Homogenization of 3D-dielectric photonic crystals and artificial magnetism.

Guy Bouchitté and Christophe Bourel , Université de Toulon, France

Didier Felbacq, Université de Montpellier, France

Maxwell System in harmonic regime: $\exp(-i\omega t)$

ω : angular frequency: $\frac{\omega}{c} = \frac{2\pi}{\lambda} = k_0$ (wave number)

$\varepsilon(\omega)$: permittivity (nonnegative imaginary part)

$\mu(\omega)$: permeability (real close to 1)

Optical parameters

Propagation in n-index homogeneous medium: $\Delta u + k_0^2\, n^2\, u \;=\; 0.$

Usual cases in optics: $\mu(\omega) \sim 1 \quad , \quad \varepsilon(\omega) = \varepsilon' + i\,\varepsilon''$

1. Metal: $\varepsilon' \in \mathbb{R} \quad , \quad \varepsilon'' > 0 \quad (\varepsilon'' \gg 1)$
2. Dielectric: $\varepsilon' > 0 \quad (\varepsilon' = 1$ if perfect dielectric, $\varepsilon'' = 0$ if lossless dielectric)

GOAL: Design a periodic structure with small metallic $(\varepsilon'' \gg 1)$ or dielectric $(\varepsilon' \gg 1)$ components yielding artificially:

I- $\varepsilon^{eff}(\omega) < 0$ (for some range of frequencies)

II- $\mu^{eff}(\omega) < 0$ (artificial magnetic activity)

III- Both I and II (negative index)

Singular limit of 3D- Maxwell system

We consider the diffraction of a monochromatic electromagnetic wave (ω is fixed) by a **bounded** 3D- obstacle Ω made of periodic inclusions.

• The period η is a small parameter (In practice $\eta \sim \frac{\text{wavelength}}{10}$)

• The relative permittivity $\varepsilon_\eta(x)$ can be very large on subsets (connected or not) with a possibly small filling ratio θ_η.

• The electromagnetic field (E_η, H_η) satisfies the system (in the distibutional sense) on all $\mathbb{R}^3$:

$$\begin{cases} \operatorname{curl} E_\eta &= i\omega\mu_0 \, H_\eta \\ \operatorname{curl} H_\eta &= -i\omega\varepsilon_0 \, \varepsilon_\eta \, E_\eta \end{cases} \tag{1}$$

$$(E_\eta - E^i, H_\eta - H^i) \quad \text{satisfies the O.W.C.} \tag{2}$$

where O.W.C. means 'outgoing radiation condition at infinity' (Silver Müller)

Mathematical issues

- Vector problems with L^2-estimates only on $\operatorname{curl} E_\eta$ and L^1 estimate on $\varepsilon_\eta E_\eta$.

- Non trivial differential geometry in the torus: (E_η, H_η) oscillates at order zero in a finite dimensional space.

- High contrast $|\varepsilon_\eta| \gg 1$ produces non local effects.

- **Complex geometries (split rings, reiterated homogenization) lead to resonance effects.**

Smith, D.R. and Pendry, J.B. and Wiltshire, M.C.K. Metamaterials and Negative Refractive Index, Science, 2004, 305, pp 788–792

Urzhumov, Y.A. / Shvets, G. Optical magnetism and negative refraction in plasmonic metamaterials Solid State Communications, 146 (5),2008, p.208-220.

Case I- Arrays of metallic nanorods

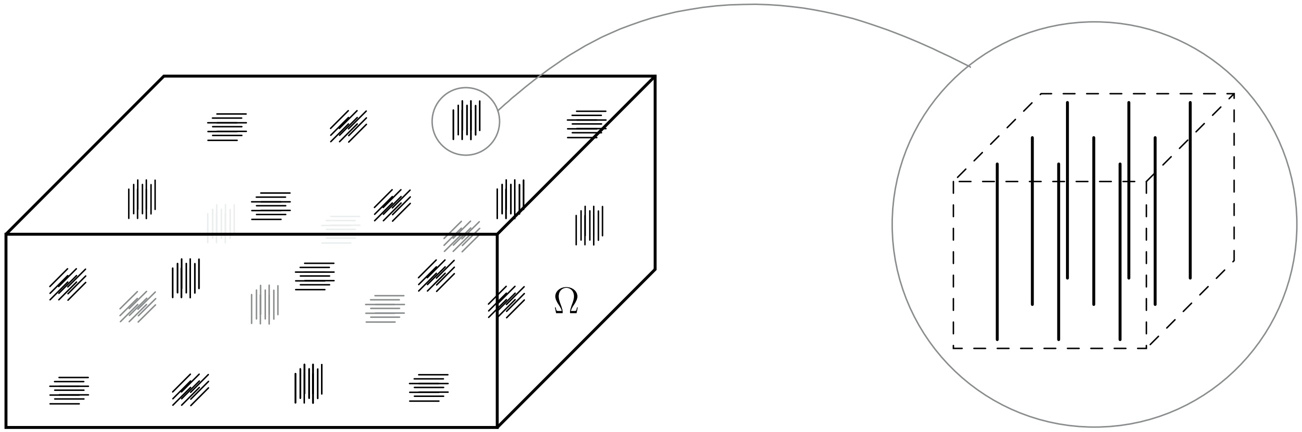

Figure 1: bloks of parallel metallic fibers, length $\sim$ period, diameter $\ll$ period

The scatter is in a bounded domain of $\mathbb{R}^3$, the volume fraction of rods vanishes

