Systemic Risk and Stochastic Games with Delay

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Coupled Diffusions: Toy Model for Liquidity Model

$X_t^{(i)}, i = 1, \ldots, N$ denote log-monetary reserves of N banks

$$dX_t^{(i)} = b_t^{(i)} dt + \sigma_t^{(i)} dW_t^{(i)} \qquad i = 1, \ldots, N,$$

which are **non-tradable quantities**.
Assume **independent Brownian motions** $W_t^{(i)}$
and **identical constant volatilities** $\sigma_t^{(i)} = \sigma$.

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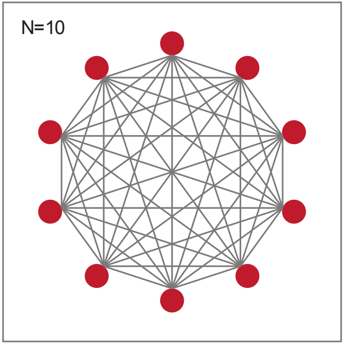
Assume **independent Brownian motions** $W_t^{(i)}$
and **identical constant volatilities** $\sigma_t^{(i)} = \sigma$.
Model **borrowing and lending** through the drifts:

$$dX_t^{(i)} = \frac{a}{N} \sum_{j=1}^{N} (X_t^{(j)} - X_t^{(i)})\, dt + \sigma dW_t^{(i)}, \quad i = 1, \dots, N.$$

The overall **rate of borrowing and lending** a/N has been normalized by the number of banks.

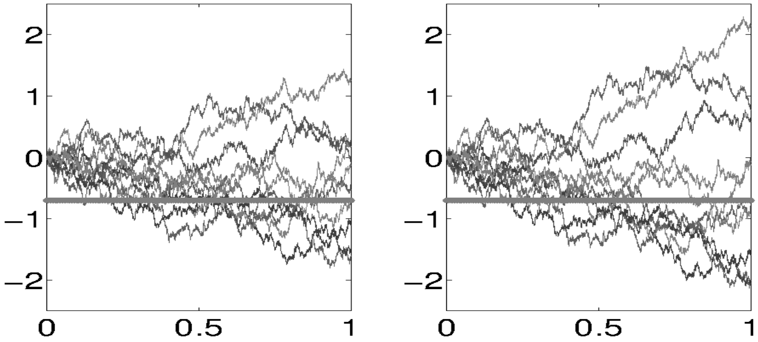
Fully Connected Symmetric Network

$$dX_t^{(i)} = \frac{a}{N}\sum_{j=1}^{N}(X_t^{(j)} - X_t^{(i)})\,dt + \sigma dW_t^{(i)}\,, \quad i = 1,\dots,N.$$



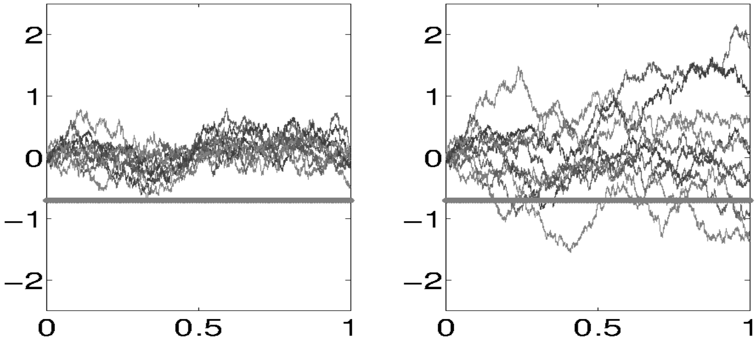
Denote the **default level** by $D < 0$ and simulate the system for various values of **a**: $\mathbf{0}, \mathbf{1}, \mathbf{10}, \mathbf{100}$ with fixed $\sigma = 1$

Weak Coupling: $a = 1$



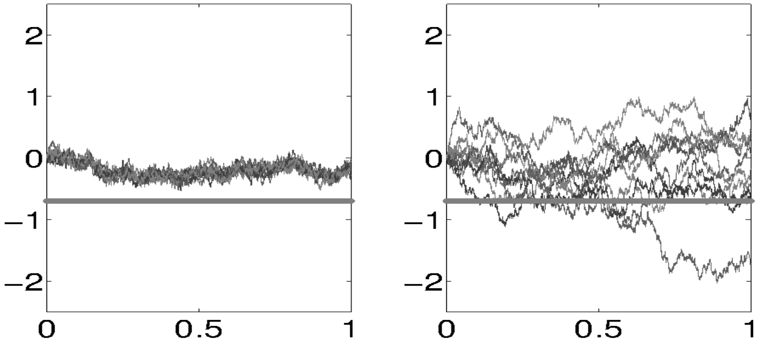
One realization of the trajectories of the coupled diffusions $X_t^{(i)}$ with **$\mathbf{a = 1}$** (left plot) and trajectories of the independent Brownian motions ($a = 0$) (right plot) using the same Gaussian increments. Solid horizontal line: default level $D = -0.7$

Moderate Coupling: $a = 10$



One realization of the trajectories of the coupled diffusions $X_t^{(i)}$ with **a = 10** (left plot) and trajectories of the independent Brownian motions ($a = 0$) (right plot) using the same Gaussian increments. Solid horizontal line: default level $D = -0.7$

Strong Coupling: $a = 100$



One realization of the trajectories of the coupled diffusions $X_t^{(i)}$ with $\mathbf{a = 100}$ (left plot) and trajectories of the independent Brownian motions ($a = 0$) (right plot) using the same Gaussian increments. Solid horizontal line: default level $D = -0.7$

Loss Distributions

These simulations "show" that STABILITY is created by increasing the rate of borrowing and lending.

Next, we compare the **loss distributions** for the coupled and independent cases. We compute these loss distributions by Monte Carlo method using 10^4 simulations, and with the same parameters as previously.

Loss Distributions

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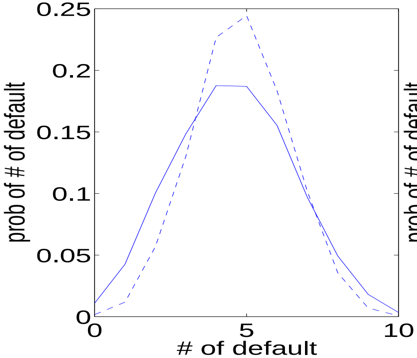
In the independent case, the loss distribution is Binomial(N, p) with parameter p given by

$$\begin{aligned} p &= \boldsymbol{P}\left(\min_{0 \le t \le T}(\sigma W_t) \le D\right) \\ &= 2\Phi\left(\frac{D}{\sigma\sqrt{T}}\right), \end{aligned}$$

where Φ denotes the $\mathcal{N}(0,1)$ cdf.

