Conformal Mapping of Brain Surfaces: Circle Packing and the Riemann Mapping Theorem

IPAM — Mathematics in Brain Imaging

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Acknowledgments

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Thanks to Bernhard Riemann.

All the errors, mathematical and scientific, are mine.

Outline

- The Brain Mapping Setting

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- Example Maps/Manipulations

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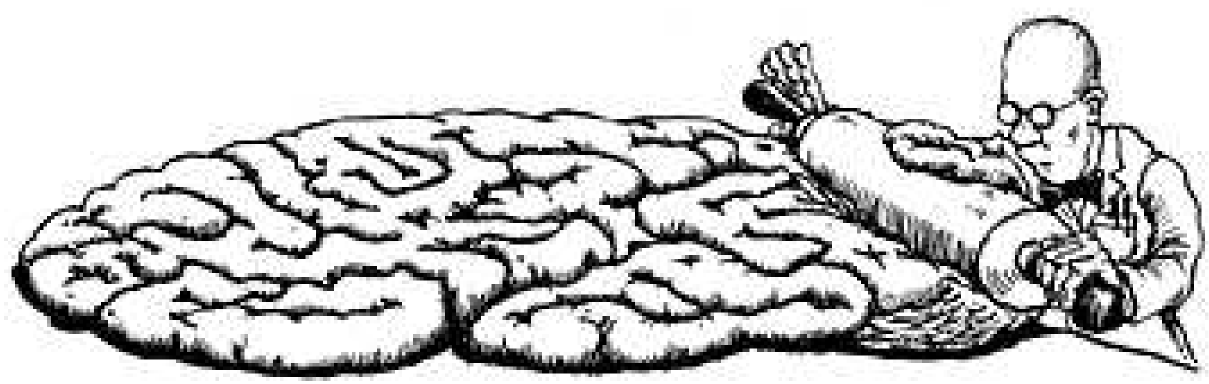

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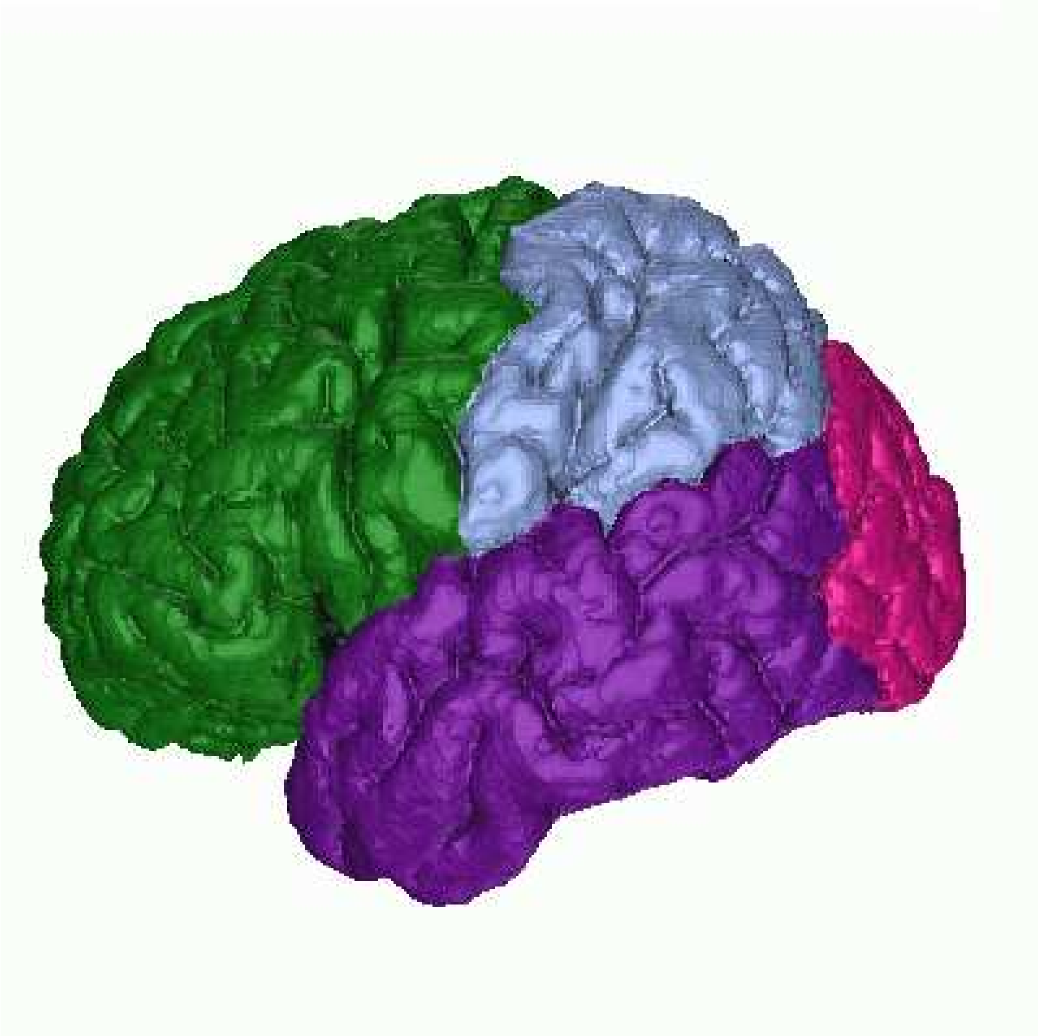
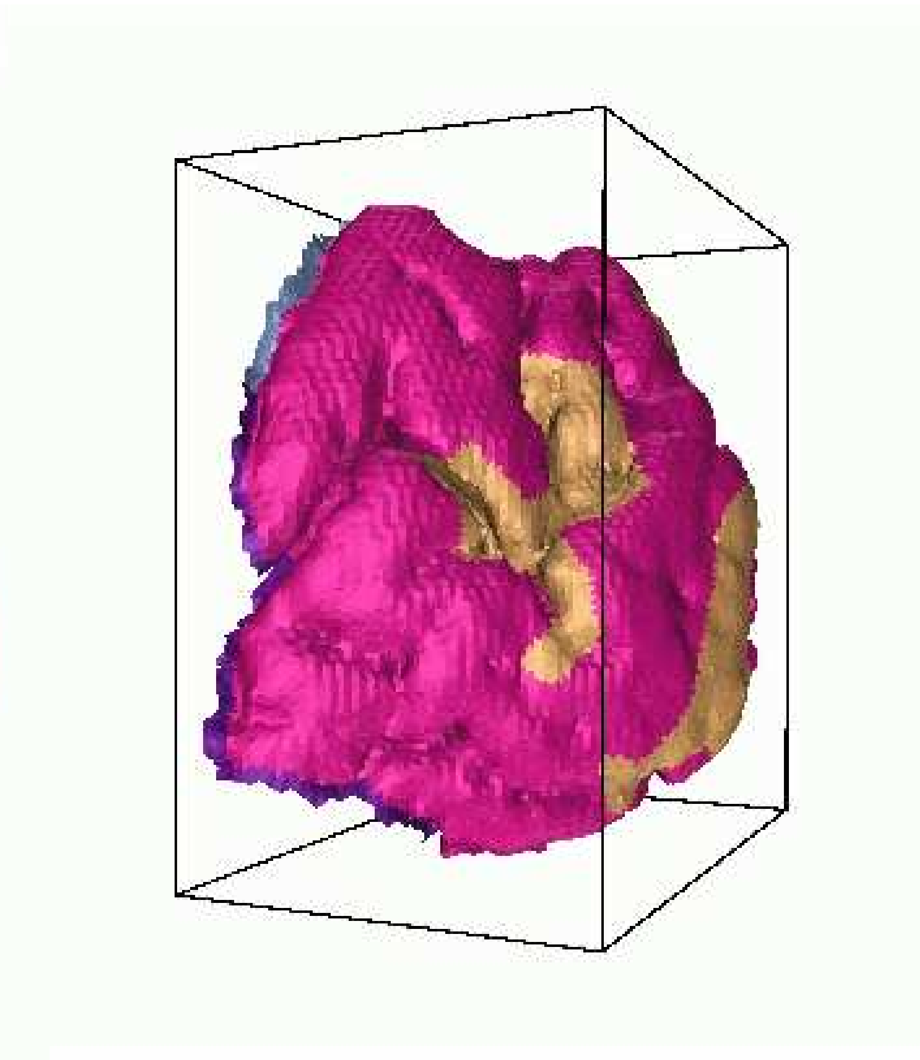
Our starting point: A topologically correct triangulation of the desired surface, typically a topological sphere or disc.

Sample Cerebrum

Left cerebral hemisphere, lateral view, color coded by lobe; the occipital lobe (visual cortex) has been isolated and is marked by (simulated) functional activity.





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Target: A **flat map** of a surface or partial surface $\mathcal{S}$ is a 1-to-1 continuous function $f : \mathcal{S} \longrightarrow \mathbb{G}$ to one of the standard three geometries $\mathbb{G}$: the plane $\mathbb{C}$, the unit sphere $\mathbb{P}$, or the unit disc $\mathbb{D}$ (as the "hyperbolic" plane).

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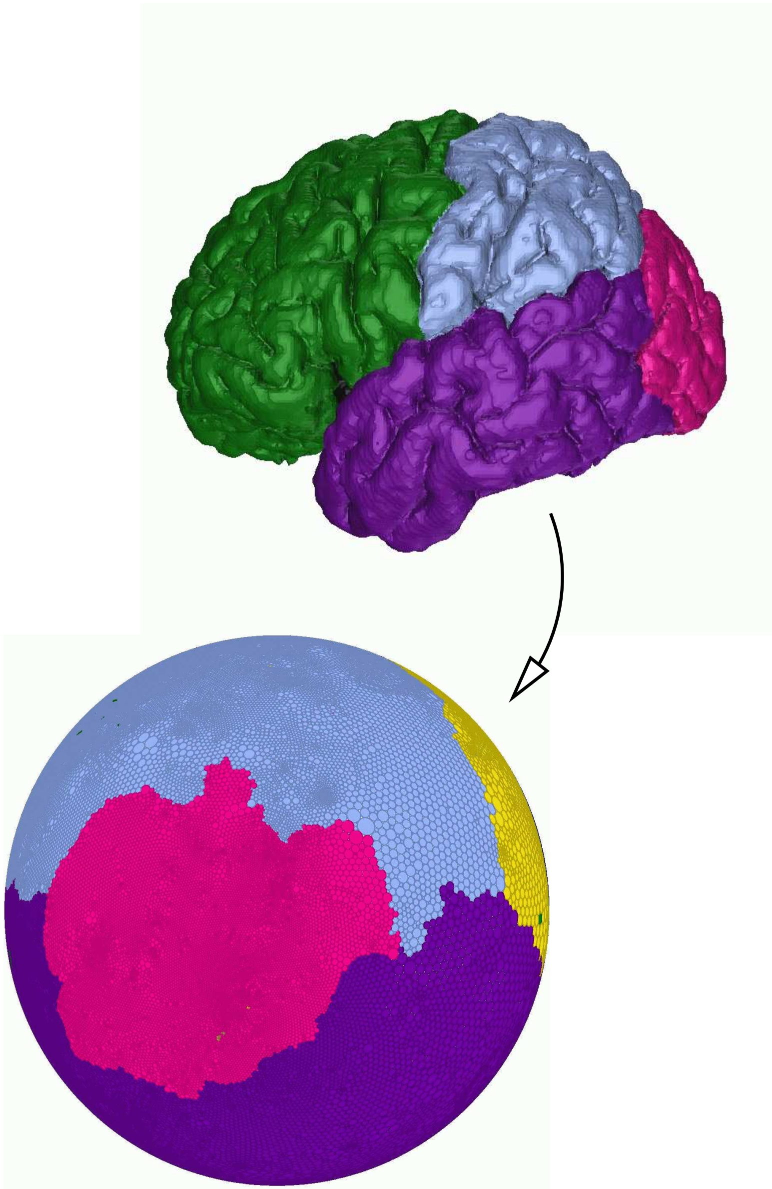
Two important distinctions for this talk:

- full surfaces $\leftrightarrow$ partial surfaces
- visualization $\leftrightarrow$ analysis

Flat Maps of a Left Cerebrum



Sphere

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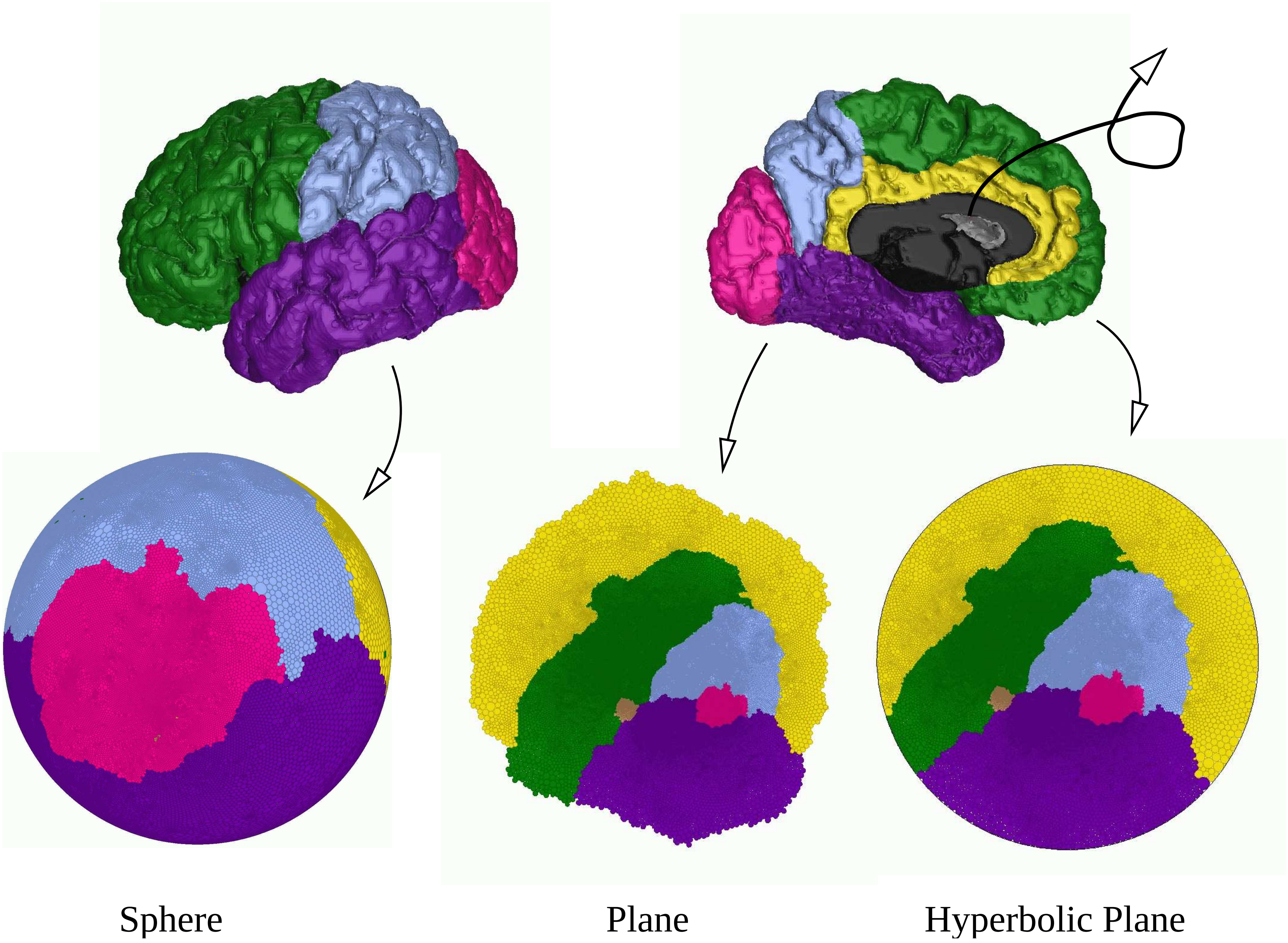

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The spherical and euclidean geometries are familiar, but hyperbolic geometry is new in this scientific setting.

