Homogenization results for viscoelastic bodies

at fixed frequency

P. Seppecher (in collaboration with G. Milton)

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1. Introduction
2. Variational form of wave propagation equations
3. Epi-hypo-convergence
4. General strategy
5. A first homogenization result
6. A second homogenization result
7. Response matrices of elastodynamic networks
8. Discretization
9. Regularization
10. Conclusion

Introduction

The question

- What can be the homogenized behavior of non-homogenous high-contrasted materials?
  - The answer was given (*Camar-Eddine and Seppecher, M3AS, 2002*) for the diffusion equation : the limit equation results from the minimization of some Dirichlet form (a l.s.c. quadratic energy satisfying the maximum principle) vanishing on constant fields.
  - The answer was given (*Camar-Eddine and Seppecher, ARMA, 2003*) for the elastostatic equation : the limit equation results from the minimization of some l.s.c. non-negative objective quadratic form. The set of possible behavior is much richer than in the diffusion case.
- Can we recover these limit behaviors in the dynamic case ?
- Can we find more ?

On a domain $\mathcal{D} \subset \mathbb{R}$, we consider the viso-elastic problem

$$\rho \frac{\partial^2 U}{\partial t^2} = \operatorname{div} \left(E \cdot e(U) + D \cdot e(\frac{\partial U}{\partial t}) \right)$$

- ρ : (positive) volume mass density
- U : displacement field (a vector valued function)
- $e(U)$ (or $e(\frac{\partial U}{\partial t})$) : strain (or rate-of-strain) tensor $2e(U) = \nabla U + \nabla^T U$.
- E, D : elasticity and viscosity tensors (positive with appropriate symmetries)

In the harmonic regime (at fixed frequency ω) we set

$$U(x,t) = \mathrm{Re}(\hat{U}(x)e^{-i\omega t}) \quad \text{and} \quad C = E - i\omega D$$

Equation of dynamics becomes

$$\rho\omega^2 \hat{U} + \mathrm{div}\left(C \cdot e(\hat{U})\right) = 0$$

In non-homogeneous case ρ and C depends on x.

Assume that we have at our disposal materials as soft or hard and as light or heavy as we need.
Let (ρ_n, C_n) a sequence of such (hight contrasted) materials.

What type of homogenized material can we obtain ? (a “closure” question)

Is it possible to get

- negative mass densities ?
- non scalar mass tensors ?
- negative elasticity tensors ?
- negative viscosity tensors ?
- non local effects ?
- higher gradients materials ?
  (involving for instance div(div $\left(H \cdot \nabla\nabla(\hat{U})\right)$?)

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- non scalar mass tensors ? yes
- negative elasticity tensors ? yes
- negative viscosity tensors ? no
- non local effects ? yes
- higher gradients materials ? yes
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Variational form of wave propagation equations

Setting $\hat{U} = u + iv$, the quantity $\int_\Omega \mathrm{Im}\left(\omega^2 \rho \hat{U}\cdot\hat{U} - e(\hat{U})\cdot C\cdot e(\hat{U})\right)$ reads

$$Q_{\rho,C}(u,v) = \int_\Omega 2\omega^2\rho u\cdot v - 2e(u)\cdot E\cdot e(v) + \omega\, e(u)\cdot D\cdot e(u) - \omega\, e(v)\cdot D\cdot e(v)$$

- $Q_{\rho,C}$ is quadratic
- $Q_{\rho,C}$ has the symmetry property : $Q_{\rho,C}(u,v) = -Q_{\rho,C}(-v,u)$.
- $u \longrightarrow Q_{\rho,C}(u,v)$ is convex, $v \longrightarrow Q_{\rho,C}(u,v)$ is concave
- any saddle point $(\bar{u},\bar{v})$ of $Q_{\rho,C}$ (submitted to some boundary conditions) satisfies

$$Q_{\rho,C}(\bar{u},\bar{v}) = \min_u \max_v Q_{\rho,C}(u,v) = \max_v \min_u Q_{\rho,C}(u,v)$$

whose Euler equations read :

$$\begin{cases} 2\omega^2 v + \mathrm{div}(2E \cdot e(v)) - \mathrm{div}(2\omega D \cdot e(v)) & = 0 \\ 2\omega^2 u + \mathrm{div}(2E \cdot e(u)) + \mathrm{div}(2\omega D \cdot e(u)) & = 0 \end{cases}$$

which coincide with the imaginary and real parts of the complex equation of dynamics.