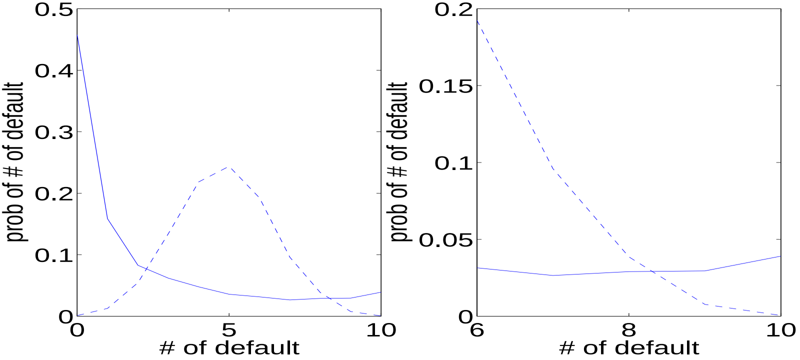
With our choice of parameters, we have $p \approx 0.5$

Loss Distribution: weak coupling



On the left, we show plots of the loss distribution for the coupled diffusions with $\mathbf{a = 1}$ (solid line) and for the independent Brownian motions (dashed line). The plots on the right show the corresponding tail probabilities.

Loss Distribution: moderate coupling



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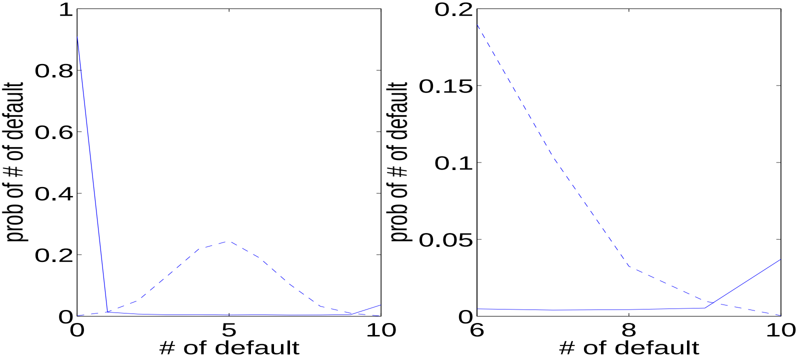
Loss Distribution: strong coupling



On the left, we show plots of the loss distribution for the coupled diffusions with $\mathbf{a = 100}$ (solid line) and for the independent Brownian motions (dashed line). The plots on the right show the corresponding tail probabilities.

Mean Field Limit

Rewrite the dynamics as:

$$
\begin{aligned}
dX_t^{(i)} &= \frac{a}{N}\sum_{j=1}^{N}(X_t^{(j)} - X_t^{(i)})\,dt + \sigma dW_t^{(i)} \\
&= a\left[\left(\frac{1}{N}\sum_{j=1}^{N} X_t^{(j)}\right) - X_t^{(i)}\right]\,dt + \sigma dW_t^{(i)}.
\end{aligned}
$$

The processes $X^{(i)}$'s are "OUs" **mean-reverting** to the **ensemble average** which satisfies

$$
d\left(\frac{1}{N}\sum_{i=1}^{N} X_t^{(i)}\right) = d\left(\frac{\sigma}{N}\sum_{i=1}^{N} W_t^{(i)}\right).
$$

Mean Field Limit

Assuming for instance that $x_0^{(i)} = 0$, $i = 1, \ldots, N$, we obtain

$$\frac{1}{N}\sum_{i=1}^{N} X_t^{(i)} = \frac{\sigma}{N}\sum_{i=1}^{N} W_t^{(i)}, \quad \text{and consequently}$$

$$dX_t^{(i)} = a\left[\left(\frac{\sigma}{N}\sum_{j=1}^{N} W_t^{(j)}\right) - X_t^{(i)}\right] dt + \sigma dW_t^{(i)}\,.$$

Note that the ensemble average is distributed as a Brownian motion with diffusion coefficient $\sigma/\sqrt{N}$.

In the limit $N \to \infty$, the strong law of large numbers gives

$$\frac{1}{N}\sum_{j=1}^{N} W_t^{(j)} \to 0 \quad a.s.\,,$$

and therefore, the processes $X^{(i)}$'s converge to independent OU processes with long-run mean zero.

Mean Field Limit

In fact, $X_t^{(i)}$ is given explicitly by

$$X_t^{(i)} = \frac{\sigma}{N}\sum_{j=1}^N W_t^{(j)} + \sigma e^{-at}\int_0^t e^{as}dW_s^{(i)} - \frac{\sigma}{N}\sum_{j=1}^N\left(e^{-at}\int_0^t e^{as}dW_s^{(j)}\right),$$

and therefore, $X_t^{(i)}$ converges to $\sigma e^{-at}\int_0^t e^{as}dW_s^{(i)}$ which are independent OU processes.

This is a simple example of a **mean-field limit** and propagation of chaos studied in general by Sznitman (1991).

Systemic Risk

Using classical equivalent for the Gaussian cumulative distribution function, we obtain the *large deviation estimate*

$$\lim_{N\to\infty} -\frac{1}{N}\log \boldsymbol{P}\left(\min_{0\le t\le T}\left(\frac{\sigma}{N}\sum_{i=1}^{N} W_t^{(i)}\right)\le D\right) = \frac{D^2}{2\sigma^2 T}.$$

For a large number of banks, the probability that the ensemble average reaches the default barrier is of order $\exp\left(-\frac{D^2 N}{2\sigma^2 T}\right)$

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For a large number of banks, the probability that the ensemble average reaches the default barrier is of order $\exp\left(-\frac{D^2 N}{2\sigma^2 T}\right)$

$$\text{Since}\quad \frac{1}{N}\sum_{i=1}^{N} X_t^{(i)}=\frac{\sigma}{N}\sum_{i=1}^{N} W_t^{(i)},\quad \text{we identify}$$

$$\left\{\min_{0\le t\le T}\left(\frac{1}{N}\sum_{i=1}^{N} X_t^{(i)}\right)\le D\right\}$$ as a **systemic event**

This event does not depend on a. In fact, once in this event, increasing a creates more defaults by **"flocking to default"**.