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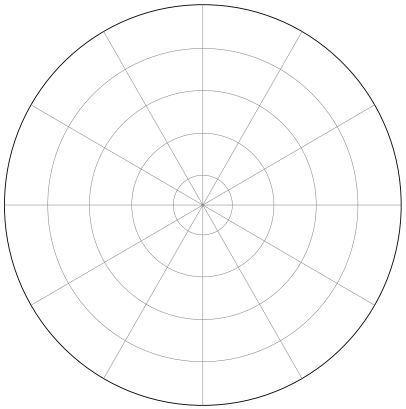

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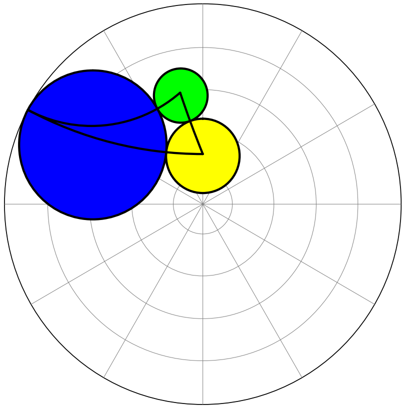

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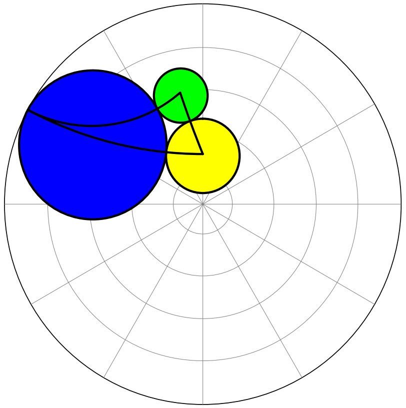

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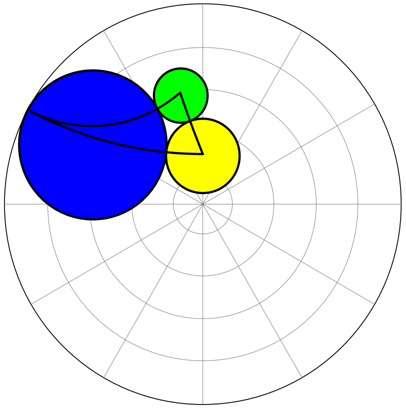

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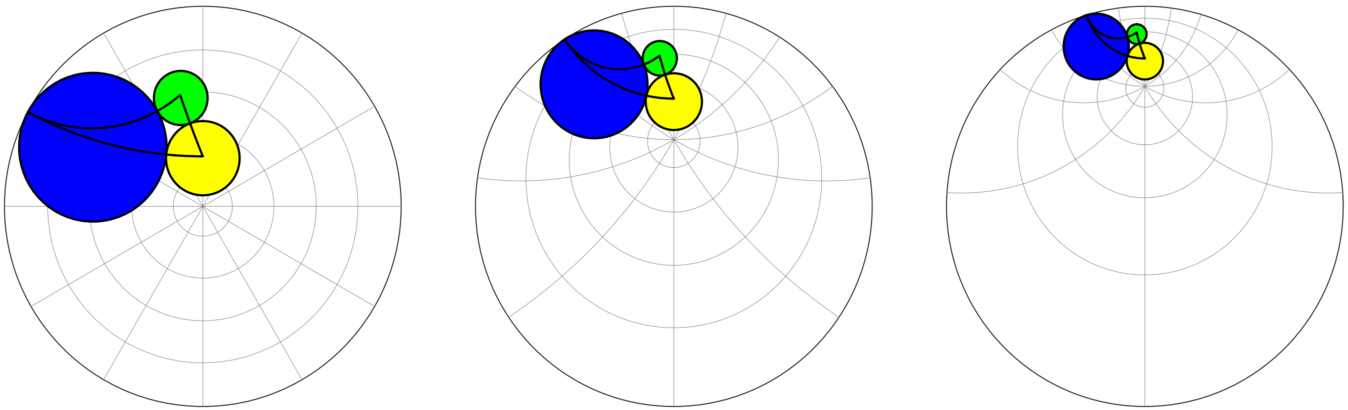

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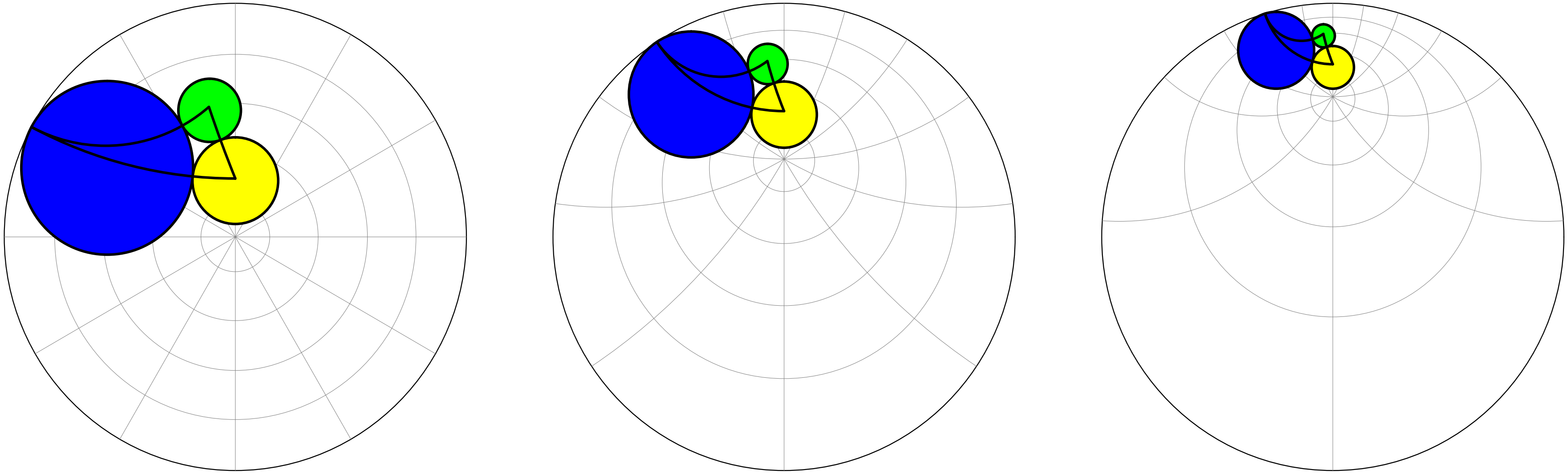
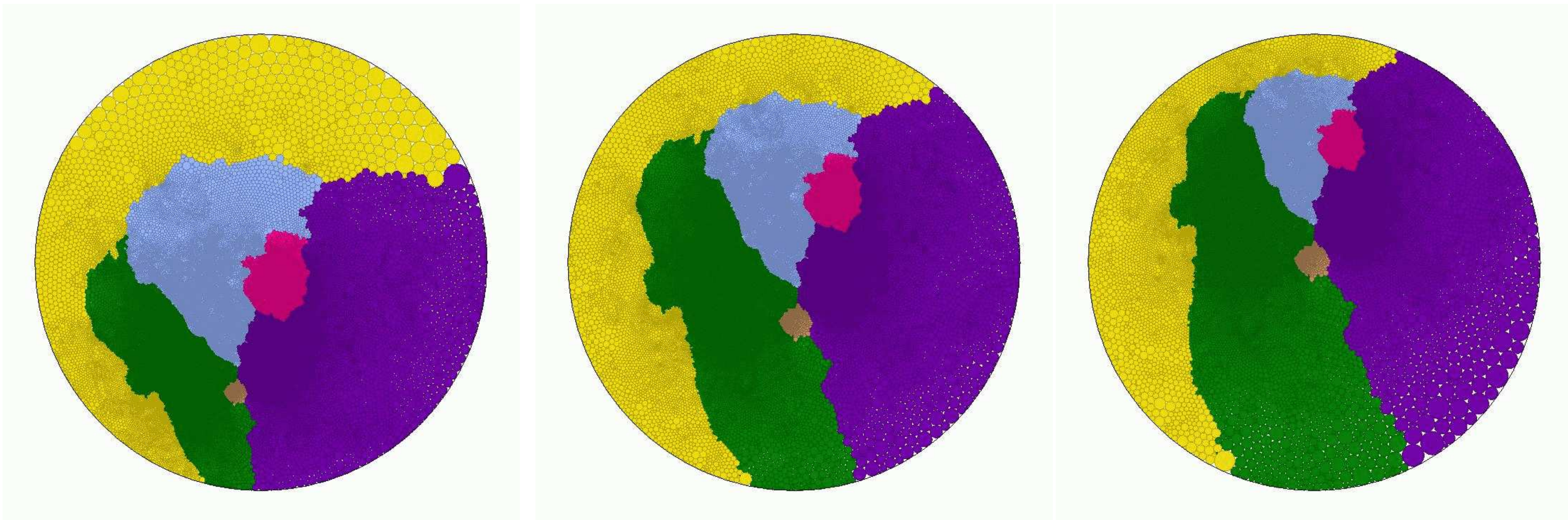
- **Isometries** = Möbius transformations of $\mathbb{D}$ given by $z \mapsto e^{i\theta}\dfrac{(z-\alpha)}{(1-\overline{\alpha}z)}$.

  These preserve circles, geodesics, hyperbolic distance/area





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- In practice, cortical surfaces and triangulations approximating cortical surfaces may be treated as Riemann surfaces.

Riemann Surfaces — Conformal Structures

A **Riemann surface** is one having a consistent way to measure angle. Its "conformal structure" is given by an atlas $\mathcal{A} = \{(U_j, \phi_j)\}$ of charts, that is, continuous 1-to-1 maps $\phi_j : U_j \longrightarrow \mathbb{C}$ from open sets U_j to the plane.

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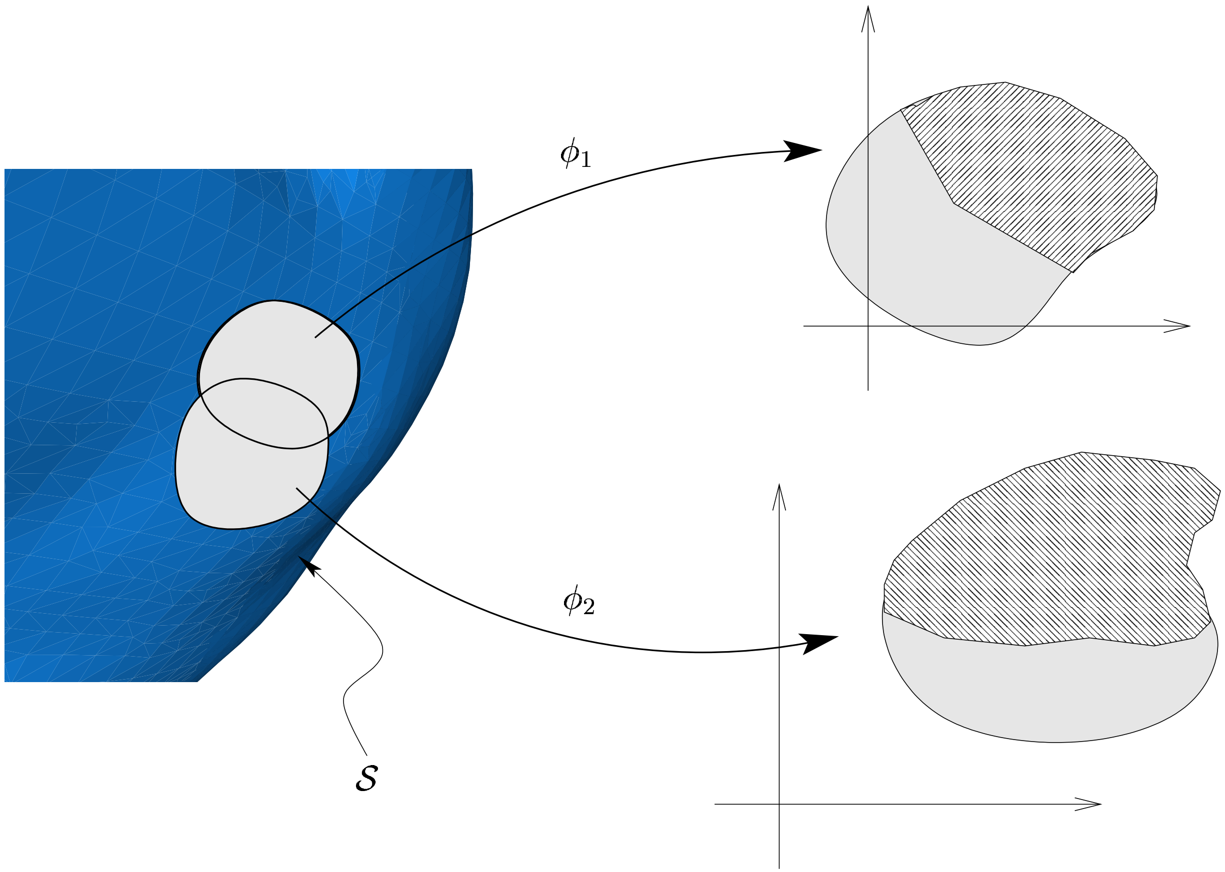

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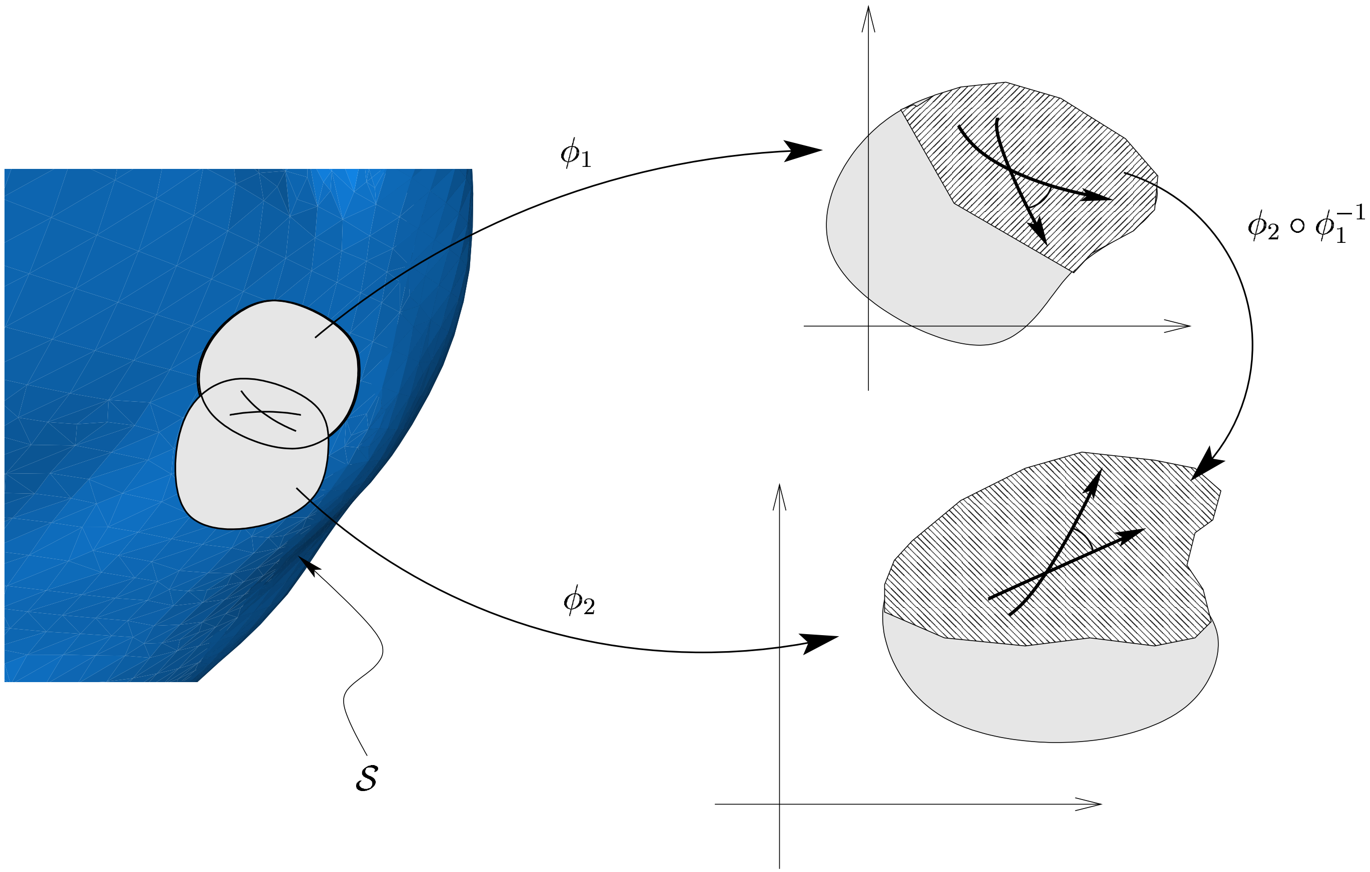

Transition maps $\phi_j \circ \phi_k^{-1}$ in the plane must be analytic, hence **conformal**; that is, they preserve angles (magnitude and orientation) at which curves intersect.

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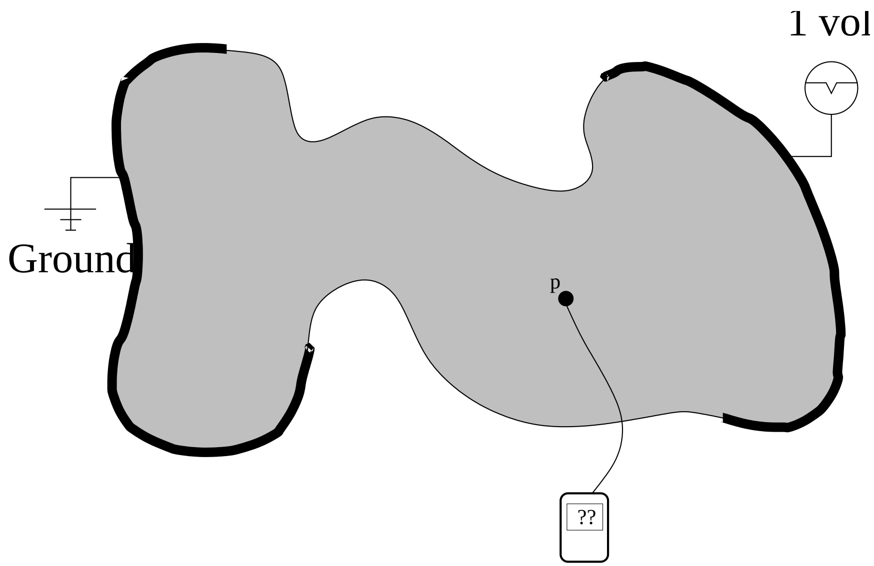
- However, only in the last decade have methods been developed to approximate conformal maps for general **non-planar** surfaces.