- At first, $Q_{\rho,C}$ is defined on $H^1(\Omega,\mathbb{R}^3) \times H^1(\Omega,\mathbb{R}^3)$.
  We extend it on $L^2(\Omega,\mathbb{R}^3) \times L^2(\Omega,\mathbb{R}^3)$ by setting

$$Q_{\rho,C}(u,v) = \begin{cases} +\infty & \text{if } u \notin H^1(\Omega,\mathbb{R}^3), \\ -\infty & \text{if } u \in H^1(\Omega,\mathbb{R}^3) \text{ and } v \notin H^1(\Omega,\mathbb{R}^3). \end{cases}$$

without changing the saddle points.

Remarks :

- Using one of the partial Fenchel transforms

$$F_{\rho,C}(u,v^*) = \sup_v \{< v^*, v >_{L^2} + Q_{\rho,C}(u,v)\}$$

$$G_{\rho,C}(u^*,v) = \sup_u \{< u^*, u >_{L^2} - Q_{\rho,C}(u,v)\}$$

we could transform the saddle point problem into a convex minimisation problem. But the symmetry $u \leftrightarrow v$ would be broken.

- $F_{\rho,C}$ and $G_{\rho,C}$ are conjugate from each other:

$$F_{\rho,C}(u,v^*) = \sup_{(u^*,v)} \{< u^*, u >_{L^2} + < v^*, v >_{L^2} - G_{\rho,C}(u^*,v)\}.$$

The functional $Q_{\rho,C}$ is said to be closed.

Epi-hypo-convergence

For any convex-concave function Q, we define

$$\underline{Q}(x,y) := \inf\{\liminf_n Q(x_n,y);\ x_n \to x\}$$

$$\overline{Q}(x,y) := \sup\{\limsup_n Q(x,y_n);\ y_n \to y\}$$

with the convention that, if $\underline{Q}(x,y) = -\infty$ for some x, then $\underline{Q}(x,y) := -\infty$ for all x (similar convention for $\overline{Q}(x,y)$).

For any sequence of convex-concave functions Q_n, we define (following *Attouch and Wets, Trans.AMS, 1983*)

$$li\, Q_n(x,y) := \begin{cases} \inf_{x_n \overset{\mathrm{L}^2}{\rightharpoonup} x} \sup_{y_n \overset{\mathrm{H}^1}{\longrightarrow} y} \liminf Q_n(x_n,y_n) & \text{if } y \in \mathrm{H}^1 \\ \inf_{x_n \overset{\mathrm{L}^2}{\rightharpoonup} x} \sup_{y_n \overset{\mathrm{L}^2}{\longrightarrow} y} \liminf Q_n(x_n,y_n) & \text{otherwise} \end{cases}$$

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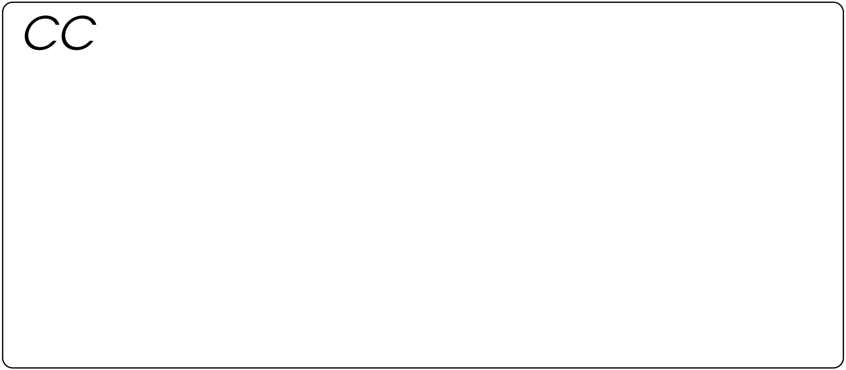
We say that $Q_n \xrightarrow{e-h} Q$ if $\underline{ls\, Q_n}(x,y) \leq Q \leq \overline{li\, Q_n}(x,y)$

- This a reinforced notion of epi-hypo convergence
  - $Q_n \xrightarrow{e-h} Q \Rightarrow$ extended epi-hypo convergence of (Q_n) for the L^2 topology.
  - $Q_n \xrightarrow{e-h} Q \Rightarrow$ extended epi-hypo convergence for the H^1 topology of the restrictions of (Q_n) to H^1.
  - $Q_n \xrightarrow{e-h} Q \Rightarrow$ convergence of the saddle points of Q_n to a saddle pt of Q.
- Reitered homogenization is possible
  - Extended epi-hypo convergence is equivalent to the Mosco-convergence of partial Fenchel transforms,
  - $Q_n \xrightarrow{e-h} Q \Rightarrow$ is a metrizable notion.
  - Diagonalization procedure is possible.