$\rightsquigarrow$ negative ε^{eff} (all symmetric tensors can be reached !) through reiterated homogenization

joined work with C. Bourel submitted CICP

Case II- Infinitely long dielectric fibers

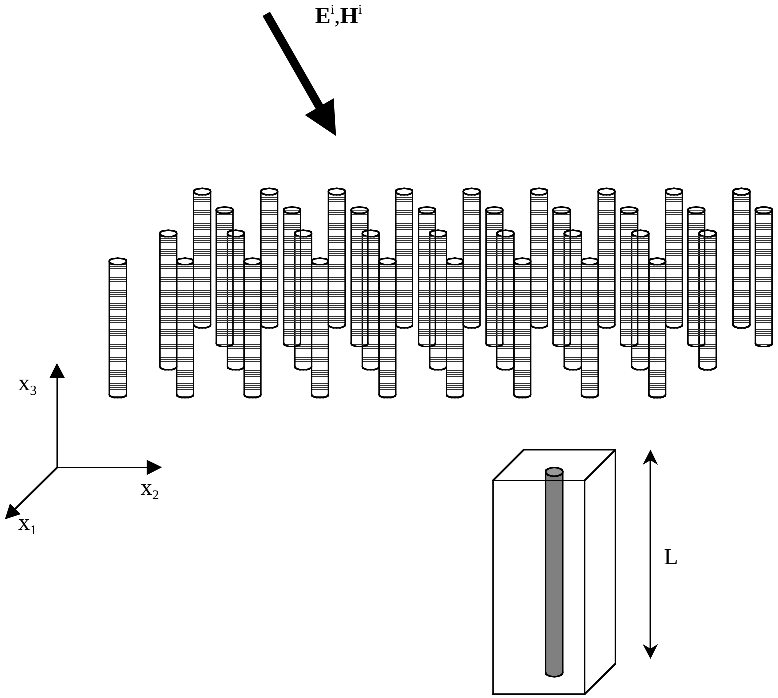

Figure 2: Fibers of length $L = \infty$, period η, diameter $a = O(\eta)$

- The filling ratio is positive
- The magnetic field is assumed e_3-parallel $\rightsquigarrow$ 2D-analysis.

Joint work with D. Felbacq (prl 2005): negative effective permeability μ^{eff}

Case III- The Pendry split ring structure

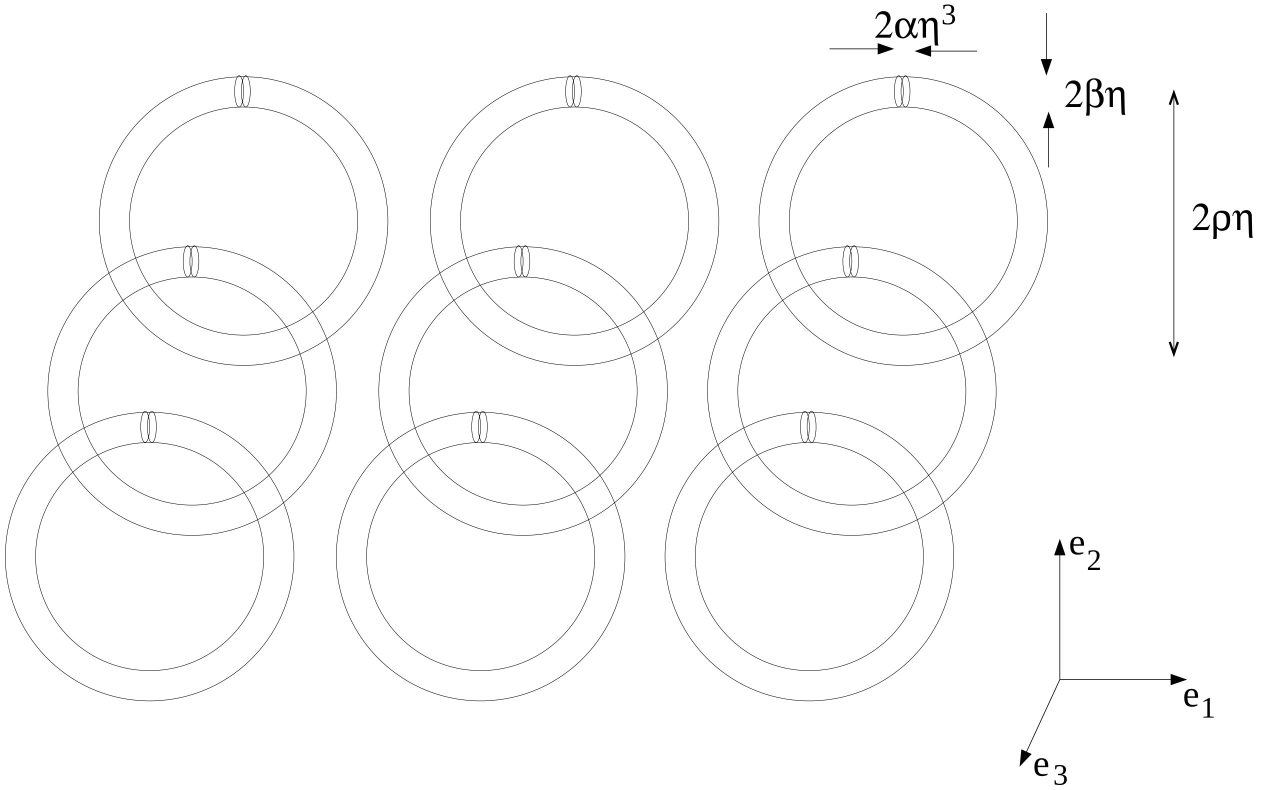

Figure 3: Sketch of the geometry, showing one layer of rings.

- The scatter $\Omega \subset \mathbb{R}^3$ contains $O(\eta^{-3})$ thin metallic split rings of diameter $O(\eta)$.
- We assume positive filling ratio.

