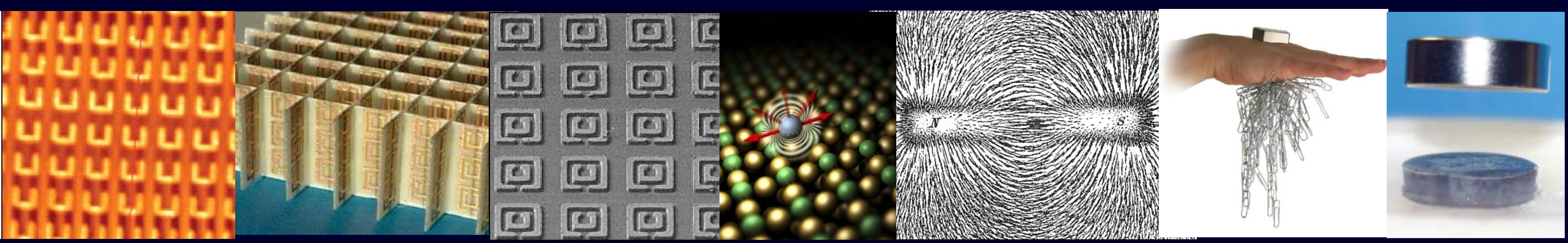
January
2010

Air
Force
Office of
Scientific
Research

Multidisciplinary
University
Research
Initiative

High-Frequency Magnetism in Metamaterials and the Landau-Lifshitz Permeability Argument



Roberto Merlin
University of Michigan

MAXWELL'S EQUATIONS

$$\vec{\nabla} \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}$$

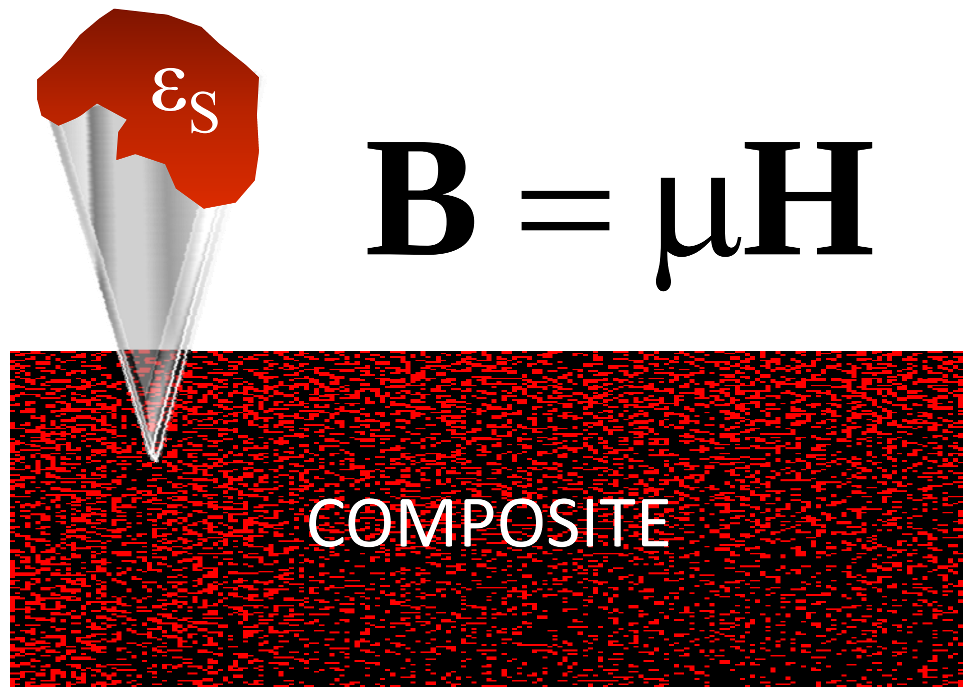
$$\vec{\nabla} \times \mathbf{H} = \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t}$$

$$\vec{\nabla} \cdot \mathbf{D} = 0$$

$$\vec{\nabla} \cdot \mathbf{B} = 0$$

$$\mathbf{D} = \varepsilon \mathbf{E}$$

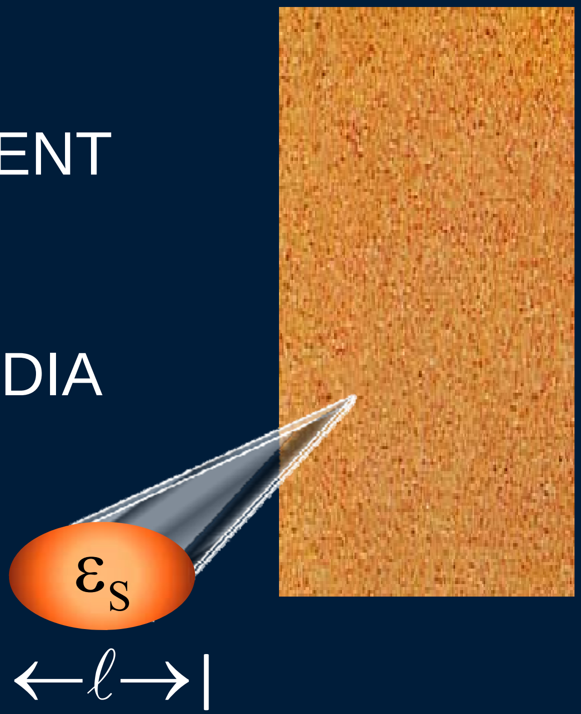
$$\mathbf{B} = \mu \mathbf{H}$$



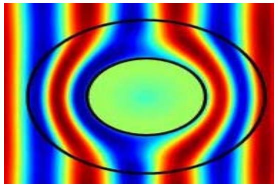
WHAT IS μ?