- The framework
  - Any e-h-limit is a convex-concave closed function.
  - Any e-h-limit of a sequence of quadratic functions is quadratic.
  - The symmetry property $Q_n(u,v) = -Q_n(-v,u)$ is preserved when passing to the limit.
  - We denote $\mathcal{CC}$ the set of closed quadratic convex-concave functions on L^2 satisfying $Q(u,v) = -Q(-v,u)$.
- Our goal : prove that the set $\mathcal{A}$ of all "achievable" functions (those which can be obtained as the limit of a sequence (Q_{ρ_n,C_n})) coincides with $\mathcal{CC}$.
- Some consequences of such a result:
  - All function $(Q_{\rho,C})$ can be obtained under the only restriction $-D = \mathrm{Im}(C) < 0$ (positive dissipation).
  - There is no restriction on the sign of ρ or $E = \mathrm{Re}(C)$.
  - Many other homogenized responses are possible (non scalar mass tensors, non local transmission, higher gradients . . .).

General strategy

A step by step approach using reiterated homogenization



- $\mathcal{CC}$: whole set of potentially achievable functionals

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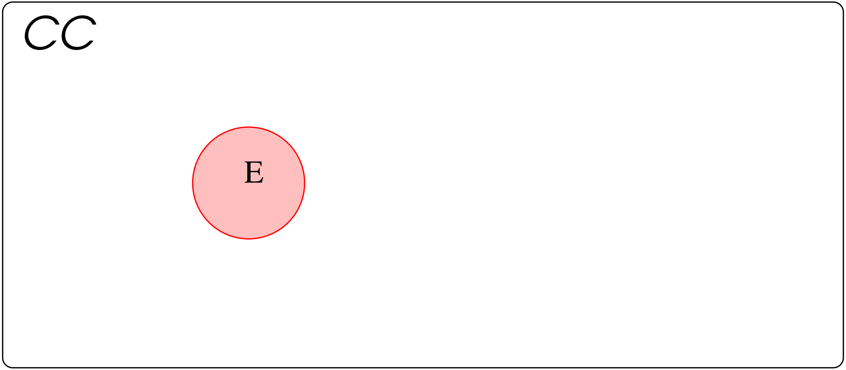

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- E : Starting set of functionals (positive mass density, positive isotropic elasticity, uniformely elliptic viscosity tensor).

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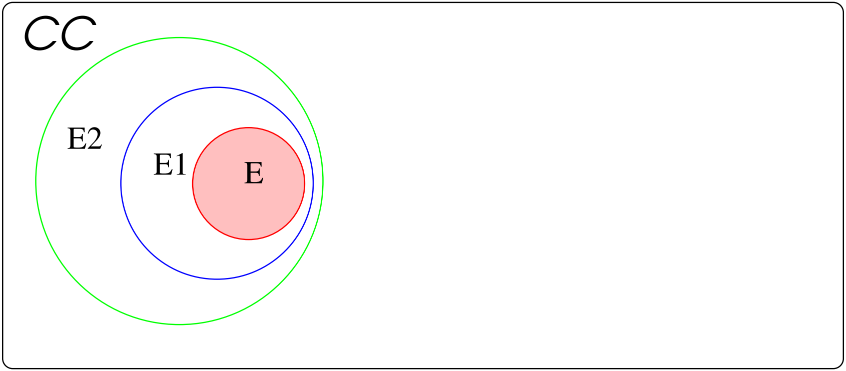

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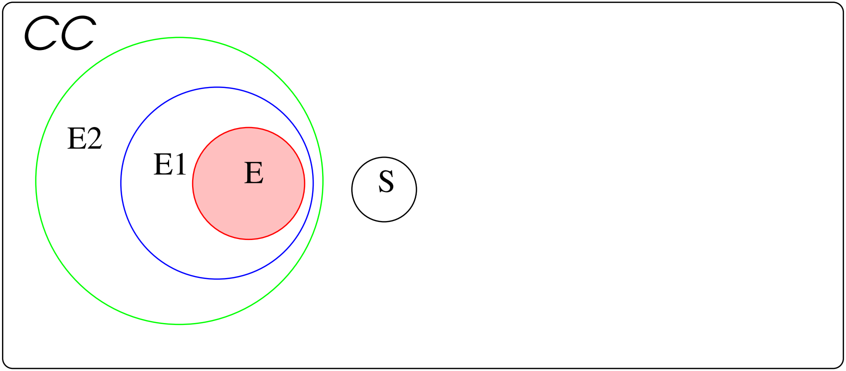