Joint work with Ben Schweizer: $\rightsquigarrow$ negative effective permeability μ^{eff}

AIM of this talk: Disconnected dielectric inclusions

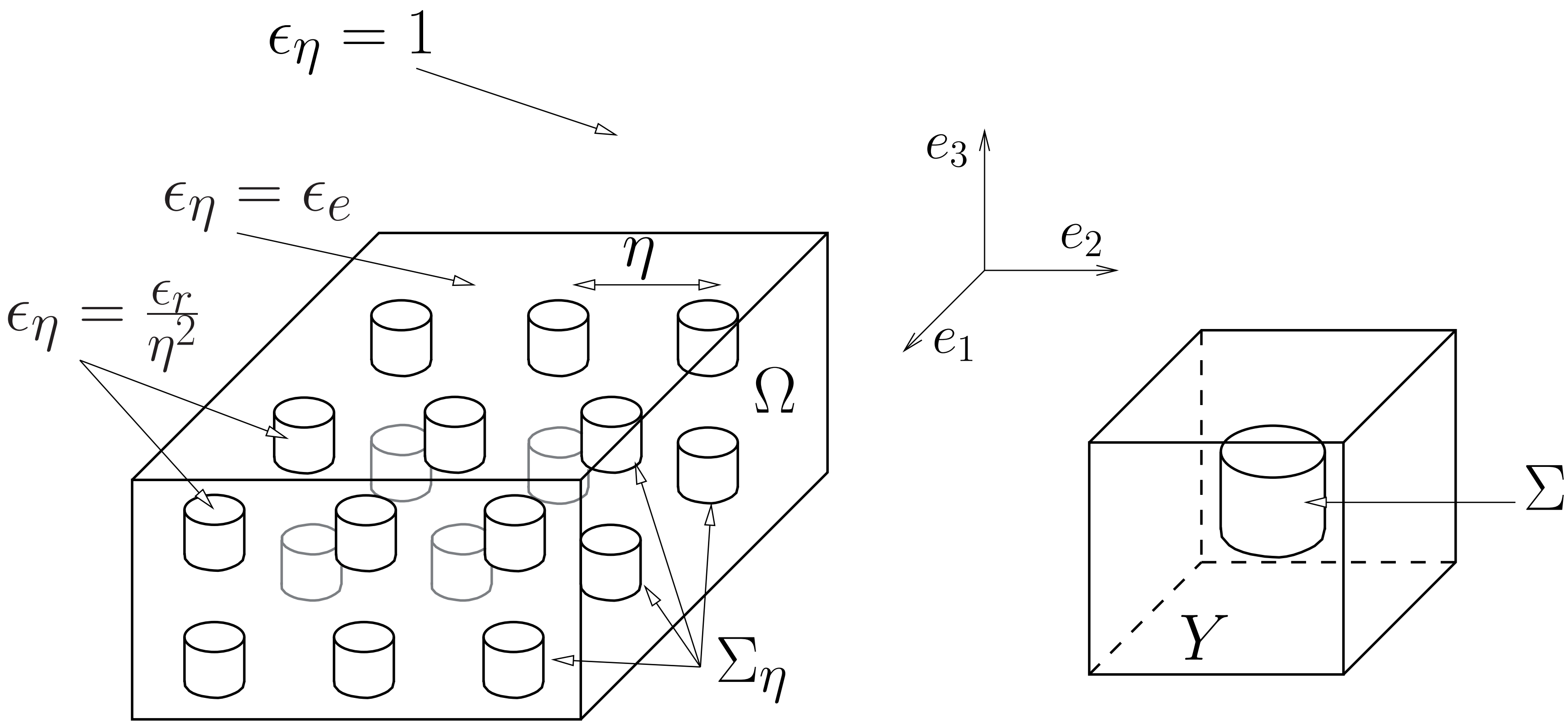

Figure 4: Disconnected inclusions of size $\sim$ period .

The macroscopic domain $\Omega \subset \mathbb{R}^3$ contains $O(\eta^{-3})$ periodic inclusions of diameter $O(\eta)$ filled with high permittivity dielectric (positive filling ratio)

Geometry and scaling

The heterogeneous structure is placed in a bounded domain $\Omega \subset \mathbb{R}^3$. It consists of periodic high permittivity inclusions (period η) embedded in a lossless dielectric matrix. The inclusions occupy a subregion

$$\Sigma_\eta := \bigcup_{i \in I_\eta} \eta(i + \Sigma) \ , \ I_\eta = \{i \in \mathbb{Z}^2 : \eta(i + \Sigma) \subset\subset \Omega\}.$$

Here $\Sigma \subset\subset Y := (-1/2, 1/2)^3$ is a regular connected domain whose complement $Y^* := Y \setminus \Sigma$ is assumed to be simply connected. The structure, whose relative permeability is assumed to be equal to 1, is characterized by its relative permittivity:

$$\varepsilon_\eta(x) := \begin{cases} \dfrac{\varepsilon_r}{\eta^2} & \text{if } \ x \in \Sigma_\eta \\ \varepsilon_e & \text{if } \ x \in \Omega \setminus \Sigma_\eta \\ 1 & \text{if } \ x \in \mathbb{R}^3 \setminus \Omega \end{cases} \tag{3}$$

We assume that $\varepsilon_r = \varepsilon_r' + i\,\varepsilon_r''$ with $\varepsilon_r' > 0$ and ε_r'' ‘small”

We start with the energy bound assumption

$$\sup_{\eta} \int_{\mathcal{B}} (|H^{\eta}|^2 + |E_{\eta}|^2) \;<\; +\infty \tag{4}$$

where $\mathcal{B}$ is a big ball containing Ω.

As (E_{η}, H_{η}) solves $\Delta u + k_0^2 u = 0$ in $\mathbb{R}^3 \setminus \Omega$, by Stratton-Shu integral identities, the flux of Poynting vector $E_{\eta} \wedge \overline{H_{\eta}}$ across $\partial\mathcal{B}$ remains bounded as well. Thus:

• Improved estimate:

$$\sup_{\eta} \int_{\mathcal{B}} (|H^{\eta}|^2 + |\varepsilon_{\eta}||E_{\eta}|^2) \;<\; +\infty \tag{5}$$

• The rescaled displacement vectors $J_{\eta} := \;\eta\varepsilon_{\eta}\, E_{\eta}$ is also bounded $L^2(\mathcal{B})$.
(in particular $\int_{\Sigma_{\eta}} |E_{\eta}|^2 \to 0$)