OUTLINE

- HIGH-FREQUENCY MAGNETISM
- LANDAU-LIFSHITZ PERMEABILITY ARGUMENT
- HOMOGENIZATION OF METAMATERIALS: ELECTRODYNAMICS OF CONTINUOUS MEDIA
- ATOMS vs. SPLIT RINGS AND SPHERES
- PLASMON RESONANCES



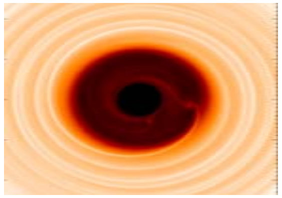
$$\mathrm{Im}\sqrt{\varepsilon_S} = \kappa_S >> \lambda / \ell >> 1$$



PERFECT DIAMAGNET

$\mu_S \approx 0$

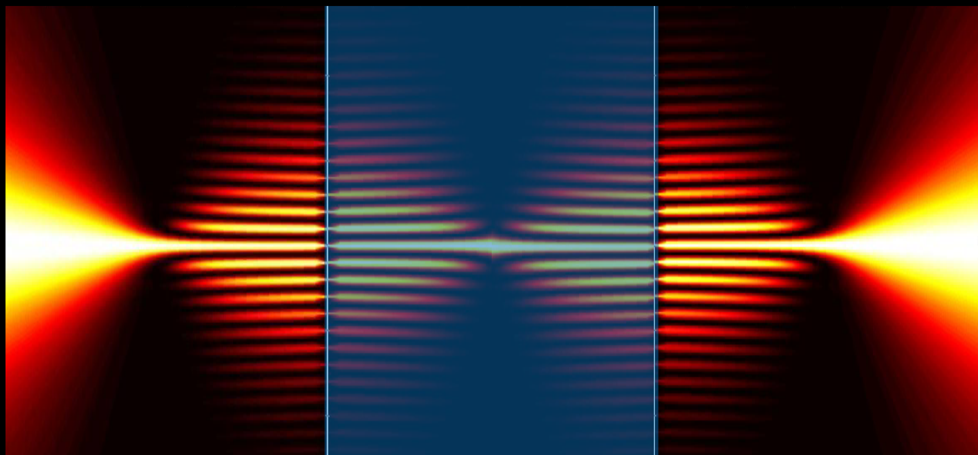
$$\mathrm{Re}\sqrt{\varepsilon_S} = n_S \sim \lambda / \ell >> 1$$



