The Interior Transmission Problem

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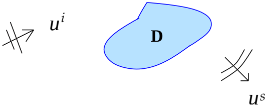
Research supported in part by AFOSR

Scattering by an Isotropic Medium



$$\Delta u + k^2 n(x) u = 0 \quad \text{in } \mathbb{R}^2$$
$$u = u^s + u^i \quad \text{in } \mathbb{R}^2$$
$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0$$

where $u^i(x) := e^{ikx\cdot d}$, $|d| = 1$ and $u \in H^1_{loc}(\mathbb{R}^2)$.
We assume that $n - 1$ has compact support $\overline{D}$, $n(x) > 0$ for $x \in \overline{D}$ and n is piecewise continuous. The scattered field u^s has the asymptotic behavior

$$u^s(x) = \frac{e^{ikr}}{\sqrt{r}} u_\infty(\hat{x}, d) + O\left(r^{-3/2}\right)$$

as $r \to \infty$ where $\hat{x} = x/|x|$, $r = |x|$ and $k > 0$ is the wave number.
$u_\infty(\hat{x}, d)$ is the far field pattern of the scattered field u^s.

The Far Field Operator

Let $\Omega := \{x : |x| = 1\}$ and define the far field operator $F : L^2(\Omega) \to L^2(\Omega)$ by

$$(Fg)(\hat{x}) := \int_{\Omega} u_\infty(\hat{x}, d) g(d) ds(d).$$

For $z \in D$ the far field equation is

$$(Fg)(\hat{x}) = \Phi_\infty(\hat{x}, z), \quad g \in L^2(\Omega)$$

where

$$\Phi_\infty(\hat{x}, z) = \frac{e^{i\pi/4}}{\sqrt{8\pi k}} e^{-ik\hat{x}\cdot z}$$

is the far field pattern of the fundamental solution

$$\Phi(x, z) := \frac{i}{4} H_0^{(1)}(k|x - z|).$$

Towards the Interior Transmission Problem

Suppose $z \in D$ and the far field equation has a solution g. By Rellich's uniqueness theorem

$$\int_{\Omega} u^s(x,d)g(d)ds(d) = \Phi(x,z)$$

for all $x \notin D$. On Γ, $u^s(x,d) = u(x,d) - \exp(ikx \cdot d)$ so for $x \in \Gamma$

$$\int_{\Omega} (u(x,d) - \exp(ikx \cdot d))g(d)\, ds(d) = \Phi(x,z)$$

The function $w(x) = \int_{\Omega} u(x,d)g(d)\, ds(d)$ satisfies

$$\Delta w + k^2 n(x) w = 0 \text{ in } D$$

Further towards the Interior Transmission Problem

The Herglotz wave function

$$v(x) = \int_{\Omega} \exp(ikx \cdot d) g(d)\, ds(d)$$

is an entire solution of the Helmholtz equation

$$\Delta v + k^2 v = 0 \text{ in } \mathbb{R}^2$$

Since $w - v = \Phi$ outside D we obtain the boundary conditions

$$\begin{aligned} w - v &= \Phi \quad \text{on } \Gamma \\ \frac{\partial}{\partial \nu}(w - v) &= \frac{\partial}{\partial \nu}\Phi \end{aligned}$$

The Interior Transmission Problem (ITP)

Given $f \in H^{3/2}(\Gamma)$ and $g \in H^{1/2}(\Gamma)$ we seek $w, v \in L^2(\Omega)$ with $w - v \in H^2(\Omega)$ such that

$$\begin{aligned} \Delta w + k^2 n w &= 0 \text{ in } D \\ \Delta v + k^2 v &= 0 \text{ in } D \\ w - v &= f \text{ on } \Gamma \\ \frac{\partial}{\partial \nu}(w - v) &= g \text{ on } \Gamma \end{aligned}$$

Study of the ITP

Study of the existence and continuous dependence of solutions of the ITP are part of the analysis of the "Linear Sampling Method" for inverse scattering.
For such studies see

- Colton, Coyle and Monk, SIAM Review (2000)
- *Qualitative Methods in Inverse Scattering Theory*, Colton & Cakoni (2006)

In the cases studied so far (including Maxwell!) the ITP has a unique solution for suitable data and a sufficiently regular domain, except possibly at a countable discrete set of Transmission Eigenvalues that we shall define shortly.

