Metrics on the space of shapes

IPAM, July 2004

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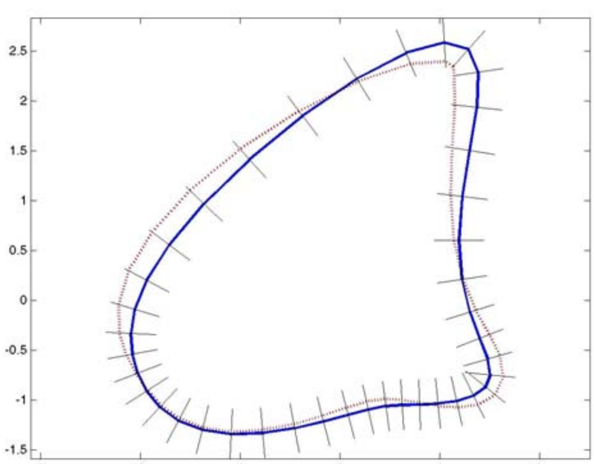
What is the space of shapes?

- S_2 = set of all "smooth" connected plane curves, no self-intersections ("simple closed curves")

- Infinite dimensional!
- Not a vector space
- BUT, locally linear, i.e. a manifold

$$\psi_a(s) = \phi(s) + a(s).\dot{\phi}^{\perp}(s)$$



$$S_2 = \bigcup_{\phi} U_{\phi},$$

$$U_{\phi} = \{a \mid \psi_a \text{ smooth}\}$$

$$\subset (\text{v.sp.of fcns. } a)$$

There are many other spaces of shapes

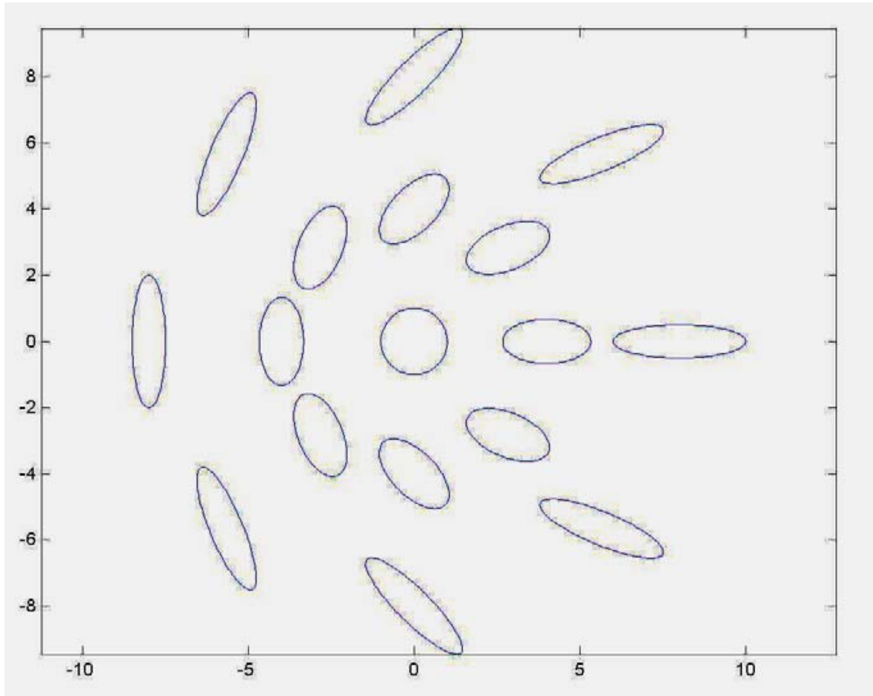
- The core space S_2 has many completions, adding non-smooth limits, giving a zoo of Banach manifolds
- And S_2 has higher dimensional versions:
- S_n = smooth (n-1)-spheres in R^n, i.e. bdries of 'blobs' $D^n \subset R^n$
- or even, fixing an ambient manifold N^n and submanifold type M^m,

