Mean-Field-Type Games

Tembine Hamidou

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NEW YORK UNIVERSITY

L&G lab

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- Collaborators:
  - Quanyan Zhu, Jian Gao, Yida Xu, Michail Smyrnakis, Massa Ndong, Julian Barreiro-Gomez,
  - Eitan Altman (INRIA), Tamer Başar (UIUC), Jean-Yves LeBoudec (EPFL), Alain Bensoussan (UT), Boualem Djehiche (KTH), Daurio Bauso (Sheffield)
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Games in Strategic/Normal Form

Strategic-form games[1]

- set of agents
- set of actions for each agent
- objective (pay-off) functions for each agent

the **payoff function** is determined by the **realized action profile**.

$$\textbf{Payoff:} \quad r_j(a_1, a_2, \ldots), \; j \in \{1, 2, \ldots\}$$

$$\textbf{Equilibrium:} \quad r_j(a_1^*, a_2^*, \ldots) = \max_{a_j \in \mathcal{A}_j(a_{-j}^*)} r_j(a_1^*, \ldots, a_{j-1}^*, a_j, a_{j+1}^*, \ldots),$$

$$j \in \{1, 2, \ldots\}$$

[1] *Cournot 1838, Bertrand 1883, Borel 1921, Edgeworth'25, von Neumann'28, Fisher'30, Stackelberg'34, Ville'38, Nash'50, Isaacs'65, Hamilton'67, etc*

State dependence

State-dependent strategic-form games:

- set of **states**,
- set of agents,
- set of actions for each agent (may depend on the state),
- objective (pay-off) functions for each agent.

The **payoff function** is determined by the realized **state-action** profile.

$$\textbf{Payoff:} \quad r_j(s, a_1, a_2, \ldots), \; j \in \{1, 2, \ldots\}$$

Shapley, L. S. (1953). Stochastic games. PNAS 39 (10): 1095-1100

Mean-Field Games

- Infinite number of agents: Borel 1921, Volterra'26, von Neumann'44, Nash'51, Wardrop'52, Aumann'64, Selten'70, Schmeidler'73, Dubey et al.'80, etc
- Discrete-time/state mean-field games:
  - Jovanovic'82, Jovanovic & Rosenthal'88, Bergins & Bernhardt'92, Weibull& Benaïm'03-, Weintraub, Benkard, Van Roy'05-, Sandholm '06-, Adlaska, Johari, Goldsmith'08-, Benaïm & Le Boudec'08-, Gast, Gaujal'09, Bardenave'09-, Gomes, Mohr & Souza'10-, Elliott'12-, etc
- Continuous-time mean-field games
  - Benamou & Brenier'00-, Huang, Caines, Malhame'03-, Lasry & Lions'06-, Kotelenez & Kurtz'07-, Li & Zhang'08-, Buckdahn, Djehiche, Li and Peng'09-, Gueant'09-, Gomes et al.'09-, Yin, Mehta, Meyn, and Shanbhag'10, Djehiche et al' 10, Feng et al.'10-, Dogbe'10-, Achdou et al.'10-, LaChapelle'10-, Zhu, Başar'11, Bardi'12, Bensoussan,Sung,Yam,Yung'12-, Kolokoltsov'12-, Carmona & Delarue'12-,Gangbo & Swiech'14-,etc

common assumptions: **Indistinguishability per class+large number of agents+regularity**

What is a Mean-Field-Type Game ?

Mean-Field-Type Game

Any game in which the payoffs and/or the state dynamics coefficient functions involve not only the [state and action profiles] but also the distribution of state-action pairs.

payoff(state,action, distribution)

$$r_i(s, a_1, \ldots, a_n, D_{(s,a)}), \text{ kernel : } \mathcal{K}(s, a, D_{(s,a)}; ds')$$

The number of agents is not necessarily large.

Quantity-of-Interest

variance, skewness, kurtosis, value-at-risk, success probability, mean-variance payoff etc

Agenda

Mean-Field-Type Control

Some recent works on Mean-Field-Type Control:

M. Lauriere, O. Pironneau, Dynamic programming for mean-field type control, C. R. Acad. Sci. Paris, 2014

A. Bensoussan and J. Frehse and S.C.P. Yam, Mean-Field Games and Mean-field Type Control, Springer Briefs in Mathematics, 2014.

