Computational Anatomy and EPDiff
A "Reconnection" Path

Alain Trouvé
Ecole Normale Supérieure de Cachan

and

University Paris 13

July 15, 2004

Plan of the talk

- I. Grenander's deformable templates and examples.
- II. Geodesic motion. (Miller-Trouvé-Younes 2003)

Plan of the talk

- I. Grenander's deformable templates and examples.
- II. Geodesic motion. (Miller-Trouvé-Younes 2003)

Central Framework: Grenander's Deformable Template

Main ingredients:

- Object set: $\mathcal{O}$
- Group of transformations G
- Group action $G \times \mathcal{O} \to \mathcal{O}$

$$o_{\text{obs}} = \underbrace{g o_{\text{ref}}}_{\text{deformed template}} + \underbrace{n}_{\text{noise}} \tag{1}$$

Examples–Landmarks

- $\mathcal{O} = \{\mathbf{x} = (x_i)_{1\le i\le N} \mid x_i \in \mathbb{R}^d\}$.
- Finite dimensional case: G rotation group, scale and translation group, affine group...
- Infinite dimensional case: G group of non rigid transformation $\varphi : \mathbb{R}^d \to \mathbb{R}^d$

$$\mathbf{x}_{\mathsf{obs}} = \varphi \mathbf{x}_{\mathsf{ref}} + \mathbf{n} \text{ where } \varphi \mathbf{x} \doteq (\varphi(x_i))_{1\le i\le N} \tag{2}$$

Examples–Images

- D a domain in $\mathbb{R}^d$, $\mathcal{O} = \{\ I : D \to \mathbb{R}\ \}$

- G group of diffeomorphisms on D

$$\varphi I \doteq I \circ \varphi^{-1}$$

$$I_{\mathsf{obs}} = \underbrace{\varphi I_{\mathsf{ref}}}_{\text{deformed template}} + \underbrace{n}_{\text{noise}} \tag{3}$$

Template Matching Problem

Bayesian or variational point of view

$$\hat{g} = \operatorname*{argmin}_{g \in G} \left(\underbrace{d_{\mathcal{O}}(o_{\mathsf{ref}}, g o_{\mathsf{ref}})^2}_{\text{distance on object space}} + \underbrace{|o_{\mathsf{obs}} - g o_{\mathsf{ref}}|_{\mathsf{n}}^2}_{\text{noise intensity}} \right) \tag{4}$$

Idea: $d_{\mathcal{O}}((o_{\mathsf{ref}}, g o_{\mathsf{ref}})$ should be depend on the amount of deformation g linking o_{ref} and $g o_{\mathsf{ref}}$. Build a model on $\hat{g}$ and n instead of o_{obs}.

Metric Transfer Through Group Action

- Differentiate the action $g \rightarrow go$ at Id.

$$\begin{array}{ccc} T_{\mathsf{Id}}G & \longrightarrow & T_o\mathcal{O} \\ v & \longrightarrow & \delta o \doteq v.o \end{array} \tag{5}$$

- Define a metric $\|v\|_{\mathsf{Id}}$ for small $g \simeq \mathsf{Id} + v$

- Transfer the metric on $T_o\mathcal{O}$

$$\|\delta o\|_o = \inf_{\delta o = vo} \|v\|_{\mathsf{Id}}$$

Metric Transfer Through Group Action (cont)

$\mathcal{O}$ inherits a Riemannian structure
For $o, o' \in \mathcal{O}$:

- $\mathcal{O}_{o,o'}$: smooth paths $t \to o_t$ from o to o'.
- $d_{\mathcal{O}}(o, o') \doteq \inf_{\mathcal{O}_{o,o'}} \int_0^1 \|\frac{do}{dt}\|_{o_t} dt$

We get a Riemannian distance on $\mathcal{O}$

Associated Right Invariant Dist. on G

Choose $\mathcal{O} = G$, G acts onto itself: $(g, g_0) \to gg_0$
Apply metric transfer:

$$\|\delta g_0\|_{g_0} = \inf_{\delta g_0 = v.g_0} \|v\|_{\mathsf{Id}}\,.$$

But $g \to gg_0$ is one to one. Hence $v \to vg_0$ is an isomorphism.

Consequence: $T_{g_0}G = \{\ v.g_0 \mid v \in T_{\mathsf{Id}}G\}$ and

$$\|vg_0\|_{g_0} = \|v\|_{\mathsf{Id}} \text{ (Right invariance)}\,.$$

Associated Right Invariant Dist. on G (Cont)

Geodesic distance on G: Take infimum on sufficiently smooth paths from g_0 to g_1

$$d_G(g_0, g_1) \doteq \inf_{g_0 \to g_1} \int_0^1 \|\frac{dg_t}{dt}\|_{g_t} dt = \inf_{g_0 \to g_1} \int_0^1 \|\frac{dg_t}{dt} g_t^{-1}\|_{\mathsf{Id}} dt$$

Right invariance property:

$$d_G(g_0, g_1) = d_G(g_0 g, g_1 g)$$

$$\boxed{\text{Theorem: } d_{\mathcal{O}}(o, o') = \inf\{d_G(\mathsf{Id}, g) \mid go = g'\}.}$$

Rigorous construction for non rigid deformations

- D a d-dimensional domain, G subgroup of Hom(D), $g \to \varphi$
- Start from V be a Hilbert space of vector fields on D with norm $\| \ \|_V$ (to be identified with $T_{\mathrm{Id}}G$)
- Assume $V \overset{\text{cont}}{\hookrightarrow} C^1(D, \mathbb{R})$ (admissibility)

For $v \in L^2([0,1], V)$, φ^v solution of $\begin{cases} \frac{d\varphi_t^v}{dt} = v_t \circ \varphi_t^v \\ \varphi_0^v = \mathrm{Id} \end{cases}$

$$G \doteq \{ \ \varphi_1^v \mid v \in L^2([0,1], V) \ \}$$

$$d_G(\mathrm{Id}, \varphi) = \inf\{\int_0^1 \|v_t\|_V dt \mid \varphi_1^v = \varphi\}$$

Rigorous construction (non-rigid deformation) (Cont)

Usual framework: $\|v_t\|_V^2 = \int \langle Lv(x), v(x)\rangle dx$ with

$$L : D(L) \subset V \to H = L^2(D, \mathbb{R}^d)$$

ex: $L = (I - \Delta)^s$.