$$\left\{\phi : M \to N \,\middle|\, \phi \text{ an embedding}\right\}$$

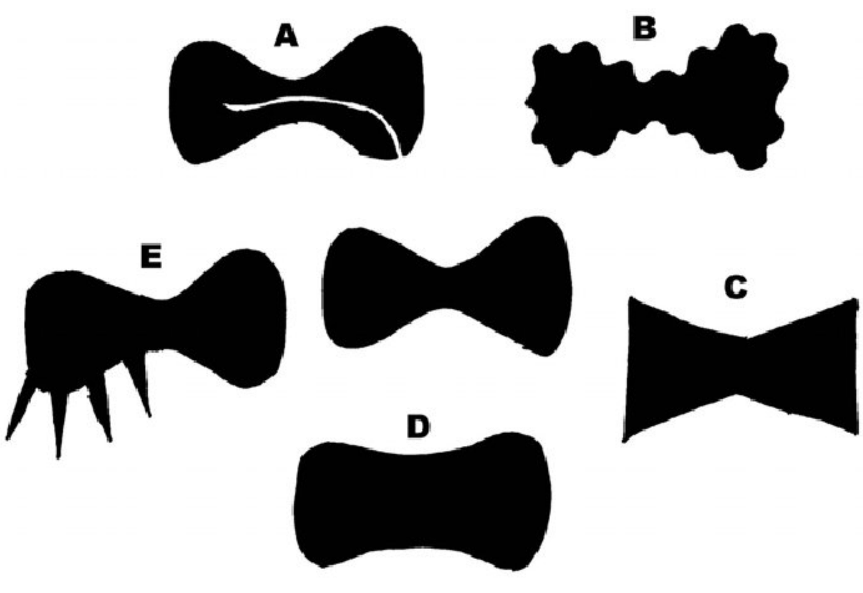
Globally, S_2 and S_3 are known to be top. trivial (contractible) but S_n for $n \geq 7$? is not

For $n = 2$, can contract with the geometric heat equation.

The set of ellipses sits in S_2 as a surface:



'Similarity' between shapes can mean many things, A,B,C,D and E are all similar to the central shape in different ways:



A Road Map to the many metrics

(C^∞-curves, the core space)

$\cap$
⋮
$\cap$

(C^2-curves)

$\cap$

(C^1-curves, w. $\theta(s)$ abs.cont. hence curvature $\kappa(s)$ exists a.e.)
(can ask $|\kappa|$ bnded, $\int \kappa^2 ds < \infty, \int |\kappa| < \infty$)

$\cap$ $\cap$

(curves of fin.length w. fin.amount of bending, i.e. $\theta(s)$ exists a.e. and is BV)

(C^1-curves, i.e. $\theta(s)$ continuous)

$\cap$

Michor Riemannian metric

$\cap$ $\cap$

(curves of finite length, i.e. *rectifiable* curves)

$\cap$ $\cap$

(closed integral currents: cntble unions of Lipschitz maps $I \to R^2$ of finite total length)

(continuous images of circle "Frechet curves")

$\cap$

(subsets $S \subset R^2$ meas. w. fin. area)

$$C^\infty \cap \cdots \cap C^2 \cap \begin{pmatrix} W^{2,p} \\ p=1,2,\infty \end{pmatrix}$$

$\cap$ $\cap$

BV^1 C^1

$\cap$

$W^{1,2}$?

⋮

$\cap$ $\cap$

⋮

$W^{1,1}$ C^0

$\cap$

L^1

Two simple metrics

- *L^1-metric* leading to set of meas. subsets $S \subset R^2$:

infinitesimally: $$\|a\| = \int_C |a(s)|\, ds$$

leads to path length: $$|\text{path}\{C_t\}| = \int_0^1 \left(\int_{C_t} |a(s,t)|\, ds \right)$$

$$= \text{ area swept out}$$

leads to global metric: $$d(S_1, S_2) = \text{area}(S_1 \Delta S_2)$$

- *Frechet metric* (like Hausdorff metric) on cont. maps $f{:}S^1 \rightarrow R^2$:

infinitesimally: $$\|a\| = \sup_{s \in C} |a(s)|$$

leads to path length: $$|\text{path}\{C_t\}| = \int_0^1 \sup_{s \in C_t} |a(s,t)|\, dt$$

$$\geq \text{ max. dist. moved}$$

leads to global metric:

$$d(f_1, f_2) = \inf_{\textit{diffeo } h} \sup_{x \in S^i} \| f_2(x) - f_1(h(x)) \|$$