ITP and Metamaterials

Consider a simple boundary value problem for $\nabla \cdot A\nabla u + k^2 nu = 0$ for negative refractive index materials. Suppose $D = D_1 \cup D_2$ and D_1 and D_1 meet at an edge Σ. Then if $A = 1$ and $n = 1$ in domain 1 and $A = -a_2 I$, $a_2 > 0$ and $n = -n_2$, $n_2 > 0$ in domain 2 the solution of the solution u_1 in domain D_1 and u_2 in D_2 satisfies

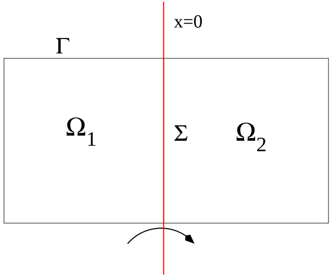
$$\begin{aligned} \Delta u_1 + k^2 u_1 &= f_1 \text{ in } D_1 \\ \Delta u_2 + k^2 n_2/a_2 u_2 &= f_2 \text{ in } D_2 \\ u_1 - u_2 &= 0 \text{ on } \Sigma \\ \frac{\partial u_1}{\partial \nu_1} - a_2 \frac{\partial u_2}{\partial \nu_2} &= 0 \text{ on } \Sigma \\ u &= 0 \text{ on } \Gamma \end{aligned}$$

where ν_j is the unit outward normal to D_j.

Change of variables for a special domain

Suppose Ω is the symmetric domain shown. Let $\hat{x} = -x$. Let $\hat{u}_1(\hat{x}, y) = u_1(-x, y)$, $x > 0$, etc. Then

$$\begin{aligned}
\hat{\Delta}\hat{u}_1 + k^2\hat{u}_1 &= \hat{f}_1 \text{ in } D_2 \\
\Delta u_2 + k^2 n_2/a_2 u_2 &= f_2 \text{ in } D_2 \\
\hat{u}_1 - u_2 &= 0 \text{ on } \Sigma \\
\frac{\partial \hat{u}_1}{\partial \nu_2} - a_2 \frac{\partial u_2}{\partial \nu_2} &= 0 \text{ on } \Sigma \\
u &= 0 \text{ on } \Gamma \cap \partial D_2
\end{aligned}$$



On Σ we see the ITP boundary condition.

Comments on negative index materials

The observation of the connection of negative index materials and the ITP is due to Bonnet-BenDhia and Ciarlet Jr. They then applied variational methods used in the analysis of the ITP to the negative index problem and derive well posed variational formulations:

- Bonnet-BenDhia, Ciarlet Jr and Zwolf (2006) - Helmholtz Equation
- Bonnet-BenDhia, Ciarlet Jr and Zwolf (2006) - Maxwell
- The poster of Chesnel here (requires backwards in time travel)

We shall not pursue this line of research further, but turn instead to study transmission eigenvalues.

Transmission Eigenvalues

Definition: $k > 0$ is a transmission eigenvalue if there exists a nontrivial solution $v \in L^2(D)$, $w \in L^2(D)$, $v - w \in H_0^2(D)$ of the interior transmission problem

$$
\begin{aligned}
\Delta w + k^2 n(x) w = 0 & \quad \text{in} \quad D \\
\Delta v + k^2 v = 0 & \quad \text{in} \quad D \\
w = v & \quad \text{on} \quad \partial D \\
\frac{\partial w}{\partial \nu} = \frac{\partial v}{\partial \nu} & \quad \text{on} \quad \partial D
\end{aligned}
$$

But what are they physically?

The transmission eigenvalues are wave numbers for which there exists an incident field that generates an arbitrarily small scattered field (if v is a Herglotz wave function, the scattered field vanishes precisely).

Note that the incident field is very special.