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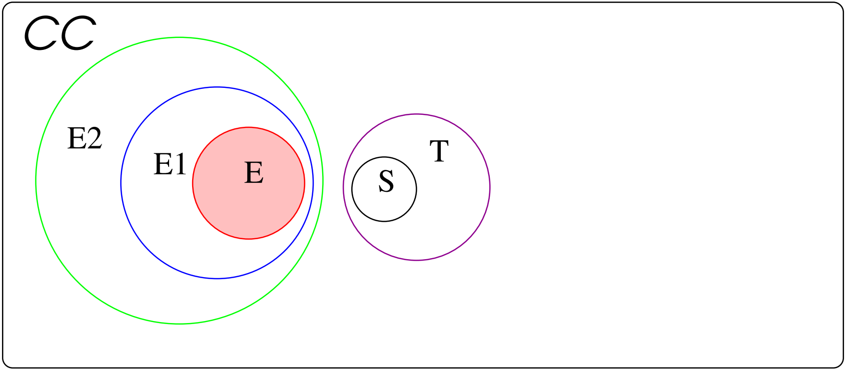

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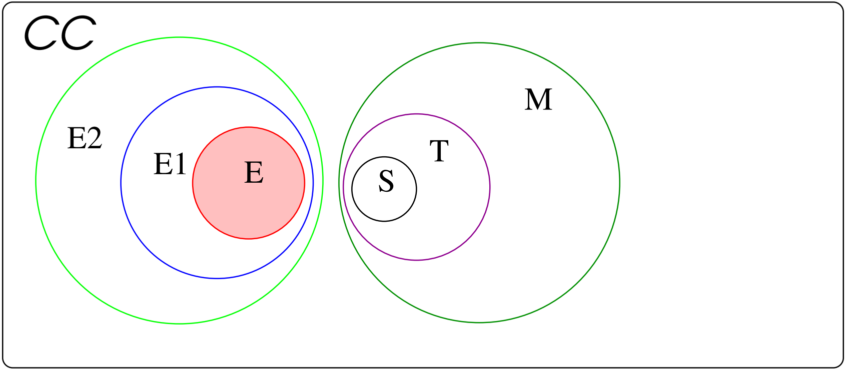

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- M : Mechanism-like non-local functionals (multi-points interaction with free internal nodes)

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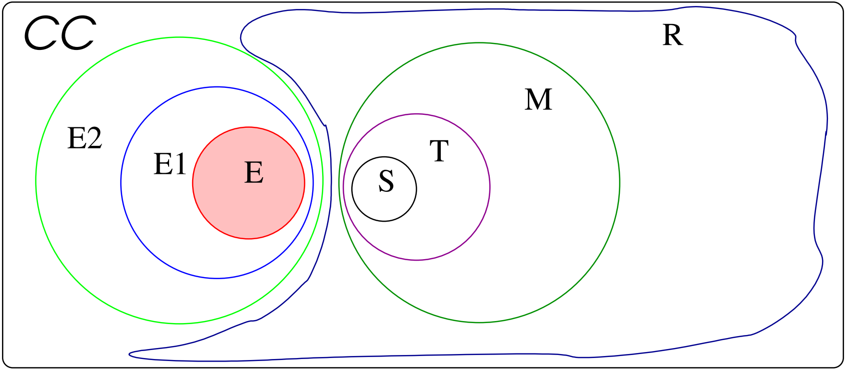

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- R : Continuous functions

Consequences of the symmetry of the considered bivariate functions :

- $Q_n \xrightarrow{e-h} Q$ will be proved if we prove that

$$\forall u_n \xrightarrow{\mathrm{L}^2} u,\ \exists v_n \xrightarrow{\mathrm{L}^2} v\ ;\ \liminf Q_n(u_n, v_n) \geq Q(u, v)$$

  and that the convergence can be reinforced in $v_n \xrightarrow{\mathrm{H}^1} v$ when $v \in \mathrm{H}^1$.

- The domain of any $Q \in \mathcal{CC}$ has the form $\mathcal{D} \times \mathcal{D}$. We define the sum $Q_1 + Q_2$ as usual on $(\mathcal{D}_1 \cap \mathcal{D}_2) \times (\mathcal{D}_1 \cap \mathcal{D}_2)$ and we extend it by $\pm\infty$ outside.