R. Buckdahn, B. Djehiche, J. Li, (2011) A general stochastic maximum principle for SDEs of mean-field type, Appl. Math. Optim., 64: 197-216.

D. Andersson and B. Djehiche, A Maximum Principle for SDEs of Mean-Field Type, Appl. Math. Optim. 63 (2011) 341-356.

R. Buckdahn, B. Djehiche, J. Li, S. Peng, (2009) Mean-field backward stochastic differential equations: a limit approach, The Annals of Probability, 37(4): 1524-1565.

How to find a best response strategy and payoff?

Solution approaches

- DPP: Dynamic programming principle
- SMP: Stochastic maximum principle
- PCE: Polynomial chaos expansion
- DM: Direct method

For $\gamma = 0, \delta = 0$ $g_i, r_i, b, \phi, \sigma$ deterministic functions, relationship between DPP and SMP can be found in

A. Bensoussan, B. Djehiche, H. Tembine, P. Yam: Risk-Sensitive Mean-Field-Type Control, to appear, CDC 2017.

S. Choutri, B. Djehiche and H. Tembine (2017): Optimal Control and Zero-Sum Games for Markov Chains of Mean-Field Type (arXiv:1606.04244).

A simple class of Mean-Field-Type Games

Full cooperative case: $(\Sigma, \mathcal{F}^{W,\tilde{N},\tilde{M}}, \mathbb{P})$
Finite set $\mathcal{I}$ (of agents). $a := (a_i)_{i\in\mathcal{I}}$ action profile.
Value of the grand coalition $\mathcal{V}(\mathcal{I}; t, s, \tilde{s})$:

$$\begin{cases} \mathcal{R}_{0,T} := g_0(s(T), \mathbb{E}\phi_1(s(T), \tilde{s}(T)) + \int_0^T r_0(t, s, \mathbb{E}\phi_2(s,a), a, \tilde{s})dt \\ \sup_{(a_i)_{i\in\mathcal{I}}} \frac{1}{\theta} \log \mathbb{E}[e^{\theta \mathcal{R}_{0,T}}], \\ ds = b(t, s, \mathbb{E}\phi_3(s,a), a, \tilde{s})dt + \sigma(t, s, \mathbb{E}\phi_4(s,a), a, \tilde{s})dW(t) \\ + \int_\Theta \gamma(t_-, s, \mathbb{E}\phi_5(s,a), a, \tilde{s}, \theta)\tilde{N}(dt, d\theta, \tilde{s}), \\ + \sum_{\tilde{s}} \delta(t_-, s, \mathbb{E}\phi_6(s,a), a, \tilde{s})\tilde{M}(dt, \tilde{s}), \\ s(0) = s_0, \\ m(t,.) = \mathbb{P}_{s(t)}, \quad m(0,.) := m_0, \ \tilde{s}(0) = \tilde{s}_0 \\ s(t) := s^{s_0,\tilde{s}_0,a}(t). \end{cases}$$

Pro: the aggregate output is in FINITE dimension !

A simple class of Mean-Field-Type Games

Non-Cooperative case:
Finite set $\mathcal{I}$ (of agents).
$a := (a_i)_{i\in\mathcal{I}}$ action profile of the agents.
Equilibrium payoff of i: $\hat{V}_i(t, s, \tilde{s})$:

$$\begin{cases} \mathcal{R}_{i,T} := g_i(s(T), \mathbb{E}\phi_1(s(T), \tilde{s}(T)) + \int_0^T r_i(t, s, \mathbb{E}\phi_2(s, a), a, \tilde{s})dt \\ \sup_{a_i} \frac{1}{\theta_i} \log \mathbb{E}[e^{\theta_i \mathcal{R}_{i,T}}], \\ s(t) := s^{s_0, \tilde{s}_0, a}(t), \\ s(0) = s_0, \\ m(t,.) = \mathbb{P}_{s(t)}, \quad m(0,.) := m_0, \ \tilde{s}(0) = \tilde{s}_0 \end{cases}$$

Pro: the aggregate output is in FINITE dimension !