Transmission Eigenvalues

Theorem

Let $m = 1 - n$ and assume $|m(x)| > 0$ for $x \in \overline{D}$. Then transmission eigenvalues exist and form a discrete set whose only accumulation point is infinity.

Proof: Colton-Monk (1988), Päivärinta-Sylvester (2008), Kirsch (2009), Cakoni-Gintides-Haddar (to appear).

Definition: A solution of the Helmholtz equation of the form

$$v_g(x) := \int_\Omega e^{ikx \cdot d} g(d) ds(d), \qquad g \in L^2(\Omega)$$

is called a Herglotz wave function with kernel g.

Transmission eigenvalues can be measured!

Theorem

Let $m = 1 - n$ and assume $|m| > 0$ for $x \in \overline{D}$. Let $u^{\delta}_{\infty}(\hat{x}, d)$ be the measured "noisy" far field pattern with noise parameter δ and for $z \in D$ let $g_{z,\delta}$ be the Tikhonov regularized solution of the far field equation $(F^{\delta} g)(\hat{x}) = \Phi_{\infty}(\hat{x}, z)$.

- *If k is not a transmission eigenvalue then $\lim_{\delta \to 0} \|v_{g_{z,\delta}}\|_{L^2(D)}$ exists.*
- *If k is a transmission eigenvalue and the far field operator F has dense range then for almost every $z \in D$*

$$\lim_{\delta \to 0} \|v_{g_{z,\delta}}\|_{L^2(D)} = \infty.$$

Proof: Arens (2004), Cakoni-Colton-Haddar (to appear).

Note: $\|v_{g_{z,\delta}}\|_{L^2(D)} \to \infty$ implies $\|g_{z,\delta}\|_{L^2(D)} \to \infty$.

What do transmission eigenvalues say about $n(x)$?

Colton-Päivärinta-Sylvester 2007 have proved the following Faber-Krahn type inequality.

Let $k_{1,n(x)}$ be the first transmission eigenvalue and let $\lambda_1(D)$ be the first Dirichlet eigenvalue for $-\Delta_2$ in D.

Theorem

If $n(x) > \alpha > 1$ for $x \in \overline{D}$. Then

$$k_{1,n(x)}^2 \geq \frac{\lambda_1(D)}{\sup_D n}.$$

A Sharper Faber-Krahn type Inequality

Let $n_* = \inf_D(n)$ and $n^* = \sup_D(n)$.

Theorem

If $1 + \alpha \le n_* \le n(x) \le n^* < \infty$ *for* $x \in \overline{D}$ *then*

$$0 < k_{1,n^*} \le k_{1,n(x)} \le k_{1,n_*}.$$

Proof: Cakoni-Gintides-Haddar (to appear).

As we shall see, this result allows us to estimate $n(x)$ from interior eigenvalues.

Numerical methods for Transmission Eigenvalues

To implement schemes based on the previous theorem, and assess the accuracy of estimates we need ways to compute transmission eigenvalues.

In (Colton-Monk-Sun, 2009) we have investigated three ways motivated by different variational formulations of the ITP. Recall the eigenvalue problem is to find k and $(w, v) \neq 0$ such that

$$\begin{aligned} \Delta w + k^2 n(x) w = 0 \quad & \text{in} \quad D \\ \Delta v + k^2 v = 0 \quad & \text{in} \quad D \\ w = v \quad & \text{on} \quad \partial D \\ \frac{\partial w}{\partial \nu} = \frac{\partial v}{\partial \nu} \quad & \text{on} \quad \partial D \end{aligned}$$

Note that $k = 0$ is an eigenvalue with an infinite dimensional eigenspace.