Theorem (T. 1995): Let $V \overset{\text{cont}}{\hookrightarrow} C^1(D, \mathbb{R})$ (admissibility)
* G is a subgroup of homeomorphisms
* G is complete for the right invariant metric $d_G(\, , \,)$
* Existence of geodesics

If $V \overset{\text{cpct}}{\hookrightarrow} C^1(D, \mathbb{R})$, G is a subgroup of diffeomorphisms (T. 1995, Dupuis et al. 1998)

Template Matching Revisited

Energetic formulation: $d_G(\mathsf{Id}, \varphi)^2 = \inf_{\varphi_1^v = \varphi} \int_0^1 \|v_t\|_V^2 dt$

Template matching problem:

$$\begin{aligned} \hat{\varphi} &= \operatorname*{argmin}_{\varphi \in G} \left(d_{\mathcal{O}}(o_{\mathsf{ref}}, \varphi o_{\mathsf{ref}})^2 + |o_{\mathsf{obs}} - \varphi o_{\mathsf{ref}}|_{\mathsf{n}}^2 \right) \\ &= \operatorname*{argmin}_{\varphi \in G} \left(d_G(\mathsf{Id}, \varphi)^2 + |o_{\mathsf{obs}} - \varphi o_{\mathsf{ref}}|_{\mathsf{n}}^2 \right) \end{aligned}$$

becomes:

$$\hat{v} = \operatorname*{argmin}_{v \in L^2([0,1],V)} \left(\frac{1}{2} \int_0^1 \|v_t\|_V^2 dt + |o_{\mathsf{obs}} - \varphi_1^v o_{\mathsf{ref}}|_{\mathsf{n}}^2 \right), \ \hat{\varphi} \doteq \varphi_1^{\hat{v}}$$

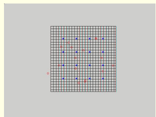
$\longrightarrow$ Existence of solutions (T. 1995, Dupuis et al. 1998)

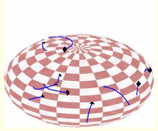
Application to Landmark Matching

- $\mathcal{O} = \{ \ \mathbf{x} = (x_i)_{1 \le i \le N} \mid x_i \in \mathbb{R}^d \ \}$
- $g\mathbf{x} \doteq (g(x_i))_{1 \le i \le N} \Rightarrow v\mathbf{x} = (v(x_i))_{1 \le i \le N}$.

$$\hat{v} = \underset{v \in L^2([0,1],V)}{\operatorname{argmin}} \int \|v_t\|_V^2 dt + \frac{1}{\sigma^2} \sum_{i=1}^{N} |\mathbf{x}^{\mathsf{obs}} - g_1^v(x_i^{\mathsf{ref}})|^2$$

Gives Large Deformation Extension of Bookstein Splines.

* 2d example: (Joshi, Miller (00), Camion,Younes (00)) 

* Landmark on the sphere, (Glaunes,Vaillant (02)) 

Application to Image Matching

- $\mathcal{O} = \{ \ I : D \to \mathbb{R} \ \}$
- V satisfying admissibility conditions
- Action : $\varphi I = I \circ \varphi^{-1}$

$$\hat{v} = \underset{v \in L^2([0,1],V)}{\operatorname{argmin}} \int \|v_t\|_V^2 dt + \frac{1}{\sigma^2} \int |I^{\mathsf{obs}}(x) - I^{\mathsf{ref}}(\varphi_1^v(x))|^2 dx$$

Large deformation extension of Amit-Grenander-Piccioni model:

$$\underset{v \in V}{\operatorname{argmin}} \|v\|_V^2 + \frac{1}{\sigma^2} \int |I^{\mathsf{obs}}(x) - I^{\mathsf{ref}}(x - v(x))|^2 dx$$

Application To Submanifold Matching

- Point x
  $\longrightarrow \delta_x$ Dirac measure at x.

- M k-dimensional submanifold
  $\longrightarrow \mu_M$ uniform probability measure on M

- $M = \cup_{i=1}^r M_i$ union of manifold with different dimensions
  $\longrightarrow \mu_M = \sum_{i=1}^r \mu_{M_i}$

- More general situations (noisy representations)
  $\longrightarrow$ arbitrary distributions

Glaunes et al. (CVPR 04)

Action on measure

Action $(\varphi, \mu) \to \varphi\mu$ given by the mass transport

$$\int f \, d(\varphi\mu) = \int f \circ \varphi \, d\mu$$

Action on measure

Action $(\varphi, \mu) \to \varphi\mu$ given by the mass transport

$$\int f \; d(\varphi\mu) = \int f \circ \varphi \; d\mu$$

- Points: $\varphi(\sum_{x \in S} \delta_x) = \sum_{x \in S} \delta_{\varphi(x)}$

Action on measure

Action $(\varphi, \mu) \rightarrow \varphi\mu$ given by the mass transport

$$\int f \; d(\varphi\mu) = \int f \circ \varphi \; d\mu$$

- Points: $\varphi(\sum_{x \in S} \delta_x) = \sum_{x \in S} \delta_{\varphi(x)}$
- Abs. cont. distribution: $\varphi(g d\lambda) = g \circ \varphi^{-1} |d\varphi^{-1}| d\lambda$

Action on measure

Action $(\varphi, \mu) \to \varphi\mu$ given by the mass transport

$$\int f \; d(\varphi\mu) = \int f \circ \varphi \; d\mu$$

- Points: $\varphi(\sum_{x \in S} \delta_x) = \sum_{x \in S} \delta_{\varphi(x)}$
- Abs. cont. distribution: $\varphi(g d\lambda) = g \circ \varphi^{-1} |d\varphi^{-1}| d\lambda$

$$\mathsf{supp}(\varphi\mu) = \varphi(\mathsf{supp}\mu)$$

Comparison between two distributions μ and ν

Principle: μ is defined by its behavior on test functions:

$$\langle \mu, f \rangle \stackrel{\text{def}}{=} \int f \, d\mu$$

W Hilbert space of space functions:

$$|\mu - \nu|_{W^*} = \sup_{|f|_W = 1} |\langle \mu, f \rangle - \langle \nu, f \rangle|$$

Complete Model for Submanifold Matching

- D a bounded domain in $\mathbb{R}^d$. The object space is $\mathcal{O} = W^*$ where W is a Hilbert space of functions on D.

- $(\varphi\mu, f) \doteq (\mu, f \circ \varphi)$

$$\hat{v} = \operatorname*{argmin}_{v \in L^2([0,1],V)} \int \|v_t\|_V^2 dt + \frac{1}{\sigma^2} |\varphi_1^v \mu_0 - \mu_1|_{W^*}^2$$

Importantly, we have existence and consistancy results

Existence Result

Let μ, $\nu \in \mathcal{M}_s$, and let for any $v \in L^2([0,1], V)$

$$J_{\mu,\nu}(v) \doteq \int \|v_t\|_V^2 dt + \frac{1}{\sigma^2} |\varphi_1^v \mu - \nu|_{W^*}^2$$

Theorem 1. [Existence] *Under suitable regularity conditions on V (admissibility condition), there exists a minimiser v_* in $L^2([0,1], V)$ of $J_{\mu,\nu}$ for any μ, ν in $\mathcal{M}_s$.*

Existence Result

Let μ, $\nu \in \mathcal{M}_s$, and let for any $v \in L^2([0,1], V)$

$$J_{\mu,\nu}(v) \doteq \int \|v_t\|_V^2 dt + \frac{1}{\sigma^2} |\varphi_1^v \mu - \nu|_{W^*}^2$$

Theorem 1. [Existence] *Under suitable regularity conditions on V (admissibility condition), there exists a minimiser v_* in $L^2([0,1], V)$ of $J_{\mu,\nu}$ for any μ, ν in $\mathcal{M}_s$.*

For v_* is a minimiser, φ_* is the associated diffeomorphism.