Proposition

Let $Q_{\rho,C} \in E$ and $Q_n \xrightarrow{e-h} Q$ in $\mathcal{CC}$. Then

$$Q_n + Q_{\rho,C} \xrightarrow{e-h} Q + Q_{\rho,C}.$$

Consequence

If Q_1 and Q_2 are two achievable functions, so is $Q_1 + Q_2$.

A first homogenization result

In the domain Ω (convex) x_1, x_2 are two distinct points in Ω, $\psi = g - ih$ is a complex function with negative imaginary part. We denote

$$s_{x_1,x_2,\psi}(u_1,v_1,u_2,v_2) := \begin{pmatrix} (u_1 - u_2)\cdot \frac{x_1-x_2}{\|x_1-x_2\|} \\ (v_1 - v_2)\cdot \frac{x_1-x_2}{\|x_1-x_2\|} \end{pmatrix} \cdot \begin{pmatrix} h & -g \\ -g & -h \end{pmatrix} \cdot \begin{pmatrix} (u_1 - u_2)\cdot \frac{x_1-x_2}{\|x_1-x_2\|} \\ (v_1 - v_2)\cdot \frac{x_1-x_2}{\|x_1-x_2\|} \end{pmatrix}$$

$$S_{x_1,x_2,\psi}(u,v) = \int_\Omega s_{x_1,x_2,\psi(x)}(u(x_1+x), v(x_1+x), u(x_2+x), v(x_2+x))\,dx$$

which corresponds to visco-elastic spring-like interaction.

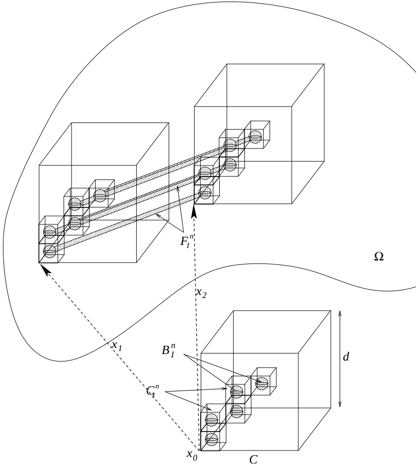
Proposition

There exists a sequence $Q_{\rho_n,C_n} \in E$ which e-h-converges to $S_{x_1,x_2,\psi}$.

Proof is constructive : we describe a suitable sequence of (very!) high contrasted materials.

- ψ has support in a small enough centered cube K
- K is divided in n^3 small cubes K_j^n
- In each cube K_j^n a point c_j^n is chosen
- B_j^n is the ball $B(c_j^n, n^{-2})$
- r_n is a parameter such that $\lim_{n\to\infty} n^{-3}|\ln r_n| = +\infty$
- T_j^n is the cylinder of axis $[x_1 + c_j^n, x_2 + c_j^n]$ and radius r_n.
- $F_j^n = (x_1 + B_j^n) \cup (x_2 + B_j^n) \cup T_j^n$
- $\rho_n = n^{-1/2}$, $C_n = \alpha_n \mathrm{Id}$ where

$$\alpha_n(x) := \begin{cases} n^{-1/2}(1-i) + \frac{\|x_2 - x_1\|}{\pi r_n^2} \int_{C_i^n} \psi & \text{if } x \in F_i^n \\ n^{-1/2}(1-i) & \text{if } x \text{ is out of the fibers} \end{cases}$$



F_I^n
Ω
x_2
x_1
B_I^n
d
C_I^n
x_0
C

Let $u_n \overset{L^2}{\rightharpoonup} u$ and $v \in L^2$. Let us estimate $\liminf Q_{\rho_n, C_n}(u_n, v_n)$ in different parts of the domain.

We can assume $u_n \in H^1$ and $v \in H^1$. By density we can even assume that $v \in C^\infty$.

- Outside of a vicinity V_n of the fibers (precised later) we set $v_n = v$ and we have

$$X_1 = \liminf \int_{V_n^c} \mathrm{Im}\left(\omega^2 \rho_n \hat{U}_n \cdot \hat{U}_n - e(\hat{U}_n) \cdot C_n \cdot e(\hat{U}_n)\right) = \liminf n^{-1/2} \int_{V_n^c} \mathrm{Im}\left(\omega^2 \hat{U}_n \cdot \hat{U}_n - (1-i) e(\hat{U}_n) \cdot e(\hat{U}_n)\right)$$

$$= \liminf n^{-1/2} \int_{V_n^c} 2\omega^2 u_n \cdot v + \|e(u_n)\|^2 - \|e(v)\|^2 - 2e(u_n) \cdot e(v) \geq \liminf n^{-1/2} \int_{\Omega} 2\omega^2 u_n \cdot v + \|e(u_n)\|^2 - \|e(v)\|^2 - 2e(u_n) \cdot e(v$$

$$\geq \liminf \int_{\Omega} 2\omega^2 (n^{-1/4} u_n) \cdot (n^{-1/4} v) + \|e(n^{-1/4} u_n)\|^2 - \|e(n^{-1/4} v)\|^2 - 2e(n^{-1/4} u_n) \cdot e(n^{-1/4} v)$$