PERFECT PARAMAGNET

$\mu_S \approx \infty$

NEGATIVE REFRACTION



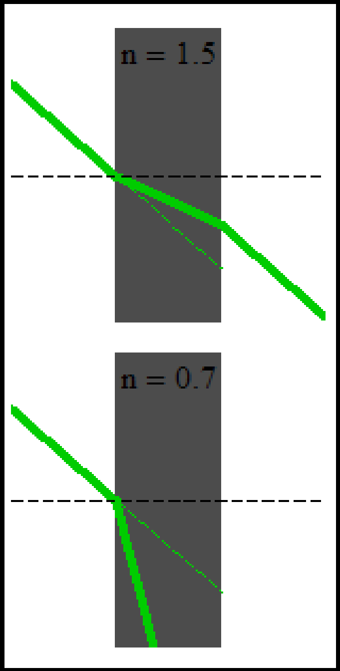
$\mu_{MM} < 0$

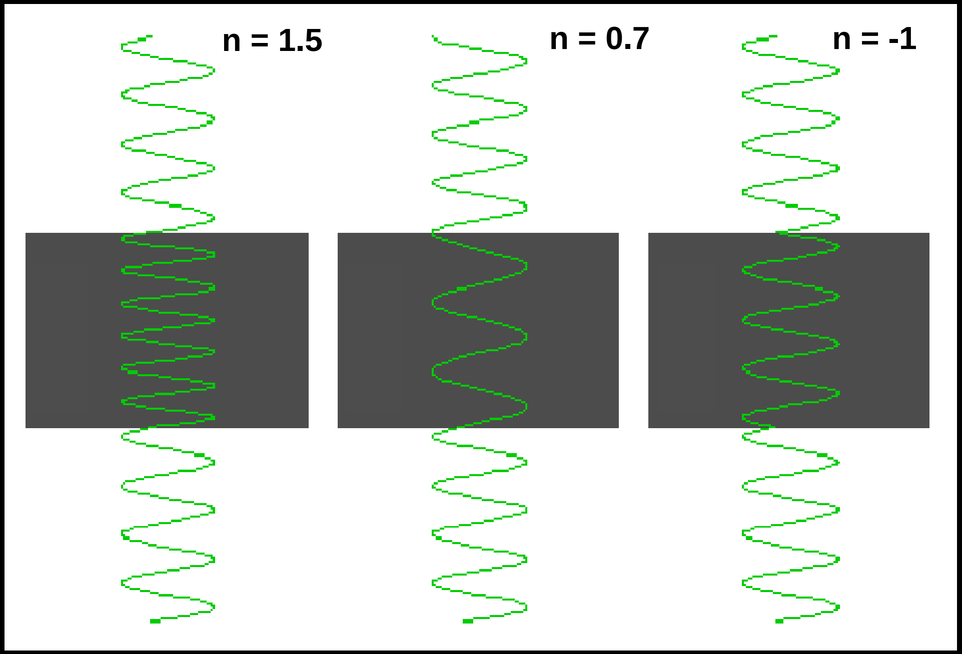
$\varepsilon_{MM} < 0$

PERFECT ABSORBER

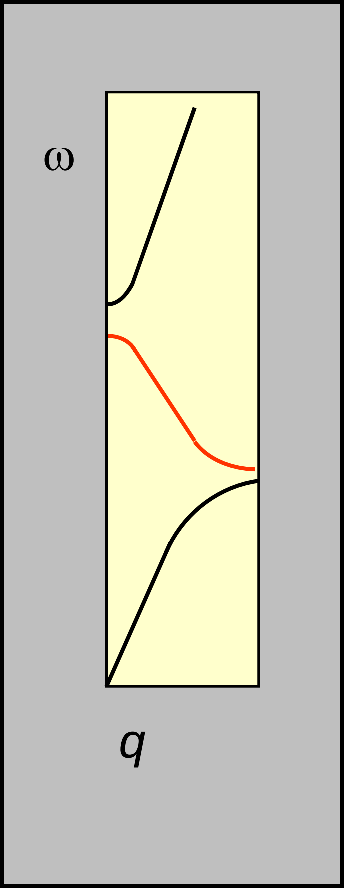
$\mu_{MM} \equiv \varepsilon_{MM}$

NEGATIVE REFRACTION





© D. Schurig



$\varepsilon<0$

$\mu<0$

V. G. Veselago, Sov. Phys. Usp. **10**, 509 (1968)

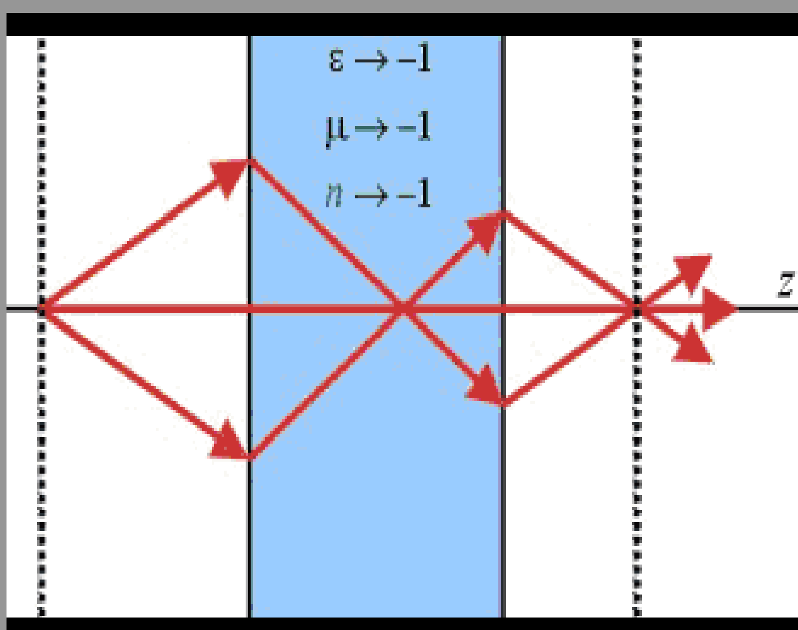
BEYOND THE DIFFRACTION LIMIT: NEGATIVE REFRACTION

VOLUME 85, NUMBER 18 PHYSICAL REVIEW LETTERS 30 OCTOBER 2000

Negative Refraction Makes a Perfect Lens

J. B. Pendry
Condensed Matter Theory Group, The Blackett Laboratory, Imperial College, London SW7 2BZ, United Kingdom
(Received 25 April 2000)

With a conventional lens sharpness of the image is always limited by the wavelength of light. An unconventional alternative to a lens, a slab of negative refractive index material, has the power to focus all Fourier components of a 2D image, even those that do not propagate in a radiative manner. Such "superlenses" can be realized in the microwave band with current technology. Our simulations show that a version of the lens operating at the frequency of visible light can be realized in the form of a thin slab of silver. This optical version resolves objects only a few nanometers across.



$$\varepsilon \equiv -1$$

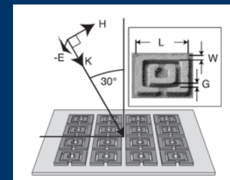
$$\mu \equiv -1$$

SINGULARITY

λ= 3 cm

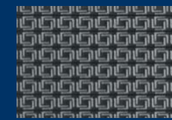
Shelby, Smith and Schultz,
Science **292**, ***77*** (2001)

λ= 0.025 cm



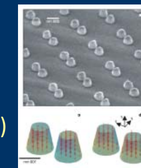
Yen et al.
Science **303**, 1494 (2004)

λ= 5 μm



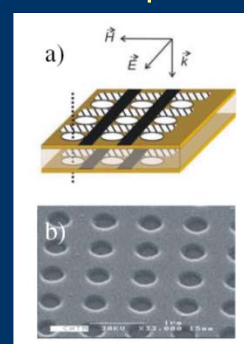
Wu et al.
APL **90**, 063107 (2007)

λ= 0.5 μm



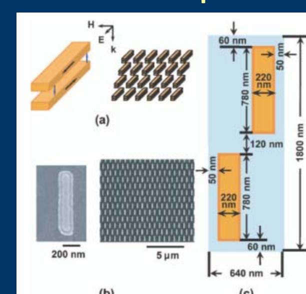
Grigorenko et al.
Nature **438**, 335 (2005)

λ= 2μm



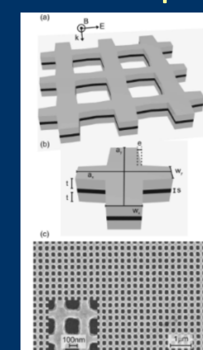
Zhang et al.
PRL ***95***, 137404 (2005)

λ= 1.5μm



Shalaev et al.
Optics Lett. **30**, 3356 (2005)

λ= 0.78μm



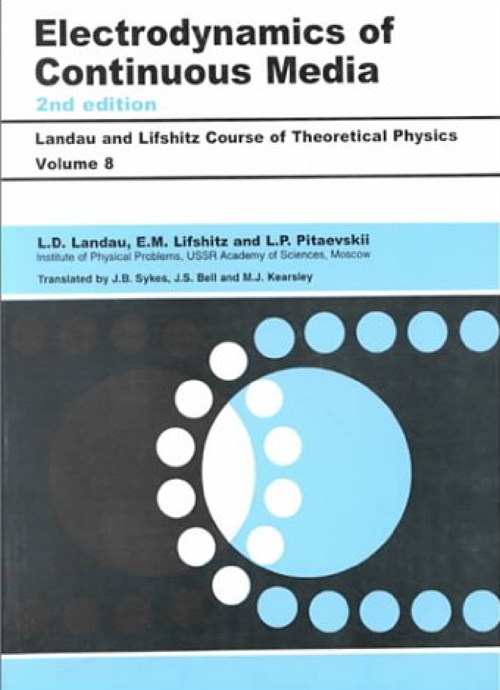
Dolling et al.
Optics Lett. **32**, 53 (2007)

$n < 0$?