Recap

So far we have seen that:
"Lending and borrowing improves stability but also contributes to systemic risk"

But how about if the banks compete?
(minimizing costs, maximizing profits,...)

- Can we find an equilibrium in which the previous analysis can still be performed?
- Can we find and characterize a Nash equilibrium?

What follows is from
Mean Field Games and Systemic Risk
by R. Carmona, J.-P. Fouque and L.-H. Sun (2015)

Stochastic Game/Mean Field Game

Banks are borrowing from and lending to a central bank:

$$dX_t^i = \alpha_t^i dt + \sigma dW_t^i, \quad i = 1, \cdots, N$$

where α^i is the control of bank i which wants to **minimize**

$$J^i(\alpha^1, \cdots, \alpha^N) = \boldsymbol{E}\left\{\int_0^T f_i(X_t, \alpha_t^i)dt + g_i(X_T)\right\},$$

with **running cost**

$$f_i(x, \alpha^i) = \left[\frac{1}{2}(\alpha^i)^2 - q\alpha^i(\overline{x} - x^i) + \frac{\epsilon}{2}(\overline{x} - x^i)^2\right], \; q^2 < \epsilon,$$

and **terminal cost** $\quad g_i(x) = \frac{c}{2}\left(\overline{x} - x^i\right)^2.$

This is an example of **Mean Field Game (MFG)** studied extensively by P.L. Lions and collaborators, R. Carmona and F. Delarue, ...

Nash Equilibria (FBSDE Approach)

The Hamiltonian (with Markovian feedback strategies):

$$\begin{aligned} &H^i(x, y^{i,1}, \cdots, y^{i,N}, \alpha^1(t,x), \cdots, \alpha^i_t, \cdots, \alpha^N(t,x)) \\ &= \sum_{k \neq i} \alpha^k(t,x) y^{i,k} + \alpha^i y^{i,i} \\ &+ \frac{1}{2}(\alpha^i)^2 - q\alpha^i(\overline{x} - x^i) + \frac{\epsilon}{2}(\overline{x} - x^i)^2, \end{aligned}$$

Minimizing H^i over α^i gives the choices:

$$\hat{\alpha}^i = -y^{i,i} + q(\overline{x} - x^i), \qquad i = 1, \cdots, N,$$

Ansatz:

$$Y_t^{i,j} = \eta_t \left(\frac{1}{N} - \delta_{i,j} \right) (\overline{X}_t - X_t^i),$$

where η_t is a deterministic function satisfying the terminal condition $\eta_T = c$.

Forward-Backward Equations

Forward Equation:

$$
\begin{aligned}
dX_t^i &= \partial_{y^{i,i}} H^i \, dt + \sigma dW_t^i \\
&= \left[q + (1 - \frac{1}{N})\eta_t \right] (\overline{X}_t - X_t^i) dt + \sigma dW_t^i,
\end{aligned}
$$

with initial conditions $X_0^i = x_0^i$.

Backward Equation:

$$
\begin{aligned}
dY_t^{i,j} &= -\partial_{x^j} H^i \, dt + \sum_{k=1}^{N} Z_t^{i,j,k} dW_t^k \\
&= (\frac{1}{N} - \delta_{i,j})(\overline{X}_t - X_t^i) \left[q\eta_t - \frac{1}{N}(\frac{1}{N} - 1)\eta_t^2 + q^2 - \epsilon \right] dt \\
&\quad + \sum_{k=1}^{N} Z_t^{i,j,k} dW_t^k, \qquad Y_T^{i,j} = c(\frac{1}{N} - \delta_{i,j})(\overline{X}_T - X_T^i).
\end{aligned}
$$

Solution to the Forward-Backward Equations

By summation of the forward equations: $d\overline{X}_t = \frac{\sigma}{N}\sum_{k=1}^{N} dW_t^k$.

Differentiating the ansatz $Y_t^{i,j} = \eta_t\left(\frac{1}{N} - \delta_{i,j}\right)(\overline{X}_t - X_t^i)$, we get:

$$\begin{aligned} dY_t^{i,j} &= \left(\frac{1}{N} - \delta_{i,j}\right)(\overline{X}_t - X_t^i)\left[\dot{\eta}_t - \eta_t\left(q + (1 - \frac{1}{N})\eta_t\right)\right]dt \\ &\quad + \eta_t(\frac{1}{N} - \delta_{i,j})\sigma\sum_{k=1}^{N}(\frac{1}{N} - \delta_{i,k})dW_t^k. \end{aligned}$$

Identifying with the backward equations:

$$Z_t^{i,j,k} = \eta_t\sigma(\frac{1}{N} - \delta_{i,j})(\frac{1}{N} - \delta_{i,k}) \text{ for } \quad k = 1, \cdots, N,$$

and η_t **must satisfy the Riccati equation**

$$\dot{\eta}_t = 2q\eta_t + (1 - \frac{1}{N^2})\eta_t^2 - (\epsilon - q^2),$$

with the terminal condition $\eta_T = c$, solved explicitly.

Financial Implications

1. Once the function η_t has been obtained, bank i implements its strategy by using its control

$$\hat{\alpha}^i_t = -Y^{i,i}_t + q(\overline{X}_t - X^i_t) = \left[q + (1 - \frac{1}{N})\eta_t\right](\overline{X}_t - X^i_t),$$

It requires its own log-reserve X^i_t but also the average reserve $\overline{X}_t$ which may or may not be known to the individual bank i.
Observe that the average $\overline{X}_t$ is given by $d\overline{X}_t = \frac{\sigma}{N}\sum_{k=1}^N dW^k_t$, and is identical to the average found in the uncontrolled case.
Therefore, systemic risk occurs in the same manner as in the case of uncontrolled dynamics.

Financial Implications

2. In fact, the controlled dynamics can be rewritten

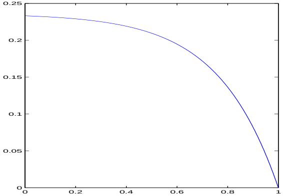
$$dX_t^i = \left(q + (1 - \frac{1}{N})\eta_t\right) \frac{1}{N} \sum_{j=1}^{N} (X_t^j - X_t^i) dt + \sigma dW_t^i.$$

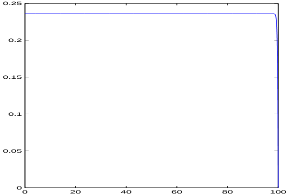
The effect of the banks using their optimal strategies corresponds to inter-bank borrowing and lending at the **effective rate**

$$A_t := q + (1 - \frac{1}{N})\eta_t.$$

Under this equilibrium, the central bank is simply a **clearing house**, and the system is operating as if banks were borrowing from and lending to each other at the rate A_t, and the net effect is **creating liquidity** quantified by the rate of lending/borrowing.