Two scale approach

We are going to identfy the zero order tem in the expansions

$$\begin{aligned} E_\eta(x) &= E_0(x, x/\eta) + \eta E_1(x, x/\eta) + \eta^2 E_2(x, x/\eta) \\ H_\eta(x) &= H_0(x, x/\eta) + \eta H_1(x, x/\eta) + \eta^2 H_2(x, x/\eta) \\ J_\eta(x) &= J_0(x, x/\eta) + \eta J_1(x, x/\eta) + \eta^2 J_2(x, x/\eta) \end{aligned}$$

where $x/\eta := y$ represents the periodic variable ($y \in Y = (-\frac{1}{2}, \frac{1}{2})^3$)

By (5), it is rigorous to assume that (possibly after extracting a subsequence)

$$E_\eta \rightharpoonup\rightharpoonup E_0(x, y) \quad , \quad H_\eta \rightharpoonup\rightharpoonup H_0(x, y) \quad , \quad J_\eta(x, y) \rightharpoonup\rightharpoonup J_0(x, y) \quad ,$$

where $E_0(x, \cdot), H_0(x, \cdot), J_0(x, \cdot)$ are Y-periodic and $E_\eta \rightharpoonup\rightharpoonup E_0(x, y)$ means

$$\int_{\mathcal{B}} E_\eta \cdot \varphi(x, \frac{x}{\eta}) \, dx \ \rightarrow \ \int_{\mathcal{B}} \int_Y E_0(x, y) \cdot \varphi(x, y) \, dx dy$$

being φ smooth in x and Y-periodic in y.

Cell problem for $E_0(x,\cdot)$

From first equation in (1), it is easy to show that the periodic field $E_0(x,\cdot)$ satisfies:

$$\mathrm{curl}_y\ E_0 = 0 \quad \text{in } Y \quad , \quad \mathrm{div}_y\ E_0 = 0 \quad \text{in } Y\setminus\overline{\Sigma} \quad , \quad E_0 = 0 \quad \text{in } \Sigma \quad (6)$$

By the curl-free condition and letting $E(x) = \int_Y E_0(x,y)\,dy$, we search a solution $E_0(x,y) = E(x) + \nabla_y \chi$ for a suitable periodic $\chi \in W^{1,2}_\sharp(Y)$. We are led to:

$$E_0(x,) = \sum_{i=1}^{3} E_i(x)\, E^i(y) \quad \text{for } x\in\Omega \quad , \quad E_0(x,y) = E(x) \quad \text{for } x \in \mathcal{B}\setminus\Omega \quad (7)$$

where the periodic shape functions E^i are characterized by

$$E^i(y) = e_i + \nabla_y \chi_i\ , \quad \Delta\chi_i = 0 \quad \text{on } Y^*\ , \quad \chi_i = -y_i \quad \text{on } \Sigma \qquad (8)$$

Effective permittivity tensor $\varepsilon^{\rm eff}$ $\boxed{\varepsilon^{\rm eff} := \varepsilon_e\, A^{\rm hom}}$

where the homogenized matrix $A^{\rm hom}$ is given by

$$A^{\rm hom}_{i,j} := \int_Y E^i \cdot E^j\, dy = \int_{Y^*} (e_i + \nabla_y\chi_i)\cdot(e_j + \nabla_y \chi_j)\, dy \qquad (9)$$

NB: $A^{\rm hom}$ is **real symmetric positive and independent of the frequency.**

Cell problem for $H_0(x,\cdot)$

By using Maxwell eq. (1), the periodic fields $H_0(x,\cdot)$ and $J_0(x,\cdot)$ satisfy

$$\mathrm{curl}_y\ H_0 + i\omega\varepsilon_0 J_0 = 0 \quad \text{in } Y \quad , \quad \mathrm{div}_y\, H_0 = 0 \quad \text{in } Y \tag{10}$$

$$\mathrm{curl}_y\ J_0 + i\varepsilon_r\omega\mu_0 H_0 = 0 \quad \text{in } \Sigma \quad , \quad J_0 = 0 \quad \text{in } Y\setminus\Sigma \tag{11}$$

Observations :

- By (10), $H_0(x,\cdot)$ belongs to the Sobolev space $W^{1,2}_\sharp(Y;\mathbb{C}^3)$.
- In contrast $J_0(x,\cdot)$ (supported in Σ) which may have a tangential jump across $\partial\Sigma$.
- Exploiting (10)(11), on subset Σ: $\Delta_y H_0 + k_0^2\,\varepsilon^r H_0 = 0$.

QUESTION: No tangential trace condition prescribed on $\partial\Sigma$.
How many degrees of freedom for solutions of (10)(11) ??

The analysis of the full system relies **on the simple connectedness of $Y\setminus\Sigma$.**

Geometric averaging

DEFINITION Let $u \in W^{1,2}_\sharp(Y;\mathbb{C}^3)$ such that

$$\operatorname{curl} u = 0 \quad \text{on } Y \setminus \Sigma\ .$$

We associate the *circulation vector* $\displaystyle\oint u \in \mathbb{C}^3$ which is characterized by the identity

$$\int_Y u \cdot \varphi \, dy \;=\; \left(\int_Y \varphi \, dy\right) \cdot \left(\oint u\right) \tag{12}$$

for all φ periodic, $\operatorname{div} \varphi = 0$ and $\varphi = 0$ on Σ.

When u is smooth, the components of $\oint u$ represent the circulation of u along any curve in $Y \setminus \Sigma$ connecting opposite points on the faces of ∂Y.

Remark • The key point is that in general $\oint u \neq \int_Y u dy$
(DIFFERS FROM THE BULK AVERAGE !)

