Computational Anatomy and EPDiff
A “Reconnection” Path

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Plan of the talk

• I. Grenander’s deformable templates and examples.

• II. Geodesic motion. (Miller-Trouvé-Younes 2003)
Plan of the talk

• I. Grenander’s deformable templates and examples.

• II. Geodesic motion. (Miller-Trouvé-Younes 2003)
Central Framework: Grenander’s Deformable Template

Main ingredients:

- **Object set**: $\mathcal{O}$
- **Group of transformations**: $G$
- **Group action**: $G \times \mathcal{O} \rightarrow \mathcal{O}$

$$o_{\text{obs}} = g_{\text{ref}} + n$$

(deformed template + noise) (1)
Examples–Landmarks

• $\mathcal{O} = \{ \mathbf{x} = (x_i)_{1 \leq i \leq N} \mid x_i \in \mathbb{R}^d \}$.

• Finite dimensional case: $G$ rotation group, scale and translation group, affine group...

• Infinite dimensional case: $G$ group of non rigid transformation $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}^d$

$$\mathbf{x}_{\text{obs}} = \varphi \mathbf{x}_{\text{ref}} + \mathbf{n} \text{ where } \varphi \mathbf{x} = (\varphi(x_i))_{1 \leq i \leq N} \quad (2)$$
Examples–Images

- \( D \) a domain in \( \mathbb{R}^d \), \( \mathcal{O} = \{ I : D \to \mathbb{R} \} \)

- \( G \) group of diffeomorphisms on \( D \)

\[
\varphi I \doteqdot I \circ \varphi^{-1}
\]

\[
I_{\text{obs}} = \varphi I_{\text{ref}} + n
\]

(deformed template + noise) (3)
Template Matching Problem

Bayesian or variational point of view

\[ \hat{g} = \arg\min_{g \in G} \left( d_O(o_{\text{ref}}, g o_{\text{ref}})^2 + |o_{\text{obs}} - g o_{\text{ref}}|^2_n \right) \]  \hspace{1cm} (4)

Idea: \(d_O(o_{\text{ref}}, g o_{\text{ref}})\) should be depend on the amount of deformation \(g\) linking \(o_{\text{ref}}\) and \(g o_{\text{ref}}\). Build a model on \(\hat{g}\) and \(n\) instead of \(o_{\text{obs}}\).
Metric Transfer Through Group Action

- Differentiate the action \( g \to go \) at \( \text{Id} \).
  \[
  T_{\text{Id}}G \quad \longrightarrow \quad T_o\mathcal{O}
  \]
  \[
  v \quad \rightarrow \quad \delta_o = v.o \quad (5)
  \]

- Define a metric \( \|v\|_{\text{Id}} \) for small \( g \simeq \text{Id} + v \)

- Transfer the metric on \( T_o\mathcal{O} \)
  \[
  \|\delta_o\|_o = \inf_{\delta_o=v_o} \|v\|_{\text{Id}}
  \]
\( \mathcal{O} \) inherits a Riemannian structure

For \( o, o' \in \mathcal{O} \):

- \( \mathcal{O}_{o,o'} \): smooth paths \( t \rightarrow o_t \) from \( o \) to \( o' \).

- \( d_{\mathcal{O}}(o, o') = \inf_{\mathcal{O}_{o,o'}} \int_{0}^{1} \| \frac{do}{dt} \|_{o_t} dt \)

We get a Riemannian distance on \( \mathcal{O} \)
Associated Right Invariant Dist. on $G$

Choose $\mathcal{O} = G$, $G$ acts onto itself: $(g, g_0) \rightarrow gg_0$

Apply metric transfer:

$$\|\delta g_0\|_{g_0} = \inf_{\delta g_0 = v.g_0} \|v\|_{\text{Id}}.$$ 

But $g \rightarrow gg_0$ is one to one. Hence $v \rightarrow vg_0$ is an isomorphism.