Classical Engineering Example



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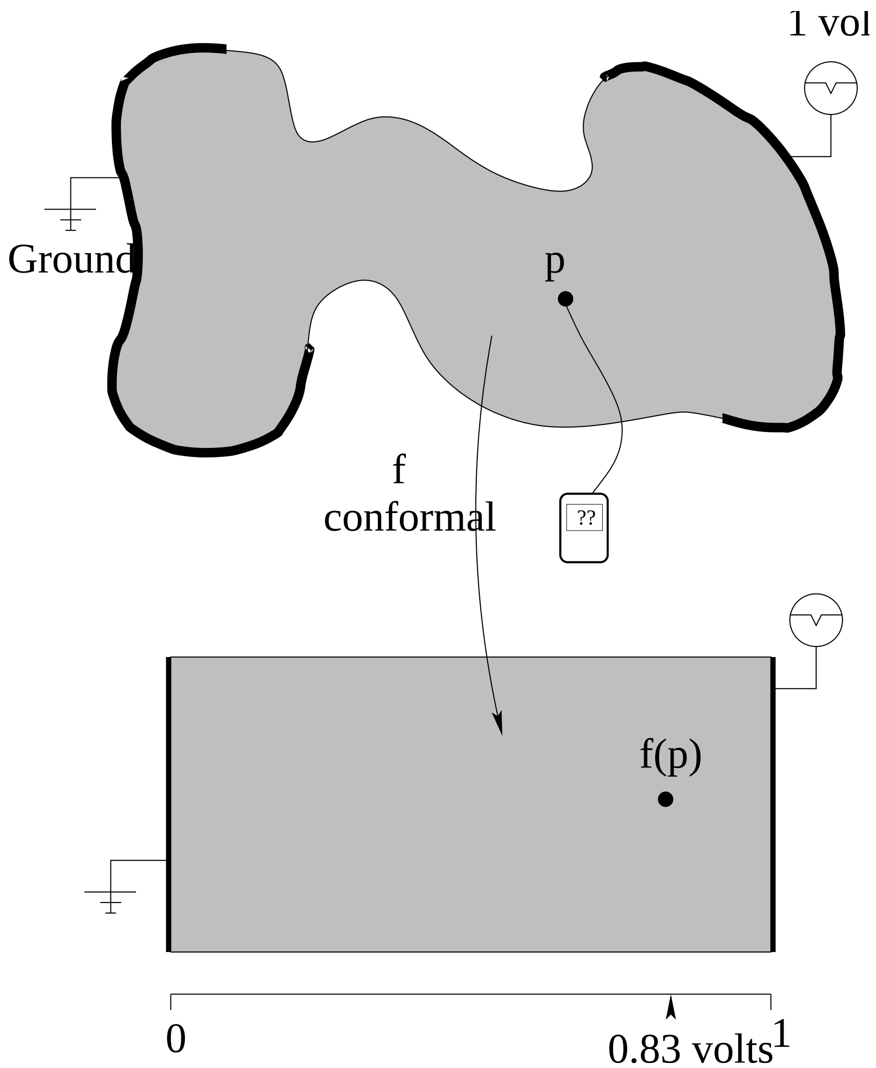

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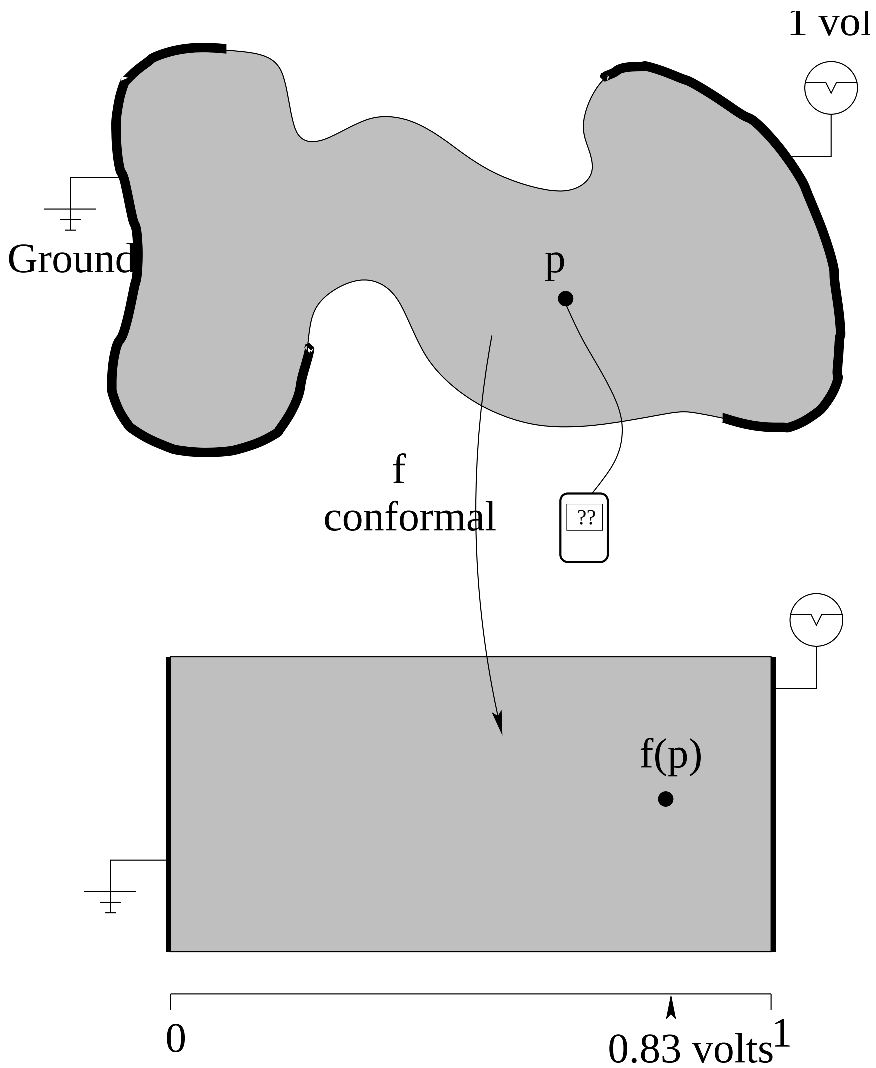

Note: There's no claim that "angle" has some intrinsic **meaning** vis-a-vis brain mapping — it simply has a rich theory to exploit!

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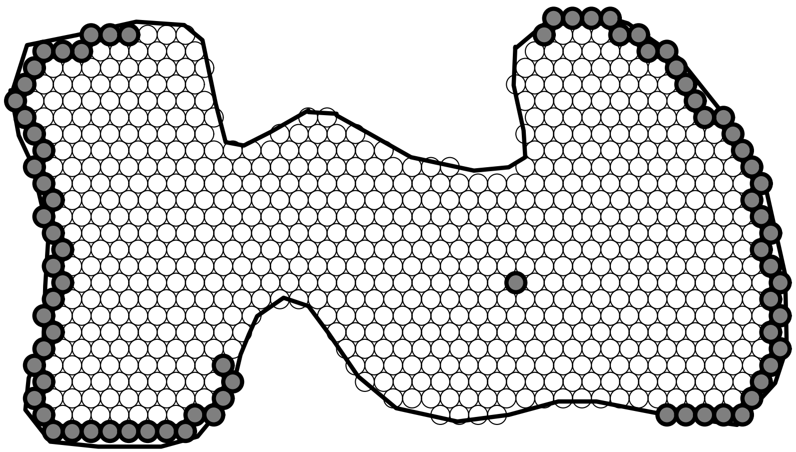

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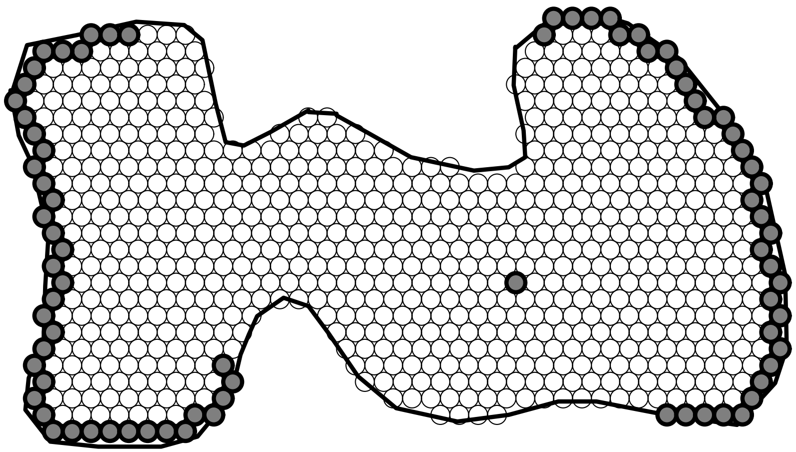

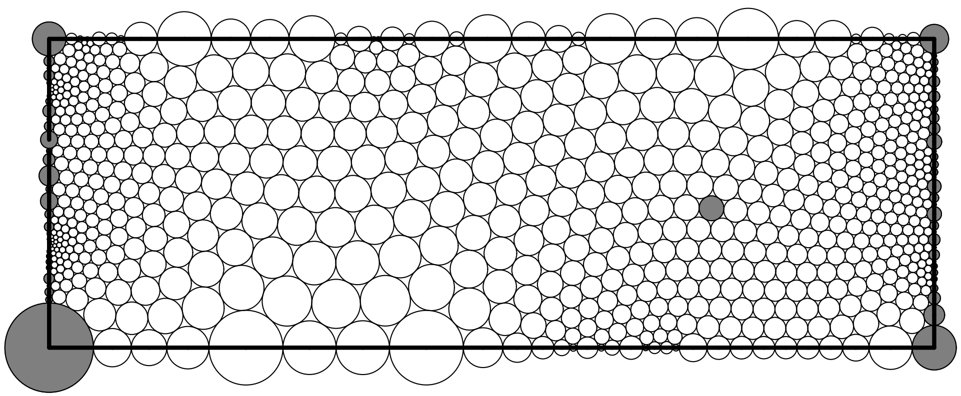

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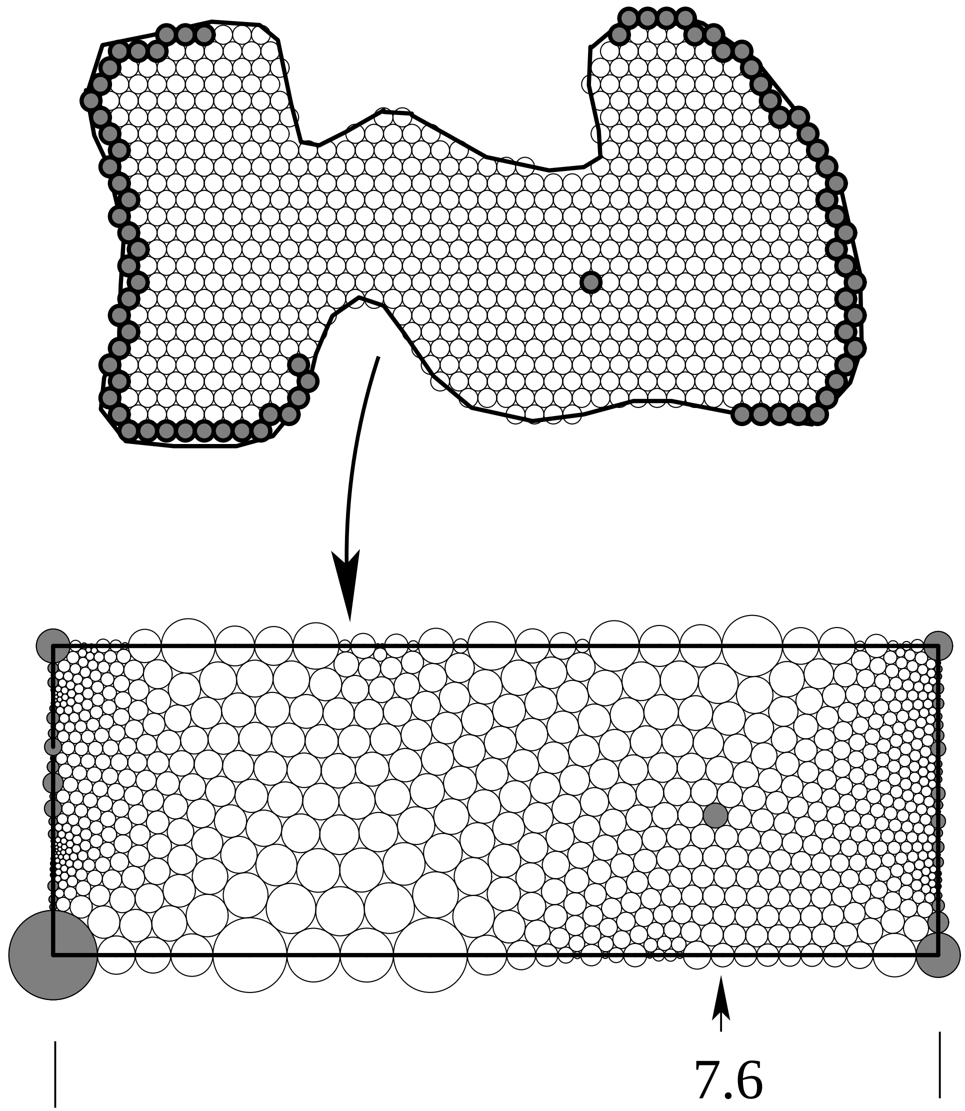

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Koebe-Andreev-Thurston: *Given a triangulation T of a topological sphere, there exists a (univalent) circle packing P_T in the round sphere $\mathbb{P}$ having the pattern prescribed by T. This packing is unique up to inversions and essentially unique (i.e., up to Möbius transformations).*

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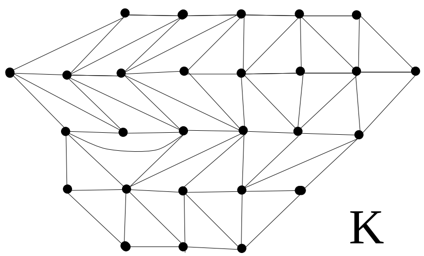
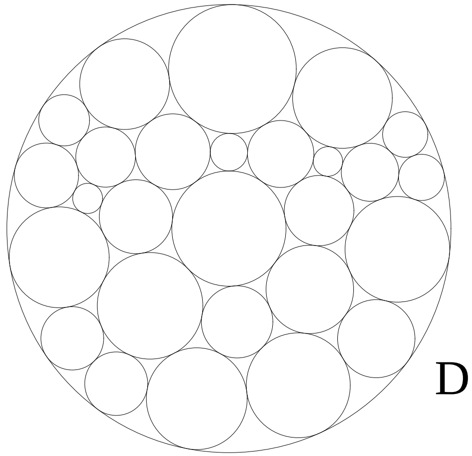
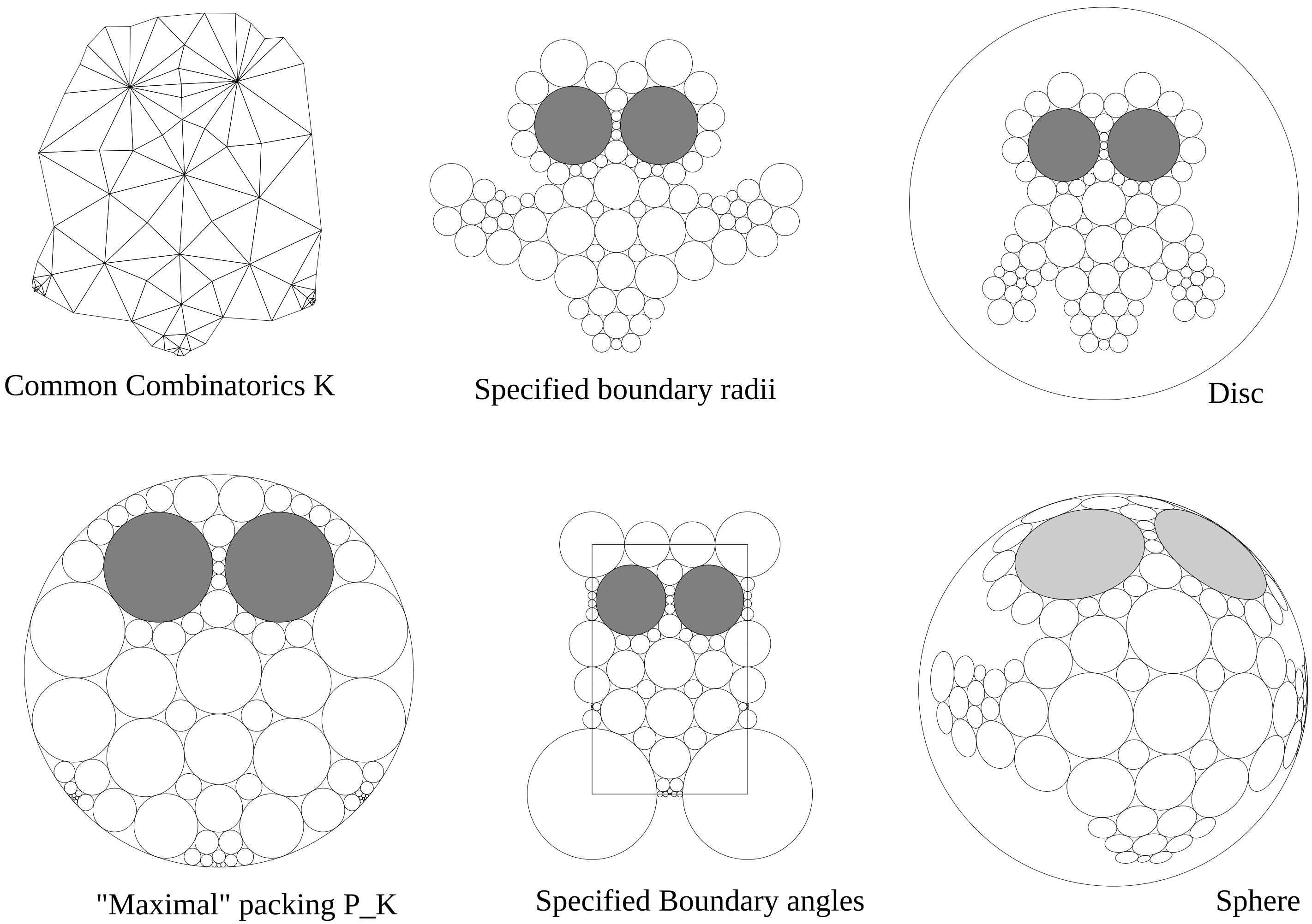




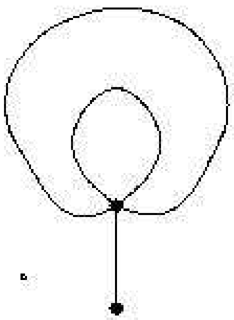
• Each vertex $v \in K$ has a corresponding circle C_v. • if $\langle u, v\rangle$ is an edge of K, then C_u and C_v are tangent. • if $\langle u, v, w\rangle$ is an oriented face of K, then $\langle C_u, C_v, C_w\rangle$ is an oriented triple of circles.