Choose a test function f and $y(t) := f(t, s(t), \tilde{s}(t))$

Itô's formula for jump-diffusion process with switching regime:

$$
\begin{aligned}
& f(T, s(T), \tilde{s}(T)) - f(0, s(0), \tilde{s}(0)) \\
& = \int_0^T [f_t + b.f_s + \frac{\sigma^2}{2} f_{ss} + J[f] + \tilde{S}[f]]dt \\
& + \int_0^T \sigma f_s dB \\
& + \int_0^T \int_\Theta \{f(t, s(t_-) + \gamma(t_-, ., \theta), \tilde{s}(t_-)) - f(t, s, \tilde{s})\} \tilde{N}(dt, d\theta), \\
& + \int_0^T \sum_{k \neq \tilde{s}} [f(t, s(.) + \delta(.), k) - f(t, s, \tilde{s})] \tilde{M}(dt),
\end{aligned}
$$

$$
J[f] := \int_\Theta \{f(t, s(t_-) + \gamma(t_-, ., \theta)) - f(t, s(t_-), .) - f_s.\gamma(t_-, ., \theta)\} \nu(d\theta),
$$

$$
\tilde{S}[f] := \sum_{k \neq \tilde{s}} q_{\tilde{s}k} [f(t, s(t_-) + \delta(., k), k) - f(t, s(t_-), \tilde{s}) - f_s.\delta].
$$

see Cox & Miller'75, Kurtz, Oksendal, Protter.

Adjoint operator

$$\langle O[f], m\rangle = \langle f, O^*[m]\rangle$$

Kolmogorov-Fokker-Planck equation

$$m_t = -(bm)_s + \frac{(\sigma^2 m)_{ss}}{2} + J^*[m] + \tilde{S}^*[m],$$

$$m(0,.) = m_0(.)$$

Setup for Dynamic Programming Principle

Finding a right state space for DPP

The expected value of the objective functions $\mathbb{E}r, g$ can be expressed as a function of $m^{a,m_0}(t,s,\tilde{s})$.
$\hat{r}_i(t,m,\tilde{s}) := \mathbb{E}^s r_i(t,\tilde{s}) = \int m(t,ds,\tilde{s}) r_i(t,s,a,m,\tilde{s})$.

Reformulation of the Problem

$$\begin{cases} \inf_{(a_i)_{i\in\mathcal{I}}} \ \hat{g}_0(m) + \int_0^T \hat{r}_0(t,m)dt, \\ m_t = -(bm)_s + \frac{(\sigma^2 m)_{ss}}{2} + J^*[m] + \tilde{S}^*[m] =: O[m], \\ m(0,.) := m_0, \end{cases}$$

Dynamic Programming Principle

Finding a suitable state space for DPP

$$\hat{H} = \int m(t, ds, .) \sup_a [\hat{r}(m, a) + \langle O[m], \hat{V}_m \rangle]$$

Bellman [infinite dimensions]

$$\hat{V}_t + \hat{H}(t, m, \hat{V}_m, \hat{V}_{sm}, \hat{V}_{ssm}) = 0,$$

$$\hat{V}(T, m) = \hat{g}(m).$$

$$\begin{cases} \langle O[m], \hat{V}_m \rangle = \\ -\langle (bm)_s, \hat{V}_m \rangle + \langle \frac{(\sigma^2 m)_{ss}}{2}, \hat{V}_m \rangle + \langle J^*[m], \hat{V}_m \rangle + \langle \tilde{S}^*[m], \hat{V}_m \rangle \\ = \langle m, b\hat{V}_{sm} + \frac{\sigma^2}{2}\hat{V}_{ssm} + J[\hat{V}_m] + \tilde{S}[\hat{V}_m] \rangle \\ \\ \hat{H} = \langle m, \sup_a r + b\hat{V}_{sm} + \frac{\sigma^2}{2}\hat{V}_{ssm} + J[\hat{V}_m] + \tilde{S}[\hat{V}_m] \rangle = \langle m, H \rangle \end{cases}$$

Dual Functional $\hat{v} := \hat{V}_m$

$$\hat{v}_t + H(t, s, \tilde{s}, m, \hat{v}, \hat{v}_s, \hat{v}_{ss}) + \mathbb{E} H_m = 0$$