PROBLEMS

$$\mu_{MM} \neq 1?$$

(at high frequencies)



he displacement $\mathbf{r}$ of the electron due to the field is given by $\dot{\mathbf{r}} = \mathbf{v}'$, $E/m\omega^2$. The polarization $\mathbf{P}$ of the body is the dipole moment per unit r all electrons, we find $\mathbf{P} = \Sigma e\mathbf{r} = -e^2 N\mathbf{E}/m\omega^2$, where N is the all the atoms in unit volume of the substance. By the definition of we have $\mathbf{D} = \varepsilon\mathbf{E} = \mathbf{E} + 4\pi\mathbf{P}$. We thus have the formula

$$\varepsilon(\omega) = 1 - 4\pi Ne^2/m\omega^2. \tag{78.1}$$

ncies over which this formula is applicable begins, in practice, at the ht elements and at the X-ray region for heavier elements.

he significance which it has in Maxwell's equations, the frequency condition $\omega \ll c/a$. We shall see later (§124), however, that the e allotted a certain physical significance even at higher frequencies.

of the magnetic permeability

agnetic permeability $\mu(\omega)$ ceases to have any physical meaning at cies. To take account of the deviation of $\mu(\omega)$ from unity would then be an unwarrantable refinement. To show this, let us investigate to what extent the physical meaning of the quantity $\mathbf{M} = (\mathbf{B} - \mathbf{H})/4\pi$, as being the magnetic moment per unit volume, is maintained in a variable field. The magnetic moment of a body is, by definition, the integral

Subtracting the equation $\mathbf{curl\ H} = (1/c)\,\partial\mathbf{D}/\partial t$, we obtain

$$\overline{\rho\mathbf{v}} = c\ \mathbf{curl\ M} + \partial\mathbf{P}/\partial t. \tag{79.3}$$

The integral (79.1) can, as shown in §29, be put in the form $\int \mathbf{M}\,dV$ only if $\overline{\rho\mathbf{v}} = c\,\mathbf{curl\ M}$ and $\mathbf{M} = 0$ outside the body.

Thus the physical meaning of $\mathbf{M}$, and therefore of the magnetic susceptibility, depends on the possibility of neglecting the term $\partial\mathbf{P}/\partial t$ in (79.3). Let us see to what extent the conditions can be fulfilled which make this neglect permissible.

For a given frequency, the most favourable conditions for measuring the susceptibility are those where the body is as small as possible (to increase the space derivatives in $\mathbf{curl\ M}$) and the electric field is as weak as possible (to reduce $\mathbf{P}$). The field of an electromagnetic wave does not satisfy the latter condition, because $E \sim H$. Let us therefore consider a variable magnetic field, say in a solenoid, with the body under investigation placed on the axis. The electric field is due only to induction by the variable magnetic field, and the order of magnitude of E inside the body can be obtained by estimating the terms in the equation $\mathbf{curl\ E} = -(1/c)\partial\mathbf{B}/\partial t$, whence $E/l \sim \omega H/c$ or $E \sim (\omega l/c)H$, where l is the dimension of the body. Putting $\varepsilon - 1 \sim 1$, we have $\partial P/\partial t \sim \omega E \sim \omega^2 l H/c$. For the space derivatives of

the magnetic moment $\mathbf{M} = \chi\mathbf{H}$ we have $c\ \mathbf{curl\ M} \sim c\chi\mathbf{H}/l$. If $|\partial\mathbf{P}/\partial t|$ is small compared with $|c\ \mathbf{curl\ M}|$, we must have

$$l^2 \ll \chi c^2/\omega^2. \tag{79.4}$$

It is evident that the concept of magnetic susceptibility can be meaningful only if this inequality allows dimensions of the body which are (at least) just macroscopic, i.e. if it is compatible with the inequality $l \gg a$, where a is the atomic dimension. This condition is certainly not fulfilled for the optical frequency range; for such frequencies, the magnetic susceptibility is always $\sim v^2/c^2$, where v is the electron velocity in the atom;† but the optical frequencies themselves are $\sim v/a$, and therefore the right-hand side of the inequality (79.4) is $\sim a^2$.

Thus there is no meaning in using the magnetic susceptibility from optical frequencies onward, and in discussing such phenomena we must put $\mu = 1$. To distinguish between $\mathbf{B}$ and $\mathbf{H}$ in this frequency range would be an over-refinement. Actually, the same is true for many phenomena even at frequencies well below the optical range.‡

The presence of a considerable dispersion of the permeability makes possible the existence of quasi-steady oscillations of the magnetization in ferromagnetic bodies. In order to exclude the possible influence of the conductivity, we shall consider ferrites, which are non-metallic ferromagnets.

The term "quasi-steady" means, as usual (§58), that the frequency is assumed to satisfy the condition $\omega \ll c/l$, where l is the characteristic dimension of the body (or the "wavelength" of the oscillation). In addition, we shall neglect the exchange energy related he oscillations; that is, the unimportant. For this, the ngth for the inhomogeneity (43.1). $\mathbf{B}'$, where $\mathbf{H}_0$ and $\mathbf{B}_0$ are the $\mathbf{B}'$ the variable parts in the se variable parts satisfy the equations

$$\mathbf{curl\ H}' = 0, \quad \mathrm{div}\ \mathbf{B}' = 0, \tag{79.5}$$

which differ from the magnetostatic equations only in that the permeability is now (for a monochromatic field $\propto e^{-i\omega t}$) a function of the frequency, not a constant.§ A ferromagnetic medium is magnetically anisotropic, and its permeability is therefore a tensor $\mu_{ik}(\omega)$, which determines the linear relation between the variable parts of the induction and the field.

† This estimate relates to the diamagnetic susceptibility; the relaxation times of any paramagnetic or ferromagnetic processes are certainly long compared with the optical periods. It must be emphasized, however, that the estimates are made for an isotropic body, and must be used with caution when applied to ferromagnets. In particular, the gyrotropic terms in the tensor μ_{ik} which decrease only slowly (as $1/\omega$) with increasing frequency (see Problem 1) may be important even at fairly high frequencies.

‡ This is discussed from a somewhat different standpoint in §103 below; see the second footnote to that section.