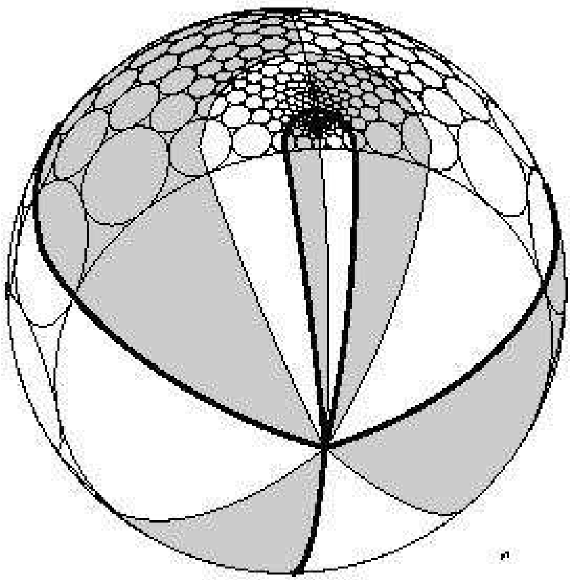
Packing Plasticity

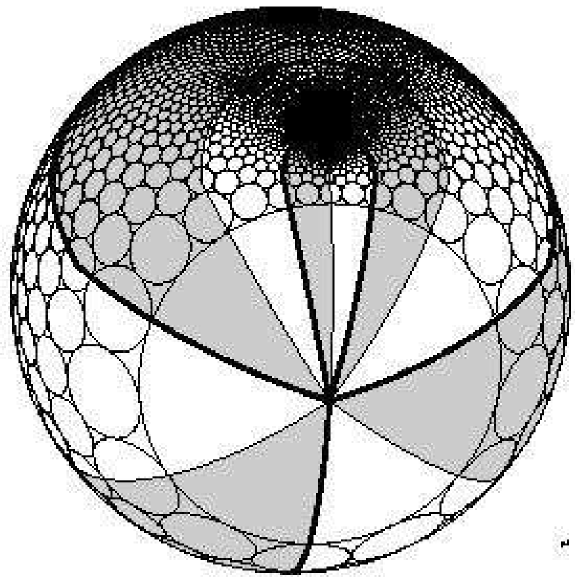
Extensions of the theory give an infinite variety of different circle packings for the same combinatorics K: different boundary radii, boundary angle sums, geometries, etc.



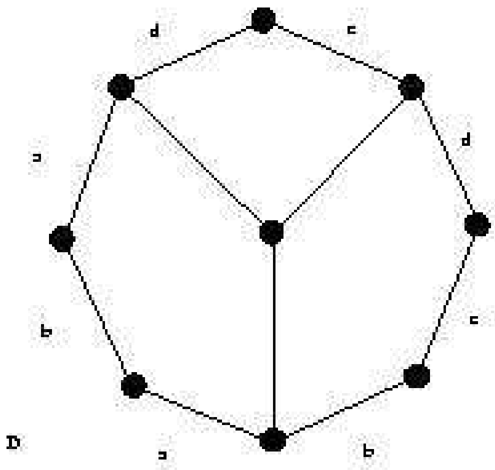
Genus 0 “Dessin”

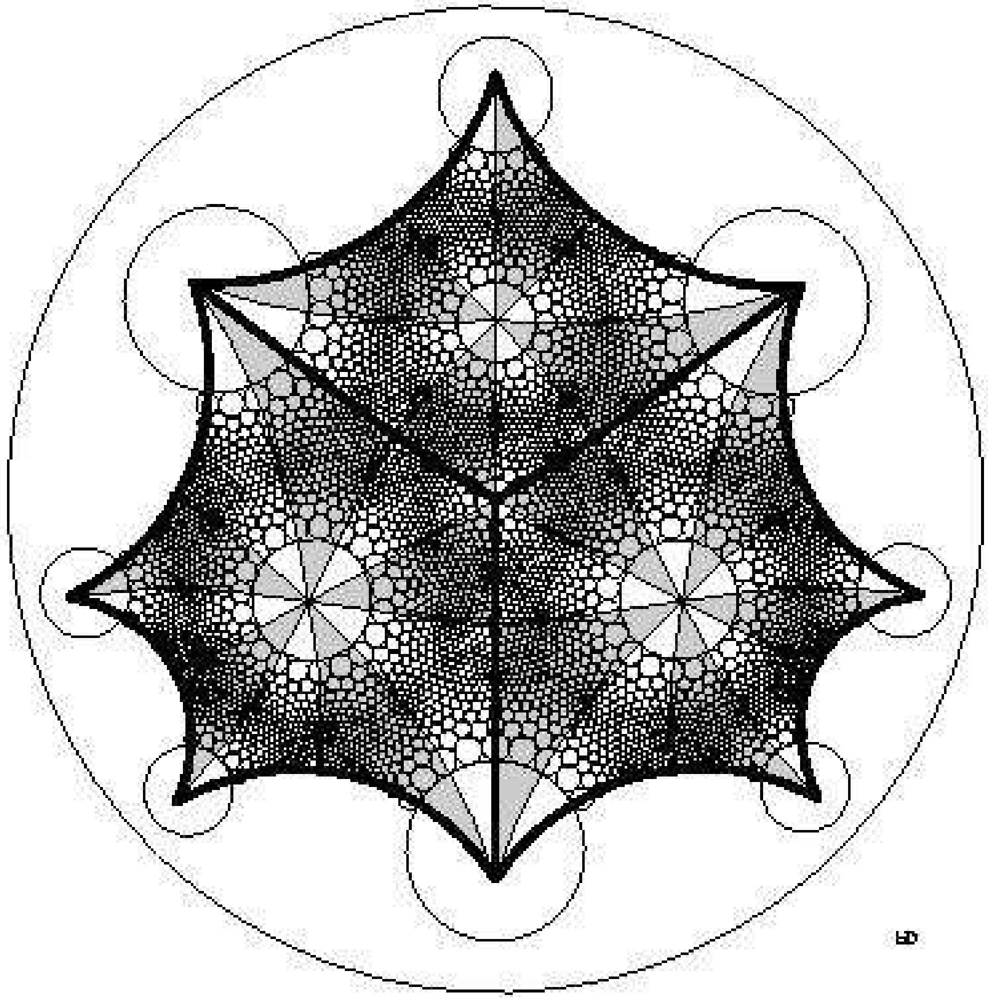




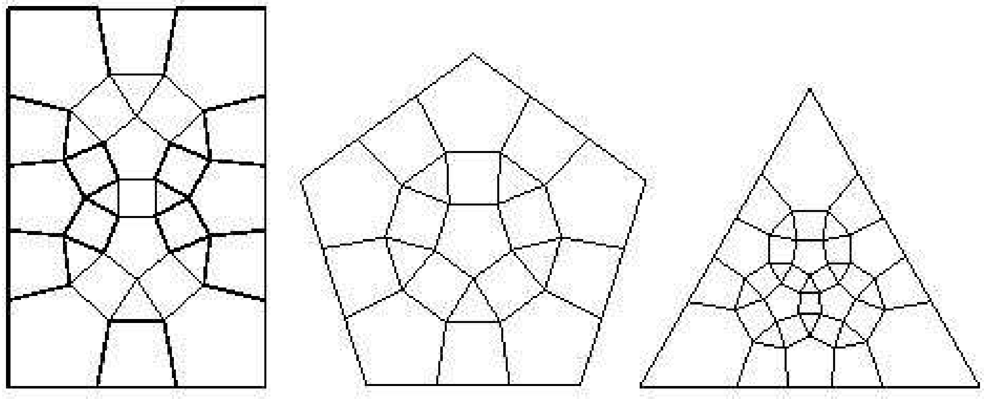


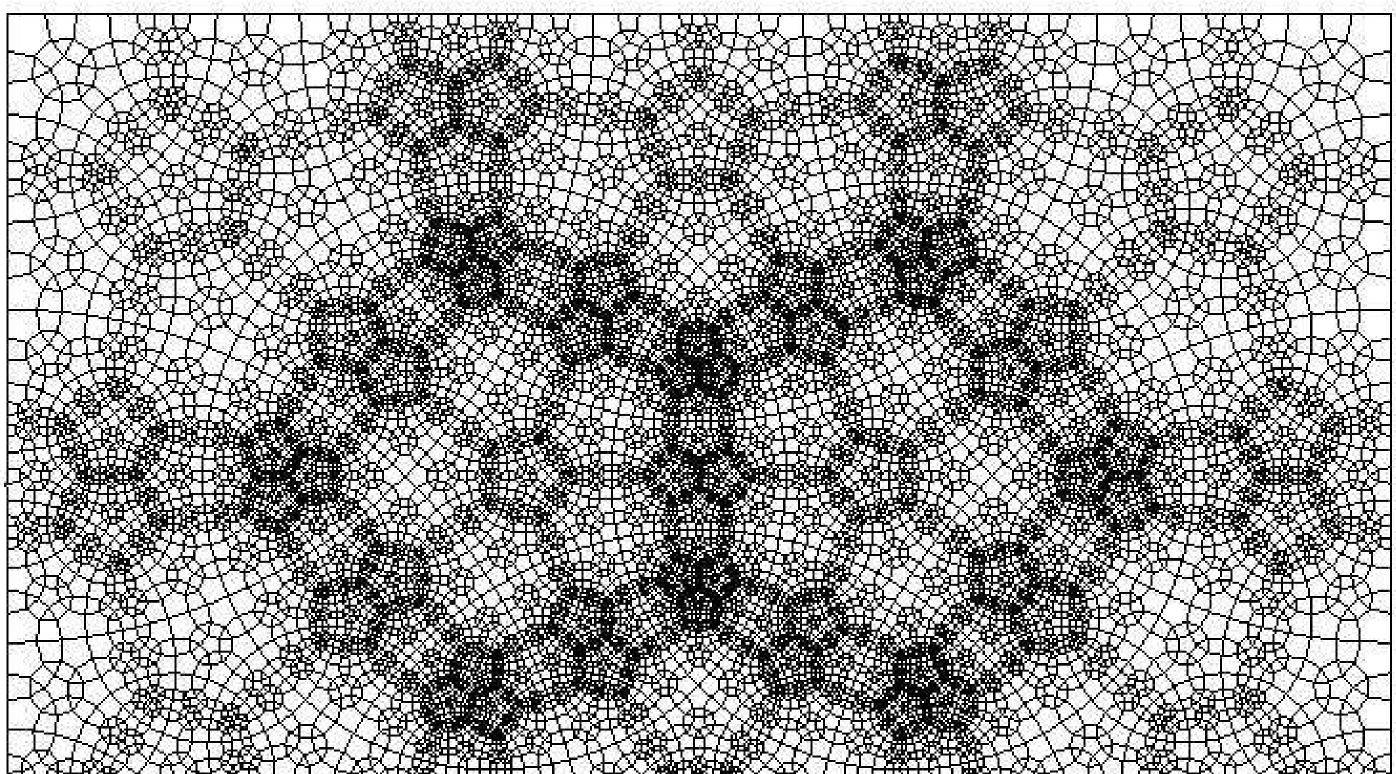
Genus 2 “Dessin”