Method 1: Fourth order problem

Following (Rynne and Sleeman, 1992), assuming $n \neq 1$, let $z = w - v$ then using the differential equations

$$\Delta z + k^2 n(x) z = -(\Delta v + k^2 n(x) v) = k^2 (1 - n(x)) v$$

so we arrive at the problem of finding k and $z \neq 0$, $z \in H^2(\Omega)$ such that

$$\begin{aligned} (\Delta + k^2)(1-n)^{-1}(\Delta z + k^2 n(x) z) &= 0 \text{ in } D \\ z = 0 \text{ and } \frac{\partial z}{\partial \nu} = 0 \quad &\text{on} \quad \Gamma \end{aligned}$$

Method 1 continued

This can be put into the variational problem of finding k and $z \neq 0$, $z \in H_0^2(\Omega)$ such that

$$\int_D (1-n)^{-1}(\Delta z + k^2 n(x) z)\overline{(\Delta\xi + k^2\xi)}\, dA = 0 \text{ for all } \xi \in H_0^2(D).$$

Note:

- We can “easily” discretize this problem using Argyris elements.
- The problem is now a quadratic eigenvalue probem. The bilinear form is not Hermitian.
- $k = 0$ is not an eigenvalue.
- At the discrete level we can convert back to a generalized matrix eigenvalue problem.

Method 2: Mixed method

Introducing $\mathbf{u} = \nabla v$ and rewriting the ITP, we obtain

$$\begin{aligned}
\Delta w + k^2 n w &= 0 \text{ in } D,\\
\mathbf{u} - \nabla v &= 0 \text{ in } D,\\
\nabla \cdot \mathbf{u} + k^2 v &= 0 \text{ in } D,\\
w - v &= 0 \text{ on } \partial D,\\
\frac{\partial w}{\partial \nu} - \mathbf{u} \cdot \nu &= 0 \text{ on } \partial D.
\end{aligned}$$

Let

$$H(\mathrm{div}; D) = \{\mathbf{u} \in (L^2(D))^2 \mid \nabla \cdot \mathbf{u} \in L^2(D)\}.$$

The mixed method continued

The weak formulation for this problem is to find $k \in \mathbb{C}$, $w \in H^1(D)$, $\mathbf{u} \in H(\mathrm{div}, D)$, and $v \in L^2(D)$ such that

$$
\begin{aligned}
\int_D -\nabla w \cdot \nabla \phi + k^2 n w \phi \, dx + \int_{\partial D} \mathbf{u} \cdot \nu \phi \, ds &= 0, \quad \forall \phi \in H^1(D), \\
\int_D \mathbf{u} \cdot \tau \, dx + \int_D v \nabla \cdot \tau \, dx - \int_{\partial D} \tau \cdot \nu w \, ds &= 0, \quad \forall \tau \in H(\mathrm{div}; D), \\
\int_D \nabla \cdot \mathbf{u} q \, dx + k^2 \int_D v q \, dx &= 0, \quad \forall q \in L^2(D).
\end{aligned}
$$

Note:

- We can easily discretize this problem using Raviart-Thomas elements.
- We can eliminate $\mathbf{u}$ to obtain a generalized eigenproblem.
- Symmetry is lost between w and v complicating spectrum for $k = 0$.
- Problem of regularity assumed for w.

Method 3: Continuous elements

Let $H^1(D) = H_0^1(D) \oplus S$ where S is the H^1 orthogonal complement of $H_0^1(D)$. Set

$$\begin{aligned} w &= w_0 + w_B \text{ where } w_0 \in H_0^1(D) \text{ and } w_B \in S, \\ v &= v_0 + w_B \text{ where } v_0 \in H_0^1(D). \end{aligned}$$

Note $w = v$ on Γ

The continuous element method continued

In the usual way, by integration by parts

$$\begin{aligned} \int_D \nabla(w_0 + w_B) \cdot \nabla\xi - k^2 n(w_0 + w_B)\xi \, dx &= 0 \quad \forall\xi \in H_0^1(D) \\ \int_\Omega \nabla(v_0 + w_B) \cdot \nabla\eta - k^2 (v_0 + w_B)\eta \, dx &= 0 \quad \forall\eta \in H_0^1(D). \end{aligned}$$

To obtain one more system of equation, choosing $\gamma \in H^1(D)$

$$\begin{aligned} \int_D \nabla w \cdot \nabla\gamma - k^2 n w \gamma \, dx - \int_{\partial D} \frac{\partial w}{\partial \nu}\gamma \, ds &= 0 \\ \int_D \nabla v \cdot \nabla\gamma - k^2 v \gamma \, dx - \int_{\partial D} \frac{\partial v}{\partial \nu}\gamma \, ds &= 0. \end{aligned}$$

