Variational problems for defect evolution

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Goal: existence for quasi-static defect evolution

We want to find quasi-static (or rate independent) evolutions of defects such as cracks, damage, and plastic zones, not based on global minimality, but on local minimality. That is, we want a reasonable definition of stability such that if there exists a continuous evolution of stable defects (subject to initial data and varying Dirichlet data/loads), we find it. We will consider three types of defects:

- Griffith fracture:

$$E(u,\Gamma) = \int_\Omega W(\nabla u)dx + \mathcal{H}^{N-1}(\Gamma)$$

- Damage

$$E(u,D) = \int_\Omega [\beta\chi_{D^c} W(\nabla u) + \alpha\chi_D W(\nabla u)]dx \ (+k|D|)$$

- Cohesive fracture (elastoplasticity with softening)

$$E(u,\Gamma) = \int_\Omega W(\nabla u)dx + \int_\Gamma \phi([u])d\mathcal{H}^{N-1}$$

Discrete-time minimization procedure?

One natural idea is to discretize time and find a variational problem such that the solutions $u_n(t_n^i)$ (e.g., $t_n^i = i\frac{T}{n}$) converge as $n \to \infty$ to some $u(t)$, which is a quasi-static solution. A bonus is that this then provides a numerical method for approximations, provided these variational problems can be solved.

For example, if we seek a globally minimizing Griffith fracture evolution, to get the (approximate) solution u_n at time t_n^i we minimize

$$v \mapsto \int_\Omega W(\nabla v)dx + \mathcal{H}^{N-1}(S_v \setminus \bigcup_{j<i} S_{u_n(t_n^j)})$$

and the limits $u(t)$ minimize

$$v \mapsto \int_\Omega W(\nabla v)dx + \mathcal{H}^{N-1}(S_v \setminus \Gamma(t))$$

where $\Gamma(t) = \bigcup_{\tau \le t} S_{u(\tau)}$. But, for any physically reasonable definition of stability, these solutions can jump out of stable states.

What works?

What variational problem should be solved in order to make sure that the state stays stuck in a (strict) local minimum? To find $u_n(t_n^i)$, minimize

$$v \mapsto \int_\Omega W(\nabla v)dx + \mathcal{H}^{N-1}(S_v \setminus \Gamma_n(t_n^{i-1})),$$

where

- the minimization is done only over v that are accessible from "$u_n(t_n^{i-1})$" in the sense that there is a continuous path from $u_n(t_n^{i-1})$ to v along which the energy does not increase
- $\Gamma_n(t_n^{i-1})$ is the union of $\bigcup_{j<i} S_{u_n(t_n^j)}$ and the discontinuities that occur along the path from $u_n(t_n^{i-1})$ to $u_n(t_n^i)$.

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This does not work.

Why: the stability the $u_n(t_n^i)$ have is local minimality, and it is easy to show that this does not work with the weak convergence you have when passing to the limit $u_n \to u$.

Instead, we need to solve the following variational problem at discrete times: For $\varepsilon > 0$, minimize

$$v \mapsto \int_\Omega W(\nabla v)dx + \mathcal{H}^{N-1}(S_v \setminus \Gamma_n(t_n^{i-1})),$$

where

- the minimization is done only over v that are ε-accessible from $u_n(t_n^{i-1})$, which means that there is a continuous path from $u_n(t_n^{i-1})$ to v along which the energy does not increase by more than ε. The resulting $u_n(t_n^i)$ is a local minimizer, and the energy well has depth at least ε.
- $\Gamma_n(t_n^{i-1})$ is the union of $\bigcup_{j<i} S_{u_n(t_n^j)}$ and the cracks that occur along the path from $u_n(t_n^{i-1})$ to v.

With this, we can pass to the limit (L. (2010))

Damage

Damage evolution has been modeled by globally minimizing the energy

$$E(u, D) = \int_\Omega [\beta \chi_{D^c} W(\nabla u) + \alpha \chi_D W(\nabla u)] dx + k|D|$$

(Francfort-Marigo (1993), Francfort-Garroni (2006)). We took a different view (Garroni-L. (2009)) – the idea of a strain threshold for damage. The basic reason was simply that damage should be based on a local criterion, and the strain is the only local information. So, remove the $k|D|$ term, and replace with precise threshold criteria.