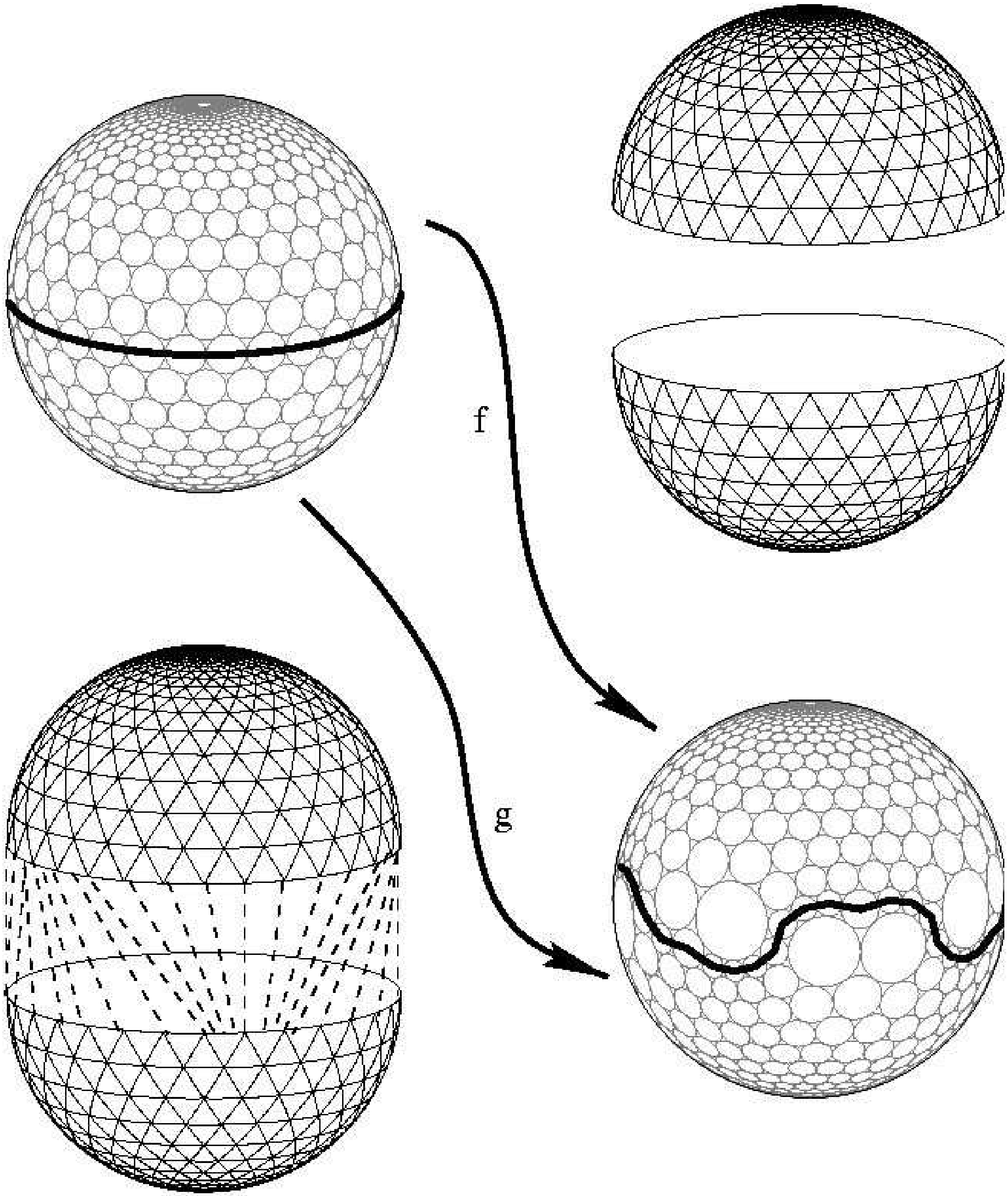


Conformal Tiling

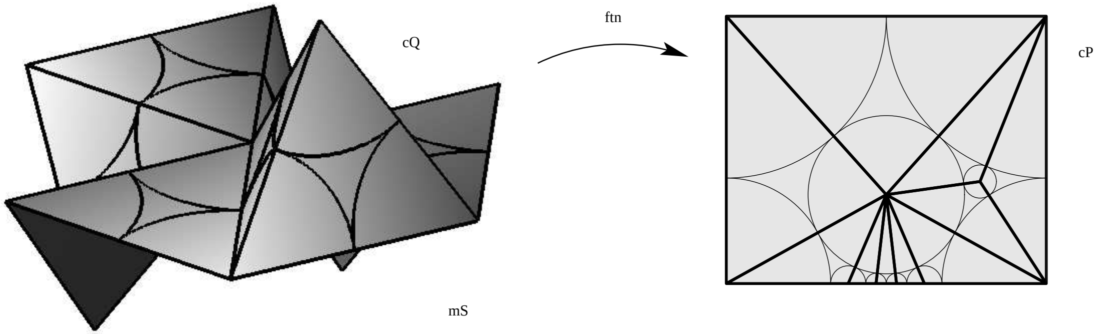


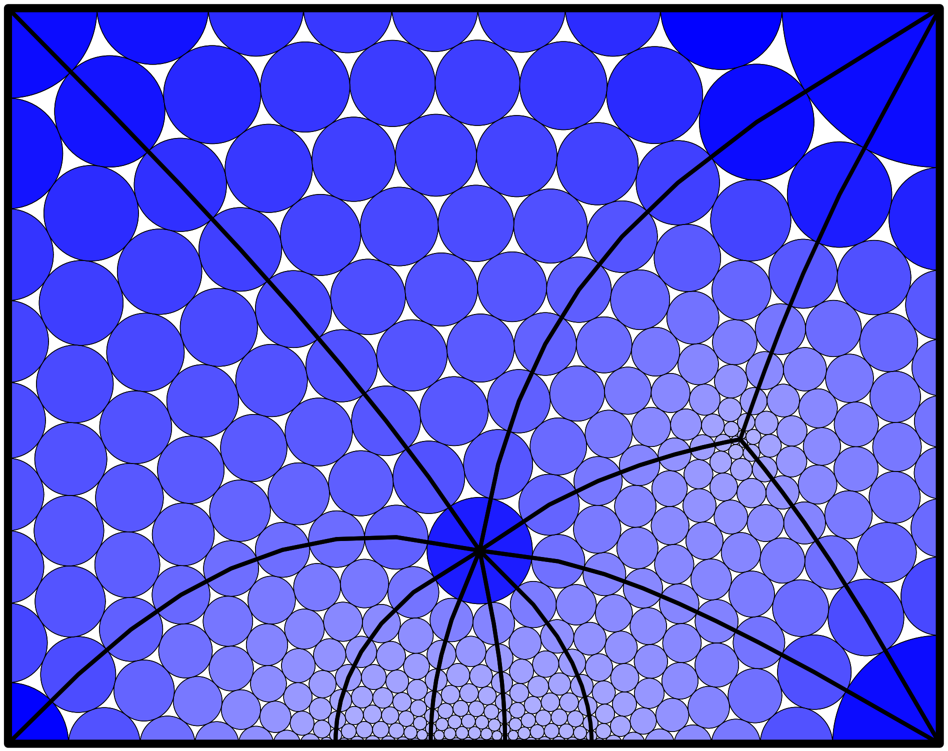


Conformal Welding



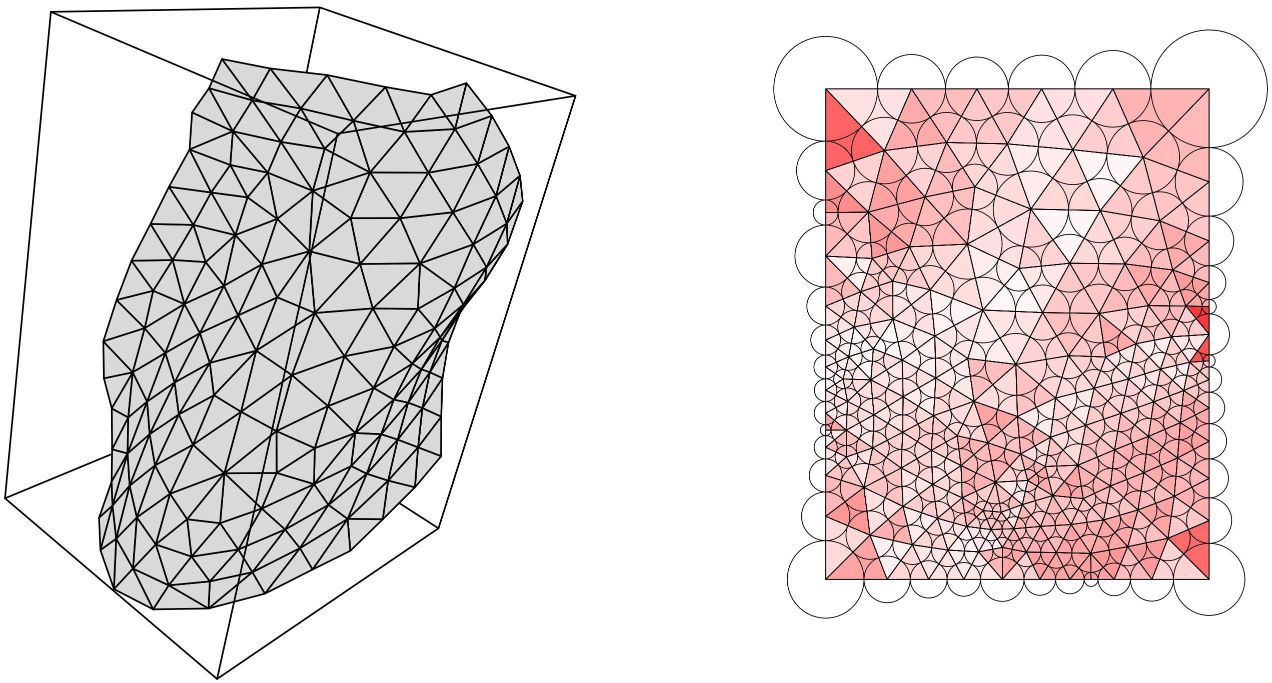
Conformal Flattening





Discrete Conformal Maps (DCM)

Def: *A* **discrete conformal map** *is one which maps a surface with triangulation* K *to the carrier underlying a circle packing* P *which has the combinatorics of* K*.*



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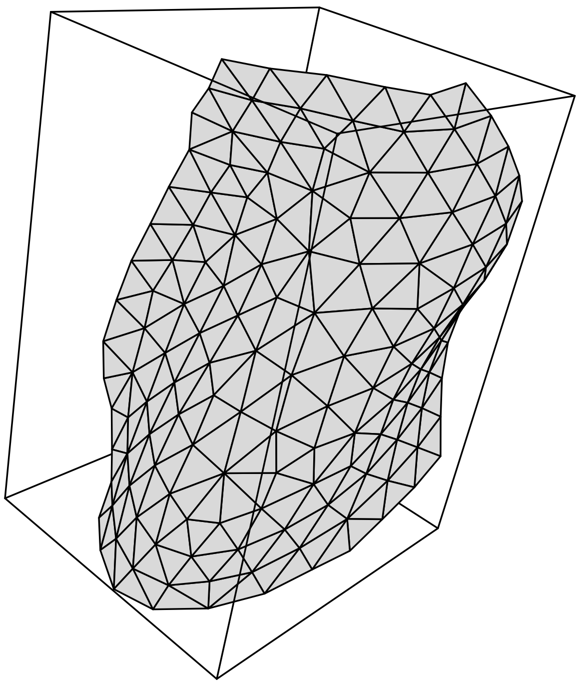

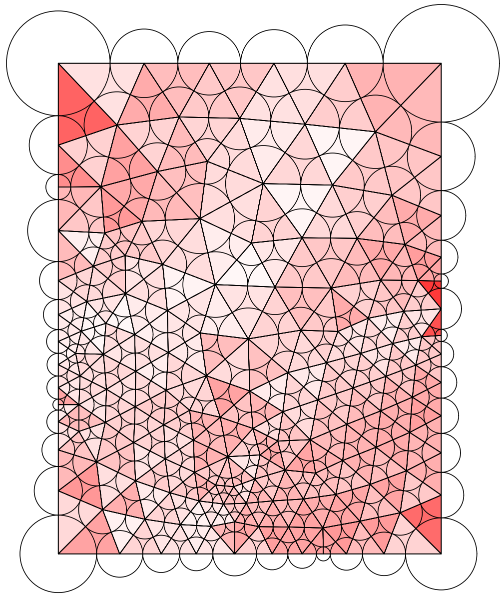

Mathematical issues remain regarding circle packing methods; I don't minimize these, but our interest is in the use of conformal information.

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Numerically computed “conformal” maps never preserve angles!

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NO. There exist non-local, “ensemble” conformal features which are (quasi)preserved under (quasi)conformal maps. These are legitimate targets for computation.

Ensemble Conformal Features

We concentrate first on classical “Extremal Lengths” (EL).

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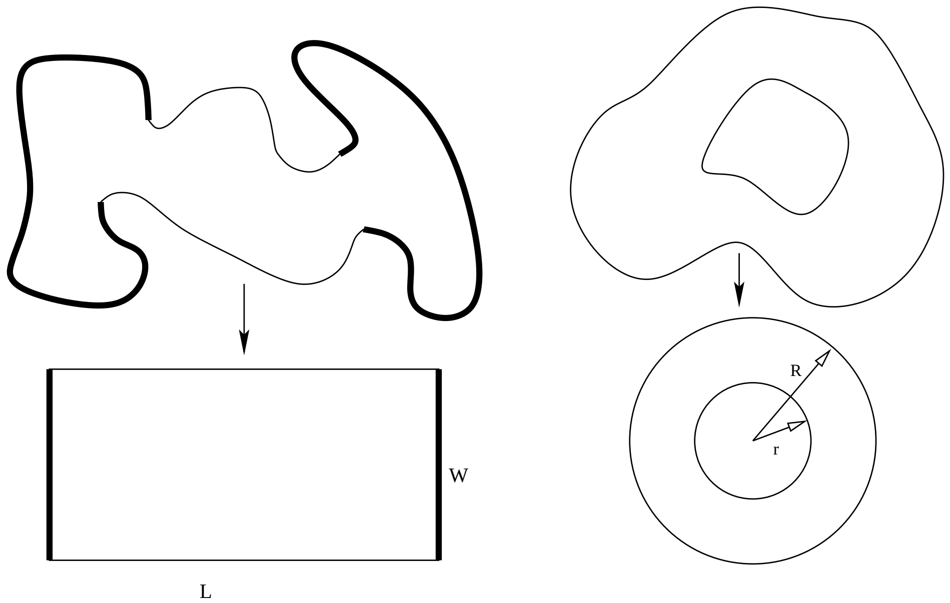

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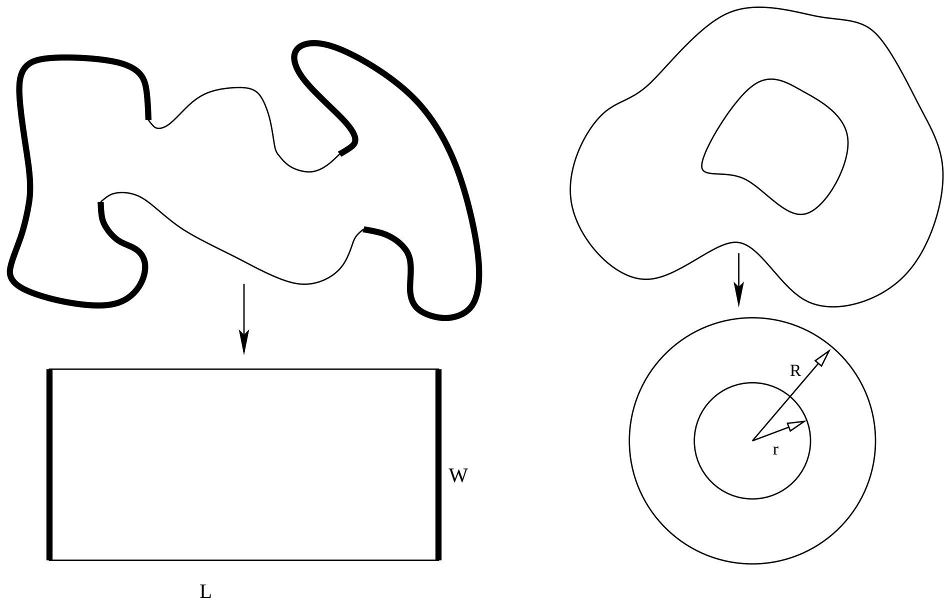

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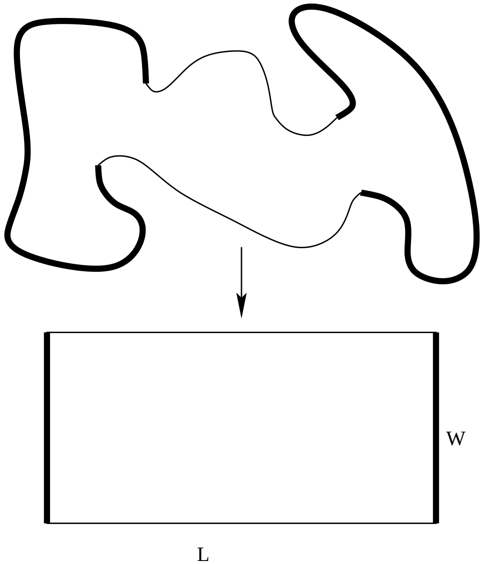