• As $Y \setminus \Sigma$ is simply connected, any $u \in X$ can be written in the form

$$u = z + \nabla_y w \quad , \quad z = \oint u \quad , \quad w \in W^{1,2}_\sharp(Y^*)\ .$$

Space of solutions $H_0(x,\cdot)$ is three dimensional

LEMMA. For $i \in \{1,2,3\}$ there is a unique solution $H^i(y)$ to (10)(11) with $\oint H^i = e_i$. Thus

$$H_0(x,) \;=\; \sum_{i=1}^{3} H_i(x)\,H^i(y) \quad \text{for } x\in\Omega \quad , \quad H_0(x,y) \;=\; H(x) \quad \text{for } x \in \mathcal{B}\setminus\Omega \tag{13}$$

The macroscopic field $H(x) = (H_i(x)) \in L^2(\mathcal{B};\mathbb{C}^3)$ is related to the weak limit $[H_0](x) := \int_Y H_0(x,\cdot)$ of (H_η) in $L^2(\mathcal{B};\mathbb{C}^3)$ by the tensorial relation

$$[H_0](x) \;=\; \mu^{\text{eff}}\,H(x) \quad , \quad \mu^{\text{eff}}_{i,j} := \int_Y (H^j\cdot e_i)\,dy \tag{14}$$

The tensor μ^{eff} is symmetric and will be written explicitly by means of a suitable spectral problem (see (20)). Eventually applying (12) to $u = H^i$ and $\varphi = E^j \wedge z$ with E^j given in (8) ($z\in\mathbb{R}^3$), we infer

$$\int_Y (H^i\wedge E^j)\,dy \quad = \quad e^i\wedge e^j \quad , \quad \text{for every } i,j\in\{1,2,3\} \tag{15}$$

The homogenization result

Recalling (9) and (14), we introduce the tensors valued functions

$$\boldsymbol{\mu}(\omega, x) = \begin{cases} \mu^{\text{eff}}(\omega) & \text{for } x \in \Omega \\ I_3 & \text{for } x \in \mathbb{R}^3 \setminus \Omega \end{cases} , \quad \boldsymbol{\varepsilon}(x) = \begin{cases} \varepsilon^{\text{eff}} & \text{for } x \in \Omega \\ I_3 & \text{for } x \in \mathbb{R}^3 \setminus \Omega \end{cases} \tag{16}$$

The limit diffraction problem as $\eta \to 0$ consists in finding $(E, H) \in L^2_{\text{loc}}(\mathbb{R}^3; \mathbb{C}^3)$ such that

$$\begin{cases} curl\, E & = & i\omega\mu_0\, \boldsymbol{\mu}(\omega, x)\, H \\ curl\, H & = & -i\omega\varepsilon_0\, \boldsymbol{\varepsilon}(x)\, E \\ (E - E^i, H - H^i) & & \text{satisfies the O.W.C} \end{cases} \tag{17}$$

Comments:

- The problem (17) is well posed provided ε_r has a positive imaginary part.
- The field H satifies the usual transmission condition on $\partial\Omega$: no tangential jump for H and no normal jump for $\boldsymbol{\mu}(\omega, x)\, H(x)$.
- H does not coincide with the weak limit of $H_\eta \sim H_0(x, x/\eta)$

The TM- case

If we consider e_3- invariant cylindrical dielectric inclusions (that is $\Sigma = D \times \mathbb{R}$ and Σ_η consists od parallel infinitely long rods) illuminated by a TM polarized incident wave, the situation becomes much simpler:

$$H_0 = u_0(x, y_1, y_2)\, e_3 \quad , \quad u_0(x, \cdot) = u(x) \quad \text{on } Y \setminus \Sigma \quad , \quad H(x) = u(x) e_3$$

($\mathrm{curl}_y\, H_0 = 0$ on $Y \setminus \Sigma$ implies that $\nabla_y u_0(x, \cdot) = 0$).

The H_0 cell problem reduces to $u_0(x, y) = u(x)\, w(y)$ where $w \in W^{1,2}_{loc}(\mathbb{R}^2)$ solves

$$\Delta_y w + k^2\, w = 0 \text{ on } D \quad , \quad w \text{ 1-periodic} \qquad w = 1 \text{ on }]0,1[^2 \setminus D$$

$\rightsquigarrow$ **former results in 2004**

D.Felbacq, GB, Homogenization near resonances and artificial magnetism from dielectrics. C. R. Math. Acad. Sci. Paris 339 (2004), no. 5, 377–382.

D.Felbacq, GB Homogenization of wire mesh photonic crystals embdedded in a medium with a negative permeability, Phys. Rev. Lett. 94, 183902 (2005)

D.Felbacq, GB, Negative refraction in periodic and random photonic crystals, New J. Phys. 7 159 10.1088/ (2005)

Variational formulation of the cell problem

The field $H^i(y)$ solution of (10)(11) with $\oint H^i = e_i$ is searched as $H^i = e_i + u_i$ where u_i solves the variational equation

$$b_0(u_i, v) - k^2\,\varepsilon_r \int u_i \cdot \bar{v}\,dy \;=\; k^2\,\varepsilon_r \int e_i \cdot \bar{v}\,dy\ , \quad \forall v \in X_0\ . \qquad (18)$$

where X_0 is the Hilbert space

$$\left\{ u \in W^{1,2}_{\sharp}(Y;\mathbb{C}^3)\ :\ \mathrm{curl}\ u = 0 \text{ on } Y \setminus \Sigma\ ,\ \oint u\ = 0 \right\}$$

(note that constant functions are ruled out) equipped with the scalar product:

$$b_0(u, v)\ :=\ \int_Y (\mathrm{curl}\, u \cdot \overline{\mathrm{curl}\, v} + \mathrm{div}\, u \cdot \overline{\mathrm{div}\, v})\, dy\ .$$

The operator B_0 on $L^2(Y;\mathbb{R}^3)$ associated with b_0 has a compact self adjoint resolvent (by the compact embedding of $W^{1,2}_{\sharp}(Y)$ in $L^2(Y)$)