Consistency Result

Theorem 2. [Consistency] *Assume the regularity condition of Thm 1. Let μ and ν be two probability distributions on $\mathbb{R}^d$ and let $x_1, \cdots, x_m$ and $y_1, \cdots, y_n$ be iid samples drawn from distribution μ and ν. Let $\hat{\mu}_m = \frac{1}{m}\sum_i \delta_{x_i}$ and $\hat{\nu}_n = \frac{1}{n}\sum_j \delta_{y_j}$ be the associated empirical measures. Then if $\varphi_*(m,n)$ is an minimiser of $J_{\hat{\mu}_m,\hat{\nu}_n}$, almost surely, $\varphi_*(m,n)$ tends uniformly (up to the extraction of a subsequence) to φ_* minimiser of $J_{\mu,\nu}$ when $m, n \to \infty$.*

Rangarajan Test Data

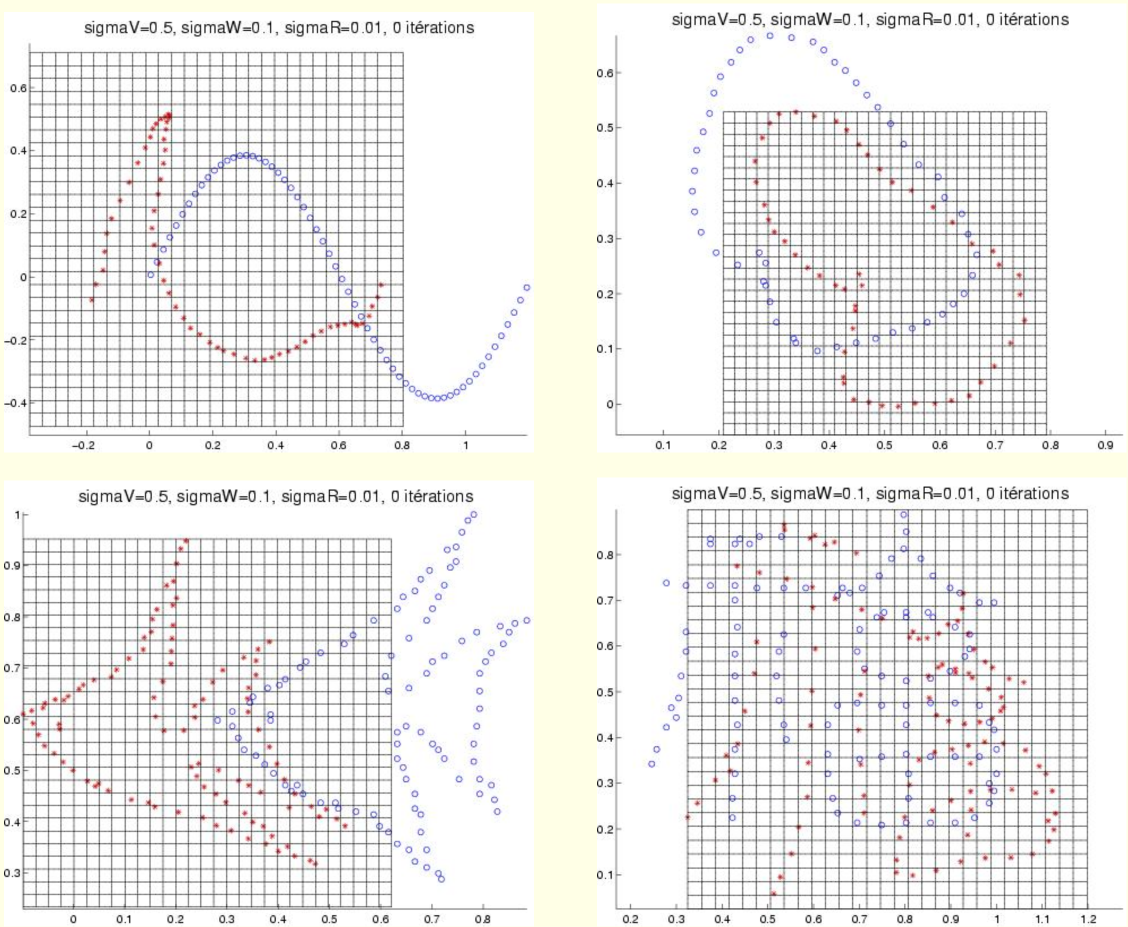


Figure 1: Rangarajan test set