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Precisely: For the threshold λ,

- (threshold) $|\nabla u| \leq \lambda$ a.e. in D^c
- (necessity of damage) For the damage set D, if $S \subset D$ with $|S| > 0$, then for the displacement $u_{D \setminus S}$ corresponding to the damage set $D \setminus S$, we have $|\{|\nabla u_{D \setminus S}| > \lambda\} \cap S| > 0$.

Discrete time variational problem

Given $u_n(t_n^{i-1})$, $D_n(t_n^{i-1})$, find $D_n(t_n^i) \supset D(t_n^{i-1})$ such that the increment $D_n(t_n^i) \setminus D_n(t_n^{i-1})$ is necessary in order to satisfy the threshold condition. $u_n(t_n^i)$ is then uniquely determined by $D_n(t_n^i)$ as the corresponding elastic equilibrium.

Precisely:

- (threshold) $|\nabla u_n(t_n^i)| \leq \lambda$ a.e. in $D_n(t_n^i)$
- (necessity of damage) If $S \subset D(t_n^i) \setminus D(t_n^{i-1})$ with $|S| > 0$, then for the displacement $u_{D(t_n^i)\setminus S}$ corresponding to the damage set $D(t_n^i) \setminus S$, we have $|\{|\nabla u_{D_n(t_n^i)\setminus S}| > \lambda\} \cap S| > 0$.

But how do we find such an increment?

Surprise: Given $\lambda > 0$, $\exists\ k > 0$ such that the threshold condition is equivalent to energy minimization with the $k|D|$ term. So, to find the increment in D, just minimize

$$E(u, D) = \int_\Omega [\beta \chi_{D^c} W(\nabla u) + \alpha \chi_D W(\nabla u)] dx + k|D|$$

with the right k, and over $D \supset D_n(t_n^i)$.

Reason for the equivalence: for the right k, the energy cannot be reduced if $|\nabla u| \leq \lambda$ in D^c, and if $|\nabla u| > \lambda$ on a subset of D^c with positive measure, a laminate of D will reduce the energy. This shows that there are no local minimizers except global ones.

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Unfortunately, there does not in general exist a solution to this minimization problem, and it needs to be relaxed. This is the most interesting part of the problem.

Elastoplasticity with softening/cohesive fracture

Basic idea of elastoplasticity: there exists a set $\mathcal{K}$ of vectors/matrices that must contain the elastic stress. Additive decomposition of the strain: $Eu = e + p$, so $\sigma(e) \in \mathcal{K}$. Softening: $\mathcal{K}$ shrinks as plasticity occurs.

Cohesive fracture:

$$E(u, \Gamma) = \int_\Omega W(\nabla u)dx + \int_{S_u} \phi([u])d\mathcal{H}^{N-1}$$

where $\phi(0) = 0$, $\lim_{x\to\infty} \phi(x) = G$, ϕ even, and concave on $\mathbb{R}^+$.

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If $\phi'(0) < \infty$, there are some serious issues in minimizing this energy. In particular, there is a failure of compactness when using the Direct Method – if $\{u_n\}$ is a minimizing sequence, then the cost of discontinuities when they are small is like $\phi'(0) \int_{S_{u_n}} |[u_n]| d\mathcal{H}^{N-1}$, which only bounds $[u_n]d\mathcal{H}^{N-1}$ as measures. This allows, for example, $u_n \to v$ for any BV function v, even if $\nabla u_n = 0$ for all $n \in \mathbb{N}$. Similar issue in minimizing an energy for elastoplasticity with softening – the energy density starts out linear, but is sublinear due to softening.

Stability

What if we don't minimize? What else can we do? Let's call $\phi'(0) =: \lambda$ and suppose $W(\cdot) = \frac{1}{2}(\cdot)^2$. First we try to find a solution to the "right" static problem. It is natural to say that if we minimize just the elastic energy and get u_e, and if $|\nabla u_e| < \lambda$, then this should be our solution (for elastoplasticity with $W(\cdot) = \frac{1}{2}(\cdot)^2$, this means $\mathcal{K} = B(0, \lambda)$ initially).