Adjoint process

$$\left\{\begin{array}{l} d\hat{v}_s = -\{r_s + b_s\hat{v}_s + \sigma_s(\sigma\hat{v}_s) + \mathbb{E}H_{sm}\}dt \\ -\int \gamma_s[\hat{v}_s(s+\gamma,.) - \hat{v}_s(s,.)]\nu(d\theta)dt \\ -\sum_{k\neq\tilde{s}} \delta_s[\hat{v}(s,k) - \hat{v}(s,\tilde{s})]dt + \sigma\hat{v}_{ss}dB \\ +\int_{\Theta}\{\hat{v}_s(s(t_-)+\gamma(t_-,\theta),\tilde{s}) - \hat{v}_s(s(t_-),\tilde{s})\}\tilde{N}(dt,d\theta), \\ +\sum_{k\neq\tilde{s}}[\hat{v}_s(s(t_-)+\delta,k) - \hat{v}_s(s(t_-),\tilde{s})]\tilde{M}(dt), \end{array}\right. \tag{1}$$

Aggregative case

$$r = r(s, \langle\phi, m\rangle, a, \tilde{s}), \;\; (r_m[s', m'])(s) = \phi(s) r_y(s', .).$$

Dynamic Programming vs Stochastic Maximum Principle:

$$\begin{cases} \bar{p} = \hat{v}_s(s(t_-), \tilde{s}(t)), \quad \bar{q} = \sigma \hat{v}_s, \\ \bar{r}(t,\theta) = [\hat{v}_s(s+\gamma, .) - \hat{v}_s(s, .)], \\ \bar{s}(t,k) = [\hat{v}(s+\delta, k) - \hat{v}(s, \tilde{s})], \\ \bar{P}_i = \hat{v}^*_{i,ss} = \hat{V}^*_{i,ssm}, \\ \bar{Q}_i = \sigma \hat{v}^*_{i,sss} = \sigma \hat{V}^*_{i,sssm}, \\ \bar{R}_i(.,\theta) = [\hat{v}^*_{i,ss}(s+\gamma, .) - \hat{v}^*_{i,ss}] \\ \bar{S}_i(.,k) = [\hat{v}^*_{i,ss}(s(t_-)+\delta, k) - \hat{v}^*_{i,ss}] \end{cases}$$

SMP : first order adjoint processes

$\bar{p}_i, \bar{q}_i, \bar{r}_i, \bar{s}_i$ adjoint processes associated to the drift, diffusion, jump terms $\bar{H}_i(t, s, m, a, \bar{p}_i, \bar{q}_i, \bar{r}_i, \bar{s}_i) :=$
$r_i + b.\bar{p}_i + \sigma.\bar{q}_i + \int_\Theta \gamma \bar{r}_i(t,\theta)\nu(d\theta) + \sum_k \delta(.,k)\bar{s}_i(.,k)q_k$

$$\begin{cases} \alpha_i := r_{i,s} + b_s.\bar{p}_i + \sigma_s.\bar{q}_i \\ + \int_\Theta \gamma_s \bar{r}_i(t,\theta)\nu(d\theta) + \sum_k \delta_s(.,k)\bar{s}_i(.,k)q_k \\ +\phi_{2,s}\mathbb{E}r_{i,y} + \phi_{3,s}\mathbb{E}[b_y.\bar{p}_i] + \phi_{4,s}\mathbb{E}[\sigma_y.\bar{q}_i] \\ +\phi_{5,s}\mathbb{E}[\int_\Theta \gamma_y \bar{r}_i(t,\theta)\nu(d\theta)] + \sum_k \phi_{6,s}\mathbb{E}[\delta_y(.,k)\bar{s}_i(.,k))q_k], \\ i \in \mathcal{I}. \end{cases}$$

$$\begin{cases} d\bar{p}_i = -\alpha_i dt + \bar{q}_i dB + \int_\Theta \bar{r}_i \tilde{N}(dt, d\theta) + \bar{s}_i.\tilde{M}(dt,.), \\ \bar{p}_i(T) = g_{i,s}(T) + \phi_{1,s}\mathbb{E}\{g_{i,y}(T)\}, \\ i \in \mathcal{I}. \end{cases}$$

Existence of first order adjoint processes

If the functions $b, \sigma, \gamma, \delta, r_i, g_i$ are continuously differentiable w.r.t (s, y), all their first-order derivatives with respect to (s, y) are continuous in (s, y, a), and bounded. Then, the first-order adjoint system is a linear SDE with a.s. bounded coefficient functions. There is a unique $\mathcal{F}-$adapted solution s.t.