Matching heterogeneous manifolds

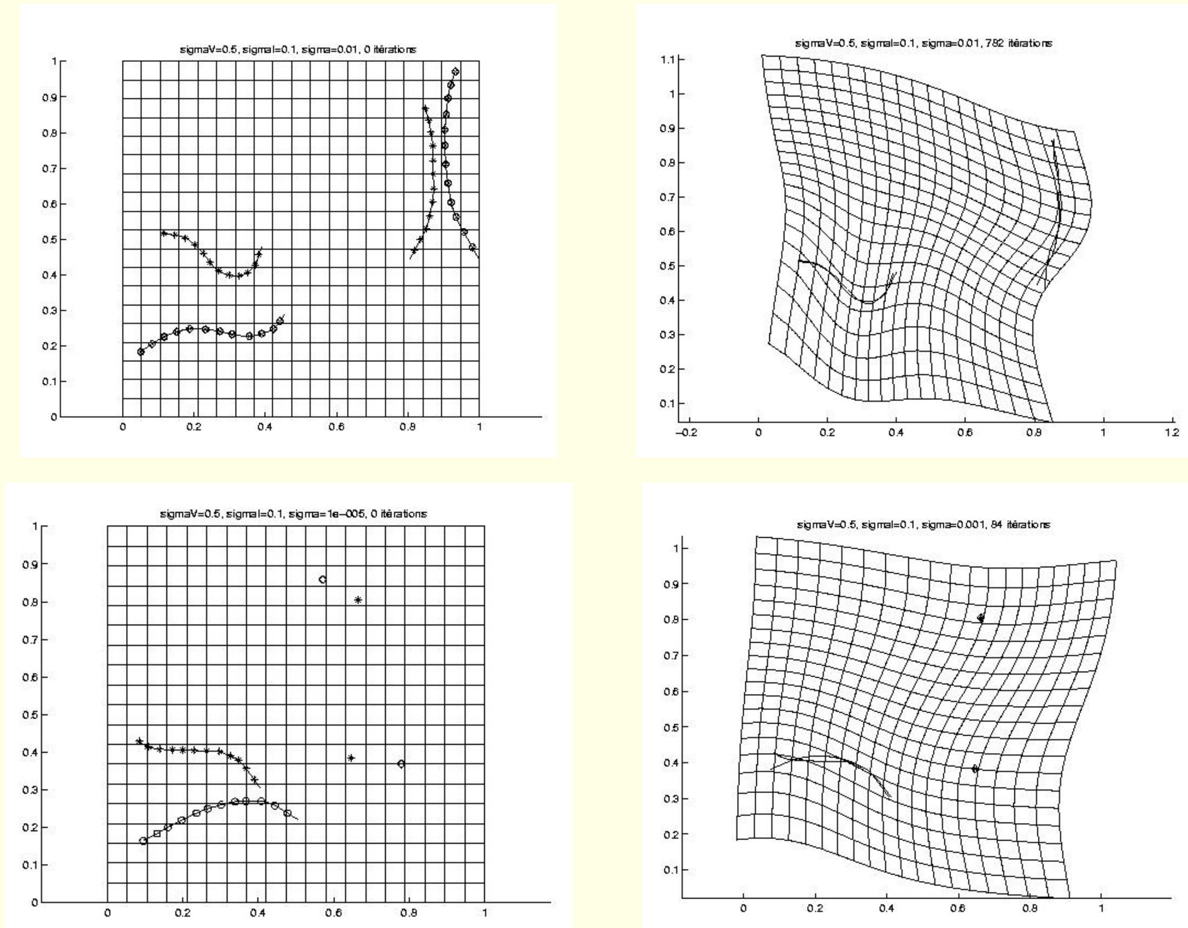


Figure 2: Left Initial configuration, Right: Final matching

Consistency

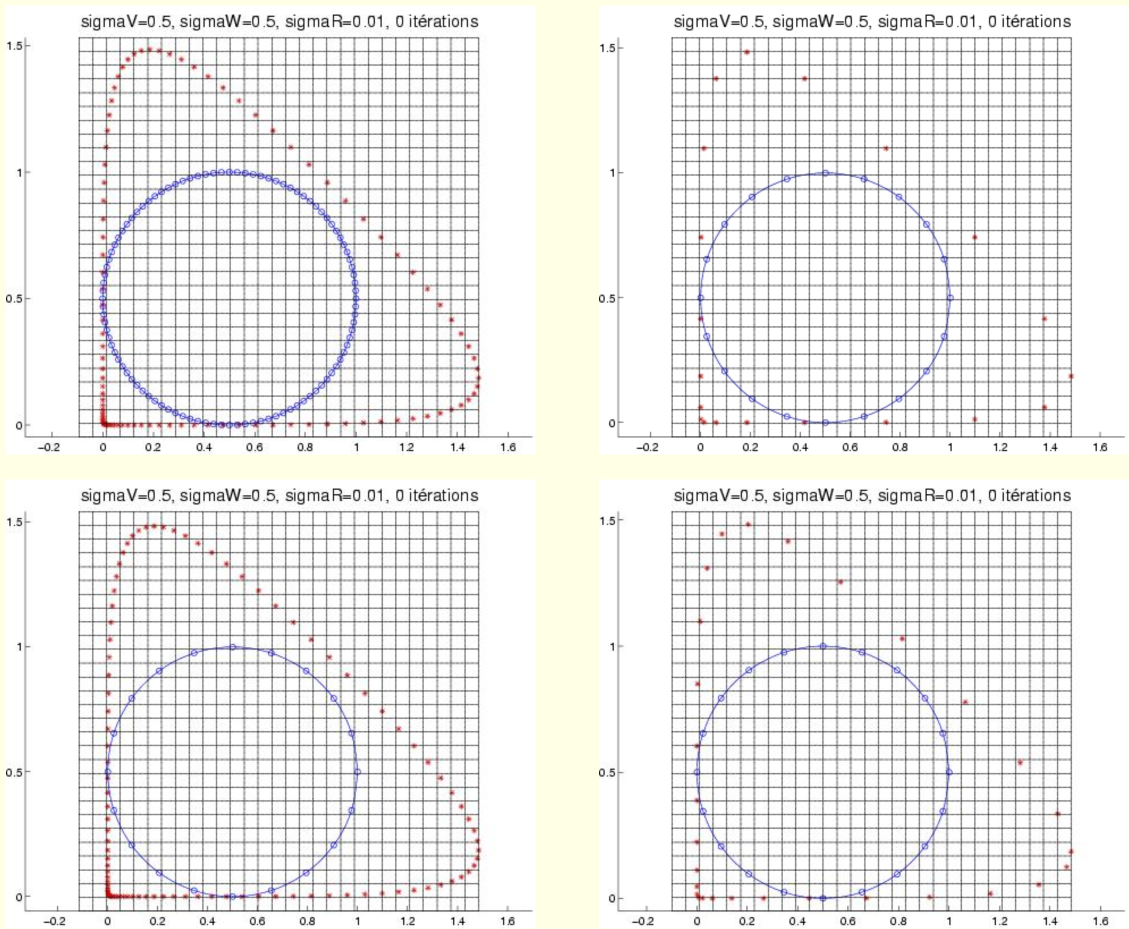


Figure 3: Various sampling rate of the same manifolds

Robustness against noise

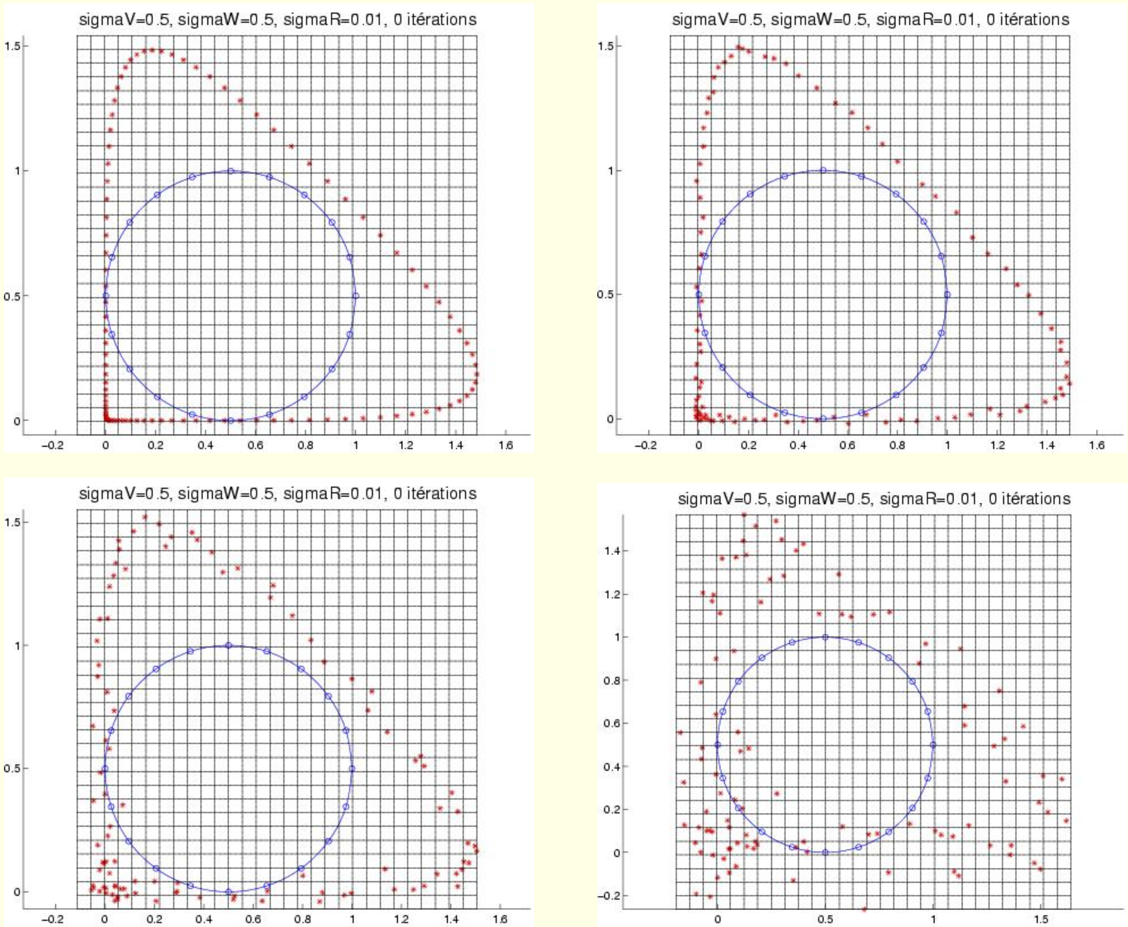
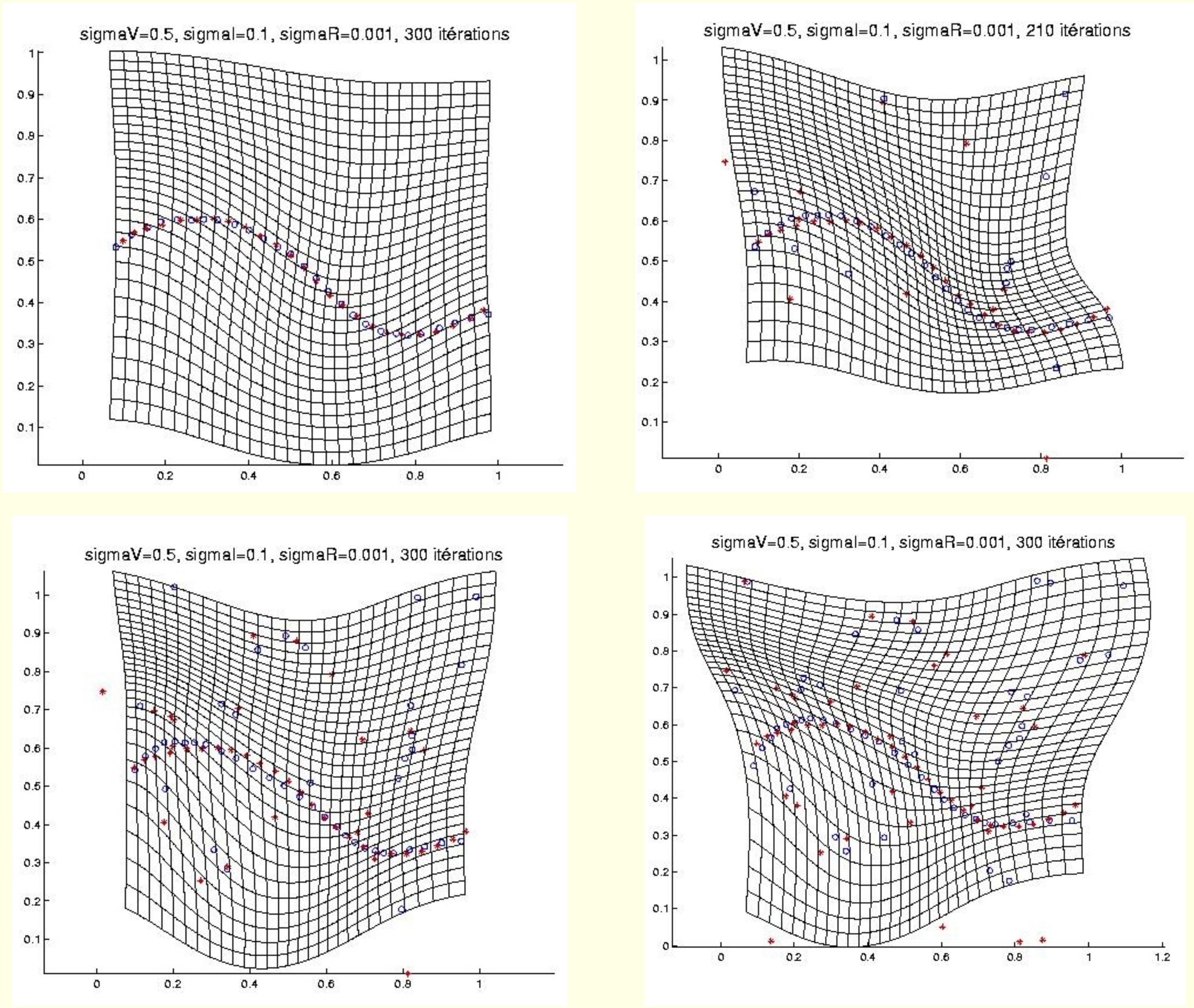