3. For T large (most of the time $T - t$ large), η_t is mainly constant. For instance, with $c = 0$, $\lim_{T\to\infty} \eta_t = \frac{\epsilon - q^2}{-\delta^-} := \overline{\eta}$.





Plots of η_t with $c = 0$, $q = 1$, $\epsilon = 2$ and $T = 1$ on the left, $T = 100$ on the right with $\overline{\eta} \sim 0.24$ (here we used $1/N \equiv 0$).

Therefore, in this infinite-horizon equilibrium, banks are borrowing and lending to each other at the constant rate

$$A := q + (1 - \frac{1}{N})\overline{\eta} = q + \overline{\eta} \quad \text{in the } \textbf{Mean Field Limit}.$$

Pause: Systemic Risk and Large Deviation

In equilibrium we have $dX_t^i = \left[-Y_t^{i,i} + q\left(\overline{X}_t - X_t^i\right)\right] dt + \sigma dW_t^i$ with $Y_t^{i,i} = \eta_t \left(\frac{1}{N} - 1\right)(\overline{X}_t - X_t^i)$.

We are interested in the **fraction of defaults** (say at time T for simplicity):

$$\boldsymbol{P}\left\{ \frac{1}{N} \sum_{1}^{N} 1_{\{X_T^i \le D\}} \ge \alpha \right\}$$

for $\alpha > \boldsymbol{P}\{X_T \le D\}$, in other words a **LDP for the empirical distribution** μ_T^N converging to $\mu_T = Law(X_T)$. In this case $X_T^i, i = 1, \cdots, N$ is Gaussian and one can compute explicitly the rate function

$$\lim_{N\to\infty} -\frac{1}{N} \log \boldsymbol{P}\left\{ \frac{1}{N} \sum_{1}^{N} 1_{\{X_T^i \le D\}} \ge \alpha \right\}.$$

Pause: Large Deviation, MFG, Master Equation

In this simple LQ stochastic game, the **decoupling field** is given explicitly:

$$Y_t^{i,j} = \eta_t \left(\frac{1}{N} - \delta_{i,j} \right) (\overline{X}_t - X_t^i).$$

In the limit $N \to \infty$, for $i = j$, it becomes $\mathbf{Y_t = -\eta_t^\infty (m_t - X_t)}$ where $m_t = \int x \mu_t(dx)$, with μ_t being the law of X_t, and $-\eta_t^\infty$ the solution to the Riccati equation $\dot{\eta}_t = 2q\eta_t + \eta_t^2 - (\epsilon - q^2), \quad \eta_T = c$.
Since $dX_t = [-Y_t + q(m_t - X_t)]\, dt + \sigma dW_t$, we simply have $m_t = m_0 = m$.
If $Y_t^{i,i}$ had not be known explicitly, we would approximate it by

$$\mathbf{\tilde{Y}_t^i = -\eta_t^\infty \left(\frac{1}{N-1} \sum_{j \neq i} \tilde{X}_t^j - \tilde{X}_t^i \right)}$$

so that μ_t^N and $\tilde{\mu}_t^N$ are **exponentially equivalent** (with common limit μ_t).

Stochastic Game/Mean Field Game with Delay

What follows is from: **Systemic Risk and Stochastic Games with Delay**
with R. Carmona, M. Mousavi, and L.-H. Sun (submitted, 2016)

Stochastic Game/Mean Field Game with Delay

Banks are borrowing from and lending to a central bank and money is returned at maturity τ:

$$dX_t^i \;=\; \left[\alpha_t^i - \alpha_{t-\tau}^i\right] dt + \sigma dW_t^i, \quad i = 1, \cdots, N$$

where α^i is the control of bank i which wants to **minimize**

$$\begin{aligned} J^i(\alpha^1, \cdots, \alpha^N) &= \boldsymbol{E}\left\{ \int_0^T f_i(X_t, \alpha_t^i) dt + g_i(X_T) \right\}, \\ f_i(x, \alpha^i) &= \left[\frac{1}{2}(\alpha^i)^2 - q\alpha^i(\overline{x} - x^i) + \frac{\epsilon}{2}(\overline{x} - x^i)^2 \right], \quad q^2 < \epsilon, \\ g_i(x) &= \frac{c}{2}\left(\overline{x} - x^i\right)^2, \\ X_0^i &= \xi^i, \qquad \alpha_t^i = 0, \quad t \in [-\tau, 0). \end{aligned}$$

Case $\tau = 0$: no lending/borrowing $\longrightarrow$ no liquidity.

Case $\tau = T$: no return/delay $\longrightarrow$ full liquidity.

Theorem. The strategy $\hat{\alpha}$ given by

$$\hat{\alpha}^i_t = q(\overline{X}_t - X^i_t) - Y^{i,i}_t + \boldsymbol{E}^{\mathcal{F}_t}(Y^{i,i}_{t+\tau}) \qquad (1)$$

is a **open-loop Nash equilibrium** where (X, Y, Z) is the unique solution to the following system of **FABSDEs**:

$$\begin{aligned}
X^i_t &= \xi^i + \int_0^t \left(\hat{\alpha}^i_s - \hat{\alpha}^i_{s-\tau}\right) ds + \sigma W^i_t, \quad t \in [0, T], && (2) \\
Y^{i,j}_t &= c\left(\frac{1}{N} - \delta_{i,j}\right)(\overline{X}_T - X^i_T) + \int_t^T \left(\frac{1}{N} - \delta_{i,j}\right)\Big\{ (\epsilon - q^2)\left(\overline{X}_s - X^i_s\right) \\
&\quad + qY^{i,j}_s - q\boldsymbol{E}^{\mathcal{F}_s}(Y^{i,j}_{s+\tau})\Big\} ds - \sum_{k=1}^N \int_t^T Z^{i,j,k}_s dW^k_s, \quad t \in [0, T], && (3) \\
Y^{i,j}_t &= 0, \quad t \in (T, T+\tau], \quad i,j = 1, \cdots, N,
\end{aligned}$$

where the processes $Z^{i,j,k}_t$, $k = 1, \cdots, N$ are adapted and square integrable, and $\boldsymbol{E}^{\mathcal{F}_t}$ denotes the conditional expectation with respect to the filtration generated by the Brownian motions.