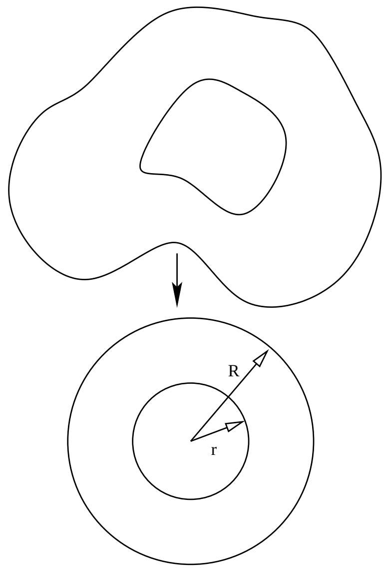

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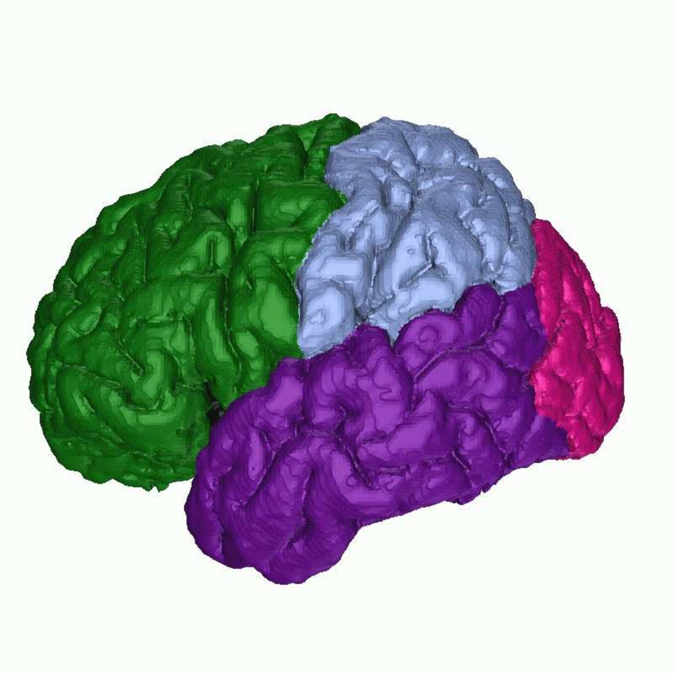
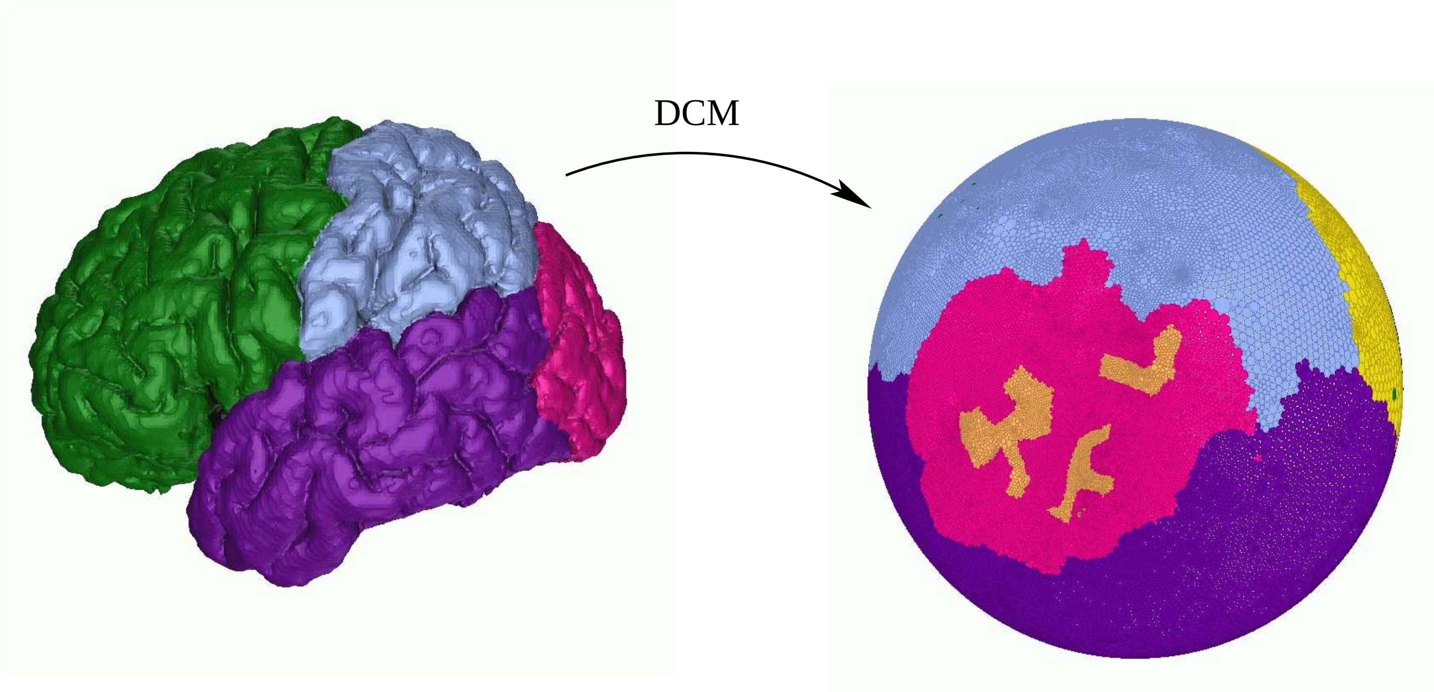
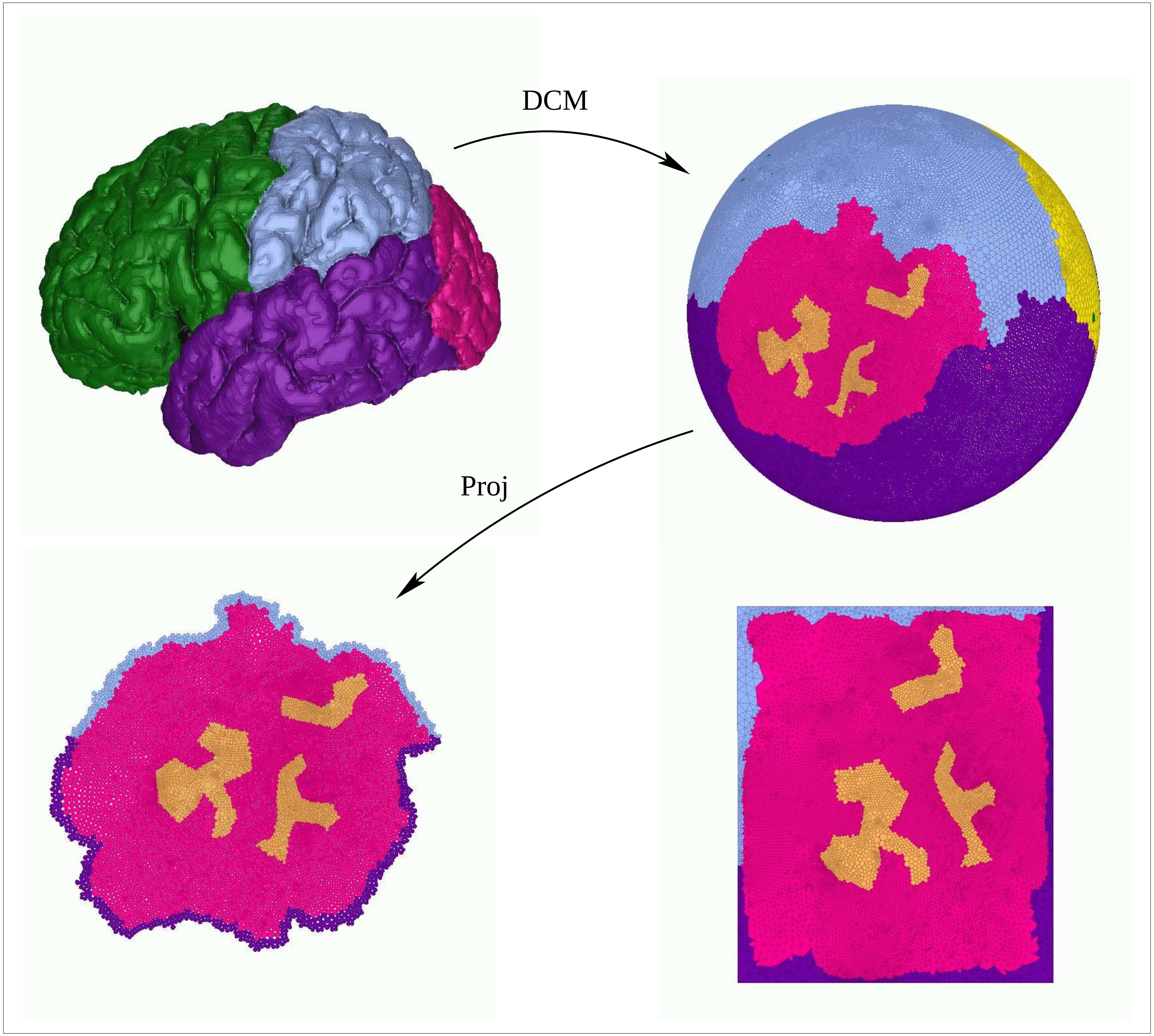
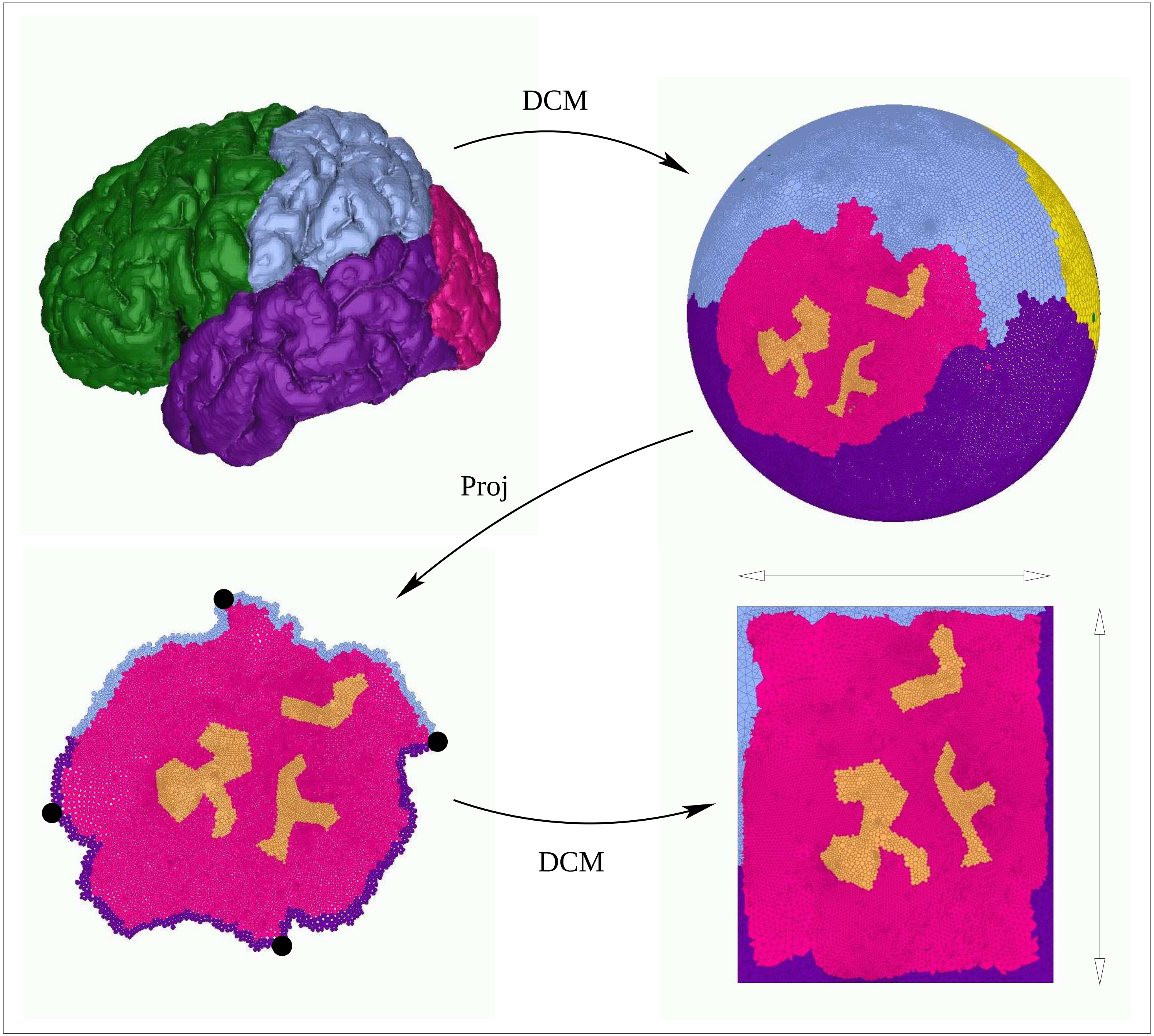
• EL’s don’t depend on lengths or areas or how the region is embedded — they reflect **conformal** information intrinsic to the surface.

Sample Mapping Experiments

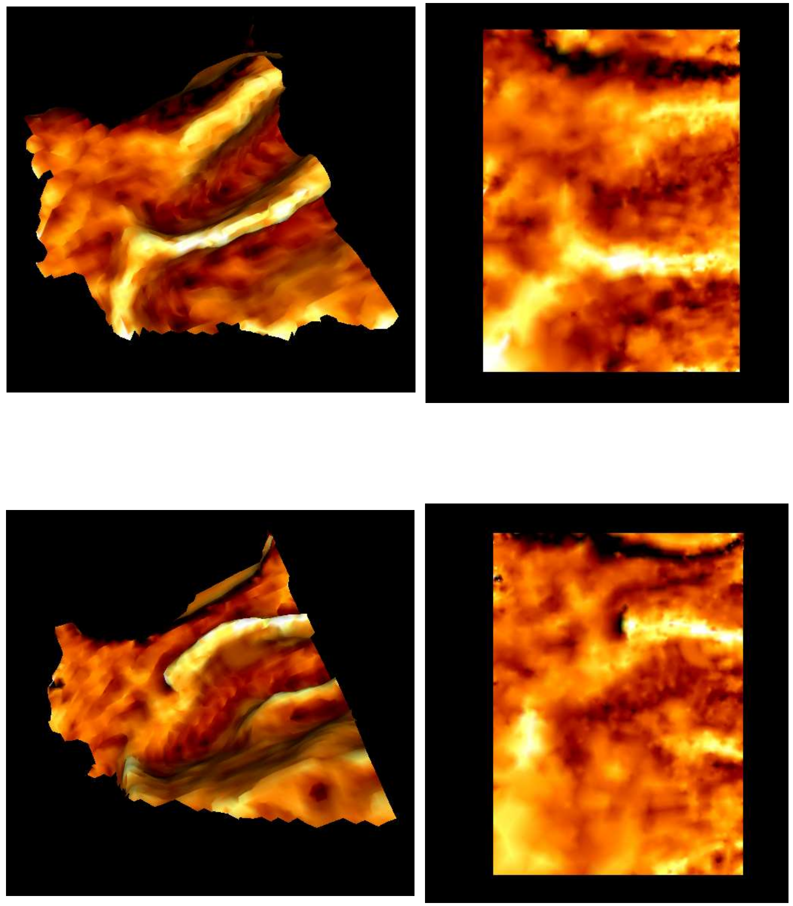



DCM


DCM
Proj

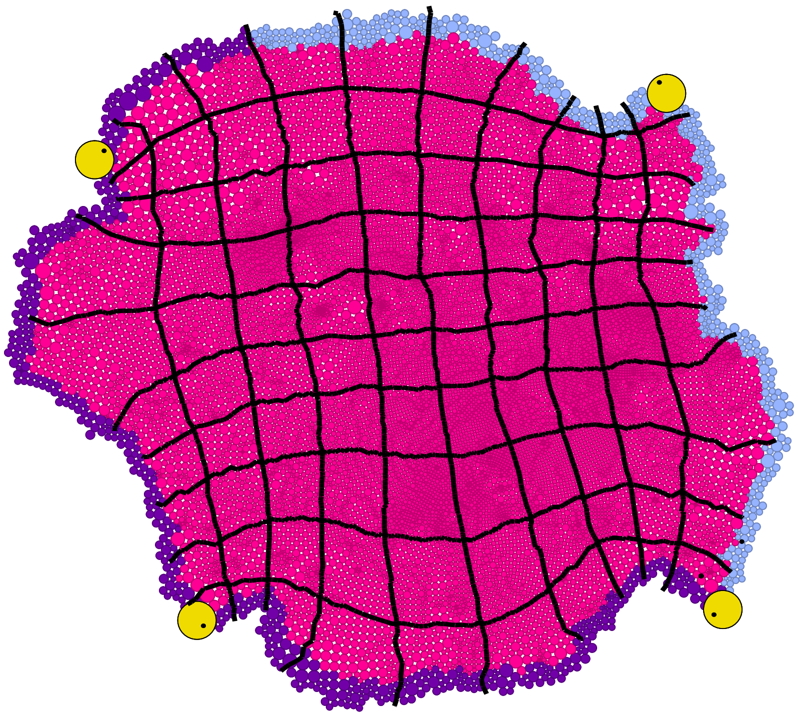

DCM
Proj
DCM

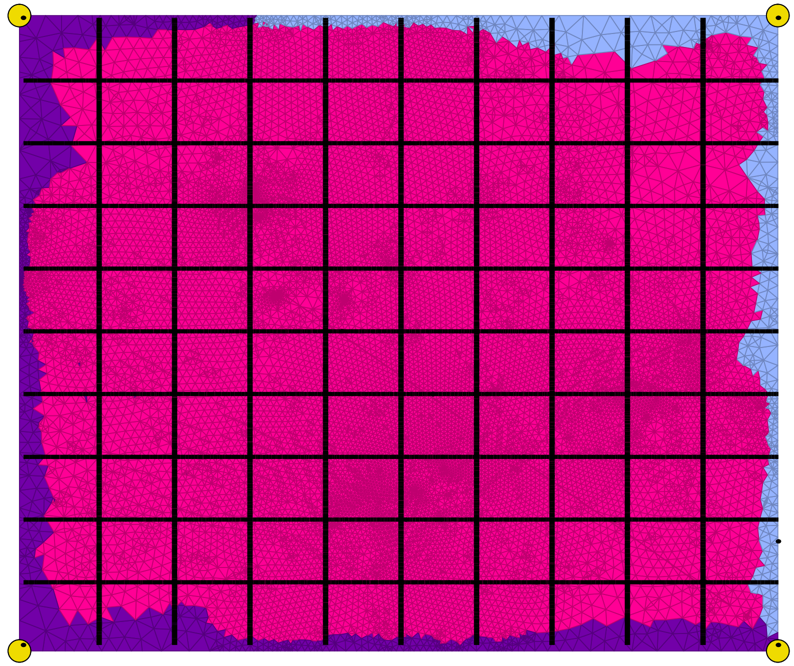
Twin Studies

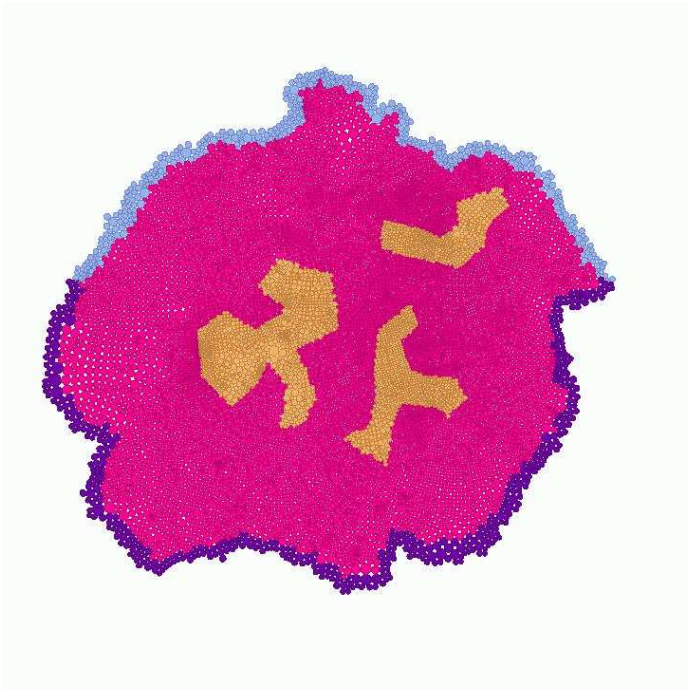


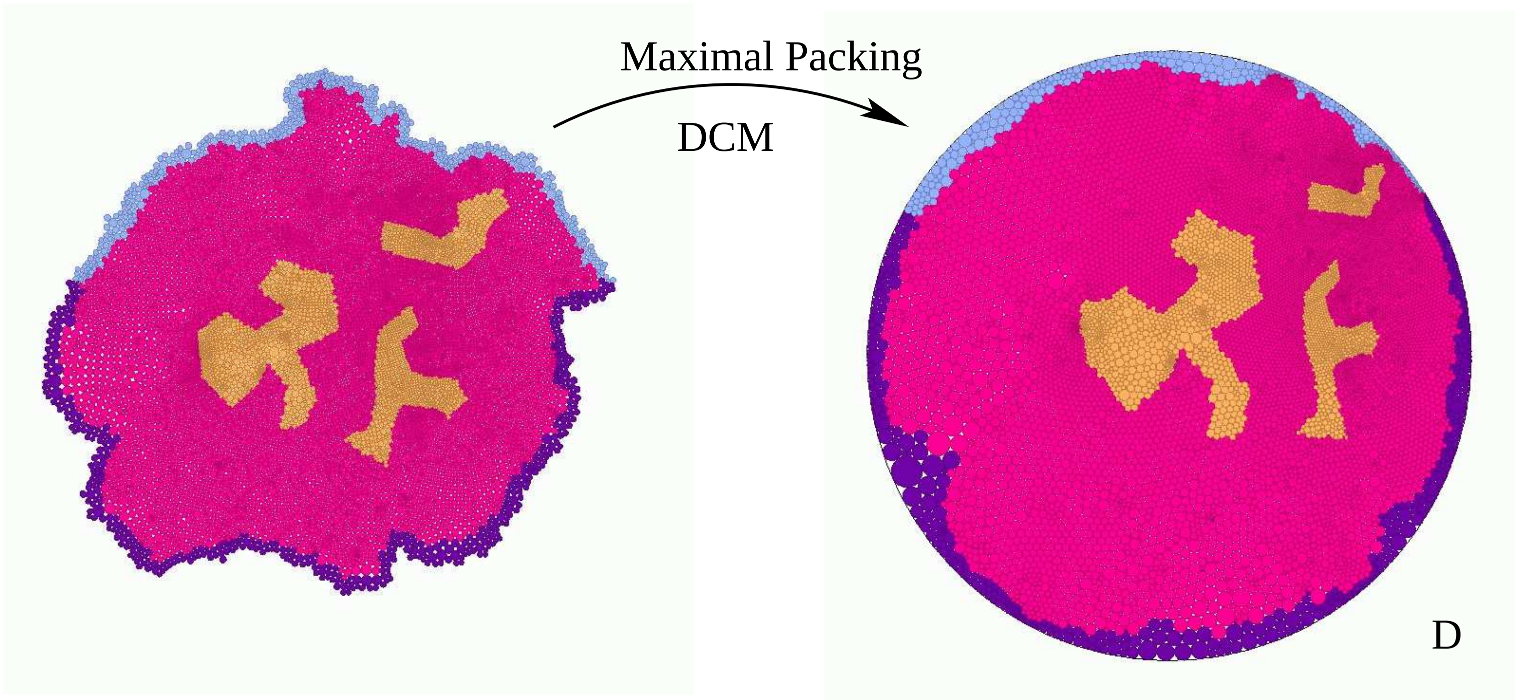
(Preliminary work by Monica Hurdal and Kelly Botteron with Michael Miller's lab at Johns-Hopkins.)