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In some sense, the material shouldn't "know" that cohesive fracture is possible, unless $|\nabla u_e| > \lambda$ somewhere. If it is, then *fracture/plasticity should only occur so as to ensure that the corresponding u satisfies* $|\nabla u| \leq \lambda$.

Reminiscent of the damage threshold condition.

Are there solutions? Is there an energy formulation?

First 1-D

Simple case: given Dirichlet data, minimize first

$$\int_\Omega \frac{1}{2}|\nabla u|^2 dx$$

and call the solution u_e. If $|\nabla u_e| < \lambda$, done. If not, there must be fracture in the region $\overline{\{|\nabla u_e| > \lambda\}}$ (1-D with W convex, this is just $\overline{\Omega}$). By the necessary condition, only one point of fracture should be added – also follows from (local) minimality.

Trivial "energy" formulation: set $\mathcal{A} := \{\Gamma \subset \overline{\Omega} : u_\Gamma \text{ satisfies } |\nabla u_\Gamma| < \lambda\}$, where u_Γ is a minimizer of

$$\int_\Omega W(\nabla u)dx + \int_{S_u} \phi([u])d\mathcal{H}^0$$

subject to the constraint that $S_u \subset \Gamma$. Then Γ^* is a minimizer of

$$\Gamma \mapsto \mathcal{H}^0(\Gamma)$$

over $\Gamma \in \mathcal{A}$, and the solution is u_{Γ^*}.

Higher dimensions

Natural extension: u_Γ is a minimizer of

$$\int_\Omega W(\nabla u)dx + \int_{S_u} \phi([u])d\mathcal{H}^{N-1}$$

subject to the constraint that $S_u \subset \Gamma$. Then Γ^* is a minimizer of

$$\Gamma \mapsto \mathcal{H}^{N-1}(\Gamma)$$

over $\Gamma \in \mathcal{A}$, and the solution is u_{Γ^*}.

Do these minimizers exist? What happened to the compactness problem? Naturally cured, since if $\mathcal{H}^{N-1}(\Gamma) < \infty$, u_Γ exists without relaxation, and by the minimization to find Γ^*, u_{Γ^*} exists. But does Γ^* exist?

Weaker version: Necessity rather than minimality

We say that Γ^* satisfies condition (N) if

$$\Gamma^* \in \mathcal{A} := \{\Gamma \subset \bar{\Omega} : \mathcal{H}^{N-1}(\Gamma) < \infty \text{ and } |\nabla u_\Gamma| \leq \lambda\}$$

and $\forall \Gamma \subset \Gamma^*$ with $\mathcal{H}^{N-1}(\Gamma^* \setminus \Gamma) > 0$, $|\nabla u_\Gamma| > c$ on a set of positive measure (i.e., $\Gamma \notin \mathcal{A}$).

Of course, if Γ^* is minimal, then it satisfies (N). Again, is there a solution? Can show that

Theorem

If there exists $\Gamma \in \mathcal{A}$, then there exists $\Gamma^* \subset \Gamma$ that satisfies (N).

Proof not too difficult, seems to work also if Γ is $\mathcal{H}^{N-1}$-σ-finite and closed. There is also a proof for a minimal Γ^*, but this is more complicated.

Still, not so clear if there always exist admissible Γ. Maybe some (new) relaxation is required.

Stability

We also have

Theorem

If Γ^* satisfies (N), then for all $v \in SBV(\Omega)$

$$\frac{d}{d\lambda}|_{\lambda=0} E(u_{\Gamma^*} + \lambda v) \geq 0$$

and

$$\frac{d^2}{d\lambda^2}|_{\lambda=0} E(u_{\Gamma^*} + \lambda v) \geq 0.$$

In fact, the calculation only needs $\Gamma^* \in \mathcal{A}$. Further, if $\phi''(0) = 0$, then

$$\frac{d^2}{d\lambda^2}|_{\lambda=0} E(u_{\Gamma^*} + \lambda v) > 0$$

for all nontrivial v.