Consequence: $T_{g_0}G = \{ v.g_0 \mid v \in T_{\text{Id}}G \}$ and

$$\|vg_0\|_{g_0} = \|v\|_{\text{Id}} \text{ (Right invariance)}.$$
Associated Right Invariant Dist. on $G$ (Cont)

Geodesic distance on $G$: Take infimum on sufficiently smooth paths from $g_0$ to $g_1$

$$d_{G}(g_0, g_1) = \inf_{g_0 \to g_1} \int_0^1 \|\frac{dg_t}{dt}\|_{g_t} dt = \inf_{g_0 \to g_1} \int_0^1 \|\frac{dg_t}{dt} g_t^{-1}\|_{Id} dt$$

Right invariance property:

$$d_{G}(g_0, g_1) = d_{G}(g_0 g, g_1 g)$$

Theorem: $d_{G}(o, o') = \inf\{d_{G}(Id, g) \mid go = g'\}$. 
Rigorous construction for non rigid deformations

• \( D \) a \( d \)-dimensional domain, \( G \) subgroup of \( \text{Hom}(D) \), \( g \to \varphi \)

• Start from \( V \) be a Hilbert space of vector fields on \( D \) with norm \( \| \|_V \) (to be identified with \( T_{\text{Id}G} \))

• Assume \( V^{\text{cont}} \hookrightarrow C^1(D, \mathbb{R}) \) (admissibility)

For \( v \in L^2([0, 1], V) \), \( \varphi^v \) solution of

\[
\begin{align*}
\frac{d\varphi^v_t}{dt} &= v_t \circ \varphi^v_t \\
\varphi^v_0 &= \text{Id}
\end{align*}
\]

\[
G \doteq \{ \varphi^v_1 \mid v \in L^2([0, 1], V) \}
\]

\[
d_G(\text{Id}, \varphi) = \inf \{ \int_0^1 \| v_t \|_V dt \mid \varphi^v_1 = \varphi \}
\]
Rigorous construction (non-rigid deformation) (Cont)

Usual framework: \[ \|v_t\|_V^2 = \int \langle Lv(x), v(x) \rangle dx \] with

\[ L : D(L) \subset V \rightarrow H = L^2(D, \mathbb{R}^d) \]

ex: \[ L = (I - \Delta)^s. \]

Theorem (T. 1995): Let \( V^\text{cont} \hookrightarrow C^1(D, \mathbb{R}) \) (admissibility)

* \( G \) is a subgroup of homeomorphisms
* \( G \) is complete for the right invariant metric \( d_G(\ , \ ) \)
* Existence of geodesics

If \( V^\text{cpt} \hookrightarrow C^1(D, \mathbb{R}) \), \( G \) is a subgroup of diffeomorphisms (T. 1995, Dupuis et al. 1998)
Template Matching Revisited

Energetic formulation: $d_G(\text{Id}, \varphi)² = \inf_{\varphi' = \varphi} \int_0^1 \|v_t\|_V^2 dt$

Template matching problem:

$$\hat{\varphi} = \arg\min_{\varphi \in G} \left( d_O(o_{\text{ref}}, \varphi o_{\text{ref}})² + |o_{\text{obs}} - \varphi o_{\text{ref}}|_n^2 \right)$$

becomes:

$$\hat{v} = \arg\min_{v \in L^2([0,1], V)} \left( \frac{1}{2} \int_0^1 \|v_t\|_V^2 dt + |o_{\text{obs}} - \varphi_1 o_{\text{ref}}|_n^2 \right), \ \hat{\varphi} = \varphi_1$$

Application to Landmark Matching

- $\mathcal{O} = \{ x = (x_i)_{1 \leq i \leq N} \mid x_i \in \mathbb{R}^d \}$

- $g(x) \cdot (g(x_i))_{1 \leq i \leq N} \Rightarrow v(x) = (v(x_i))_{1 \leq i \leq N}$.

$$\hat{v} = \arg\min_{v \in L^2([0,1],V)} \int \| v_t \|_V^2 dt + \frac{1}{\sigma^2} \sum_{i=1}^{N} |x_{obs}^i - g_1^v(x_{ref}^i)|^2$$

Gives Large Deformation Extension of Bookstein Splines.