Subtracting these equations we get the final equation to be discretized.

$$\int_D \nabla(w - v) \cdot \nabla\gamma \, dx - k^2 \int_D (nw - v)\gamma \, dx = 0.$$

The continuous element method

Find k and non-trivial $w_0, v_0 \in H_0^1(D)$ and $w_B \in S^B$ such that

$$\begin{aligned}
\int_D \nabla(w_0 + w_B) \cdot \nabla\xi \, dx &= k^2 \int_D n(w_0 + w_B)\xi \, dx \quad \forall \xi \in H_0^1(D) \\
\int_D \nabla(v_0 + w_B) \cdot \nabla\eta \, dx &= k^2 \int_D (v_0 + w_B)\eta \, dx \quad \forall \eta \in H_0^1(D) \\
\int_D \nabla(w_0 - v_0) \cdot \nabla\gamma \, dx &= k^2 \int_D (n(w_0 - v_0) + (n-1)w_B))\gamma \, dx \\
&\qquad \forall \gamma \in S^B
\end{aligned}$$

Note:

- Similar to methods proposed by Kirsch.
- Can use easy^2 standard piecewise linear elements.
- Gives rise to a generalized eigenvalue problem $\mathcal{A}\vec{x} = k^2\mathcal{B}\vec{x}$ for real symmetric matrices, but neither $\mathcal{A}$ or $\mathcal{B}$ is positive definite.
- $k = 0$ is an eigenvalue of large multiplicity.

Numerical results 1: the circle

Table: Transmission eigenvalues of a disk D with radius 1/2. The index of refraction n is 16. Same mesh for all methods.

Exact	1.9880	2.6129	2.6129	3.2240
Argyris Method	2.0076 (0.98%)	2.6382	2.6396	3.2580
Continuous Method	1.9986 (0.53%)	2.6334	2.6343	3.2641
Mixed Method	1.9912 (0.16%)	2.6218	2.6234	3.2308

Numerical Results 2: convergence order

Table: Errors and convergence rates for the first and second transmission eigenvalues computed by the continuous finite element method. The domain D is a disk of radius $1/2$. The index of refraction n is 16. $k_i^h, i = 1, 2$ are the computed first and second eigenvalues on three meshes.

Mesh size h	$\lvert k_1^h - k_1 \rvert$	order	$\lvert k_2^h - k_2 \rvert$	order
$h \approx 0.1$	0.04215	-	0.08078	-
$h \approx 0.05$	0.01064	1.986	0.02052	1.977
$h \approx 0.025$	0.00266	2.002	0.00514	1.995

Numerical Results 3: do complex eigenvalues exist?

Table: The first pair of complex transmission eigenvalues of a disk of radius $R = 1/2$. The index of refraction n is 16.

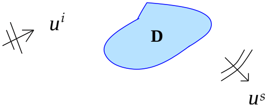
Numerical Root Finding Software	$4.9009 \pm 0.5781i$
Argyris Method	$4.9495 \pm 0.5795i$
Continuous Method	$5.0667 \pm 0.4878i$
Mixed Method	$4.9350 \pm 0.4959i$

It is highly likely that complex eigenvalues exist.

Comments on the various methods

- Only the Argyris method respects the minimum regularity. But it is difficult to extend to problems where both A and n vary.
- All three methods require to find all eigenvalues (i.e. `eig` not `eigs`) and so run out of memory quickly. The Continuous method is "cheapest".
- A fourth approach suggested by Cakoni-Gintedes-Haddar involving solving a nonlinear equation has also been examined by Sun.
- All methods predict a large number of complex eigenvalues.
- We are going to investigate the pseudo-spectrum