  When this limit is $< +\infty$, then the sequence $(n^{-1/4} u_n)$ is bounded in H^1. As $(n^{-1/4} v)$ tends to 0 in H^1, we get

$$X_1 \geq \liminf \int_{\Omega} \|e(n^{-1/4} u_n)\|^2 \geq 0.$$

- The part of $Q_{\rho_n, C_n}(u_n, v_n)$ corresponding to transition layers around the fibers ... gives the same result.
- Inside each fiber F_j^n, we choose for v_n the constant $v(x_1 + c_j^n)$, (respectively $v(x_2 + c_j^n)$) on the ball $x_1 + B_j^n$ (resp. $x_2 + B_j^n$). On the cylinder T_j^n, in a basis where $v(x_1 + c_j^n) - v(x_2 + c_j^n)$ takes the form $\xi_1 e^1 + \xi_3 e^3$, denoting $y_3 = \frac{x_3 - \delta}{\ell - 2\delta}$ with $\ell = \|x_2 - x_1\|$, $\delta = n^{-1}$, we choose

$$(v_n)(x) = \begin{pmatrix} v_1(x_1 + c_j^n) + \xi_1 \left[3y_3^2 - 2y_3^3\right] \\ v_2(x_1 + c_j^n) \\ v_3(x_1 + c_j^n) - \frac{\xi_1}{\ell - 2\delta}\left[6y_3 - 6y_3^2\right] x_1 + \xi_3 y_3 \end{pmatrix}$$

  so that $e_{33}(v_n) = \frac{\xi_3}{\ell - 2\delta} - \frac{\xi_1 x_1}{(\ell - 2\delta)^2}(6 - 12y_3)$ is the only non vanishing coefficient in $e(v_n)$. Hence ...

$$\liminf \sum_j \int_{T_j^n} \mathrm{Im}\left(\omega^2 \rho_n \hat{U}_n \cdot \hat{U}_n - e(\hat{U}_n) \cdot C_n \cdot e(\hat{U}_n)\right) \geq S_{x_1, x_2, \psi}(u, v).$$

Conclusion : $Q_{\rho_n,C_n} \in E$ and $Q_{\rho_n,C_n} \xrightarrow{e-h} S_{x_1,x_2,\psi}$. Hence

Result (Spring-like interactions)

The set S of functions of type $S_{x_1,x_2,\psi}$ is included in $\mathcal{A}$.

A second homogenization result

Using the additivity property we can reach any function Q of the type

$$Q_{q,(x_i),g}(u,v) = \int_\Omega q(u(x_1+x),\dots,u(x_p+x),v(x_1+x),\dots,v(x_p+x))\,g(x)\,dx$$

where

$$q(u_1,v_1,\dots,u_p,v_p) = \sum_{j=1}^{p}\sum_{k=1}^{p} s_{x_j,x_k,\psi_{j,k}}(u_j,v_j,u_k,v_k) \qquad (1)$$

Result (Truss-like interactions)

The set T of functions of type $Q_{q,(x_i),g}$ (where q has form (1)) is included in $\mathcal{A}$.

A second homogenization result enlarges the type of possible quadratic forms q, getting so richer multi-point interactions.

A second homogenization result

- Let $q(u_1, v_1, \dots, u_p, v_p)$ be a $2p$-variables quadratic form, convex with respect to the variables u_j and satisfying $q(u_1, v_1, \dots, u_p, v_p) = -q(-v_1, u_1, \dots, -v_p, u_p)$.
- Let M_q be the $2p \times 2p$ (symmetric) matrix associated to q.
- Let $r < p$ and M_q^r be the submatrix corresponding to the $2r$ first indices and $\widetilde{M}$ its Schur complement.
- Let $\tilde{q}$ be the quadratic form associated to $\widetilde{M}$

Proposition

Assume that, for any $g \geq 0$, $\mathcal{Q}_{q,(x_1,\dots,x_p),g}$ is achievable. Then so is $\mathcal{Q}_{\tilde{q},(x_{r+1},\dots,x_p),g}$.