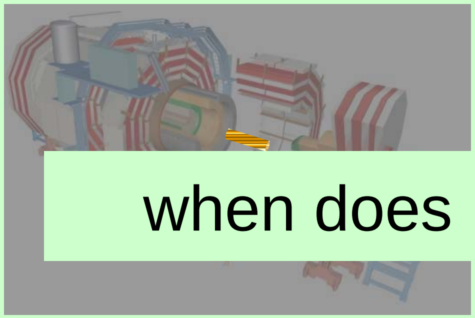
§ These oscillations are therefore called *magnetostatic oscillations*. The theory has been given by C. Kittel (1947) for homogeneous (see below) magnetostatic oscillations and by L. R. Walker (1957) for inhomogeneous ones.

Thus there is no meaning in using the magnetic susceptibility from optical frequencies onward, and in discussing such phenomena we must put $\mu = 1$. To distinguish between $\mathbf{B}$ and $\mathbf{H}$ in this frequency range would be an over-refinement. Actually, the same is true for many phenomena even at frequencies well below the optical range.‡

LANDAU-LIFSHITZ PERMEABILITY ARGUMENT

$$\mathbf{j} = c\nabla \times \mathbf{M} + \partial \mathbf{P} / \partial t$$

$$a_{\mathrm{L}} << L << \lambda$$



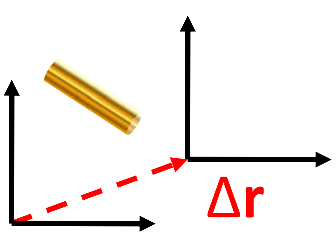
total magnetic moment $\equiv \int \left(\mathbf{M} - i \frac{\omega}{2c} \mathbf{r} \times \mathbf{P} \right) \mathrm{d}V$

when does **M** represent the magnetic-dipole density?

~~$\chi_{\mathrm{M}} \sim v^2/c^2 \sim a_L^2/\lambda^2$~~

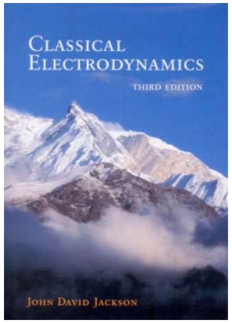
$n_S \sim \lambda/\ell$ or $\kappa_S >> \lambda/\ell$

uniqueness and significance of **M**



change in total magnetic moment = total electric moment × $-i\omega\Delta\mathbf{r}/2c$

METAMATERIALS and MOLECULAR SOLIDS



252 Chapter 6 **Maxwell Equations, Macroscopic Electromagnetism, Conservation Laws—SI**

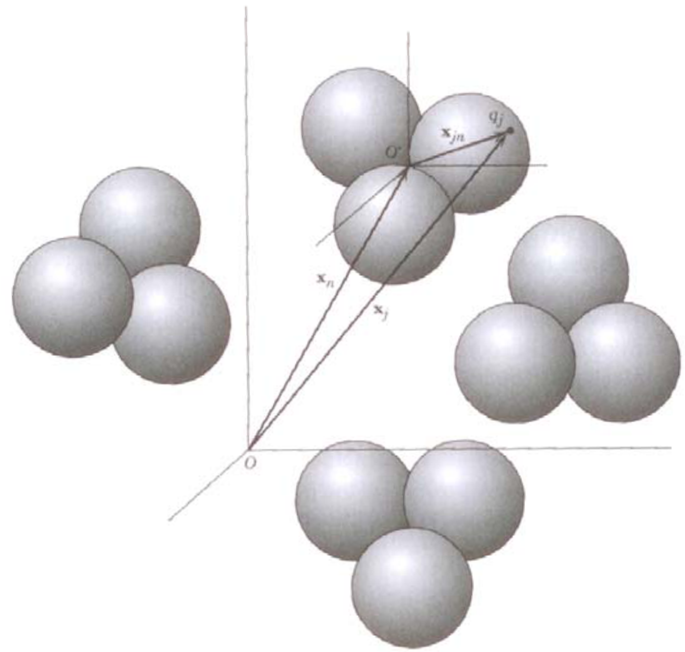


Figure 6.2 Coordinates for the nth molecule. The origin O' is fixed in the molecule (usually it is chosen at the center of mass). The jth charge has coordinate $\mathbf{x}_{jn}$ relative to O', while the molecule is located relative to the fixed (laboratory) axes by the coordinate $\mathbf{x}_n$.

$$\mathbf{M} \equiv \mathbf{m} / V_C$$

origin varies from cell to cell

coordinate change

$$\mathbf{m} \to \mathbf{m} - i\omega\Delta\mathbf{r} \times \mathbf{p} / 2c$$

ambiguity is removed if

$$|\mathbf{m}| >> (a_L / \lambda)|\mathbf{p}|$$

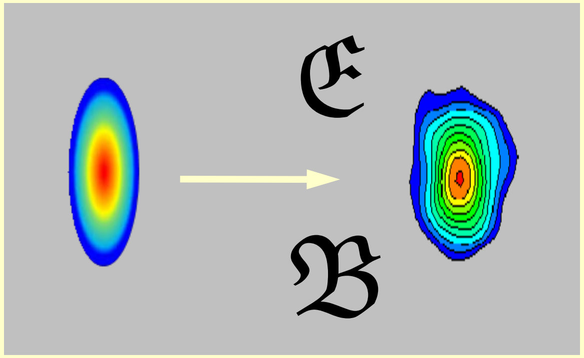
non-resonant induced moments (molecules)

$$|\mathbf{p}| \sim V_C \mathfrak{E} \qquad |\mathbf{m}| \sim (a_L / \lambda)^2 V_C \mathfrak{H}$$

high-ε_S substances

$$|\mathbf{p}| \sim V_C \mathfrak{E} \qquad |\mathbf{m}| \sim V_C \mathfrak{H}$$

First Homogenization Step: Induced Multipoles and Single Particle Scattering



$$p_i = \sum_{jk} \alpha_{ij} \mathfrak{E}_j(0) + \frac{1}{3} A_{ijk}\, \partial \mathfrak{E}_j / \partial x_k \Big|_{\mathbf{r}=0} + G_{ij} \mathfrak{B}_j(0) + \ldots$$