$$\mathbb{E}\left[\sup_{t\in[0,T]} |\bar{p}_i(t)|^2 + \int_0^T \{|\bar{q}_i(t)|^2 + \int_\theta |\bar{r}_i(t,\theta)|^2 \nu(d\theta)\} dt\right]$$

$$\mathbb{E}\int_0^T \sum |\bar{s}_i(t,k)|^2 q_k < +\infty$$

Second order adjoint processes

$$\left\{\begin{array}{l}\beta_i := r_{i,ss} + b_{ss}.\bar{p}_i + \sigma_{ss}.\bar{q}_i\\ + \int_\Theta \gamma_{ss}\bar{r}_i(.,\theta)\nu(d\theta) + \sum_k \delta_{ss}(.,k)\bar{S}_i(.,k))q_k]\\ +\phi_{2,ss}\mathbb{E}r_{i,y} + \phi_{3,ss}\mathbb{E}[b_y.\bar{p}_i] + \phi_{4,ss}\mathbb{E}[\sigma_y.\bar{q}_i]\\ +\phi_{5,ss}\mathbb{E}[\int_\Theta \gamma_y\bar{r}_i(t,\theta)\nu(d\theta)] + \sum_k \phi_{6,ss}\mathbb{E}[\delta_y(.,k)\bar{S}_i(.,k))q_k]\\ +(2b_s + \sigma_s^2 + \int_\Theta \gamma_s^2\nu(d\theta) + \sum_k \delta_s^2 q_k)\bar{P}_i\\ +2\sigma_s\bar{Q}_i + \int_\Theta(2\gamma_s + \gamma_s^2)\bar{R}_i(t,\theta)\nu(d\theta),\\ \sum_k(2\delta_s + \delta_s^2)\hat{S}_i(t,k)q_k dt,\ \ i\in\mathcal{I}.\end{array}\right.$$

Peng's equation of mean-field type

$$\left\{\begin{array}{l}d\bar{P}_i = -\beta_i dt + \bar{Q}_i dB + \int_\Theta \bar{R}_i\tilde{N}(dt,d\theta) + \hat{S}_i.\tilde{M}(dt,.),\\ \bar{P}_i(T) = g_{i,ss}(T) + \phi_{1,ss}\mathbb{E}\{g_{i,y}(T)\},\\ i\in\mathcal{I}.\end{array}\right.$$

Existence of second order adjoint process

H0: Assume the functions $b, \sigma, \gamma, \delta, r_i, g_i$ are twice continuously differentiable w.r.t (s, y), all their derivatives up to second order w.r.t (s, y) are continuous in (s, y, a), and bounded. Then, there is a unique $\mathcal{F}-$adapted solution s.t.

$$\mathbb{E}\left[\sup_{t\in[0,T]} |\bar{P}_i(t)|^2 + \int_0^T \{|\bar{Q}_i(t)|^2 + \int_\Theta |\bar{R}_i(t,\theta,.)|^2 \nu(d\theta)\}dt\right]$$

$$\mathbb{E}\int_0^T \sum_k |\bar{S}_i(t,k)|^2 q_k < +\infty$$

SMP

If (s, a) is a pair of equilibrium state and equilibrium strategy then there is a vector of $\mathbb{F}-$adapted processes $(\bar{p}_i, \bar{q}_i, \bar{r}_i, \bar{P}_i, \bar{Q}_i, \bar{R}_i, \bar{S}_i)_{i\in\mathcal{I}}$ such that

$$\left\{\begin{array}{l} \bar{H}_i(t,s,m,a,\bar{p}_i,\bar{q}_i,\bar{r}_i,\bar{s}_i) - \bar{H}_i(t,s,m,a'_i,a_{-i},\bar{p}_i,\bar{q}_i,\bar{r}_i,\bar{s}_i) \\ +\frac{1}{2}\bar{P}_i(\sigma(s,m,a'_i,a_{-i}) - \sigma(s,m,a))^2 \\ +\frac{1}{2}\int_\Theta(\gamma(s,m,a'_i,a_{-i},\theta) - \gamma(s,m,a,\theta))^2[\bar{P}_i + \bar{R}_i(t,\theta)]\nu(d\theta) \\ +\frac{1}{2}\sum_k(\delta(s,m,a'_i,a_{-i},k) - \delta(s,m,a,k))^2[\bar{P}_i + \bar{S}_i(t,k)]q_k \geq 0, \end{array}\right.$$

for all $a'_i \in \mathcal{A}_i$, almost every time $t \in [0, T]$ and $\mathbb{P}-$almost surely.