Imposing Grids

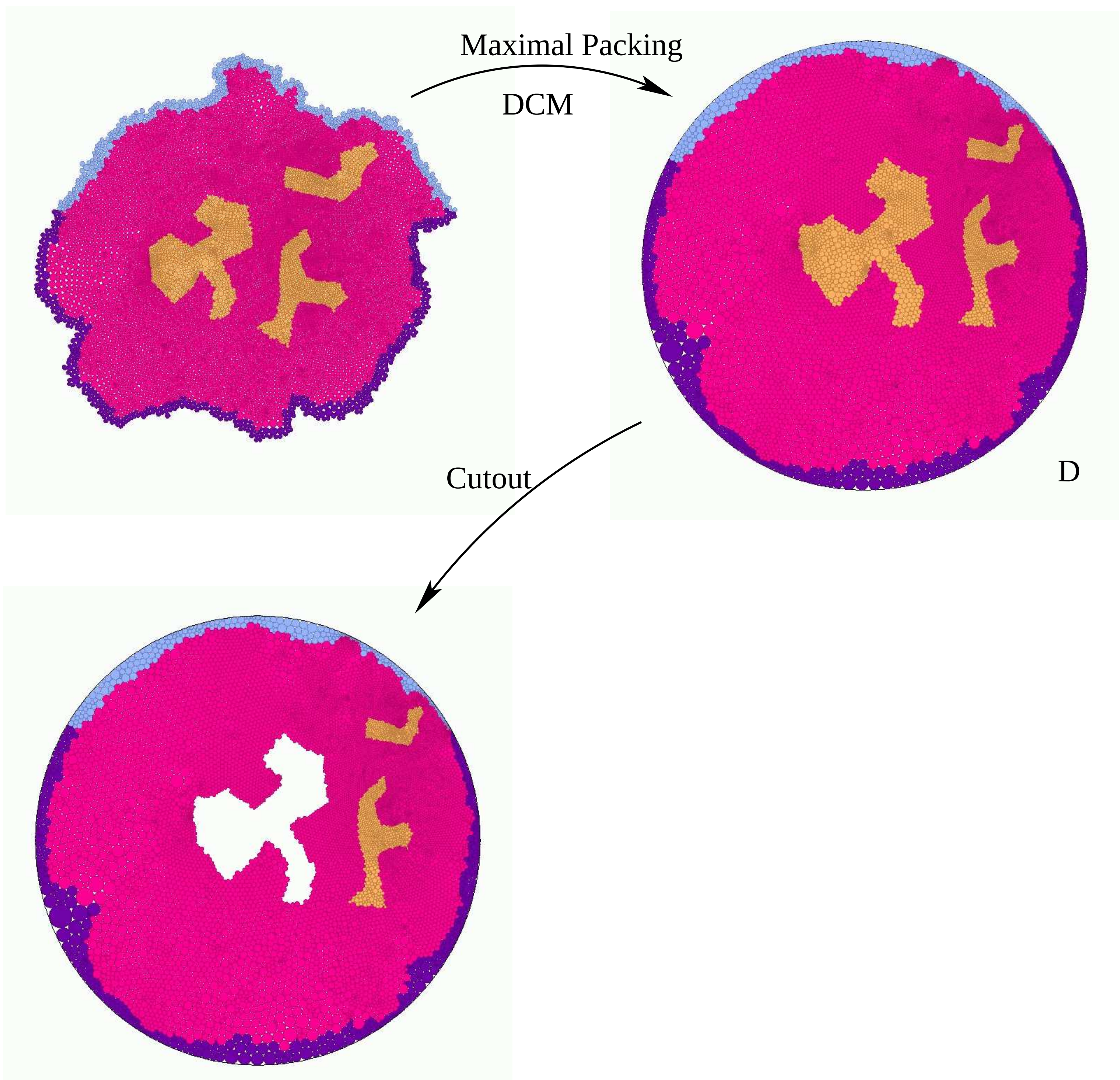


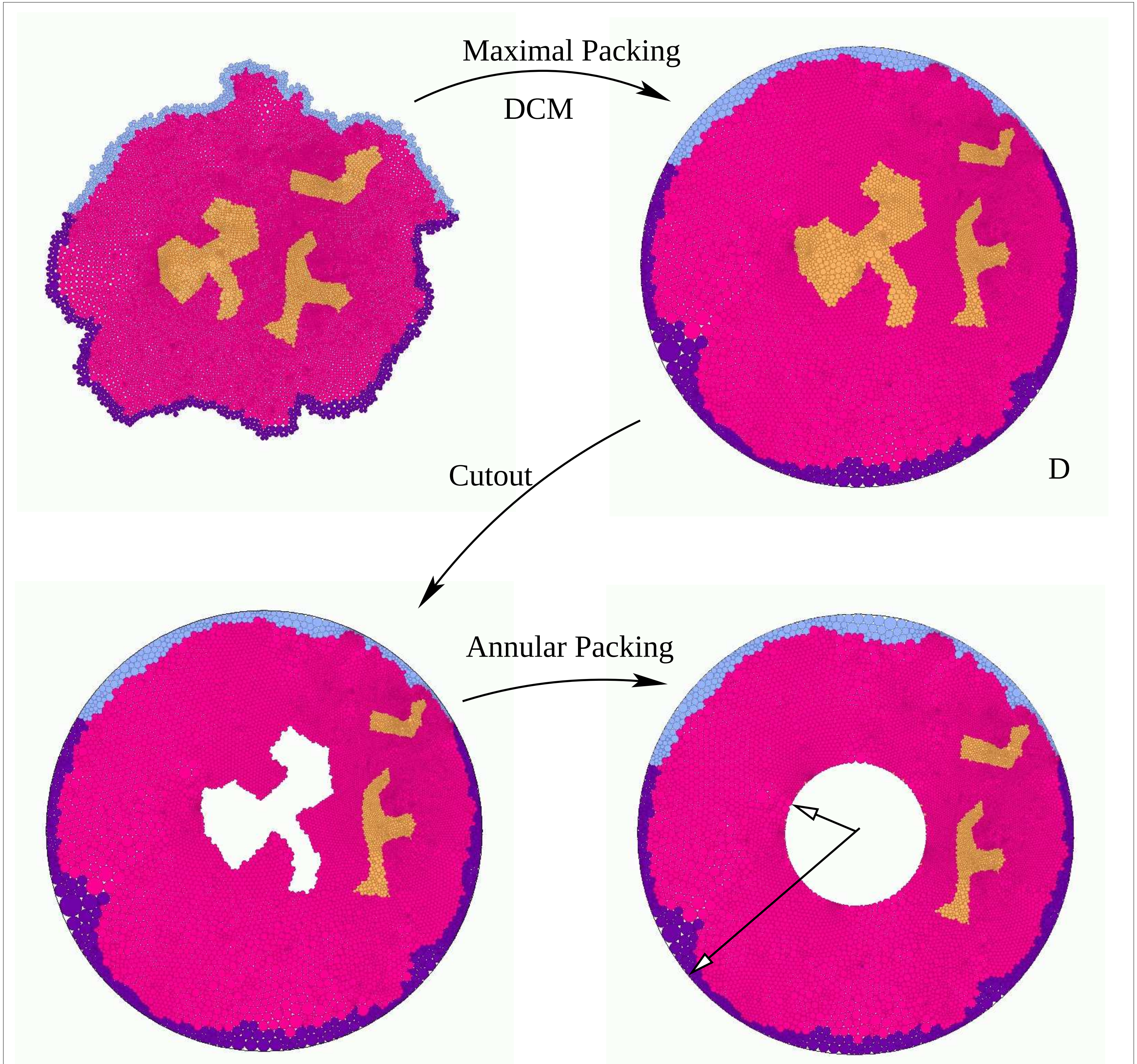




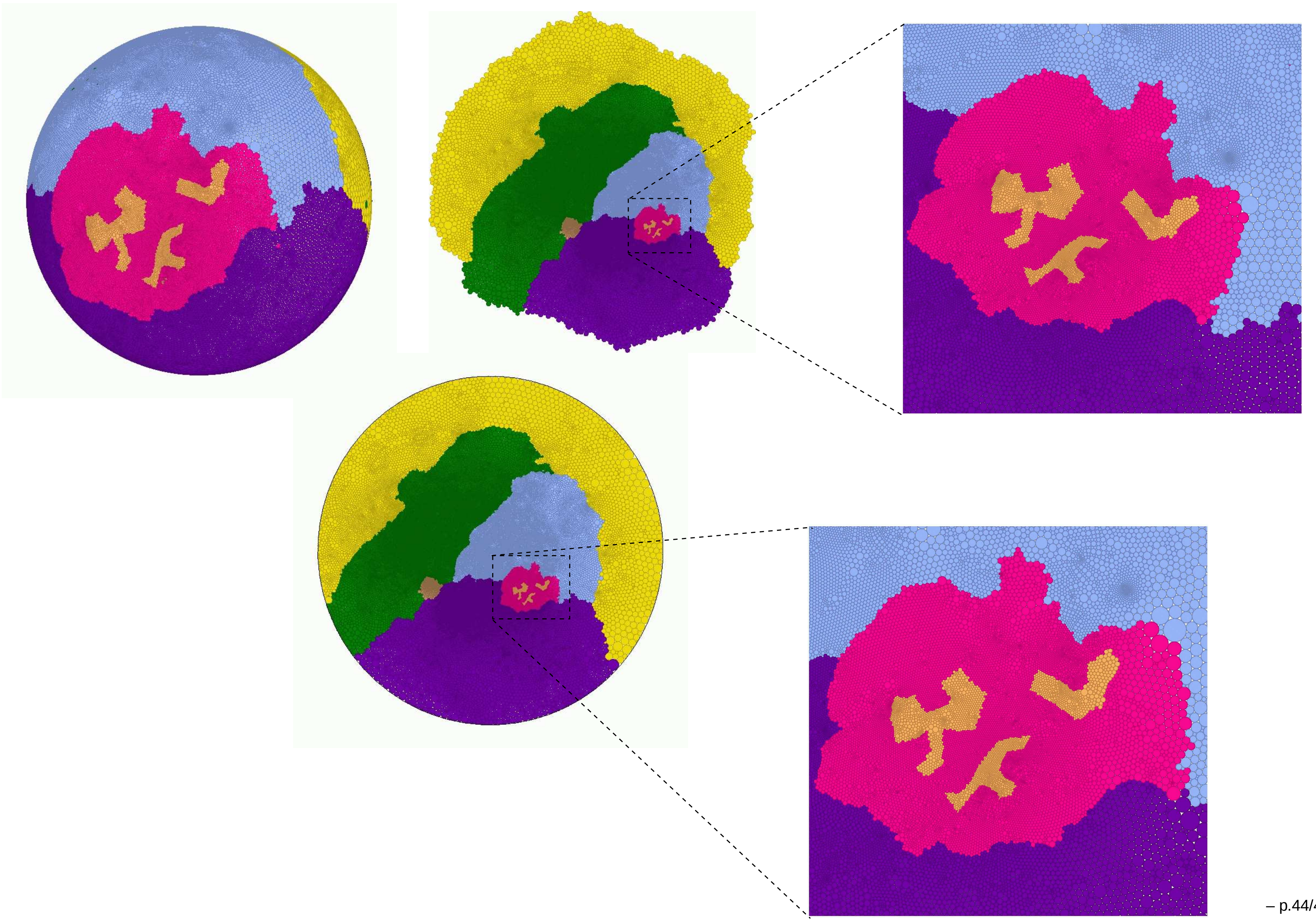


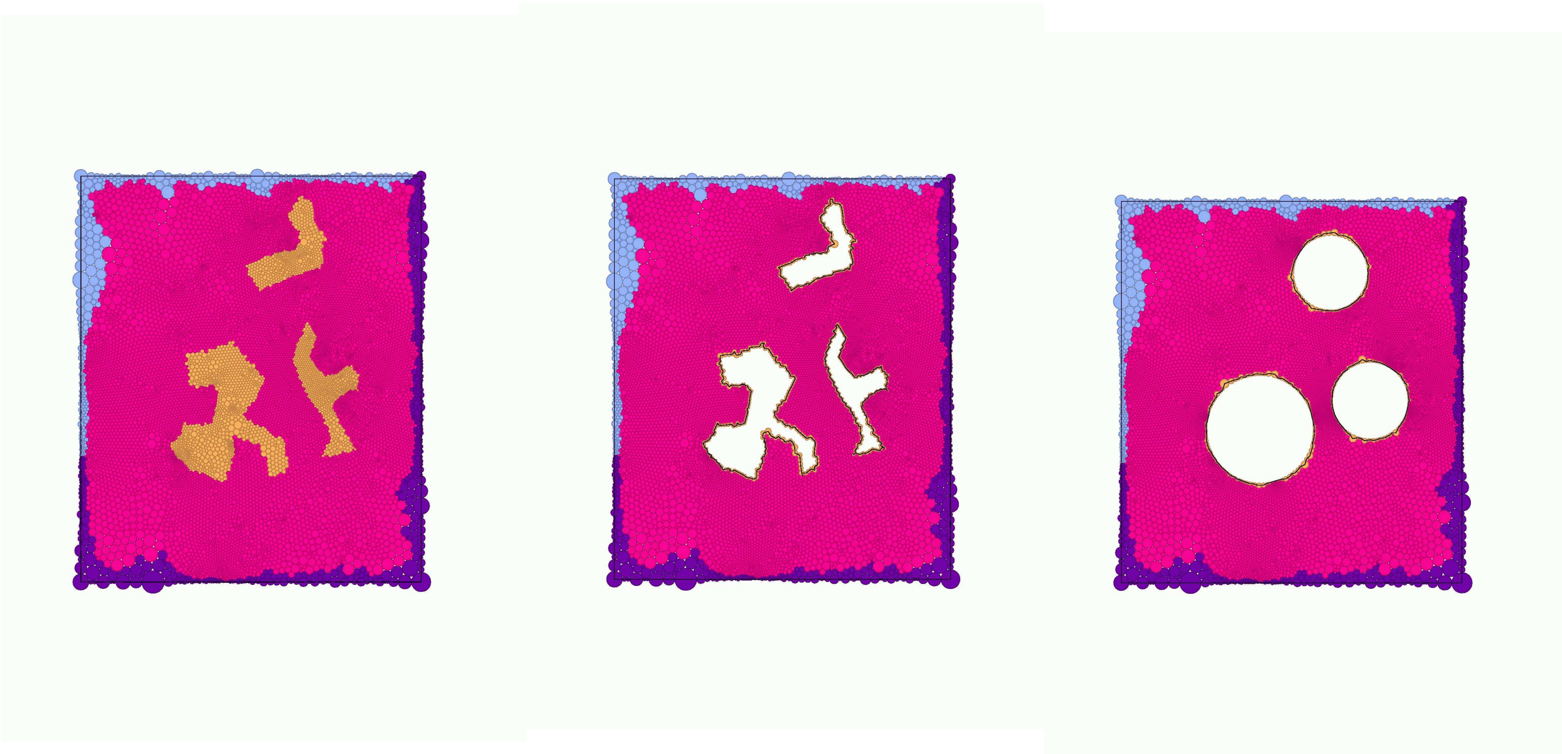
Maximal Packing
DCM
D

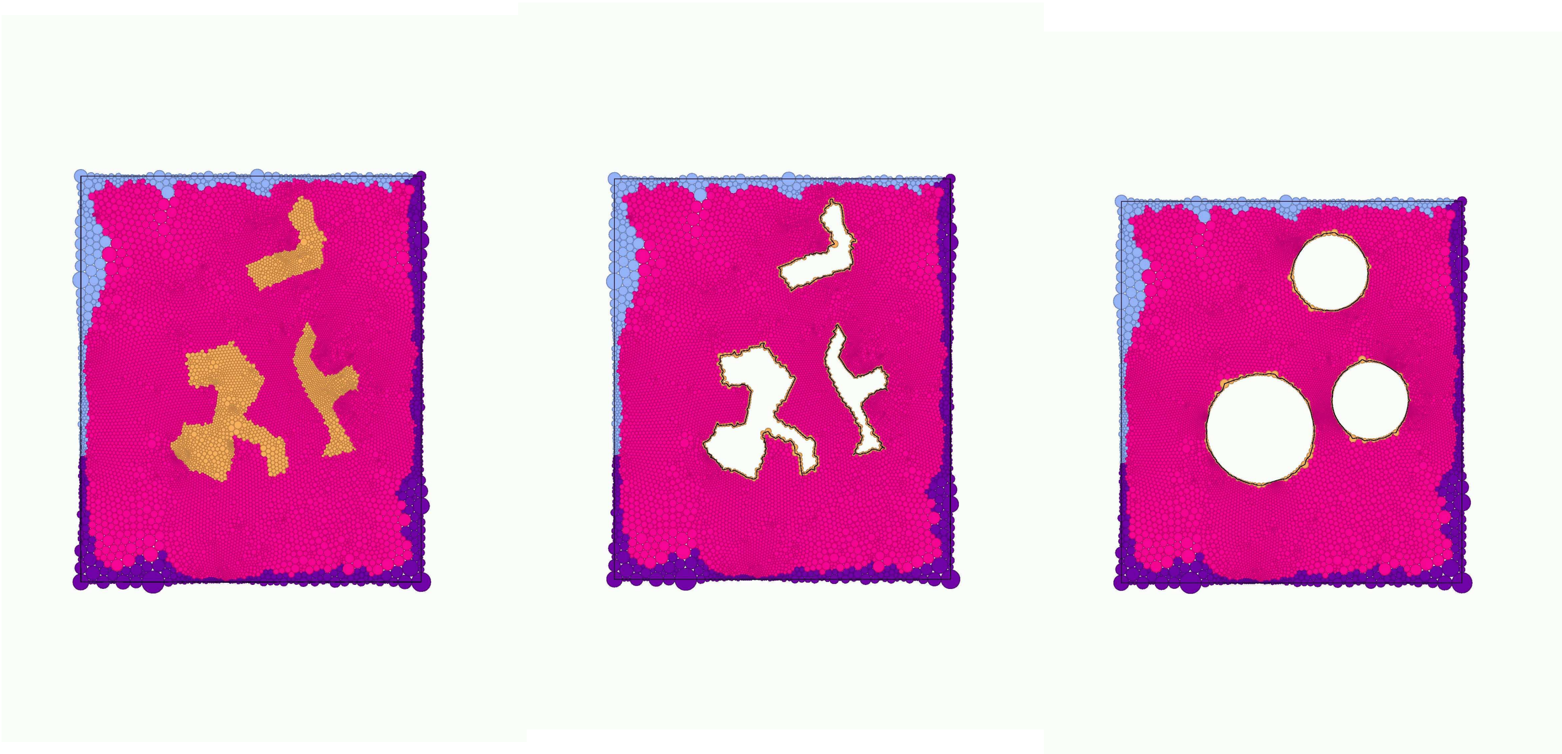

Maximal Packing
DCM
D
Cutout

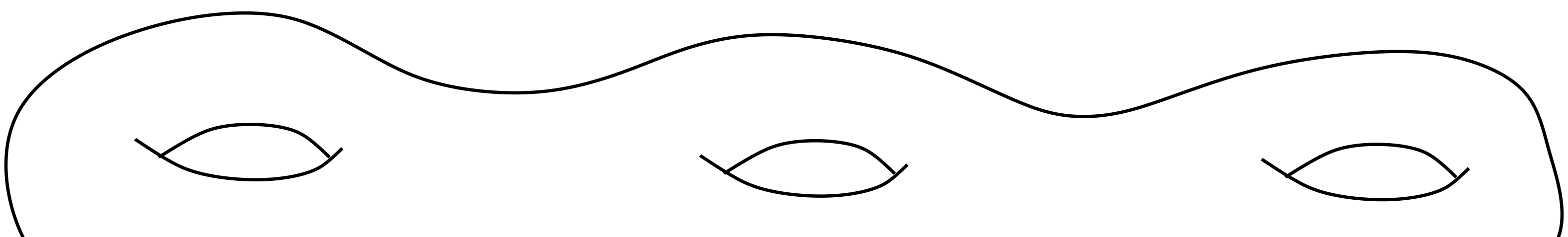

Maximal Packing
DCM
D
Cutout
Annular Packing

Notions of conformal 'Shape'