* 2d example: (Joshi, Miller (00), Camion, Younes (00))

* Landmark on the sphere, (Glaunes, Vaillant (02))
Application to Image Matching

• \( \mathcal{O} = \{ I : D \to \mathbb{R} \} \)

• \( V \) satisfying admissibility conditions

• Action : \( \varphi I = I \circ \varphi^{-1} \)

\[
\hat{v} = \arg\min_{v \in L^2([0,1],V)} \int \|v_t\|^2_V dt + \frac{1}{\sigma^2} \int |I^{\text{obs}}(x) - I^{\text{ref}}(\varphi_1^v(x))|^2 dx
\]

Large deformation extension of Amit-Grenander-Piccioni model:

\[
\arg\min_{v \in V} \|v\|^2_V + \frac{1}{\sigma^2} \int |I^{\text{obs}}(x) - I^{\text{ref}}(x - v(x))|^2 dx
\]
Application To Submanifold Matching

- **Point** $x$
  \[ \rightarrow \delta_x \text{ Dirac measure at } x. \]

- **$M$** $k$-dimensional submanifold
  \[ \rightarrow \mu_M \text{ uniform probability measure on } M \]

- **$M = \bigcup_{i=1}^{r} M_i$** union of manifold with different dimensions
  \[ \rightarrow \mu_M = \sum_{i=1}^{r} \mu_{M_i} \]

- More general situations (**noisy representations**)
  \[ \rightarrow \text{arbitrary distributions} \]

Glaunès et al. (CVPR 04)
Action on measure

Action \((\varphi, \mu) \to \varphi \mu\) given by the mass transport

\[
\int f \, d(\varphi \mu) = \int f \circ \varphi \, d\mu
\]
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- Points: \(\varphi(\sum_{x \in S} \delta_x) = \sum_{x \in S} \delta_{\varphi(x)}\)
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- Points: \(\varphi(\sum_{x \in S} \delta_x) = \sum_{x \in S} \delta \varphi(x)\)

- Abs. cont. distribution: \(\varphi(g d\lambda) = g \circ \varphi^{-1} |d\varphi^{-1}| d\lambda\)
Action on measure

Action \((\varphi, \mu) \rightarrow \varphi \mu\) given by the mass transport

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- Points: \(\varphi(\sum_{x \in S} \delta_x) = \sum_{x \in S} \delta_{\varphi(x)}\)

- Abs. cont. distribution: \(\varphi(gd\lambda) = g \circ \varphi^{-1}|d\varphi^{-1}|d\lambda\)

\[\text{supp}(\varphi \mu) = \varphi(\text{supp} \mu)\]
Comparison between two distributions $\mu$ and $\nu$

**Principle:** $\mu$ is defined by its behavior on test functions:

$$\langle \mu, f \rangle \overset{\text{def}}{=} \int f \, d\mu$$

$W$ Hilbert space of space functions:

$$|\mu - \nu|_{W^*} = \sup_{|f|_W=1} |\langle \mu, f \rangle - \langle \nu, f \rangle|$$
Complete Model for Submanifold Matching

• $D$ a bounded domain in $\mathbb{R}^d$. The object space is $\mathcal{O} = W^*$ where $W$ is a Hilbert space of functions on $D$.

• $(\varphi \mu, f) \overset{\equiv}{=} (\mu, f \circ \varphi)$

$$
\hat{v} = \arg\min_{v \in L^2([0,1],V)} \int \|v_t\|^2_V dt + \frac{1}{\sigma^2}\|\varphi^v_1\mu_0 - \mu_1\|^2_{W^*}
$$

Importantly, we have existence and consistancy results
Existence Result

Let $\mu, \nu \in \mathcal{M}_s$, and let for any $v \in L^2([0, 1], V)$

$$J_{\mu, \nu}(v) \doteq \int \|v_t\|^2_V dt + \frac{1}{\sigma^2} |\varphi^v_1 \mu - \nu|^2_{W^*}$$

**Theorem 1. [Existence]** Under suitable regularity conditions on $V$ (admissibility condition), there exists a minimiser $v_*$ in $L^2([0, 1], V)$ of $J_{\mu, \nu}$ for any $\mu, \nu$ in $\mathcal{M}_s$. 
Existence Result

Let $\mu, \nu \in \mathcal{M}_s$, and let for any $v \in L^2([0, 1], V)$

$$J_{\mu, \nu}(v) = \int \|v_t\|_V^2 dt + \frac{1}{\sigma^2} |\varphi_1^v \mu - \nu|^2_{W^*}$$

**Theorem 1. [Existence]** Under suitable regularity conditions on $V$ (admissibility condition), there exists a minimiser $v_*$ in $L^2([0, 1], V)$ of $J_{\mu, \nu}$ for any $\mu, \nu$ in $\mathcal{M}_s$.