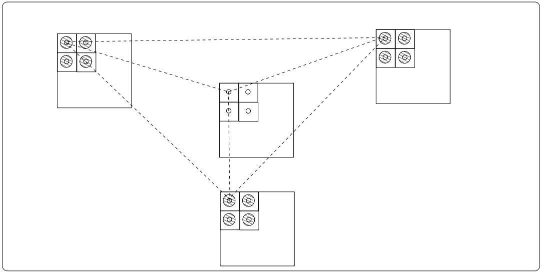
Proof : We explicit a sequence of visco-elastic materials (ρ_n, C_n) and a sequence g_n such that $\mathcal{Q}_{q,(x_1,\dots,x_p),g_n} + \mathcal{Q}_{\rho_n,C_n} \xrightarrow{e-h} \mathcal{Q}_{\tilde{q},(x_{r+1},\dots,x_p),g}$.

A second homogenization result

We set

- $g_n = \left(\sum_{j=1}^{p} \int_{C_j^n} g(x)dx \right) \frac{1}{|B(c_j^n, n^{-4})|} \mathbf{1}_{B(c_j^n, n^{-4})}$,
- $\rho_n = n^{-1/2}$ and $C_n = (\alpha_n(1-i))\,\mathrm{Id}$ where

$$\alpha_n(x) := \begin{cases} n^6 & \text{if } x \in B(c_j^n, n^{-2}) \text{ for some } j > r \\ n^{-1/2} & \text{otherwise} \end{cases}$$



An homogenization result states that :

$$\mathcal{Q}_{q,(x_1,\dots,x_p),g_n} + Q_{\rho_n,C_n} \xrightarrow{e-h} Q_{\tilde{q},(x_{r+1},\dots,x_p),g}.$$

Response matrices of elastodynamic networks

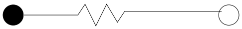
Now we face a discrete problem : what can be all Schur complements of truss-like matrices ?
What can be the response of a network of masses joined by viscoelastic springs when internal(free) nodes are present (a mechanism) ?

Proposition (Milton and Seppecher (2008))

Any quadratic form $q(u_1, v_1, \dots, u_p, v_p)$ convex with respect to the variables u_j and satisfying $q(u_1, v_1, \dots, u_p, v_p) = -q(-v_1, u_1, \dots, -v_p, u_p)$ corresponds to the response of some visco-elastic network with terminal nodes $(x_1, \dots, x_p)$.

Proof: playing with resonances, we construct successively basic network elements

- masses of tensorial nature and any sign :
  Consider the one terminal network



The terminal node has no mass, internal node has mass $m \geq 0$ and the spring stiffness is $k > 0$.
The response matrix is

$$W = (\mu n \otimes n)$$

where $\mu = -km\omega^2/(k - m\omega^2)$ is any real.
Using 3 copies in 3 perpendicular directions we see that a node can be endowed with any symmetric (eventually non positive) mass tensor.

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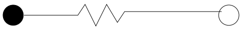

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  $W = \left(\mu n \otimes n\right)$

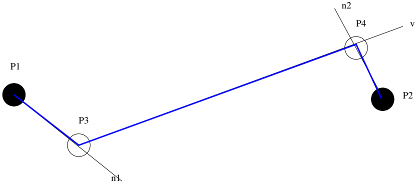
  where $\mu = -km\omega^2/(k - m\omega^2)$ is any real.

  Using 3 copies in 3 perpendicular directions we see that a node can be endowed with any symmetric (eventually non positive) mass tensor.

"Give me a place to stand (or better a negative mass), and I shall move the world."

- springs with negative stiffness and even acting in a direction different from the spring direction.

Consider the two-terminal network



- springs joining $1,3$ and $2,4$ have constant $|K|$,
- central spring has constant $k' = 2|K| - K$,
- terminals have (tensor) masses M_1 and M_2,
- internal nodes have masses $k'\omega^{-2}(v \otimes n_1 + n_1 \otimes v)$ and $k'\omega^{-2}(v \otimes n_2 + n_2 \otimes v)$

Then

$$W = \begin{pmatrix} -\omega^2 M_1 + (|K| - K) n_1 \otimes n_1 & K n_1 \otimes n_2 \\ K n_2 \otimes n_1 & -\omega^2 M_2 + (|K| - K) n_2 \otimes n_2 \end{pmatrix}$$

Choosing suitable M_1 and M_2 we can get, for any K and n,

$$W = \begin{pmatrix} -Kn \otimes n & Kn \otimes n \\ Kn \otimes n & -Kn \otimes n \end{pmatrix}$$

Note that K can be negative and that n needs not to be parallel to $x_2 - x_1$.
We conclude using a superposition argument.