Figure 4: Various noise levels

Robustness against outliers



3D examples

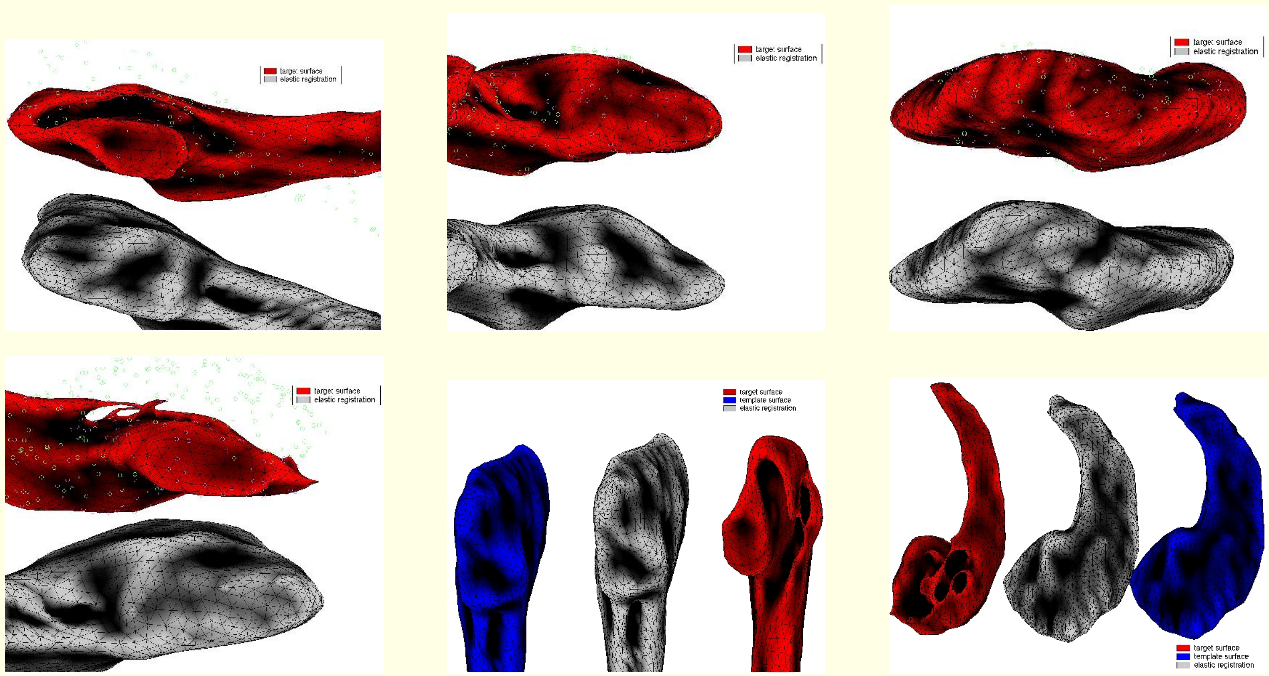


Figure 5: Left and Right Hippocampus (Surface tessellation provided by CNRS Cognitive Neuroscience and Brain Imaging Laboratory, La Salpetriere Hospital, Paris)

Morphing Metrics on Object Space

Previous construction:

$$d_{\mathcal{O}}(o, o') < +\infty \text{ iff } o \text{ and } o' \text{ on same orbit}$$

For images: $d_{\mathcal{I}}(I, I') < +\infty \Leftrightarrow I' \circ \varphi = I, \ \varphi \in G$

no apparition of new structure in one orbit

• Assume the object space $\mathcal{O} = H$ Hilbert space.
Let $\tilde{G} = \{\ a = (g, h) \mid g \in G,\ \ h \in H\ \}$

• If the action $o \to go$ is linear, $\tilde{G}$ is a group for the composition law $(g, h)(g', h') = (gg', gh' + h)$ (semi-direct product $G \ltimes H$) and $\tilde{G}$ acts (affinely) on $\mathcal{O}$ by $(g, h)o = go + h$.

$$\text{But now: } \tilde{G}o_{\mathsf{ref}} = \mathcal{O}$$

• Let $W = V \times H$ be the tangent space at $e_{\tilde{G}}$ of $\tilde{G}$ equip with the induced product metric: $\langle (v, z), (v', z') \rangle_W = \langle v, v' \rangle_V + \langle z, z' \rangle_H$. Consider the associated invariant metric on $\tilde{G}$

$$d_{\mathcal{O},\mathsf{morph}}(o, o') = \inf\{\ d_{\tilde{G}}(e_{\tilde{G}}, (g, h)) \mid (g, h)o = o'\}\,.$$

Many extensions in Laurent's talk yesterday.

Plan of the talk

- I. Grenander's deformable templates and examples.
- II. Geodesic motion.

The central role of geodesic paths in G

- **Inexact Matching**

$$\hat{g} = \operatorname*{argmin}_{g \in G} d_O(o_{\text{ref}}, g o_{\text{ref}})^2 + E(o_{\text{ref}}, o_{\text{obs}})$$

- **Exact Matching**

$$\hat{g} = \operatorname*{argmin}_{g \in G} d_O(o_{\text{ref}}, g o_{\text{ref}})^2 \text{ subject to } g o_{\text{ref}} = o_{\text{obs}}$$

If (g_t) is a geodesic path in G from Id to $\hat{g}$, then $o_t = g_t o_{\text{ref}}$ is a geodesic path in $\mathcal{O}$ from o_{ref} to $\hat{g} o_{\text{ref}}$.

A key fact is that G is always equip with a right invariant metric.

- Start with G Lie group of matrices with right invariant metric
- Define $L : T_{\mathsf{Id}}G \to T_{\mathsf{Id}}G^*$ (Inertial operator)

Lv is the Momentum

$$\|v\|_{\mathsf{Id}}^2 \doteq (Lv, v)_{T_{\mathsf{Id}}G^* \times T_{\mathsf{Id}}G}$$

Note that $d_G(g_0, g_1)^2$ is given by:

$$\min \int (Lv_t, v_t)dt \text{ subject to: } \left| \begin{array}{l} g_0^v = g_0 \text{ and } g_1^v = g_1 \\ \text{where} \\ \frac{dg^v}{gt} = v_t g_t^v \end{array} \right. \qquad \text{(LAP)}$$

This is exactly the least action principale for a Lagragian reduced to the kinetic energy.

The Euler-Poincaré equation for group of matrices

Let g_t be an extremal curve for the kinetic energy and $g_{t,\epsilon}$ be a smooth deformation around $\epsilon = 0$ with fixed end values $(g_{0,\epsilon} = g_0,\ g_{1,\epsilon} = g_1)$. Let $v_{t,\epsilon}$ and $w_{t,\epsilon}$ such that

$$\frac{\partial g}{\partial t} = vg \text{ and } \frac{\partial g}{\partial \epsilon} = wg\,.$$

Since $\frac{\partial}{\partial \epsilon}\left(\frac{\partial g}{\partial t}\right) = \frac{\partial}{\partial t}\left(\frac{\partial g}{\partial \epsilon}\right)$, we get $\frac{\partial v}{\partial \epsilon} g + vwg = \frac{\partial w}{\partial t} g + wvg$ i.e.

$$\frac{\partial v}{\partial \epsilon} = \frac{\partial w}{\partial t} - \underbrace{(vw - wv)}_{[v,w]} = \frac{\partial w}{\partial t} - [v, w] = \frac{\partial w}{\partial t} - \mathrm{ad}_v w\,.$$

Reminder: $\dfrac{\partial v}{\partial \epsilon} = \dfrac{\partial w}{\partial t} - \mathrm{ad}_v w\,.$

The curve $w_{t,\epsilon}$ can vary freely in $\mathfrak{g}$, with boundary conditions $w_{0,\epsilon} = w_{1,\epsilon} = 0$. From $\frac{d}{d\epsilon}\left(\int \langle v, v\rangle_e dt\right)_{|\epsilon=0} = 0$, we get

$$\int_0^1 \langle v, \frac{\partial v}{\partial \epsilon}\rangle_{\mathfrak{g}} = \int \langle v, \frac{\partial w}{\partial t} - \mathrm{ad}_v w\rangle_{\mathfrak{g}} = \int (Lv, \frac{\partial w}{\partial t} - \mathrm{ad}_v w) = 0$$