$$m_i = \sum_{j} \gamma_{ij} \mathfrak{B}_j(0) + G^*_{ij} \mathfrak{E}_j(0) + \ldots$$

$$q_{ij} = \sum_{k} A^*_{ijk} \mathfrak{E}_k(0) + \ldots$$

$$\alpha_{ij} = \sum_s \frac{\langle 0 | \hat{p}_i | s \rangle \langle s | \hat{p}_j | 0 \rangle}{\hbar(\omega_s - \omega)} \qquad A_{ijk} = \sum_s \frac{\langle 0 | \hat{p}_i | s \rangle \langle s | \hat{\Theta}_{jk} | 0 \rangle}{\hbar(\omega_s - \omega)}$$

$$\gamma_{ij} = \sum_s \frac{\langle 0 | \hat{m}_i | s \rangle \langle s | \hat{m}_j | 0 \rangle}{\hbar(\omega_s - \omega)} \qquad G_{ij} = \sum_s \frac{\langle 0 | \hat{p}_i | s \rangle \langle s | \hat{m}_j | 0 \rangle}{\hbar(\omega_s - \omega)}$$

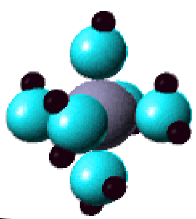
$$\alpha_{\mathrm{mol}} \sim V_{\mathrm{mol}} >> \left\{ \begin{matrix} G_{\mathrm{mol}} \\ A_{\mathrm{mol}} / \lambda_0 \end{matrix} \right\} \sim V_{\mathrm{mol}} \frac{\ell_{\mathrm{mol}}}{\lambda_0} >> \gamma_{\mathrm{mol}} \sim V_{\mathrm{mol}} \frac{\ell^2_{\mathrm{mol}}}{\lambda_0^2}$$

Small Spheres ($\lambda_0 >> R_S$) vs. Molecules

$$\mathbf{p} = \left(\frac{\varepsilon_S F - 1}{\varepsilon_S F + 2}\right) R_S^3 \mathfrak{E} \quad \mathbf{m} = \left(\frac{F - 1}{F + 2}\right) R_S^3 \mathfrak{H} \quad F = \frac{2(\sin\theta - \theta\cos\theta)}{(\theta^2 - 1)\sin\theta + \theta\cos\theta} \quad \left(\theta = kR_S\sqrt{\varepsilon_S}\right)$$

$$kR_S\sqrt{\varepsilon_S} << 1$$

$$\alpha_{SPH} = \frac{3(\varepsilon_S - 1)}{4\pi(\varepsilon_S + 2)} V_{SPH} \qquad \gamma_{SP} \approx V_{SPH} \left(\frac{R_S}{\lambda_0}\right)^2 \pi\varepsilon_S / 10$$



$$\alpha_{mol} \sim \frac{\omega_{ED}}{(\omega_{ED} - \omega)} V_{mol} \qquad \gamma_{mol} \sim V_{mol} \left(\frac{\ell_{mol}}{\lambda_0}\right)^2 \frac{\omega_{MD}}{(\omega_{MD} - \omega)}$$

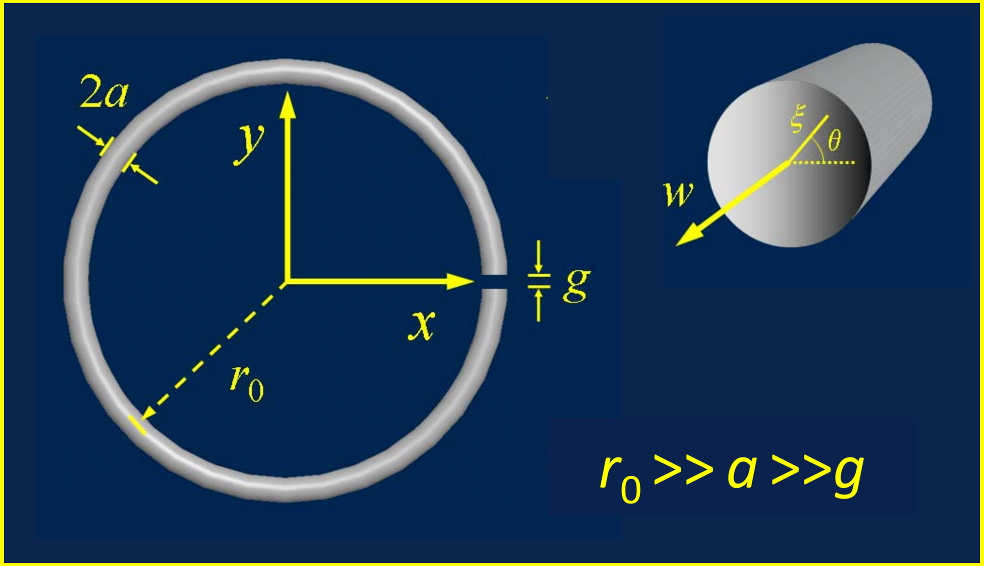
$$kR_S\kappa_S >> 1$$

$$\alpha_{SPH} = \frac{3}{4\pi} V_{SPH} \qquad \gamma_{SP} \approx -\frac{3}{8\pi} V_{SPH}$$