Outline of the Proof

Proof. Denote by $\tilde{\alpha} = (\hat{\alpha}^{-i}, \tilde{\alpha}^i)$ the strategy obtained from the strategy $\hat{\alpha}$ by replacing the ith component by $\tilde{\alpha}^i$. Denote by $\tilde{X}$, the state generated by $\tilde{\alpha}$ and observe that $X^j = \tilde{X}^j$ for all $j \neq i$ since the dynamics of X^j is only driven by $\hat{\alpha}^j$. We have

$$J^i(\hat{\alpha}) - J^i(\tilde{\alpha}) = \boldsymbol{E}\left\{ \int_0^T \left(f_i(X_t, \hat{\alpha}^i_t) - f_i(\tilde{X}_t, \tilde{\alpha}^i_t) \right) dt + g_i(X_T) - g_i(\tilde{X}_T) \right\}. \tag{4}$$

Since g_i is convex in x, we obtain that

$$\begin{aligned} g_i(x) - g_i(\tilde{x}) &\leq \partial_x g_i(x)(x - \tilde{x}) \\ &= \partial_{x^i} g_i(x)(x^i - \tilde{x}^i), \end{aligned}$$

for $\tilde{x}$ such that $\tilde{x}^j = x^j$ for $j \neq i$. Therefore,

$$\begin{aligned} \boldsymbol{E}(g_i(X_T) - g_i(\tilde{X}_T)) &\leq \boldsymbol{E}(\partial_{x^i} g_i(X_T)(X^i_T - \tilde{X}^i_T)) \\ &= \boldsymbol{E}(Y^{i,i}_T(X^i_T - \tilde{X}^i_T)). \end{aligned}$$

Outline of the Proof

Applying Itô's formula, we have

$$
\begin{aligned}
&\boldsymbol{E}(Y_T^{i,i}(X_T^i - \tilde{X}_T^i)) \\
= \boldsymbol{E}\int_0^T \Big\{ &-(X_t^i - \tilde{X}_t^i)(\frac{1}{N} - 1)\Big\{(\epsilon - q^2)(\overline{X}_t - X_t^i) + qY_t^{i,i} - q\boldsymbol{E}^{\mathcal{F}_t}(Y_{t+\tau}^{i,i})\Big\} \\
&+ Y_t^{i,i}\left(\hat{\alpha}_t^i - \tilde{\alpha}_t^i - (\hat{\alpha}_{t-\tau}^i - \tilde{\alpha}_{t-\tau}^i)\right)\Big\} dt. \qquad (5)
\end{aligned}
$$

Then, we write

$$
\begin{aligned}
\boldsymbol{E}\int_0^T Y_t^{i,i}\left(\hat{\alpha}_{t-\tau}^i - \tilde{\alpha}_{t-\tau}^i\right) dt &= \boldsymbol{E}\int_{-\tau}^{T-\tau} Y_{s+\tau}^{i,i}\left(\hat{\alpha}_s^i - \tilde{\alpha}_s^i\right) ds \\
= \boldsymbol{E}\int_0^T Y_{s+\tau}^{i,i}\left(\hat{\alpha}_s^i - \tilde{\alpha}_s^i\right) ds &= \boldsymbol{E}\int_0^T \boldsymbol{E}^{\mathcal{F}_s}(Y_{s+\tau}^{i,i})\left(\hat{\alpha}_s^i - \tilde{\alpha}_s^i\right) ds. \qquad (6)
\end{aligned}
$$

since $\hat{\alpha}_t^i = \tilde{\alpha}_t^i = 0$ for $t \in [-\tau, 0)$ and $Y_t^{i,i} = 0$ for $t \in (T, T+\tau]$.

Outline of the Proof

Plugging (6) into (5), we obtain

$$\begin{aligned}
&\boldsymbol{E}(Y_T^{i,i}(X_T^i - \tilde{X}_T^i)) \\
= \boldsymbol{E}\int_0^T &\Big\{ -(X_t^i - \tilde{X}_t^i)(\frac{1}{N} - 1)\Big\{(\epsilon - q^2)(\overline{X}_t - X_t^i) + qY_t^{i,i} - q\boldsymbol{E}^{\mathcal{F}_t}(Y_{t+\tau}^{i,i})\Big\} \\
&+ \left(Y_t^{i,i} - \boldsymbol{E}^{\mathcal{F}_t}(Y_{t+\tau}^{i,i})\right)(\hat{\alpha}_t^i - \tilde{\alpha}_t^i)) \Big\} dt. \qquad (7)
\end{aligned}$$

Using (4) and convexity of f_i in x and α^i, and $X_t^j = \tilde{X}_t^j$ for $j \neq i$, we deduce

$$\begin{aligned}
& J^i(\hat{\alpha}) - J^i(\tilde{\alpha}) \\
\leq \ & \boldsymbol{E}\int_0^T (\partial_{x^i} f_i(X_t, \hat{\alpha}_t^i))(X_t^i - \tilde{X}_t^i) + \partial_{\alpha^i} f^i(X_t, \hat{\alpha}_t^i)(\hat{\alpha}_t^i - \tilde{\alpha}_t^i)dt \\
& + \boldsymbol{E}(Y_T^{i,i}(X_T^i - \tilde{X}_T^i))
\end{aligned}$$