The two last propositions imply that

Result (Mechanisms)

The set M of quadratic form $Q_{q,(x_j),g}$ (for any (x_j), any $g \geq 0$ in L^∞ and any $q(u_1, v_1, \ldots, u_p, v_p)$ convex with respect to the variables u_j, satisfying $q(u_1, v_1, \ldots, u_p, v_p) = -q(-v_1, u_1, \ldots, -v_p, u_p)$) is included in $\mathcal{A}$.

Discretization

- Let $Q \in \mathcal{CC}$, continuous for the strong L^2 topology ($Q \in R$),
- Cover Ω by disjoint cubes $(C_j^n)_{j=1}^N$ of size n^{-1},
- To any $u \in \mathrm{L}^2$, associate the family $(\bar{u})_{j=1}^N$ defined by $\bar{u}_j = n^3 \int_{C_j^n} u(x)dx$
- To any family $(\bar{u}_1, \dots, \bar{u}_N)$, associate the piecewise constant function $T_n(\bar{u}) = \sum_{i=j}^N \bar{u}_j \mathbf{1}_{C_j^n}$.
- Define $q_n(\bar{u}, \bar{v}) = Q(T_n(\bar{u}), T_n(\bar{v}))$ and $Q_n = Q_{q_n,(c_j^n),1}$.
- $Q_n \in M \subset \mathcal{A}$

Proposition

$Q_n \xrightarrow{e-h} Q$

Proof :

- $Q(u, v)$ being continuous has the form

$$Q(u,v) = \int_\Omega (u(x)\cdot A(x)\cdot u(x) - v(x)\cdot A(x)\cdot v(x) - 2u(x)\cdot B(x)\cdot v(x))\, dx$$

where A and B are L^∞ fields of symetric matrices (A being positive).

- Let $u_n \xrightharpoonup{\mathrm{L}^2} u$ and set $v_n = v$. Clearly $v_n \xrightarrow{\mathrm{L}^2} v$ ($v_n \xrightarrow{\mathrm{H}^1} v$ if $v \in \mathrm{H}^1$).
- We have $T_n(\bar{u}_n) \xrightharpoonup{\mathrm{L}^2} u$ and $T_n(v) \xrightarrow{\mathrm{L}^2} v$. Thus

$$\liminf Q_n(u_n, v_n) = \liminf Q(T_n(\bar{u}_n), T_n(\bar{v})) \geq Q(u, v).$$

Hence $Q_n \in \mathcal{A}$ and $Q_n \xrightarrow{e-h} Q$ and we get

Result (Regular functions)

The subset R of continuous functions in $\mathcal{CC}$ is included in $\mathcal{A}$.

Regularization

- Let $Q \in \mathcal{CC}$.
- Let F be the partial Fenchel transform of Q:

$$F(u, v^*) = \sup_v \{< v^*, v >_{L^2} + Q(u, v)\}$$

- Consider the Moreau-Yosida approximation of F (which Mosco-converge to F)

$$F_n(u, v^*) = \inf_{\tilde{u}, \tilde{v}^*} \{F(\tilde{u}, \tilde{v}^*) + n\|(\tilde{u}, \tilde{v}^*) - (u, v^*)\|^2_{L^2}\}$$

- Set $G_n(u, v^*) = F_n(u, v^*) + \frac{1}{2n}\|v^*\|^2_{L^2}$. Check that G_n still Mosco-converges to F.
- Set $H_n(u, v) = \inf_{v^*}\{< v^*, v >_{L^2} + G_n(u, v^*)\}$. Check that $H_n \in R$.

- Using the equivalence between Mosco convergence of partial conjugates and e-h-convergence, we get $H_n \xrightarrow{e-h} Q$.

Result

The whole set $\mathcal{CC}$ is included in $\mathcal{A}$: $\mathcal{CC} = \mathcal{A}$

Conclusion

The strategy "continuum $->$ discrete systems $->$ continuum" is still (more) efficient in the dynamic case.

To be done

- make precise what boundary conditions can be taken into account.
- Study precisely the dependence with respect to ω. The crucial step of characterizing the response *functions* of discrete networks was very recently done by *Vasquez, Milton, Onofrei arXiv, 2009.*
- Apply this strategy to electromagnetism equations.

Questions : change of nature of the system of equations is possible but

- is it possible from a practical point of view ?
- have these exotic metamaterials interesting physical properties ?