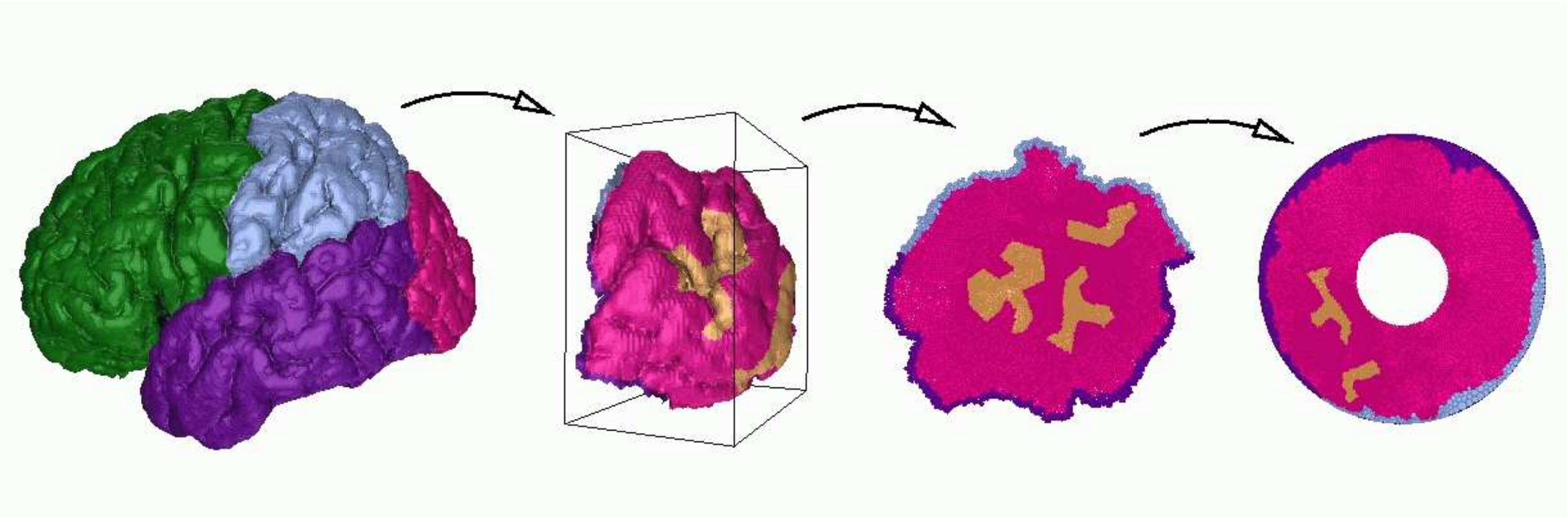








Summary



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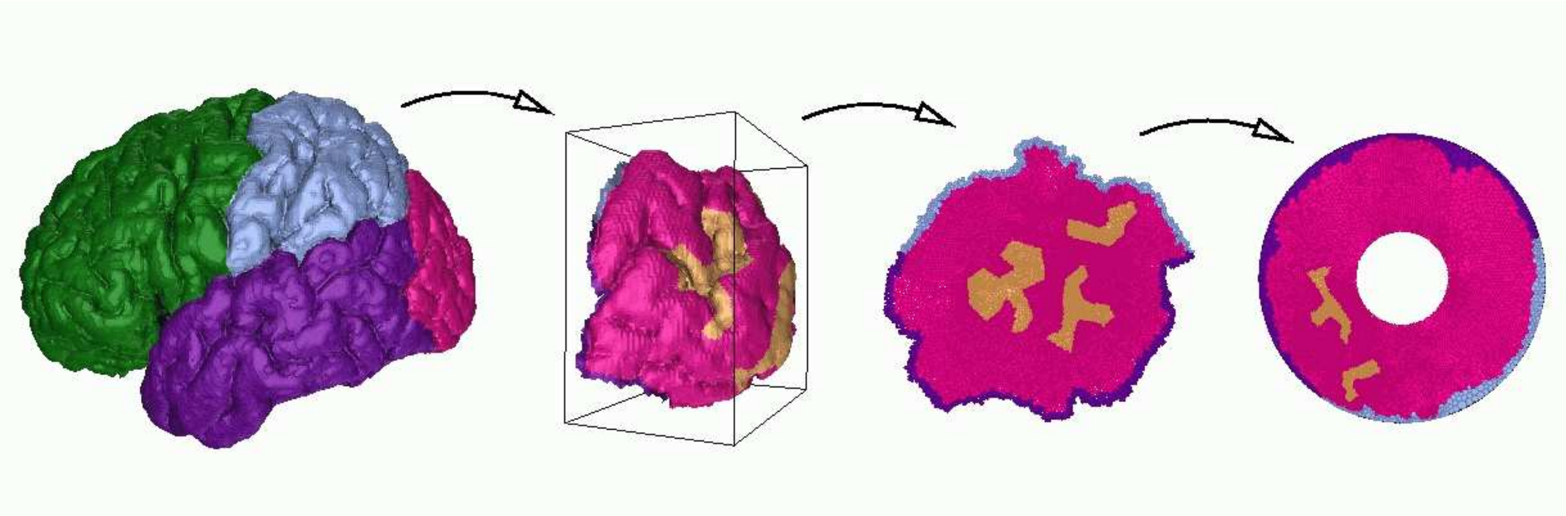

Could the “ensemble conformal features” become a part of the normal processing pipeline?

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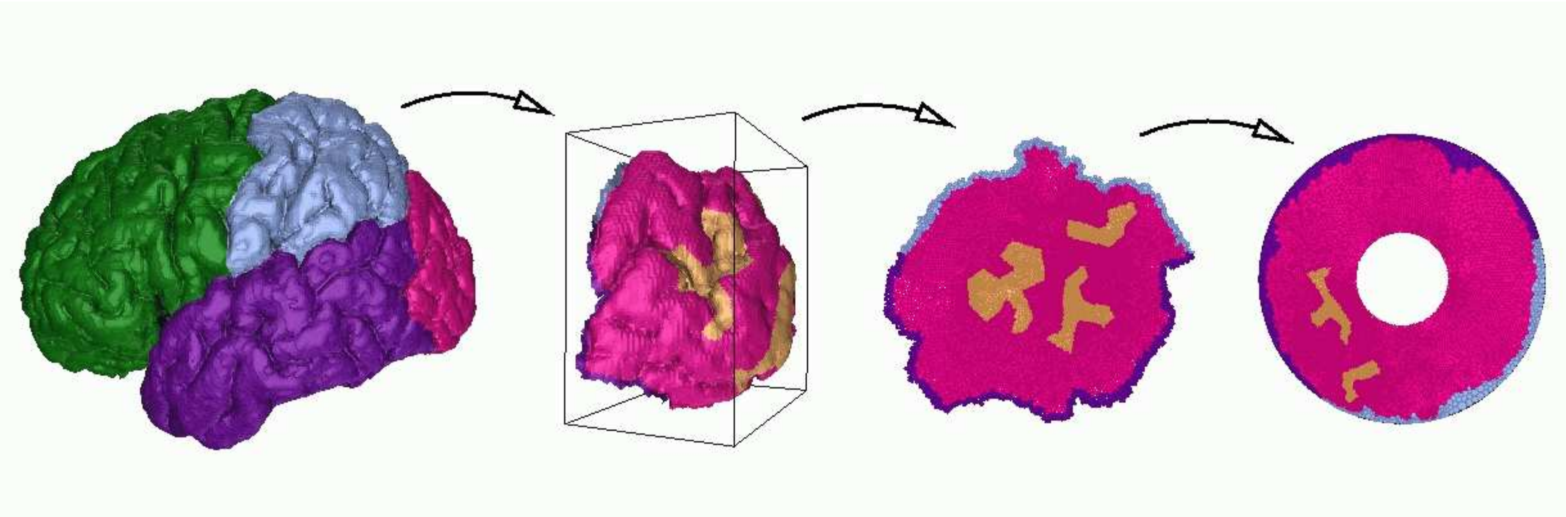

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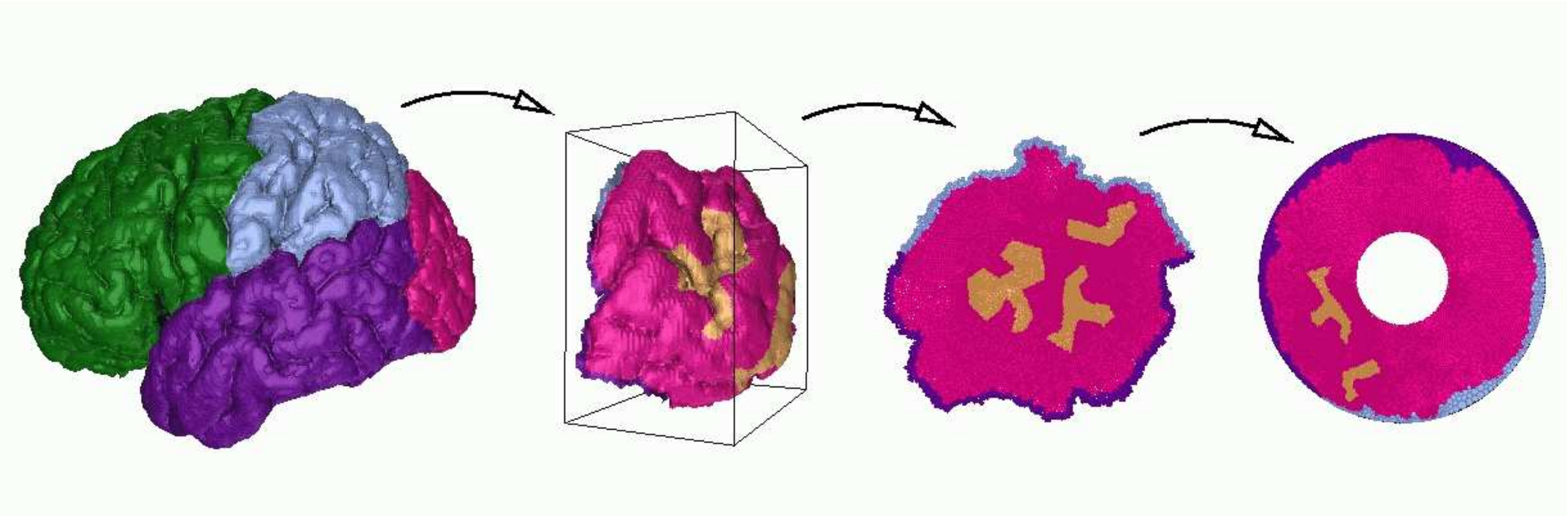

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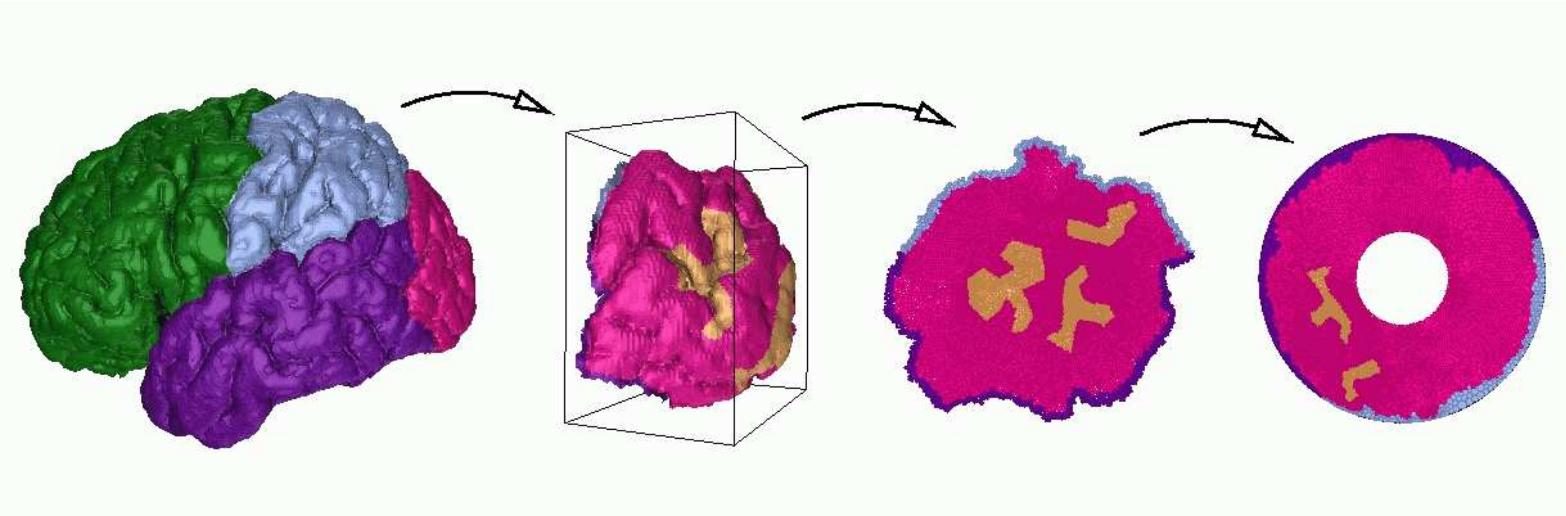

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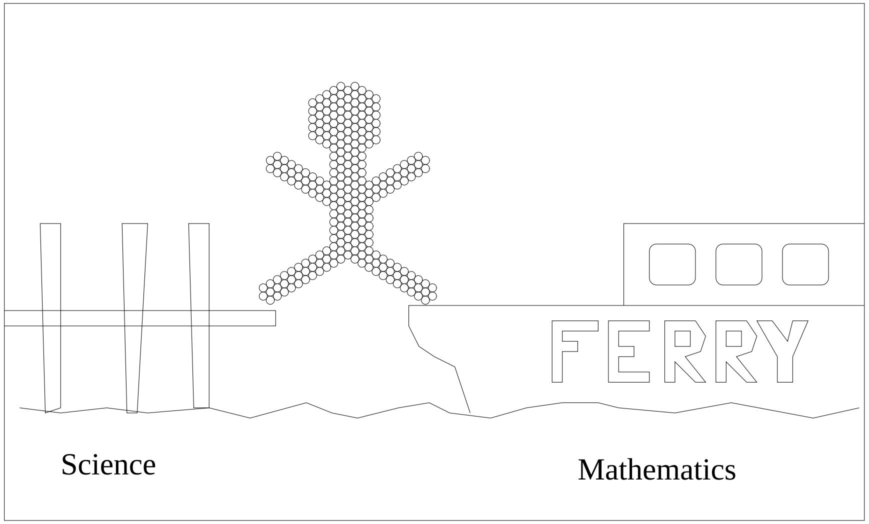

It's still a stretch!

Information

- web: http://www.math.utk.edu/ kens
  e-mail: kens@math.utk.edu
- NSF, FRG grant collaboration: Phil Bowers, Monica Hurdal, and De Witt Sumners (Florida State), Chuck Collins and Ken Stephenson (Tennessee), David Rottenberg (Minnesota).
- Sources:
  - Ahlfors, “Complex Analysis”
  - Ahlfors, “Conformal Invariants”
  - Lehto/Virtanen, “Quasiconformal mapping”
  - Circle packing surveys: see my web site
  - Forthcoming book, Cambridge University Press