Spectral problem on the unit cell

The eigenvalue problem in $L^2(Y;\mathbb{R}^3)$

$$b_0(\varphi,\psi) \;=\; \lambda \int \varphi\cdot\overline{\psi}\,dy \quad , \quad \forall\psi\in X_0\cap L^2(Y;\mathbb{R}^3). \tag{19}$$

has a sequence of real eigenvalues $0<\lambda_0<\lambda_1\leq\cdots\leq\lambda_n\leq\cdots$ $(\lambda_n\to+\infty)$ and we denote by $\{\varphi_n, n\in\mathbb{N}\}$ an orthonormal basis of $L^2(Y;\mathbb{R}^3)$ made of eigenfunctions in X_0. The solution u_i to (18) is given by

$$u_i=\sum_{n\in\mathbb{N}} c_{i,n}\,\varphi_n \quad , \quad c_{i,n}=\frac{\epsilon_r k^2}{\lambda_n-\epsilon_r k^2}\int_Y (e_i\cdot\varphi_n)\,dy$$

The tensor μ^{eff} defined in (14) can be therefore rewritten as the series

$$\mu^{\text{eff}}_{ij}(\omega)=\delta_{ij}+\sum_{n\in\mathbb{N}}\frac{\varepsilon_r k^2}{\lambda_n-\varepsilon_r k^2}\left(e_j.\int_Y\varphi_n\right)\left(e_i.\int_Y\varphi_n\right) \;. \tag{20}$$

Remark: The series (20) is absolutely convergent provided $\varepsilon_r k^2$ is away from $\{\lambda_n : n\in\mathbb{N}\}$.

About the series expansion for μ^{eff}

Remark 1 The eigenfunctions associated with $w_p = \nabla\big(\exp(2i\pi\, p \cdot y)\big)$, $p \in \mathbb{Z}^3$ (and $\lambda = 4\pi^2|p|^2$) have zero average and do not contribute in the series (20). All the other eigenvectors are divergence free as being orthogonal to $\{w_p, p \in \mathbb{Z}^3\}$. This will be used later in order to reduce to a spectral problem on Σ which fits better to numerical treatments in which the unknown is $f = \operatorname{curl}\varphi$.

Remark 2 We may rewrite (20) without repeating the eigenvalues. Denoting V_λ the (finite dimensional) eigenspace associated with λ and P_{V_λ} the orthogonal projector

$$\mu^{\text{eff}}_{ij}(\omega) = \delta_{ij} + \sum_{\lambda \in \Lambda_0} \frac{\varepsilon_r k^2}{\lambda - \varepsilon_r k^2} (M_\lambda)_{i,j} \quad , \quad (M_\lambda)_{i,j} := (P_{V_\lambda}(e_i)|P_{V_\lambda}(e_j))\,. \tag{21}$$

Here Λ_0 denotes the eigenvalues λ such that the real symmetric nonnegative matrix M_λ is not zero. Note that M_λ is of rank one for λ simple and to have M_λ of full rank, we need that λ has multiplicity ≥ 3 (this is the case if the inclusion Σ presents enough symmetries).

Band gaps and negative μ^{eff}

- By series expansion (21), the real part of the eigenvalues of $\mu^{\text{eff}}(\omega)$ change of sign due to *micro-resonances* at scale η (although $\eta \ll \lambda$).
- $H_\eta \sim H_0(x, x/\eta)$ shows very large components on some eigenspace of (19) whose oscillations produce a loop of displacement current and thus a microscopic magnetic moment. All the microscopic moments add up to a collective macroscopic moment $\Rightarrow$ *artificial magnetism.*
- By (9), if ε^e is a positive real, so is ε^{eff}. Thus for negative eigenvalues of μ^{eff}, the diffracted field will be exponentially damped (at least in some directions).

“Ideal case” $\varepsilon_r > 0$

Let:

$\mu^{\pm}(\omega)$ the maximal (resp. minimal) eigenvalue of μ^{eff}.

λ eigenvalue for (19) such that $M_\lambda \neq 0$

$\omega_\lambda := \sqrt{\lambda}(\varepsilon_0 \mu_0 \varepsilon_r)^{-1/2}$.

Then: $\mu^{-}(\omega) \searrow -\infty$ as $\omega \searrow \omega_\lambda$ whereas the same holds for $\mu^{+}(\omega)$ if in addition M_λ has full rank. Thus:

• **If M_λ is of full rank,** the existence of a band-gap (of forbidden frequencies) on which the effective permeability tensor is negative.

• **If M_λ is not of full rank,** the nearest eigenvalues have to be considered and the magnetic field may propagate in directions given by $\mathrm{Ker}(\mathrm{M}_\lambda)$.

A similar situation has been noticed by B. Miara et all. in the context of elastic waves in a composite (*weak band gaps*).

Sketch of proof

Step 1 (E_η, H_η) is assumed to be bounded in $L^2(\mathcal{B})$ (assumption (4)). Then after extracting subsequences, it has a two-scale limit $(E_0(x,y), H_0(x,y))$ characterized as before in terms of macroscopic fields $(E(x), H(x))$.

$$E_\eta \rightharpoonup\rightharpoonup \sum_i E_i(x)\, H^i(y) \quad , \quad H_\eta \rightharpoonup\rightharpoonup \sum_i H_i(x)\, H^i(y)$$

- Passing to the weak limit in the first Maxwell equation and exploiting (14):

$$\operatorname{curl} E \;=\; i\omega\mu_0\ [H_0] \;=\; i\omega\mu_0\, \mu^{\text{eff}}\ H$$

- Let $\Psi = (\Psi_j) \in \mathcal{D}(\mathcal{B}; \mathbb{C}^3)$ a smooth test function and set

$$\Psi_\eta(x) := \sum_{j=1}^{j=3} E^j(x/\eta)\, \Psi_j(x).$$

As $\int_Y E^j = e_j$, we have $\Psi_\eta \rightharpoonup \Psi$.

We multiply the second Maxwell equation (1) by Ψ_η and integrate over $\mathcal{B}$.