Introducing the adjoint operator ad^*, we get $(Lv, \mathrm{ad}_v w) = (\mathrm{ad}_v^*(Lv), w)$ so that by integration by part, we have finally the following Euler-Poincaré equation

$$\frac{\partial Lv}{\partial t} + \mathrm{ad}_v^*(Lv) = 0\,. \tag{6}$$

Euler-Poincaré equation in the case of diffeomorphisms

Let φ_t be an extremal curve for the kinetic energy and $\varphi_{t,\epsilon}$ be a perturbation of φ_t with fixed end points ($\varphi_{t,0} = \varphi_t$, $\varphi_{0,\epsilon} = \varphi_0$ and $\varphi_{1,\epsilon} = \varphi_1$ for any ϵ). Let $v_{t,\epsilon}$ and $w_{t,\epsilon}$ such that

$$\frac{\partial \varphi}{\partial t} = v \circ \varphi \text{ and } \frac{\partial \varphi}{\partial \epsilon} = w \circ \varphi \,.$$

Since $\frac{\partial}{\partial \epsilon}\left(\frac{\partial \varphi}{\partial t}\right) = \frac{\partial}{\partial t}\left(\frac{\partial \varphi}{\partial \epsilon}\right)$, we get $\frac{\partial v}{\partial \epsilon} \circ \varphi + d_\varphi v(w \circ \varphi) = \frac{\partial w}{\partial t} \circ \varphi + d_\varphi w(v \circ \varphi)$ and finally

$$\frac{\partial v}{\partial \epsilon} = \frac{\partial w}{\partial t} - \underbrace{(dv(w) - dw(v))}_{[v,w]} = \frac{\partial w}{\partial t} - \mathrm{ad}_v(w)$$

As for the case of matrices, we have obtained

$$\frac{\partial v}{\partial \epsilon} = \frac{\partial w}{\partial t} - \mathsf{ad}_v w \,.$$

and reproducing exactly the same computation we get the Euler Poincaré equation for the diffeomorphisms

$$\frac{\partial Lv}{\partial t} + \mathsf{ad}_v^*(Lv) = 0 \,.$$

or equivalently for any $u \in \mathfrak{g}$ and $m_t = Lv_t$

$$\frac{\partial}{\partial t}(m_t, u) + (m_t, \mathsf{ad}_{v_t} u) = \frac{\partial}{\partial t}(m_t, u) + (m_t, dv_t(u) - du(v_t)) = 0$$

By integration by parts: $\dfrac{\partial}{\partial t}(m_t, u) + (m_t, dv_t(u) - du(v_t))$
$= (\dfrac{\partial}{\partial t} m_t + \mathrm{div}(m_t \otimes v_t) + dv_t^*(m_t), u)$.

Hence the Euler Poincaré equation (EPDiff equation) is

$$\begin{cases} \dfrac{\partial}{\partial t} m_t + \mathrm{div}(m_t \otimes v_t) + dv_t^*(m_t) = 0 \\ \\ v_t = L^{-1} m_t \end{cases}$$

(Mumford-Vishik (98), Holm-Marsden-Ratiu (98), Miller-Trouvé-Younes (02), Holm-Trouvé-Younes (04)...).

For $Lu = u - \alpha^2 \Delta u$: Camassa-Holm Equation (93)

Momentum Map

The Euler Poincaré equation is a conservation equation: Indeed, (check for matrices): If $\mathsf{Ad}_g(u) \doteq gug^{-1} = (d_e L_{g^{-1}} d_e R_g(u))$, then

$$\frac{d}{dt}(\mathsf{Ad}_{g_t}(u)) = \mathsf{ad}_{v_t}(\mathsf{Ad}_{g_t}(u))\,.$$

Hence

$$\frac{\partial}{\partial t}(\mathsf{Ad}^*_{g_t}(m_t), u) = (\frac{\partial m_t}{\partial t}, \mathsf{Ad}_{g_t}(u)) + (m_t, \mathsf{ad}_{v_t}(\mathsf{Ad}_{g_t}(u)) = 0$$

and

$$m_t = \mathsf{Ad}^*_{g_t^{-1}}(m_0)$$

or equivalently

$$(m_t, u) = (m_0, \mathsf{Ad}_{g_t^{-1}}(u))$$

Momentum Map for EPDiff

- $(m_t, u) = (m_0, g_t^{-1} u g_t)$ translates to

$$(m_t, u) = (m_0, (d\varphi_t)^{-1} u \circ \varphi_t)$$

- Let $m_0 \doteq \nabla f_0 \, \mu_0$ (μ_0 signed measure) ie

$$(m_0, u) = \int_D \langle \nabla f_0, u \rangle d\mu_0$$

Then we get $m_t = \nabla f_t \mu_t$ with

$$f_t = \underbrace{\varphi_t f_0}_{\text{action on fonctions}} \quad \text{and } \mu_t = \underbrace{\varphi_t \mu_0}_{\text{action on measures}} .$$

Normal Momentum Motion

Let $t \to I_t$ be a geodesic path in image space between I_0 to I_1.

We know

$$I_t = \varphi_t I_0 = I_0 \circ \varphi_t^{-1}$$

with $t \to \varphi_t$ geodesic path in G

Assume that $m_0 = \nabla I_0 d\mu_0$ (Normality Constraint).
Then

$$m_t = \nabla I_t \, d\mu_t$$

with $\mu_t = \mu_0 \circ \varphi_t^{-1}$.

$$\boxed{\text{The momentum stays normal to level sets of } I_t}$$

Normality Constraint

Image case:

$$\frac{\partial}{\partial t}(\varphi_t I_0)_{|t=0} = -\langle \nabla I_0, v_0 \rangle$$

Let $V_0 = \{\ w \in V \mid \langle \nabla I_0(x), v(x) \rangle = 0,\ \forall x\ \}$. We have

$$\langle \nabla I_0, v_0 \rangle = \langle \nabla I_0, v_0 + w \rangle,\ w \in V_0$$

This leads to the constraint $v_0 \in V_0^{\perp}$.

However, if $m_0 = Lv_0 = \alpha \nabla I_0 \lambda$ then

$$\langle v_0, w \rangle_V = (m_0, w) = \int \alpha \langle \nabla I_0, w \rangle dx = 0,\ w \in V_0\,.$$

Normal Momentum Constraint in Valid for Template Matching

$$\hat{v} = \underset{v \in L^2([0,1],V)}{\operatorname{argmin}} J(v) : \int \|v_t\|_V^2 dt + \frac{1}{\sigma^2} \int |I^{\mathsf{obs}} - I^{\mathsf{ref}} \circ (\varphi_1^v)^{-1}|^2 dx$$

$I_t \doteq \varphi_t^{\hat{v}} I^{\mathsf{ref}}$ is a geodesic path from I^{ref} to $\varphi_1^{\hat{v}} I^{\mathsf{ref}}$ in image space.