SCATTERING BY A SPLIT-RING

$$\gamma = (1-\varepsilon_S^{-1})\frac{i\pi^2 r_0^4\omega}{c^2 Z_{spr}}$$

$$\alpha = -(1-\varepsilon_S^{-1})\frac{ig^2}{\omega Z_{spr}^*}$$



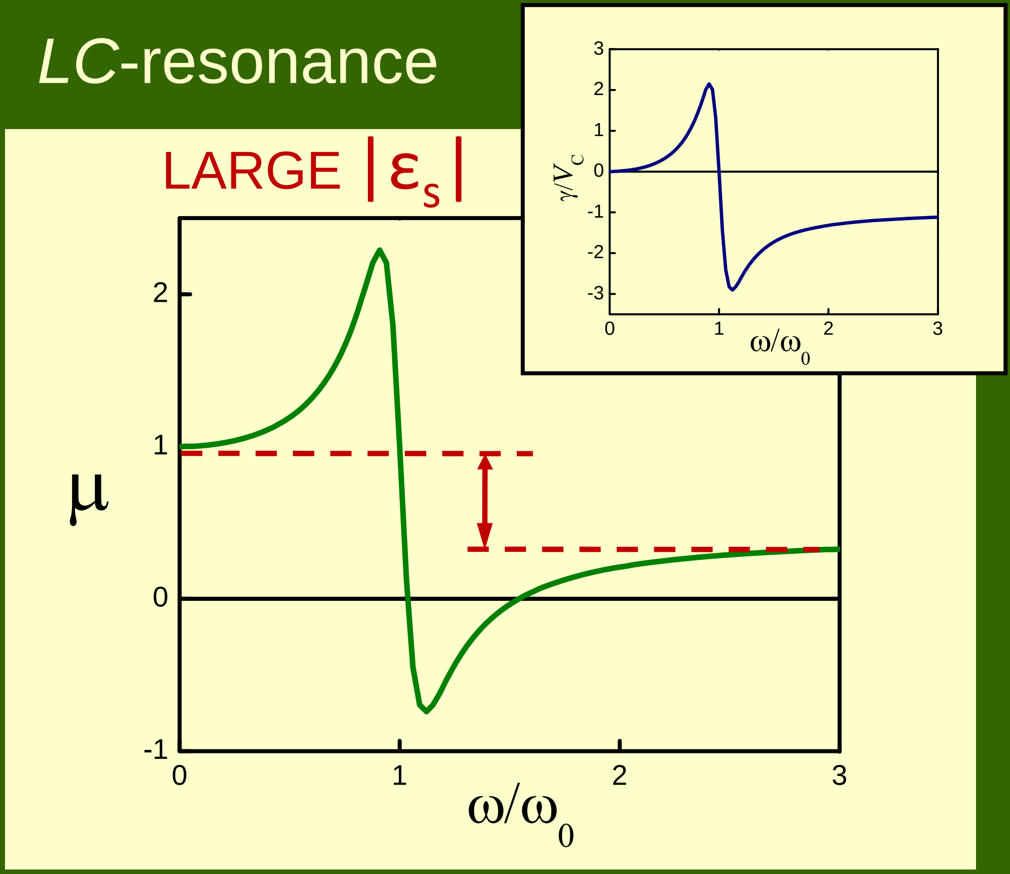
$$Z_{spr} = i\left(\frac{4\pi r_0\omega}{ka\varepsilon_S^{-1/2}c^2}\right)\frac{J_0(ka\varepsilon_S^{-1/2})}{J_1(ka\varepsilon_S^{-1/2})} - i\left(\omega L/c^2 - 1/\omega C\right)$$

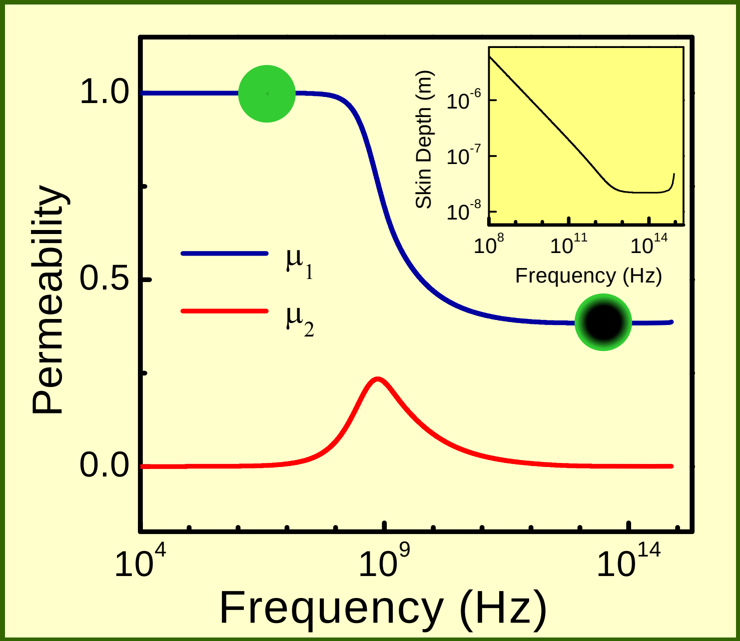
RM, *PNAS* 106, 1693 (2009)

$\kappa_S >> \lambda_0 / a >> 1$	$n_S \sim \lambda_0 / a >> 1$	$a << \lambda_0 / \lvert\varepsilon_S^{1/2}\rvert$
LC RESONANCE	CAVITY-LIKE RESONANCES	PLASMON RESONANCE
$\omega^2 = c^2/LC$	$n_S ka \approx 5\pi/4,\ 9\pi/4,\ 13\pi/4$	$\oint \mathbf{E}.d\mathbf{l} = 0$

