4\eta}.$$

An explicitly solvable case

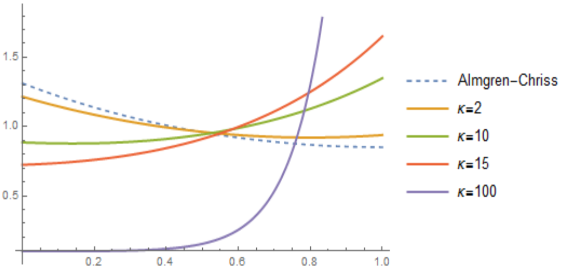

Figure: Optimal liquidation rate ξ^* corresponding to parameters $T = 1$, $X = 1$, $\lambda = 5$ and $\eta = 5$. The dashed line corresponds to $\kappa = 0$.

Outline

Conclusion

- we introduced a new MFG model for optimal liquidation
- its solution characterized by a FBSDE with singular coefficients
- CVE: global solution through a FBSDE with singular coefficients
- PVE: global solution through a BSDE

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- we introduced a new MFG model for optimal liquidation
- its solution characterized by a FBSDE with singular coefficients
- CVE: global solution through a FBSDE with singular coefficients
- PVE: global solution through a BSDE
- ongoing project: add a major player

Thank you!