Scattering by an Anisotropic Medium



$$\nabla \cdot A \nabla u + k^2 u = 0 \qquad \text{in } \mathbb{R}^2$$
$$u = u^s + u^i \qquad \text{in } \mathbb{R}^2$$
$$\lim_{r \to \infty} \sqrt{r} \left(\frac{\partial u^s}{\partial r} - iku^s \right) = 0$$

where $u^i(x) = e^{ikx \cdot d}$, $|d| = 1$ and $u \in H^1_{loc}(\mathbb{R}^2)$.
A is a positive, real valued 2×2 matrix whose entries are piecewise continuously differentiable in $\overline{D}$ and $A - I$ has compact support $\overline{D}$.
The scattered field u^s has the asymptotic behavior

$$u^s(x) = \frac{e^{ikr}}{\sqrt{r}} u_\infty(\hat{x}, d) + O\left(r^{-3/2}\right)$$

as $r \to \infty$ where $\hat{x} = x/|x|$, $r = |x|$ and $k > 0$ is the wave number.
$u_\infty(\hat{x}, d)$ is the far field pattern of the scattered field u^s.

The Far Field Operator

Let $\Omega := \{x : |x| = 1\}$ and define the far field operator $F : L^2(\Omega) \to L^2(\Omega)$ by

$$(Fg)(\hat{x}) := \int_{\Omega} u_\infty(\hat{x}, d) g(d) ds(d).$$

For $z \in D$ the far field equation is

$$(Fg)(\hat{x}) = \Phi_\infty(\hat{x}, z), \quad g \in L^2(\Omega)$$

where

$$\Phi_\infty(\hat{x}, z) = \frac{e^{i\pi/4}}{\sqrt{8\pi k}} e^{-ik\hat{x}\cdot z}$$

is the far field pattern of the fundamental solution

$$\Phi(x, z) := \frac{i}{4} H_0^{(1)}(k|x - z|).$$

Transmission Eigenvalues

k is a transmission eigenvalue if there exists a nontrivial solution $v, w \in H^1(D)$ of the interior transmission problem

$$\begin{aligned} \nabla \cdot A \nabla w + k^2 w = 0 \quad & \text{in} \quad D \\ \Delta v + k^2 v = 0 \quad & \text{in} \quad D \\ w = v \quad & \text{on} \quad \partial D \\ \frac{\partial w}{\partial \nu_A} = \frac{\partial v}{\partial \nu} \quad & \text{on} \quad \partial D. \end{aligned}$$

where $\frac{\partial w}{\partial \nu_A} = \nu \cdot A \nabla w$.

Theorem

If $\|A^{-1}\|_2 \geq \alpha > 1$ or $\|A^{-1}\|_2 \leq 1 - \alpha < 1$ then transmission eigenvalues exist and form a discrete set whose only accumulation point is infinity.

Proof: Kirsch (2009), Cakoni-Gintides-Haddar (to appear).

Transmission Eigenvalues

Let $u_\infty^\delta(\hat{x}, d)$ be the measured "noisy" far field pattern and for $z \in D$ let $g_{z,\delta}$ be the Tikhonov regularized solution of the far field equation $(F^\delta g)(\hat{x}) = \Phi_\infty(\hat{x}, z)$. Let v_g denote the Herglotz wave function

$$v_g(x) := \int_\Omega e^{ikx\cdot d} g(d) ds(d), \qquad g \in L^2(\Omega)$$

Then [Cakoni-Colton-Haddar (2009)]

- If k is not a transmission eigenvalue then $\lim_{\delta\to 0} \|v_{g_{z,\delta}}\|_{L^2(D)}$ exists.
- If k is a transmission eigenvalue and the far field operator F has dense range then for almost every $z \in D$

$$\lim_{\delta\to 0} \|v_{g_{z,\delta}}\|_{L^2(D)} = \infty$$

What do these transmission eigenvalues say about $A(x)$?

The far field pattern $u_\infty(\hat{x}, d)$ for $\hat{x}, d \in \Omega$ does not uniquely determine $A(x)$ even if u_∞ is known for an interval of values of the wave number k!

However, as we have just seen, the transmission eigenvalues corresponding to $A(x)$ can be determined from u_∞. So they must contain information about A.