LC & SKIN-DEPTH RESONANCES

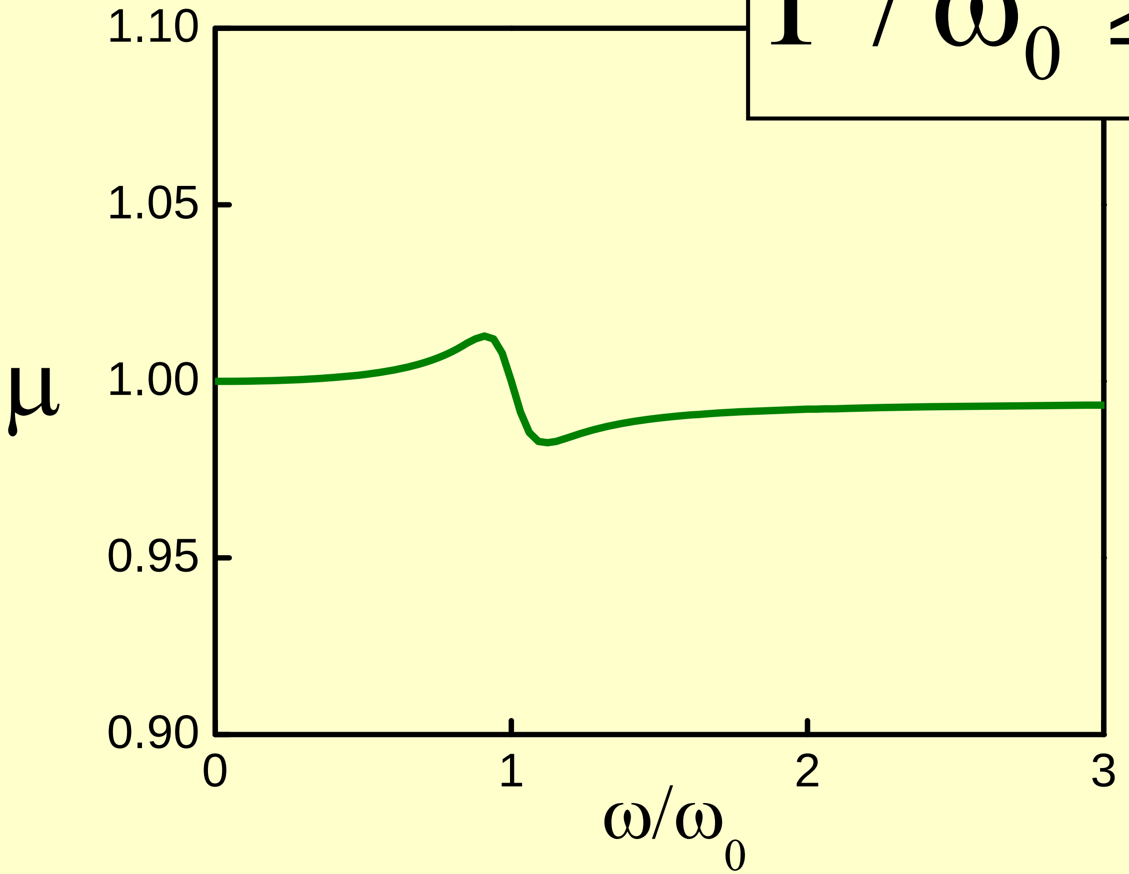
LC-resonance





$$\frac{\gamma_{\mathrm{spr}} r_0^{-3}}{\gamma_{\mathrm{mol}} \ell_{\mathrm{mol}}^{-3}} \sim \frac{\hbar c}{e^2} \times \frac{\lambda}{\ell_{\mathrm{mol}}} \approx 10^5 - 10^6$$

PLASMON RESONANCES: OPTICAL FREQUENCIES ($|\varepsilon| < 100$)



$$\Gamma / \omega_0 \geq \left(\ell / \lambda_0 \right)^2$$

$$\omega_0 \gtrsim 1\,\mathrm{eV}$$

LORENTZ-LORENZ, CLAUSIUS-MOSOTTI & LEWIN FORMULAS

$$\mathbf{P} \equiv \frac{\mathbf{p}}{V_C} = \alpha \mathfrak{E} / V_C$$

$$\mathbf{M} \equiv \frac{\mathbf{m}}{V_C} = \gamma \mathfrak{B} / V_C$$

$+$

$$\mathfrak{E} = \mathbf{E} + 4\pi \mathbf{P} / 3$$

$$\mathfrak{B} = \mathbf{H} + 4\pi \mathbf{M} / 3$$

$=$

$$\varepsilon = \varepsilon_1 + i\varepsilon_2 = \varepsilon_H \left(\frac{3V_C + 8\pi\alpha}{3V_C - 4\pi\alpha} \right)$$

$$\mu = \mu_1 + i\mu_2 = \mu_H \left(\frac{3V_C + 8\pi\gamma}{3V_C - 4\pi\gamma} \right)$$

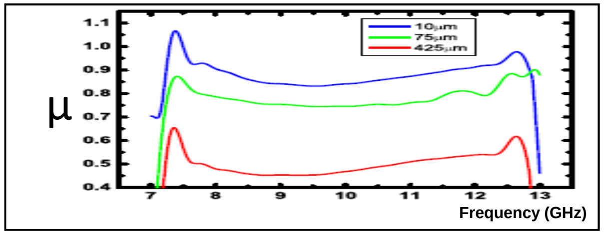


Table 1. Calculated effective-medium permeability of a system of spherical particles of radius *R* in a simple-cubic arrangement of lattice constant *d*

Material	λ, μm	$\sqrt{\varepsilon_S} = n_S + i\kappa_S$	$\mu_1 + i\mu_2$
Cu	3,000	975 + *i*975	0.382 + *i*0.005
$KTa_{0.982}Nb_{0.018}O_3$	500	17.3 + *i*0.58	3.322 + *i*1.226
PbTe	312.5	43.4 + *i*43.0	0.487 + *i*0.102
$SrTiO_3$	111.0	25 + *i*25	0.571 + *i*0.165
SiC	12.5	17 + *i*17	0.678 + *i*0.221
Sb	4.0	9.73 + *i*13.77	0.811 + *i*0.163
Ag	1.93	0.24 + *i*14.09	0.834 + *i*0.004
Ge	0.590	5.75 + *i*1.63	1.041 + *i*0.029
Si	0.288	4.09 + *i*5.39	0.978 + *i*0.053

The refractive index n_S and the extinction coefficient κ_S of the corresponding materials are room temperature values at the wavelengths shown. Results are for $d = 2R = \lambda/20$ and $\varepsilon_H = 1.96$. Note the paramagnetic response of the substances for which n_S dominates over κ_S.

CONCLUDING REMARKS

HIGH-FREQUENCY MAGNETISM

$$\mathrm{Im}\sqrt{\varepsilon_S} = \kappa_S >> \lambda / \ell >> 1$$

METALS

$\lambda > 2.5\ \mu m$

$$\lambda >> \ell >> \lambda_P$$

$$\delta \approx c / \omega_P$$

$$\mathrm{Re}\sqrt{\varepsilon_S} = n_S \sim \lambda / \ell >> 1$$

FERROELECTRICS

LOW-FREQUENCY TO PHONONS

FAR-INFRARED