Some recent works on Risk-Sensitive Mean-Field

A. Bensoussan, B. Djehiche, H. Tembine, P. Yam: Risk-Sensitive Mean-Field-Type Control, to appear, CDC 2017.

J. Moon and T. Başar. Linear Quadratic Risk-Sensitive and Robust Mean-Field Games. IEEE Transactions on Automatic Control, 62(3):1062-1077, March 2017.

Djehiche B. and Tembine H. (2016): A Stochastic Maximum Principle for Risk-Sensitive Mean-Field-Type Control under Partial observation. Book chapter in Mathematics and Statistics, Vol. 138, G. D. Nunno and F. E. Benth (Eds): Stochastic of environmental and financial economics.

H. Tembine, Risk-Sensitive Mean-Field-Type Games with Lp-norm Drifts, Elsevier Automatica, Vol. 59, 9, 2015, pp. 224-237.

H. Tembine, Q. Zhu, T. Başar, Risk-Sensitive Mean-Field Games, IEEE Transactions on Automatic Control, vol. 59, 4, 2014.

Risk Sensitive Case

$$\begin{cases} \sup_{a_i \in \mathcal{A}_i} \frac{1}{\theta_i} \log \left(\mathbb{E} e^{\theta_i \mathcal{R}_{i,T}} \right) \text{ subject to} \\ s(t) = s_0 + \int_0^t b dt' + \int_0^t \sigma dB(t') \\ + \int_0^t \int_\Theta \gamma \tilde{N}(dt', d\theta) + \int_0^t \delta \tilde{M}(dt'), \;\; t > 0 \\ s(t) = s^{s_0, a, \tilde{s}_0}(t), \;\; s(0) = s_0 \sim m_0, \tilde{s}(0) = \tilde{s}_0 \\ m(t,.) = m^{m_0, a, \tilde{s}_0}(t,.) = \mathbb{P}_{s^{s_0, a, \tilde{s}_0}(t)}. \end{cases}$$

$$\begin{cases} \sup_{a_i} \mathbb{E} e^{\theta_i [z_i(T) + g_i(s(T), m(T), \tilde{s}(T))]}, \\ z_i(t) = \int_0^t r_i \, dt', \\ z_i(0) = 0, \\ s(t) \text{ as above} \\ s(0) = s_0. \end{cases}$$

$$
\begin{aligned}
&0 = \hat{V}^{\theta}_{i,t}(t,\mu) + \int H^{\theta}_i \mu(t, dsdz),\\
&\hat{V}^{\theta}_i(T,\mu) = \int \mu(T, dsdz)\ e^{\theta_i(z_i + g_i(s, \int_{\tilde{z}} \mu(T,.,d\tilde{z})))},\\
&\mu_t = -(b\mu)_s - (f\mu)_z + \frac{1}{2}(\sigma'\sigma\mu)_{ss} + J^*[\mu] + \tilde{S}^*[\mu],\\
&\mu(0, dsdz) = m_0(ds)\delta_0(dz)
\end{aligned}
$$

$$
H^{\theta}_i = \sup_{a_i} \left\{ b\hat{V}^{\theta}_{i,s\mu} + \langle r, \hat{V}^{\theta}_{i,z\mu} \rangle + \frac{\sigma^2}{2}\hat{V}^{\theta}_{i,ss\mu} + J[\hat{V}^{\theta}_{i,m}] + \tilde{S}[\hat{V}^{\theta}_{i,m}] \right\}
$$