For $v_*$ is a minimiser, $\varphi_*$ is the associated diffeomorphism.
Theorem 2. [Consistency] Assume the regularity condition of Thm 1. Let $\mu$ and $\nu$ be two probability distributions on $\mathbb{R}^d$ and let $x_1, \cdots, x_m$ and $y_1, \cdots, y_n$ be iid samples drawn from distribution $\mu$ and $\nu$. Let $\hat{\mu}_m = \frac{1}{m} \sum_i \delta_{x_i}$ and $\hat{\nu}_n = \frac{1}{n} \sum_j \delta_{y_j}$ be the associated empirical measures. Then if $\varphi_*(m, n)$ is an minimiser of $J_{\hat{\mu}_m, \hat{\nu}_n}$, almost surely, $\varphi_*(m, n)$ tends uniformly (up to the extraction of a subsequence) to $\varphi_*$ minimiser of $J_{\mu, \nu}$ when $m, n \to \infty$. 
Figure 1: Rangarajan test set
Matching heterogeneous manifolds

Figure 2: Left Initial configuration, Right: Final matching
Consistency

Figure 3: Various sampling rate of the same manifolds
Robustness against noise

Figure 4: Various noise levels
Robustness against outliers
3D examples

Figure 5: Left and Right Hippocampus (Surface tessellation provided by CNRS Cognitive Neuroscience and Brain Imaging Laboratory, La Salpetriere Hospital, Paris)
Morphing Metrics on Object Space

Previous construction:

\[ d_{\mathcal{O}}(o, o') < +\infty \iff o \text{ and } o' \text{ on same orbit} \]

For images: \[ d_{\mathcal{I}}(I, I') < +\infty \iff I' \circ \varphi = I, \ \varphi \in G \]

no apparition of new structure in one orbit
• Assume the object space $O = H$ Hilbert space.
Let $\tilde{G} = \{ a = (g, h) \mid g \in G, \ h \in H \}$

• If the action $o \to go$ is linear, $\tilde{G}$ is a group for the composition law $(g, h)(g', h') = (gg', gh' + h)$ (semi-direct product $G \ltimes H$) and $\tilde{G}$ acts (affinely) on $O$ by $(g, h)o = go + h$.

But now: $\tilde{G}o_{ref} = O$

• Let $W = V \times H$ be the tangent space at $e_{\tilde{G}}$ of $\tilde{G}$ equip with the induced product metric: $\langle (v, z), (v', z') \rangle_W = \langle v, v' \rangle_V + \langle z, z' \rangle_H$. Consider the associated invariant metric on $\tilde{G}$

$$d_{O,morph}(o, o') = \inf \{ d_{\tilde{G}}(e_{\tilde{G}}, (g, h)) \mid (g, h)o = o' \}.$$ 

Many extensions in Laurent’s talk yesterday.
Plan of the talk

- I. Grenander’s deformable templates and examples.
- II. Geodesic motion.
The central role of geodesic paths in $G$

- **Inexact Matching**

$$\hat{g} = \arg\min_{g \in G} d_O(o_{\text{ref}}, go_{\text{ref}})^2 + E(o_{\text{ref}}, o_{\text{obs}})$$

- **Exact Matching**

$$\hat{g} = \arg\min_{g \in G} d_O(o_{\text{ref}}, go_{\text{ref}})^2 \text{ subject to } go_{\text{ref}} = o_{\text{obs}}$$

If $(g_t)$ is a geodesic path in $G$ from $\text{Id}$ to $\hat{g}$, then $o_t = g_t o_{\text{ref}}$ is a geodesic path in $O$ from $o_{\text{ref}}$ to $\hat{g} o_{\text{ref}}$.