Sketch of proof

The second equation of (17) is deduced from the following convergences:

$$\lim_{\eta\to 0}\int_{\mathcal{B}} \operatorname{curl} H_\eta \cdot \Psi_\eta \, dx = \int_{\mathcal{B}} \operatorname{curl} H \cdot \Psi \, dx \tag{22}$$

$$\lim_{\eta\to 0}\int_{\mathcal{B}} \varepsilon_\eta E_\eta \cdot \Psi_\eta \, dx = \int_{\mathcal{B}} \boldsymbol{\varepsilon}(x) E \cdot \Psi \, dx. \tag{23}$$

• To obtain (22), we integrate by parts and pass to the limit exploiting that by (13)

$$H_\eta \rightharpoonup\!\!\rightharpoonup \sum_i H_i(x)\, H^i(y) \quad , \quad \operatorname{curl} \Psi_\eta \rightharpoonup\!\!\rightharpoonup \sum_j E^j(y) \wedge \nabla_x \Psi_j(x) \ ,$$

where the last two-scale convergence is strong. The limit is computed with the help of relations (15).

• The derivation of (23) using the characterization (8) of E_0 and (9) is straightforward since Ψ_η vanishes on Σ_η.

• As usual the uniform convergence of (E_η, H_η) on compacts subsets of $\mathbb{R}^3 \setminus \Omega$ implies that the radiation condition (2) is preserved as $\eta \to 0$.

Step 2 the upperbound (5) is established a posteriori by a contradiction argument exploiting the uniqueness for the limit diffraction problem.

Numerical approach for the spectral problem

To compute eigenvectors φ_n related to (19), we transform into another spectral problem involving the unknown $f = \operatorname{curl} \varphi_n$ (supported in Σ) in the space

$$\mathcal{Z} = \left\{ f \in L^2(\Sigma, \mathbb{R}^3) \ / \ \operatorname{div} f = 0, \ f.n = 0 \text{ on } \partial\Sigma \right\}$$

Step 1. We have noticed that, in view of expansion (20) for μ^{eff}, we may restrict spectral equation (19) taking φ, φ' in $X_0^0 := X_0 \cap \{\operatorname{div} v = 0\}$ Such $\varphi \in X_0^0$ can be uniquely represented by using a periodic divergence free field ψ:

$$\varphi = \operatorname{curl} \psi - z \quad \text{with} \quad z = z(\psi) := \oint \operatorname{curl} \psi \quad (\operatorname{curl} \varphi = -\Delta \psi) \ .$$

Inserting this in (19) with $\varphi, \varphi' \in X_0^0$, we are led to

$$\int_\Sigma \Delta\psi \, \Delta\psi' = \lambda \left(- \int_\Sigma \psi \Delta\psi' + z(\psi).z(\psi') \right) \qquad (*)$$

Step 2. We rewrite (*) in term of $f := -\Delta\psi, \ g := -\Delta\psi'$ (seen as elements of $\mathcal{Z}$). Denote, for every $f \in Z$, the divergence free fields

- Hf the restriction to Σ of $\psi \in H^1_\sharp(Y)$ solution of $-\Delta\psi = f, \int_Y \psi = 0$.
- $\Gamma f(y) := \frac{1}{4} \left(\int_\Sigma y \wedge f(y) \, dy \right) \wedge y$.

Equivalent spectral problem on Σ

We define the operator $A : \mathcal{Z} \mapsto \mathcal{Z}$ by

$$A: \ f \in \mathcal{Z} \ \longrightarrow Hf + \Gamma f + Rf \ ,$$

where $Rf = \nabla \rho$, ρ being is the unique solution of

$$\Delta \rho = 0 \ , \ \frac{\partial \rho}{\partial n} = -(Hf + \Gamma f).n \text{ in } \partial\Sigma \ .$$

From (*) we are led to:

$$\int_\Sigma Af.g \ = \ \frac{1}{\lambda} \int_\Sigma f.g \ .$$

Summarizing, we need to compute the eigenvalues λ_n^{-1} and eigenfunctions of of the positive compact self adjoint operator A and the resulting μ^{eff} is recovered from (20) exploiting relation