$$\begin{cases} dp_i^\theta = -\{\bar{H}^\theta_{i,s} + \frac{1}{\eta_i}\mathbb{E}[\eta_i \bar{H}^\theta_{i,s\mu_i}]\}dt \\ -q_i^\theta l_i dt - \int_\Theta r_i^\theta \omega_i^1 \nu(d\theta)dt - \sum_k s_i^\theta \omega_i^2 q_k dt \\ q_i^\theta dB + \int_\Theta r_i^\theta.\tilde{N}(dt, d\theta, .) + s_i^\theta.\tilde{M}(dt), \\ p_i^\theta(T) = g_{i,s} + \frac{1}{\eta_i(T)}\mathbb{E}\{\eta_i(T) g_{i,sm}\}, \\ d\eta_i = \eta_i \left[l_i dB + \int \omega_i^1(t,\theta,.)\tilde{N}(dt, d\theta, \tilde{s}) + \sum_k \omega_i^2(t,k)\tilde{M}(dt,k)\right], \\ \eta_i(T) = \theta_i e^{\theta_i[z_i(T) + g_i(s(T), m(T,.))]} \end{cases} \tag{2}$$

$$\bar{H}_i^\theta = r_i + bp_i^\theta + \sigma(q_i^\theta + l_i p_i^\theta) + \int \gamma[r_i^\theta + \omega_i^1 p_i^\theta]\nu(d\theta)dt + \sum_k \delta[s_i^\theta + \omega_i^2 p_i^\theta] q_k,$$

$$
\begin{cases}
p_i^{\theta} = \frac{\hat{v}_{i,s}^*}{\hat{v}_{i,z}^*} = \frac{\hat{v}_{i,s}^*}{\eta_i} \\
q_i^{\theta} = \frac{1}{\eta_i}[\sigma \hat{v}_{i,ss}^* - \sigma p_i^{\theta} \hat{v}_{i,sz}^*] \\
r_i^{\theta} = \frac{1}{\eta_i}\{\hat{v}_{i,s}^*(s+\gamma,\tilde{s}) - \hat{v}_{i,s}^*(s,\tilde{s})\} \\
-\frac{1}{\eta_i} p_i^{\theta}\{\hat{v}_{i,z}^*(s+\gamma,\tilde{s}) - \hat{v}_{i,z}^*(s,\tilde{s})\}, \\
s_i^{\theta}(t) = \frac{1}{\eta_i}[\hat{v}_{i,s}^*(s+\delta,k) - \hat{v}_{i,s}^*(s,\tilde{s})] \\
-\frac{1}{\eta_i} p_i^{\theta}[\hat{v}_{i,z}^*(s+\delta,k) - \hat{v}_{i,z}^*(s,\tilde{s})], \\
\eta_i = \hat{v}_{i,z}^* = \hat{V}_{i,z\mu_i}^{\theta}, \\
l_i = \sigma \frac{\hat{v}_{i,sz}^*}{\hat{v}_{i,z}^*} = \sigma \frac{\hat{v}_{i,sz}^*}{\eta_i}, \\
\omega_i^1 := \frac{1}{\eta_i}(\hat{v}_{i,z}^*(s+\gamma,.) - \hat{v}_{i,z}^*), \\
\omega_i^2 := \frac{1}{\eta_i}(\hat{v}_{i,z}^*(s+\delta,k) - \hat{v}_{i,z}^*).
\end{cases}
\tag{3}
$$

Example: Non-quadratic

$\mathcal{A}_1 = \mathbb{R}_+, \ \mathcal{A}_2 = [0,1], \ b := -a_1 + (1-a_2)\kappa_1 s + a_2\kappa_2 s,$
$\kappa_j = \kappa_j(t,\tilde{s}).\ \sigma := a_2 s\tilde{\sigma}(t,\tilde{s}),\ \gamma := a_2 s\tilde{\gamma}(t,\theta,\tilde{s}),\ \ \delta = 0.$
$r := e^{-\beta t}\frac{a_1(t)^{\alpha(\tilde{s})}}{\alpha(\tilde{s})}, \quad \alpha(\tilde{s}) < 1$

$$g := e^{-\beta T}\frac{s(T)^{\alpha(\tilde{s})}}{\alpha(\tilde{s})} + e^{-\beta T}u^{\epsilon}[\frac{1}{\alpha(\tilde{s})}\mathbb{E}s(T)^{\alpha(\tilde{s})}].$$