A Faber-Krahn Inequality

Theorem

Assume that $\|A^{-1}\|_2 \geq \alpha > 1$ *for* $x \in \overline{D}$*. Then, if* k_1 *is the first transmission eigenvalue and* $\lambda_1(D)$ *is the first Dirichlet eigenvalue for* $-\Delta_2$ *in* D*,*

$$k_1^2 \geq \frac{\lambda_1(D)}{\sup_D \|A^{-1}\|_2}.$$

Proof: Cakoni-Colton-Haddar (2009).

Estimates for $\|A^{-1}\|_2$ using this inequality are rather crude. Furthermore, if $\|A^{-1}\|_2 < 1$ for $x \in \overline{D}$ all that is known is $k_1^2 \geq \lambda_1(D)$.

Towards Another Faber-Krahn Inequality

Let

$$a_*(x) := \text{smallest eigenvalue of } A^{-1}(x) \qquad \text{and}$$
$$a^*(x) := \text{largest eigenvalue of } A^{-1}(x).$$

Define $n_* := \inf_D(a_*(x))$ and $n^* = \sup_D(a^*(x))$ and denote by k_{1,n_0} be the first transmission eigenvalue of

$$\begin{aligned}
\Delta w + k^2 n_0 w = 0 &\qquad \text{in} \qquad D\\
\Delta v + k^2 v = 0 &\qquad \text{in} \qquad D\\
w = v &\qquad \text{on} \qquad \partial D\\
\frac{1}{n_0}\frac{\partial w}{\partial \nu} = \frac{\partial v}{\partial \nu} &\qquad \text{on} \qquad \partial D.
\end{aligned}$$

Faber-Krahn Inequalities

Theorem

If $\|A^{-1}\|_2 > 1 + \alpha$ *for* $x \in \overline{D}$ *then*

$$0 < k_{1,n^*} \leq k_{1,A(x)} \leq k_{1,n_*}.$$

Proof: Cakoni-Gintides-Haddar (to appear).

Numerical Examples

Given the first transmission eigenvalue $k_{1,A(x)}$ and the domain D our aim is to obtain information about $A^{-1}(x)$. From the previous theorem we have that k_{1,n_0} is a monotonic function of n_0 and from Cakoni-Colton-Monk-Sun (to appear) we can show that k_{1,n_0} is also continuous with respect to n_0.

Adjusting n_0 so that k_{1,n_0} equals the measured transmission eigenvalue now gives n_0 where $n_* \leq n_0 \leq n^*$ i.e. n_0 lies between the infimum of the smallest eigenvalue and the supremum of the largest eigenvalue of $A^{-1}(x)$.

The continuous element method

Let $H^1(D) = H^1_0(D) \oplus S$ where S is the H^1 orthogonal complement of $H^1_0(D)$. Find non trivial $u^I \in H^1(D)$, $u^B \in S$ and $u^I_0 \in H^1_0$ such that $u = u^I + u^B$ and $u_0 = u^I_0 + u^B$ and k such that

$$\int_D A\nabla(u^I + u^B) \cdot \xi - k^2(u^I + u^B)\xi \, dA = 0 \qquad \text{for all } \xi \in H^1_0(\Omega)$$

$$\int_D \nabla(u^I_0 + u^B) \cdot \chi - k^2(u^I_0 + u^B)\chi \, d = 0 \qquad \text{for all } \chi \in H^1_0(\Omega)$$

$$\int_D A\nabla(u^I + u^B) \cdot \mu - k^2(u^I + u^B)\mu, dA =$$

$$\int_D A\nabla(u^I_0 + u^B) \cdot \mu - k^2(u^I_0 + u^B)\mu \, dA \qquad \text{for all } \mu \in S.$$

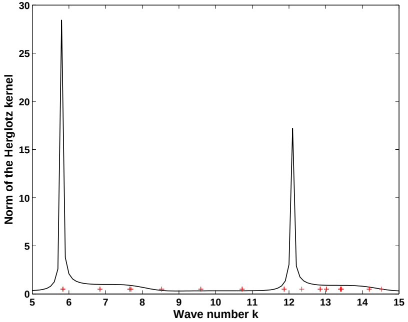
From Cakoni-Colton-Monk-Sun (2009)

A test: Eigenvalues for the circle, isotropic media

Set $z = 0$, plot $\|g_{z,\delta}\|_{L^2(\Omega)}$ against wave number k. Use

$$A = A_{iso} = \begin{pmatrix} 1/4 & 0 \\ 0 & 1/4 \end{pmatrix}$$

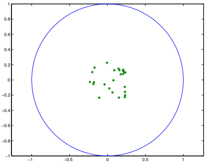
Peaks should correspond to real interior transmission eigenvalues. Crosses denote computed eigenvalues.