The gradient of J in $L^2([0,1],V)$ (Beg et al. 2002) satisfies

$$L(\nabla J)_t = 2v_t + \frac{2}{\sigma^2} |d\varphi_{t,1}| (I_t^{\mathsf{obs}} - I_t^{\mathsf{ref}}) \nabla I_t^{\mathsf{ref}}$$

where $\varphi_{s,u} = \varphi_u \circ \varphi_s^{-1}$, $I_t^{\mathsf{ref}} = \varphi_{0,t} I^{\mathsf{ref}}$, $I_t^{\mathsf{obs}} = \varphi_{t,1} I^{\mathsf{obs}}$

Euler-Lagrange Equation $\nabla J = 0$ gives at $t = 0$

$$(m_0, u) = \int_D \langle \nabla I^{\mathsf{ref}}, u \rangle \alpha dx$$

with $\alpha = \frac{1}{\sigma^2} |d\varphi_1| (\varphi_1 I^{\mathsf{obs}} - I^{\mathsf{ref}})$

Selected geodesics from I_{ref} satisfy
Normal Momentum Constraint

Geodesic Motion Equation in Image Space

If $\alpha_t = |d\varphi_t^{-1}|\alpha_0 \circ \varphi_t^{-1}$, and $I_t = I_{\mathsf{ref}} \circ \varphi_t^{-1}$, then we get

$$v_t = L^{-1}(\alpha_t \nabla I_t dx) \text{ and } \begin{cases} \dfrac{\partial \alpha_t}{\partial t} + \mathrm{div}(\alpha_t v_t) = 0 \\ \\ \dfrac{\partial I_t}{\partial t} + \langle \nabla I_t, v_t \rangle = 0 \end{cases}$$

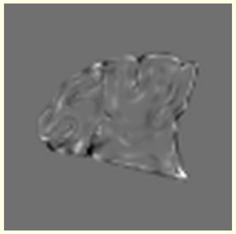
Geodesic shooting build the Exponential Map

$$\mathrm{Exp}_{I_0} : T_{I_0}\mathcal{O} \to \mathcal{O}$$

The complete geodesic coded by α_0.

Direct vs Inverse Exponential Mapping

Minimal Path vs Geodesic Shooting for macaque brain

$$I^{\mathsf{ref}} : \alpha_0 \leftrightarrow I_1$$

Given I^{ref} (Template)

- Coding Process: From any I_1, compute (Beg's Algorithm) α_0
- Decoding Process: From any α_0, compute I_1 (Geodesic Shooting)

Extension in the case of metamorphosis

Semi-directe product case

If $I_t = I_{\text{ref}} \circ \varphi_t^{-1}$, then we get for the geodesic evolution equation:

$$v_t = -L^{-1}(z_t \nabla I_t dx) \text{ and } \begin{cases} \dfrac{\partial z_t}{\partial t} + \mathrm{div}(z_t v_t) = 0 \\ \\ \dfrac{\partial I_t}{\partial t} + \langle \nabla I_t, v_t \rangle = \sigma^2 z_t \end{cases}$$

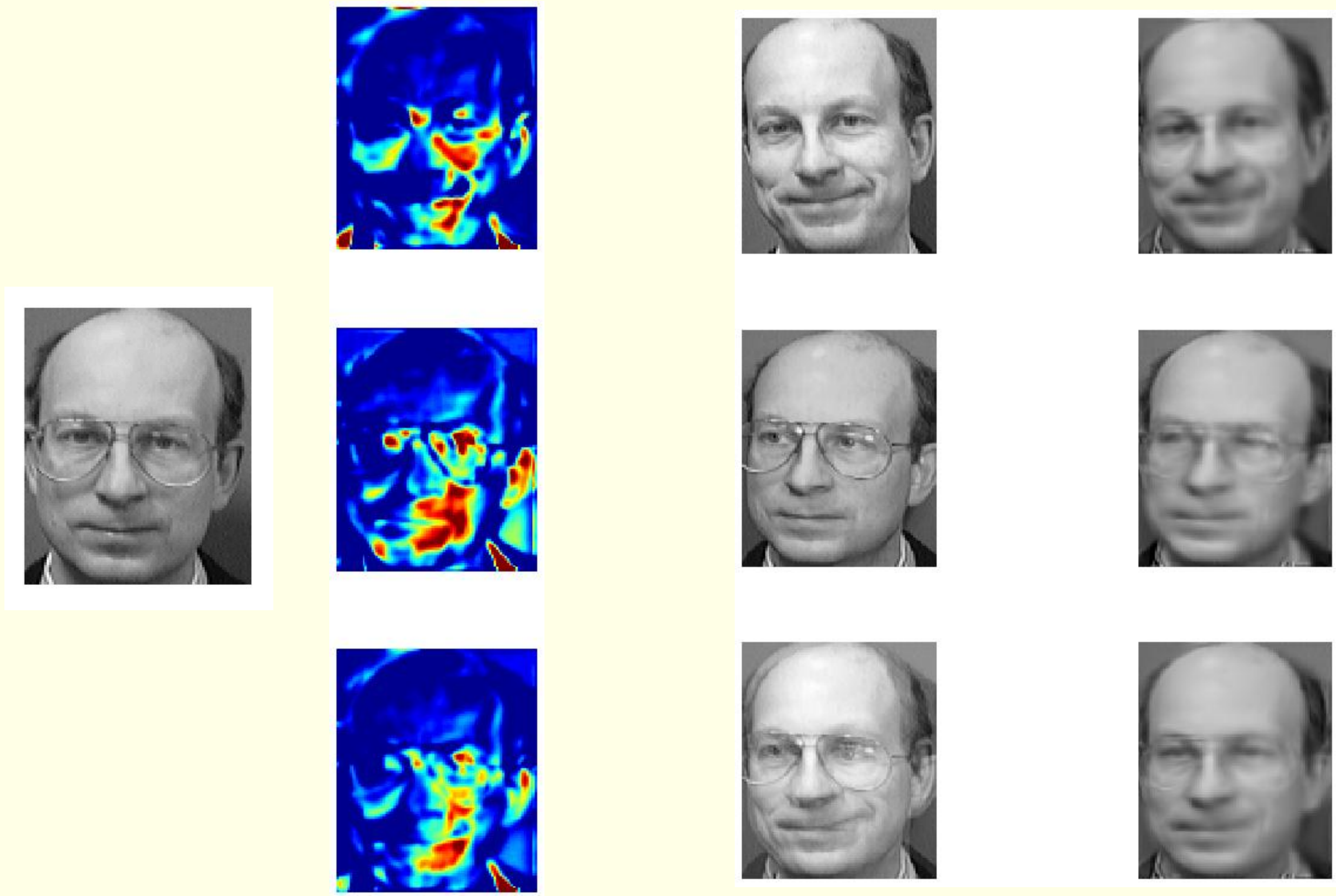
Geodesic shooting for metamorphosis

I_{ref}

z_0

I_1

Generatif model with normal coordinates



Towards New Tools For Shape Analysis ?

- Natural interplay between photometry and geometry

- Good framework the learning generative models (statistical models on α_0 or z_0 + geodesic shooting