Best response strategies

$$\hat{V}_m(t,s,\tilde{s}) = \xi_1(t)\frac{s^{\alpha(\tilde{s})}}{\alpha(\tilde{s})}+\xi_2(t)\frac{s^{\alpha(\tilde{s})}}{\alpha(\tilde{s})}u_s^{\epsilon}[\frac{1}{\alpha(\tilde{s})}\int y^{\alpha(\tilde{s})}m(t,dy)]+o(\epsilon)$$

Best response 1

$$a_1^* = s\left(e^{\beta t}[\xi_1+\xi_2 u_s^{\epsilon}(\bar{m}_\alpha)]\right)^{\frac{1}{\alpha(\tilde{s})-1}}$$

Best response 2

$$0 = [-\kappa_1(t,\tilde{s})+\kappa_2(t,\tilde{s})]+a_2\tilde{\sigma}(t,\tilde{s})^2[\alpha(\tilde{s})-1] \\ +\int_{\mathbb{R}}\tilde{\gamma}(t,\theta,\tilde{s})[(1+a_2\tilde{\gamma})^{\alpha(\tilde{s})-1}-1]\nu(d\theta)$$

Semi-explicit solution/approximation: ODE

$$\xi_{1,t} + e^{\frac{\beta}{\alpha-1}t}(1-\alpha(\tilde{s})).\xi_1^{1+\frac{1}{\alpha(\tilde{s})-1}} + \Omega.\xi_1$$
$$+\sum_{k\neq\tilde{s}} q_{\tilde{s}k}[\xi_1(t,k)-\xi_1(t,\tilde{s})] = 0,$$
$$\xi_1(T,\tilde{s}) = e^{-\beta T},$$
$$\xi_{2,t}\frac{u^{\epsilon}[\bar{m}_\alpha]}{\bar{m}_\alpha u_{\tilde{s}}^{\epsilon}(\bar{m}_\alpha)} + \xi_2[e^{\frac{\beta}{\alpha-1}t}\alpha(\tilde{s})\xi_1^{\frac{1}{\alpha(\tilde{s})-1}} + \Omega]$$
$$+\sum_{k\neq\tilde{s}} q_{\tilde{s}k}\{\xi_2(t,k)-\xi_2(t,\tilde{s})\},$$
$$\xi_2(T,\tilde{s}) = e^{-\beta T},$$
$$\frac{d}{dt}\bar{m}_\alpha = \bar{m}_\alpha\{-\alpha e^{\frac{\beta}{\alpha-1}t}(\xi_1+\xi_2 u_s^{\epsilon})^{\frac{1}{\alpha-1}} + \Omega\}$$
$$+\sum_{k\neq\tilde{s}} q_{\tilde{s}k}[\xi_2(t,k)-\xi_2(t,\tilde{s})],$$
$$\bar{m}_\alpha(0) := c_0.$$

$$\Omega := [\kappa_1 - (\kappa_1 - \kappa_2)a_2 + \frac{1}{2}[a_2\tilde{\sigma}(t,\tilde{s})]^2(\alpha(\tilde{s}) - 1)]\alpha(\tilde{s})$$
$$+ \int_{\mathbb{R}} \{(1 + a_2\tilde{\gamma})^{\alpha} - 1 - a_2\tilde{\gamma}\alpha\}\nu(d\theta).$$

Conclusion & Ongoing works

Summary

- Aggregative structure $\rightarrow$ FINITE dimension SMP
- Linear structure of SMP $\rightarrow$ existence and uniqueness (under H0)
- Price of "symmetrization" $\rightarrow$ unbounded

Ongoing works

- Semi-explicit solutions for risk-sensitive problems
- Multi-level building evacuation, traffic video analytics

Videos at: https://youtu.be/VnODWGDTQsE

THANK YOU


MD200W NEEDWOOD C-209
5.1
SW
MD200W DCKOVR APP C-208
MD200W DCKOVR INT C-207
MD200W OLDE MILL C-204
4.3
E
MD200E. W of 115 C-210
5.6
NE
MD200E DCKOVR INT C-206
MD200W N/METRO RD
MM 2.9
W
MD200W Crabbs Brch C201
2.2
E
MD200W. E of 115 C211
6.5
NW
MD200W/W of MD97 C212
8.0
W
MD200W E of 97 C-213
8.7
W
MD200W Longmead C-214
9.6
W