Outline of the Proof

Using (7) we get

$$
\begin{aligned}
&= \boldsymbol{E}\int_0^T \Big\{ (\frac{1}{N}-1)\left(-q\hat{\alpha}_t^i + \epsilon(\overline{X}_t - X_t^i)\right)(X_t^i - \tilde{X}_t^i) \\
&\qquad + \left(-q(\overline{X}_t - X_t^i) + \hat{\alpha}_t^i\right)(\hat{\alpha}_t^i - \tilde{\alpha}_t^i)\Big\} dt \\
&+\boldsymbol{E}\int_0^T \Big\{ -(X_t^i - \tilde{X}_t^i)(\frac{1}{N}-1)\Big\{(\epsilon - q^2)(\overline{X}_t - X_t^i) + qY_t^{i,i} - q\boldsymbol{E}^{\mathcal{F}_t}(Y_{t+\tau}^{i,i})\Big\} \\
&\qquad + \left(Y_t^{i,i} - \boldsymbol{E}^{\mathcal{F}_t}(Y_{t+\tau}^{i,i})\right)(\hat{\alpha}_t^i - \tilde{\alpha}_t^i) \Big\} dt \\
&= \boldsymbol{E}\int_0^T \left[q(\frac{1}{N}-1)(X_t^i - \tilde{X}_t^i) + (\hat{\alpha}_t^i - \tilde{\alpha}_t^i) \right] \\
&\qquad\qquad \times \left[-\hat{\alpha}_t^i + q(\overline{X}_t - X_t^i) - Y_t^{i,i} + \boldsymbol{E}^{\mathcal{F}_t}(Y_{t+\tau}^{i,i}) \right] dt \\
&= 0\,,
\end{aligned}
$$

where in the last step we used the form of $\hat{\alpha}_t^i$ given by (1).

Existence

Therefore, the strategy $\hat{\alpha}$ is a **Nash equilibrium for the open-loop game with delay** **provided that the FABSDE system (2)-(3) admits a solution**.
This is shown by a **continuation argument** introduced by Shige Peng in the context of stochastic control problems. This is quite technical and we refer to the Appendix in the paper.

Existence, no Uniqueness

Therefore, the strategy $\hat{\alpha}$ is a **Nash equilibrium for the open-loop game with delay provided that the FABSDE system (2)-(3) admits a solution**.
This is shown by a **continuation argument** introduced by Shige Peng in the context of stochastic control problems. This is quite technical and we refer to the Appendix in the paper.

In general, there is **no uniqueness** of Nash equilibrium for the open-loop game with delay.
We observe that in contrast with the case without delay, there is **no simple explicit formula** for the optimal strategy $\hat{\alpha}$ given by (1).

Clearing House Property

Summing over $i = 1, \cdots, N$ the equations for $Y^{i,i}$ and denoting $\overline{Y}_t = \frac{1}{N}\sum_{i=1}^N Y_t^{i,i}$, $\overline{Z}_t^k = \frac{1}{N}\sum_{i=1}^N Z_t^{i,i,k}$, gives

$$\begin{aligned} d\overline{Y}_t &= -\left(\frac{1}{N}-1\right) q\left(\overline{Y}_t - \mathbf{E}^{\mathcal{F}_t}(\overline{Y}_{t+\tau})\right) dt + \sum_{k=1}^N \overline{Z}_t^k dW_t^k, \quad t \in [0,T], \\ \overline{Y}_t &= 0, \quad t \in [T, T+\tau], \end{aligned}$$

which admits the unique solution

$$\overline{Y}_t = 0, \quad t \in [0, T+\tau], \quad \text{with} \quad \overline{Z}_t^k = 0,\ k = 1, \cdots, N,\ t \in [0,T].$$

Summing over $i = 1, \cdots, N$ the equations for $\hat{\alpha}_t^i$ gives

$$\sum_{\mathbf{i=1}}^{\mathbf{N}} \hat{\alpha}_{\mathbf{t}}^{\mathbf{i}} = \sum_{i=1}^N \left[q(\overline{X}_t - X_t^i) - Y_t^{i,i} + \mathbf{E}^{\mathcal{F}_t}(Y_{t+\tau}^{i,i}) \right] = 0.$$

Note that $\overline{X}_t = \overline{\xi} + \frac{\sigma}{N}\sum_{i=1}^N W_t^i$ as in the case with no delay.

Infinite-dimensional HJB Approach

Following Gozzi and Marinelli (2004). Let $\boldsymbol{H}^N$ be the Hilbert space defined by $\boldsymbol{H}^N = \boldsymbol{R}^N \times L^2([-\tau, 0]; \boldsymbol{R}^N)$, with the inner product $\langle z, \tilde{z} \rangle = z_0 \tilde{z}_0 + \int_{-\tau}^{0} z_1(\xi) \tilde{z}_1(\xi)\, d\xi$, where $z, \tilde{z} \in \boldsymbol{H}^N$, and z_0 and $z_1(.)$ correspond respectively to the $\boldsymbol{R}^N$-valued and $L^2([-\tau, 0]; \boldsymbol{R}^N)$-valued components (the states and the past of the strategies in our case).

In order to use the **dynamic programming principle** for stochastic game in search of a **closed-loop Nash equilibrium**, at time $t \in [0, T]$, given the initial state $Z_t = z$, bank i chooses the control α^i to minimise its objective function $J^i(t, z, \alpha)$.

$$J^i(t, z, \alpha) = \boldsymbol{E} \left\{ \int_t^T f_i(Z_{0,s}, \alpha_s^i) dt + g_i(Z_{0,T}) \mid Z_t = z \right\},$$

See also *Stochastic Control and Differential Games with Path-Dependent Controls* by Yuri Saporito (2017) for a FITO (PPDE) approach.

Coupled HJB Equations

Bank i's value function $V^i(t,z)$ is

$$V^i(t,z) = \inf_{\alpha} J^i(t,z,\alpha).$$

The set of **value functions** $V^i(t,z)$, $i = 1, \cdots, N$ is the unique solution (in a suitable sense) of the following system of **coupled HJB equations**:

$$\partial_t V^i + \frac{1}{2} Tr(Q \partial_{zz} V^i) + \langle Az, \partial_z V^i \rangle + H_0^i(\partial_z V^i) = 0,$$
$$V^i(T) = g_i,$$

$$Q = G * G, \quad G : z_0 \to (\sigma z_0, 0),$$

$$A : (z_0, z_1(\gamma)) \to (z_1(0), -\frac{dz_1(\gamma)}{d\gamma}) \quad a.e., \quad \gamma \in [-\tau, 0],$$

$$H_0^i(p^i) = \inf_{\alpha} [\langle B\alpha, p^i \rangle + f_i(z_0, \alpha^i)], \quad p^i \in \boldsymbol{H}^N,$$

$$B : u \to (u, -\delta_{-\tau}(\gamma) u), \quad \gamma \in [-\tau, 0].$$

Ansatz

By convexity of $f_i(z_0, \alpha^i)$ with respect to (z_0, α^i),

$$\hat{\alpha}^i = -\langle B, p^{i,i}\rangle - q(z_0^i - \bar{z}_0), \quad \text{and}$$

$$\begin{aligned} H_0^i(p^i) &= \langle B\hat{\alpha}, p^i\rangle + f_i(z_0, \hat{\alpha}^i), \\ &= \sum_{k=1}^{N} \langle B, p^{i,k}\rangle \left(-\langle B, p^{k,k}\rangle - q(z_0^k - \bar{z}_0) \right) \\ &+ \frac{1}{2}\langle B, p^{i,i}\rangle^2 + \frac{1}{2}(\epsilon - q^2)(\bar{z}_0 - z_0^i)^2. \end{aligned}$$