$$\int_Y \varphi_n \ = \ \frac{1}{2} \int_\Sigma y \wedge f_n.$$

Some numerical results

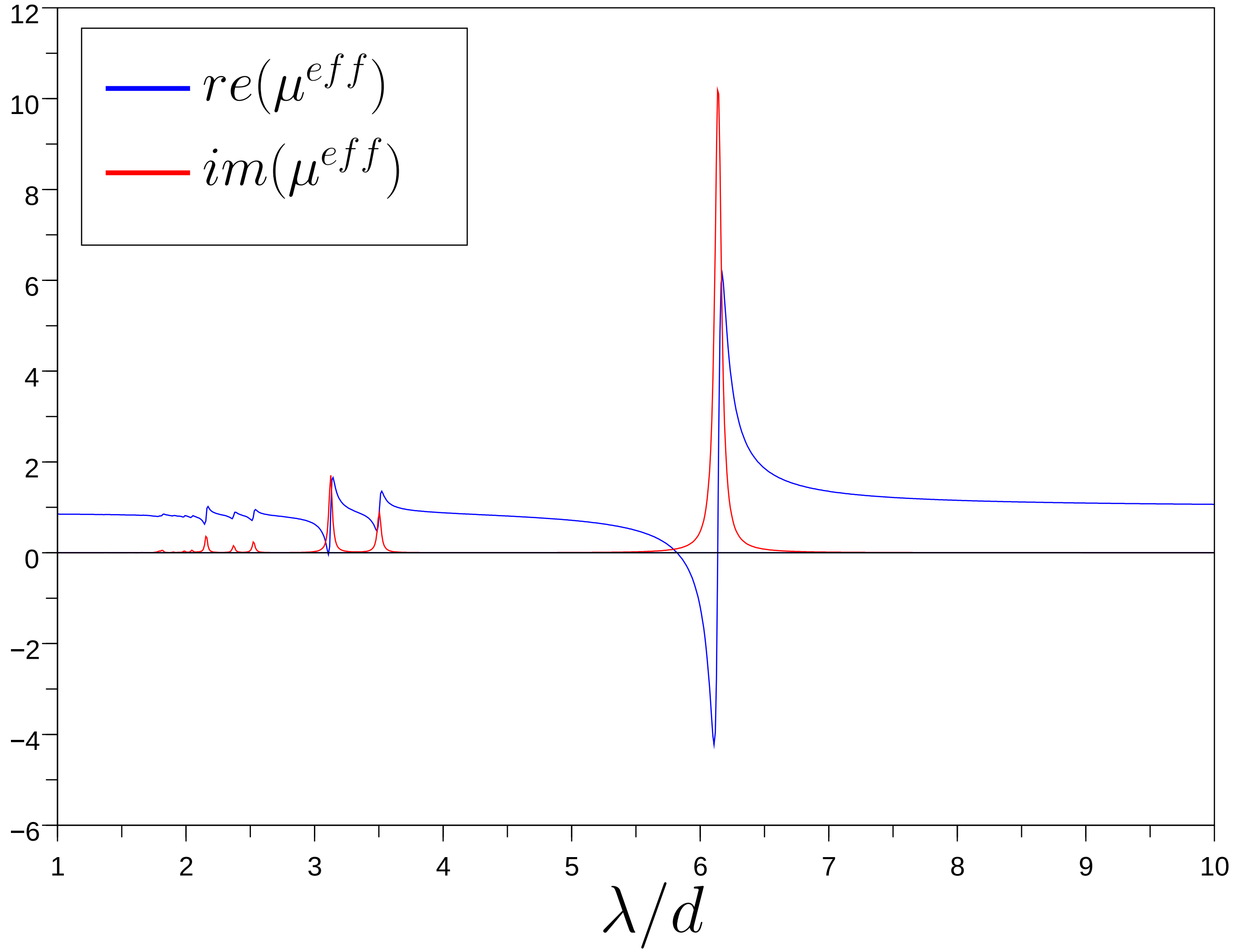

Figure 5: Case $\varepsilon_r := 100 + i$ and $\Sigma := [-0.25, 0.25]^3$.

More numerical results

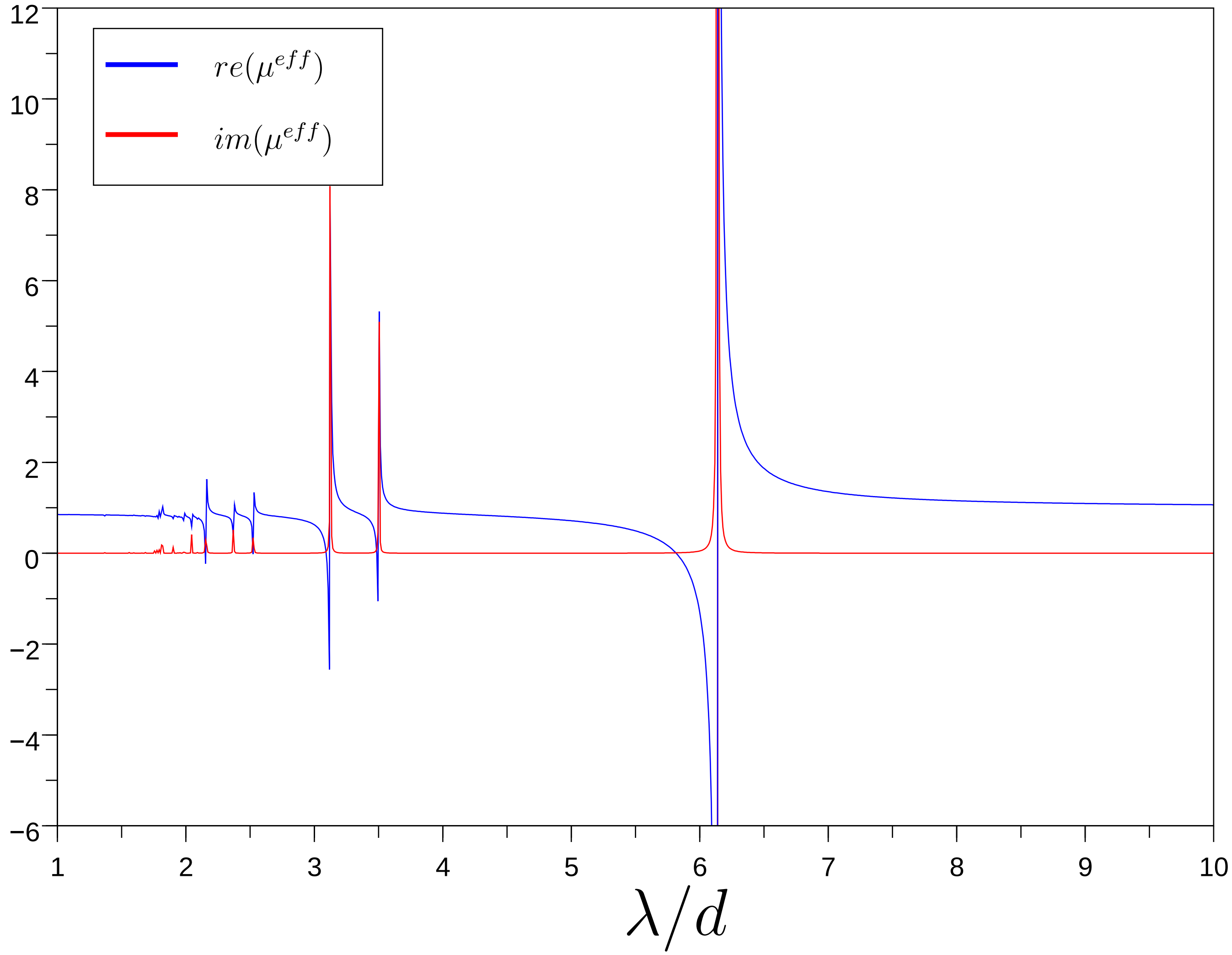

Figure 6: Case $\varepsilon_r := 100 + 0.1i$ and $\Sigma := [-0.25, 0.25]^3$.

References

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