A key fact is that $G$ is always equip with a right invariant metric.
• Start with $G$ Lie group of matrices with right invariant metric

• Define $L : T_{id}G \rightarrow T_{id}G^*$ (Inertial operator)

$Lv$ is the Momentum

$$\|v\|_{id}^2 \equiv (Lv, v)_{T_{id}G^* \times T_{id}G}$$

Note that $d_G(g_0, g_1)^2$ is given by:

$$\min \int (Lv_t, v_t)dt \text{ subject to: } \begin{cases} g_0^v = g_0 \text{ and } g_1^v = g_1 \\ \frac{dg^v}{gt} = v_t g_t^v \end{cases} \quad \text{(LAP)}$$

This is exactly the least action principale for a Lagragian reduced to the kinetic energy.
The Euler-Poincaré equation for group of matrices

Let $g_t$ be an extremal curve for the kinetic energy and $g_{t,\epsilon}$ be a smooth deformation around $\epsilon = 0$ with fixed end values $(g_{0,\epsilon} = g_0, g_{1,\epsilon} = g_1)$. Let $v_{t,\epsilon}$ and $w_{t,\epsilon}$ such that

$$\frac{\partial g}{\partial t} = vg \text{ and } \frac{\partial g}{\partial \epsilon} = wg .$$

Since $\frac{\partial}{\partial \epsilon} \left( \frac{\partial g}{\partial t} \right) = \frac{\partial}{\partial t} \left( \frac{\partial g}{\partial \epsilon} \right)$, we get

$$\frac{\partial v}{\partial \epsilon} g + vw g = \frac{\partial w}{\partial t} g + wvg \text{ i.e.}$$

$$\frac{\partial v}{\partial \epsilon} = \frac{\partial w}{\partial t} - (vw - wv) = \frac{\partial w}{\partial t} - [v, w] = \frac{\partial w}{\partial t} - \text{ad}_v w .$$
Reminder: \[ \frac{\partial v}{\partial \epsilon} = \frac{\partial w}{\partial t} - \text{ad}_v w. \]

The curve \( w_{t,\epsilon} \) can vary freely in \( g \), with boundary conditions \( w_{0,\epsilon} = w_{1,\epsilon} = 0 \). From \( \frac{d}{d\epsilon} \left( \int \langle v, v \rangle_e dt \right) |_{\epsilon=0} = 0 \), we get

\[
\int_0^1 \langle v, \frac{\partial v}{\partial \epsilon} \rangle_g = \int \langle v, \frac{\partial w}{\partial t} - \text{ad}_v w \rangle_g = \int (Lv, \frac{\partial w}{\partial t} - \text{ad}_v w) = 0
\]

Introducing the adjoint operator \( \text{ad}^* \), we get \( (Lv, \text{ad}_v w) = (\text{ad}_v^* (Lv), w) \) so that by integration by part, we have finally the following Euler-Poincaré equation

\[
\frac{\partial Lv}{\partial t} + \text{ad}_v^* (Lv) = 0. \tag{6}
\]
Euler-Poincaré equation in the case of diffeomorphisms

Let $\varphi_t$ be an extremal curve for the kinetic energy and $\varphi_{t,\epsilon}$ be a perturbation of $\varphi_t$ with fixed end points ($\varphi_{t,0} = \varphi_t$, $\varphi_{0,\epsilon} = \varphi_0$ and $\varphi_{1,\epsilon} = \varphi_1$ for any $\epsilon$). Let $v_{t,\epsilon}$ and $w_{t,\epsilon}$ such that

$$\frac{\partial \varphi}{\partial t} = v \circ \varphi \text{ and } \frac{\partial \varphi}{\partial \epsilon} = w \circ \varphi.$$ 