We then make the **ansatz**

$$\begin{aligned} V^i(t,z) &= E_0(t)(\bar{z}_0 - z_0^i)^2 - 2(\bar{z}_0 - z_0^i)\int_{-\tau}^{0} E_1(t, -\tau-\theta)(\bar{z}_{1,\theta} - z_{1,\theta}^i)d\theta \\ &+ \int_{-\tau}^{0}\int_{-\tau}^{0} E_2(t, -\tau-\theta, -\tau-\gamma)(\bar{z}_{1,\theta} - z_{1,\theta}^i)(\bar{z}_{1,\gamma} - z_{1,\gamma}^i)d\theta d\gamma + E_3(t). \end{aligned}$$

Partial Derivatives

$$\partial_t V^i = \frac{dE_0(t)}{dt}(\bar{z}_0 - z_0^i)^2 - 2(\bar{z}_0 - z_0^i)\int_{-\tau}^{0} \frac{\partial E_1(t, -\tau-\theta)}{\partial t}(\bar{z}_{1,\theta} - z_{1,\theta}^i)d\theta$$

$$+\int_{-\tau}^{0}\int_{-\tau}^{0} \frac{\partial E_2(t, -\tau-\theta, -\tau-\gamma)}{\partial t}(\bar{z}_{1,\theta} - z_{1,\theta}^i)(\bar{z}_{1,\gamma} - z_{1,\gamma}^i)d\theta d\gamma + \frac{dE_3(t)}{dt},$$

$$\partial_{z^j} V^i =$$

$$\left[\begin{array}{c} 2E_0(t)(\bar{z}_0 - z_0^i) - 2\int_{-\tau}^{0} E_1(t, -\tau-\theta)(\bar{z}_{1,\theta} - z_{1,\theta}^i)d\theta \\ -2(\bar{z}_0 - z_0^i)E_1(t,\theta) + 2\int_{-\tau}^{0} E_2(t, -\tau-\theta, -\tau-\gamma)(\bar{z}_{1,\gamma} - z_{1,\gamma}^i)d\gamma \end{array}\right]\left(\frac{1}{N} - \delta_{i,j}\right),$$

$$\partial_{z^j z^k} V^i = \left[\begin{array}{cc} 2E_0(t) & -2E_1(t, -\tau-\theta) \\ -2E_1(t, -\tau-\theta) & 2E_2(t, -\tau-\theta, -\tau-\gamma) \end{array}\right]\left(\frac{1}{N} - \delta_{i,j}\right)\left(\frac{1}{N} - \delta_{i,k}\right),$$

and plug in the HJB equation.

PDEs for the coefficients E_i, $i = 0, 1, 2, 3, 4$

The equation corresponding to the **constant terms** is

$$\frac{dE_3(t)}{dt} + (1 - \frac{1}{N})\sigma^2 E_0(t) = 0,$$

The equation corresponding to the $(\bar{z}_0 - z_0^i)^2$ **terms** is

$$\frac{dE_0(t)}{dt} + \frac{\epsilon}{2} = 2(1 - \frac{1}{N^2})(E_1(t,0) + E_0(t))^2 + 2q(E_1(t,0) + E_0(t)) + \frac{q^2}{2}.$$

The equation corresponding to the $(\bar{z}_0 - z_0^i)(\bar{z}_1 - z_1^i)$ **terms** is

$$\frac{\partial E_1(t,\theta)}{\partial t} - \frac{\partial E_1(t,\theta)}{\partial \theta} = \left[2(1 - \frac{1}{N^2})\left(E_1(t,0) + E_0(t)\right) + q\right]\left(E_2(t,\theta,0) + E_1(t,\theta)\right).$$

The equation corresponding to the $(\bar{z}_1 - z_1^i)(\bar{z}_1 - z_1^i)$ **terms** is

$$\frac{\partial E_2(t,\theta,\gamma)}{\partial t} - \frac{\partial E_2(t,\theta,\gamma)}{\partial \theta} - \frac{\partial E_2(t,\theta,\gamma)}{\partial \gamma} =$$
$$2(1 - \frac{1}{N^2})\left(E_2(t,\theta,0) + E_1(t,\theta)\right)\left(E_2(t,\gamma,0) + E_1(t,\gamma)\right).$$

Boundary Conditions

$$
\begin{aligned}
&E_0(T) = \frac{c}{2}, \\
&E_1(T, \theta) = 0, \\
&E_2(T, \theta, \gamma) = 0, \\
&E_2(t, \theta, \gamma) = E_2(t, \gamma, \theta), \\
&E_1(t, -\tau) = -E_0(t), \quad \forall t \in [0, T), \\
&E_2(t, \theta, -\tau) = -E_1(t, \theta), \quad \forall t \in [0, T), \\
&E_3(T) = 0.
\end{aligned}
$$

We have existence and uniqueness for this system of PDEs

Optimal Strategies

$$
\begin{aligned}
\hat{\alpha}^i_t \quad & = -\langle B, \partial_{z^i} V^i \rangle - q(z^i_0 - \bar{z}_0), \\
& = 2\left(1 - \frac{1}{N}\right) \left[\left(E_1(t,0) + E_0(t) + \frac{q}{2\left(1-\frac{1}{N}\right)} \right) (\bar{z}_0 - z^i_0) \right. \\
& \left. - \int_{-\tau}^{0} \left(E_2(t, -\tau-\theta, 0) + E_1(t, -\tau-\theta) \right) (\bar{z}_{1,\theta} - z^i_{1,\theta}) d\theta \right].
\end{aligned}
$$