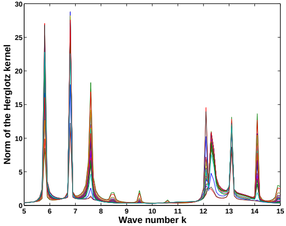
Clearly many eigenvalues are missing from the scan. Why?

Eigenvalues for the circle, isotropic media, continued

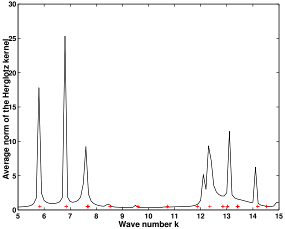
For $z = 0$ the solution $g_{z,\delta}$ only involves J_0 and so eigenvalues from other orders are not see. Solution: use several randomly chosen source points z



25 random points z_i



$\|g_{z_i}\|_{L^2(\Omega)}$ against k



The average of $\|g_{z_i}\|_{L^2(\Omega)}$

Now the eigenvalues are present!

Numerical Examples: Homogeneous Anisotropic Media

We consider D to be the unit square $[-1/2, 1/2] \times [-1/2, 1/2]$ and

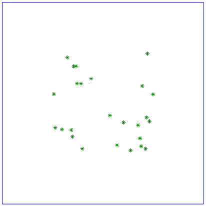
$$A_1^{-1} = \begin{pmatrix} 2 & 0 \\ 0 & 8 \end{pmatrix} \quad A_2^{-1} = \begin{pmatrix} 6 & 0 \\ 0 & 8 \end{pmatrix}$$

and

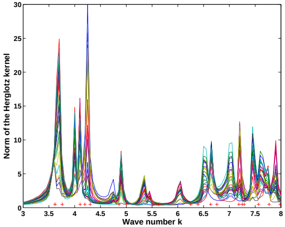
$$A_{2r}^{-1} = \begin{pmatrix} 7.4136 & -0.9069 \\ -0.9069 & 6.5834 \end{pmatrix}$$

In each case we can find $k_{1,A}$ from the far field equation, and then compute n_0 that gives the same interior transmission eigenvalue.

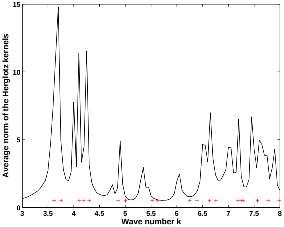
Computing the first transmission eigenvalue for the square



25 random points z_i



$\|g_{z_i}\|_{L^2(\Omega)}$ against k



The average of $\|g_{z_i}\|_{L^2(\Omega)}$

Computation of the transmission eigenvalues from the far field equation for the square D and A_{2r}^{-1}.

Computed values of n_0

Domain	Matrix	E-values (A^{-1})	$k_{1,D,A(x)}$	Predicted n_0
Circle	A_{iso}	4,4	5.8	4.03
	A_1	2,8	4.81	5.32
	A_2	6,8	3.95	7.46
	A_{2r}	6,8	3.95	7.46
Square	A_{iso}	4,4	5.3	4.03
	A_1	2,8	4.1	5.81
	A_2	6,8	3.55	7.41
	A_{2r}	6,8	3.7	6.90
L shape	A_{iso}	4,4	6.45	4.38
	A_1	2,8	5.2	5.49
	A_2	6,8	4	8
	A_{2r}	6,8	4.1	7.69

Conclusion

- The ITP is a novel interior problem which manifests itself in physical scattering measurements
- Interior transmission eigenproblems are currently under study for the acoustic, elastic and Maxwell systems
- An efficient method for computing the interior transmission eigenvalues is yet to be developed
- Much of this material already extends to Maxwell's equations