Since $\frac{\partial}{\partial \epsilon}\left(\frac{\partial \varphi}{\partial t}\right) = \frac{\partial}{\partial t}\left(\frac{\partial \varphi}{\partial \epsilon}\right)$, we get

$$\frac{\partial w}{\partial t} \circ \varphi + d\varphi v(w \circ \varphi) = \frac{\partial v}{\partial \epsilon} \circ \varphi + d\varphi w(v \circ \varphi)$$

and finally

$$\frac{\partial v}{\partial \epsilon} = \frac{\partial w}{\partial t} - \left(dv(w) - dw(v)\right) = \frac{\partial w}{\partial t} - [v, w].$$
As for the case of matrices, we have obtained

$$\frac{\partial v}{\partial \epsilon} = \frac{\partial w}{\partial t} - \text{ad}_v w.$$ 

and reproducing exactly the same computation we get the Euler Poincaré equation for the diffeomorphisms

$$\frac{\partial Lv}{\partial t} + \text{ad}^*_v (Lv) = 0.$$ 

or equivalently for any $u \in \mathfrak{g}$ and $m_t = Lv_t$

$$\frac{\partial}{\partial t} (m_t, u) + (m_t, \text{ad}_{v_t} u) = \frac{\partial}{\partial t} (m_t, u) + (m_t, dv_t(u) - du(v_t)) = 0.$$
By integration by parts: 

\[ \frac{\partial}{\partial t} (m_t, u) + (m_t, dv_t(u) - du(v_t)) \]

\[ = (\frac{\partial}{\partial t} m_t + \text{div}(m_t \otimes v_t) + dv^*_t(m_t), u). \]

Hence the Euler Poincaré equation (EPDiff equation) is

\[ \begin{cases} 
\frac{\partial}{\partial t} m_t + \text{div}(m_t \otimes v_t) + dv^*_t(m_t) = 0 \\
v_t = L^{-1} m_t
\end{cases} \]

(Mumford-Vishik (98), Holm-Marsden-Ratiu (98), Miller-Trouvé-Younes (02), Holm-Trouvé-Younes (04)…).

For \( Lu = u - \alpha^2 \Delta u \): Camassa-Holm Equation (93)
Momentum Map

The Euler Poincaré equation is a conservation equation: Indeed, (check for matrices): If \( \text{Ad}_g(u) \triangleq gug^{-1} = (d_eL_{g^{-1}}d_eR_g(u)) \), then

\[
\frac{d}{dt}(\text{Ad}_g(t)(u)) = \text{ad}_v(t)(\text{Ad}_g(t)(u)).
\]

Hence

\[
\frac{\partial}{\partial t}(\text{Ad}^*_g(m_t), u) = (\frac{\partial m_t}{\partial t}, \text{Ad}_g(t)(u)) + (m_t, \text{ad}_v(t)(\text{Ad}_g(t)(u))) = 0
\]

and

\[
m_t = \text{Ad}^*_{g^{-1}_t}(m_0)
\]

or equivalently

\[
(m_t, u) = (m_0, \text{Ad}^{-1}_{g_t}(u))
\]
Momentum Map for EPDiff

- \((m_t, u) = (m_0, g_t^{-1}ug_t)\) translates to
  \[(m_t, u) = (m_0, (d\varphi_t)^{-1}u \circ \varphi_t)\]

- Let \(m_0 = \nabla f_0 \mu_0\) (\(\mu_0\) signed measure) ie
  \[(m_0, u) = \int_D \langle \nabla f_0, u \rangle d\mu_0\]

Then we get \(m_t = \nabla f_t \mu_t\) with

\[f_t = \varphi_t f_0\] \(\text{action on functions}\)

\[\mu_t = \varphi_t \mu_0\] \(\text{action on measures}\)
Normal Momentum Motion

Let $t \to I_t$ be a geodesic path in image space between $I_0$ to $I_1$.

We know

$$I_t = \phi_t I_0 = I_0 \circ \phi_t^{-1}$$

with $t \to \phi_t$ geodesic path in $G$.

Assume that $m_0 = \nabla I_0 d\mu_0$ (Normality Constraint).

Then

$$m_t = \nabla I_t d\mu_t$$

with $\mu_t = \mu_0 \circ \phi_t^{-1}$.

The momentum stays normal to level sets of $I_t$.
Normality Constraint

Image case:
\[
\frac{\partial}{\partial t}(\varphi t I_0)|_{t=0} = -\langle \nabla I_0, v_0 \rangle
\]

Let \( V_0 = \{ w \in V \mid \langle \nabla I_0(x), v(x) \rangle = 0, \forall x \} \). We have

\[
\langle \nabla I_0, v_0 \rangle = \langle \nabla I_0, v_0 + w \rangle, \quad w \in V_0
\]

This leads to the constraint \( v_0 \in V_0^\perp \).