In terms of the original system of coupled diffusions, the **closed-loop Nash equilibrium** corresponds to

$$
\begin{aligned}
\hat{\alpha}^i_t \quad & = \left[2\left(1 - \frac{1}{N}\right) \left(E_1(t,0) + E_0(t) \right) + q \right] (\bar{X}_t - X^i_t) \\
& +2 \int_{t-\tau}^{t} \left[E_2(t, \theta - t, 0) + E_1(t, \theta - t) \right] (\bar{\hat{\alpha}}_\theta - \hat{\alpha}^i_\theta) d\theta, \quad i = 1, \cdots, N.
\end{aligned}
$$

Clearing house property: $\sum_{i=1}^N \hat{\alpha}^i_t = 0.$

Closed-loop Nash Equilibria: Verification Theorem

At time $t \in [0, T]$, given $X_t = x$ and $\alpha_{[t)} = (\alpha_\theta)_{\theta \in [t-\tau, t)}$, bank i chooses the strategy α^i to minimise its objective function

$$J^i(t, x, \alpha, \alpha^i) = \boldsymbol{E}\left\{ \int_t^T f_i(X_s, \alpha_s^i) ds + g_i(X_T) \mid X_t = x, \alpha_{[t)} = \alpha \right\}.$$

Bank i's value function $V^i(t, x, \alpha)$ is

$$V^i(t, x, \alpha) = \inf_{\alpha^i} J^i(t, x, \alpha, \alpha^i).$$

Guessing that the value function should be quadratic in the state and in the past of the control, we make the following **ansatz** for the value function:

Ansatz (from HJB formal derivation)

$$
\begin{aligned}
V^i(t,x,\alpha) &= E_0(t)(\bar{x}-x^i)^2 + 2(\bar{x}-x^i)\int_{t-\tau}^{t} E_1(t,\theta-t)(\bar{\alpha}_\theta-\alpha^i_\theta)d\theta \\
&+ \int_{t-\tau}^{t}\int_{t-\tau}^{t} E_2(t,\theta-t,\gamma-t)(\bar{\alpha}_\theta-\alpha^i_\theta)(\bar{\alpha}_\gamma-\alpha^i_\gamma)d\theta d\gamma + E_3(t),
\end{aligned}
$$

where $E_0(t)$, $E_1(t,\theta)$, $E_2(t,\theta,\gamma)$, $E_3(t)$, are deterministic functions satisfying the particular system of partial differential equations for $t \in [0,T]$ and $\theta, \gamma \in [-\tau, 0]$ obtained before.

Itô's formula

Applying Itô's formula to $V^i(t, X_t, \alpha_{[t)})$, we obtain the following expression for the nonnegative quantity

$$\boldsymbol{E} V^i(T, X_T, \alpha_{[T)}) - V^i(0, \xi^i, \alpha_{[0)}) + \boldsymbol{E} \int_0^T f^i(X_s, \alpha_s^i) dt$$

A long computation and use of the system of PDEs for the E_i's $\longrightarrow$

Outline of proof

$$
\begin{aligned}
&\boldsymbol{E}\int_0^T \Bigg\{ \frac{1}{2}\Bigg(\alpha_t^i - 2\left(E_1(t,0) + E_0(t) + \frac{q}{2}\right)(\bar{X}_t - X_t^i) \\
&-2\int_{t-\tau}^{t} \left[E_2(t,\theta - t, 0) + E_1(t,\theta - t)\right](\bar{\alpha}_\theta - \alpha_\theta^i)d\theta \Bigg)^2 \\
&+2(\bar{\alpha}_t - \bar{\alpha}_{t-\tau})\left[E_0(t)(\bar{X}_t - X_t^i) + \int_{t-\tau}^{t} E_1(t,\theta - t)(\bar{\alpha}_\theta - \alpha_\theta^i)d\theta\right] \\
&+2\bar{\alpha}_t\left[(E_1(t,0) - \frac{q}{2})(\bar{X}_t - X_t^i) + \int_{t-\tau}^{t} E_2(t,\theta - t, 0)(\bar{\alpha}_\theta - \alpha_\theta^i)d\theta\right] \\
&-2\bar{\alpha}_{t-\tau}\left[E_1(t,-\tau)(\bar{X}_t - X_t^i) + \int_{t-\tau}^{t} E_2(t,\theta - t, -\tau)(\bar{\alpha}_\theta - \alpha_\theta^i)d\theta\right] \Bigg\} dt.
\end{aligned}
$$

Result

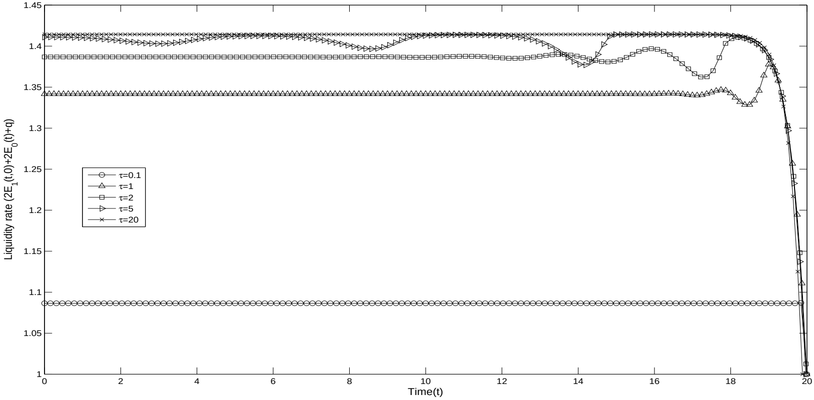
An optimal strategy can be characterized as the strategy $\hat{\alpha}$ which makes the previous quantity equal to zero.
Therefore, if all the other banks choose their optimal strategies, bank i's optimal strategy $\hat{\alpha}^i$ should satisfy

$$\begin{aligned}\hat{\alpha}^i_t &= 2\left[E_1(t,0) + E_0(t) + \frac{q}{2}\right](\bar{X}_t - X^i_t) \\ &+ 2\int_{t-\tau}^{t}\left[E_2(t,\theta-t,0) + E_1(t,\theta-t)\right](\bar{\hat{\alpha}}_\theta - \hat{\alpha}^i_\theta)d\theta, \\ &\qquad\qquad\qquad \text{for} \quad i = 1,\cdots,N,\end{aligned}$$

since, with that choice, the square term in the integral is zero, and the three other terms vanish because $\bar{\alpha}_t = \bar{\alpha}_{t-\tau} = 0$ (by summing over i).

Effect of delay on liquidity

$T = 20, q = 1, \varepsilon = 2, c = 0$



Work in progress with Zhaoyu Zhang
To be continued ...