However, if \( m_0 = L v_0 = \alpha \nabla I_0 \lambda \) then

\[
\langle v_0, w \rangle_V = (m_0, w) = \int \alpha \langle \nabla I_0, w \rangle dx = 0, \quad w \in V_0.
\]
Normal Momentum Constraint in Valid for Template Matching

\[ \hat{v} = \arg\min_{v \in L^2([0,1], V)} \ J(v) : \int \|v_t\|_V^2 dt + \frac{1}{\sigma^2} \int |I_{\text{obs}} - I_{\text{ref}} \circ (\varphi_1^v)^{-1}|^2 dx \]

\[ I_t = \varphi_t^\hat{v} I_{\text{ref}} \] is a geodesic path from \( I_{\text{ref}} \) to \( \varphi_1^\hat{v} I_{\text{ref}} \) in image space.

The gradient of \( J \) in \( L^2([0,1], V) \) (Beg et al. 2002) satisfies

\[ L(\nabla J)_t = 2v_t + \frac{2}{\sigma^2} |d\varphi_{t,1}| (I_{\text{obs}}^t - I_{\text{ref}}^t) \nabla I_{\text{ref}}^t \]

where \( \varphi_{s,u} = \varphi_u \circ \varphi_s^{-1} \), \( I_{\text{ref}}^t = \varphi_{0,t} I_{\text{ref}} \), \( I_{\text{obs}}^t = \varphi_{t,1} I_{\text{obs}} \).
Euler-Lagrange Equation $\nabla J = 0$ gives at $t = 0$

$$(m_0, u) = \int_{D} \langle \nabla I^{\text{ref}}, u \rangle \alpha dx$$

with $\alpha = \frac{1}{\sigma^2}|d\varphi_1|(\varphi_1 I^{\text{obs}} - I^{\text{ref}})$

Selected geodesics from $I_{\text{ref}}$ satisfy
Normal Momentum Constraint
If \( \alpha_t = |d\varphi_t^{-1}| \alpha_0 \circ \varphi_t^{-1} \), and \( I_t = I_{\text{ref}} \circ \varphi_t^{-1} \), then we get

\[
v_t = L^{-1}(\alpha_t \nabla I_t dx) \quad \text{and} \quad \begin{cases} 
\frac{\partial \alpha_t}{\partial t} + \text{div}(\alpha_t v_t) = 0 \\
\frac{\partial I_t}{\partial t} + \langle \nabla I_t, v_t \rangle = 0
\end{cases}
\]

Geodesic shooting build the Exponential Map

\[
\text{Exp}_{I_0} : T_{I_0} \mathcal{O} \to \mathcal{O}
\]

The complete geodesic coded by \( \alpha_0 \).
Direct vs Inverse Exponential Mapping
Minimal Path vs Geodesic Shooting for macaque brain

\[ I^\text{ref} : \alpha_0 \leftrightarrow I_1 \]

Given \( I^\text{ref} \) (Template)

- **Coding Process:** From any \( I_1 \), compute (Beg’s Algorithm) \( \alpha_0 \)

- **Decoding Process:** From any \( \alpha_0 \), compute \( I_1 \) (Geodesic Shooting)
Extension in the case of metamorphosis

Semi-directe product case

If $I_t = I_{\text{ref}} \circ \varphi_t^{-1}$, then we get for the geodesic evolution equation:

$$v_t = -L^{-1}(z_t \nabla I_t dx)$$

and

$$\begin{cases}
\frac{\partial z_t}{\partial t} + \text{div}(z_t v_t) = 0 \\
\frac{\partial I_t}{\partial t} + \langle \nabla I_t, v_t \rangle = \sigma^2 z_t
\end{cases}$$
Geodesic shooting for metamorphosis

$I_{\text{ref}}$  $z_0$  $I_1$
Generatif model with normal coordinates
Towards New Tools For Shape Analysis?

- Natural interplay between photometry and geometry

- Good framework the learning generative models (statistical models on $\alpha_0$ or $z_0 +$ geodesic shooting)