Neither metric has good geodesics – balls are like boxes, but they stack well, can measure 'volume' (K.Leonard, using ε-entropy)

The Michor metric – the simplest Riemannian metric

- Infinitesimally:

$$\|a\|^2 = \int_C a(s)^2.\left(1 + A\kappa_C^2(s)\right) ds$$

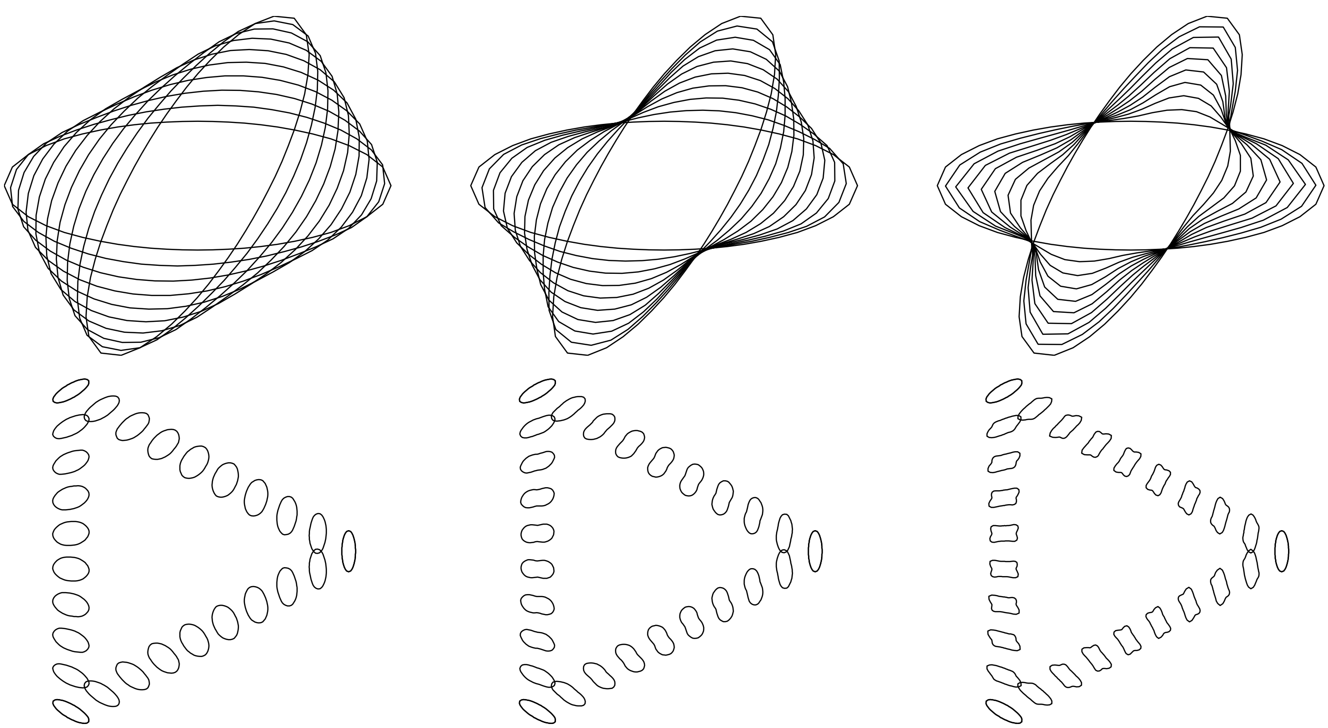
- Globally:

$$\left|\text{path}\{C_t\}\right| = \int_0^1 \sqrt{\int_{C_t} |a(s,t)|^2 \, ds} \, .dt$$

- If A=0, get 'geodesic spray', positive curvature, but infimum of path lengths is zero
- If A>0, the metric controls |C| and gives interesting geodesics – not always unique.

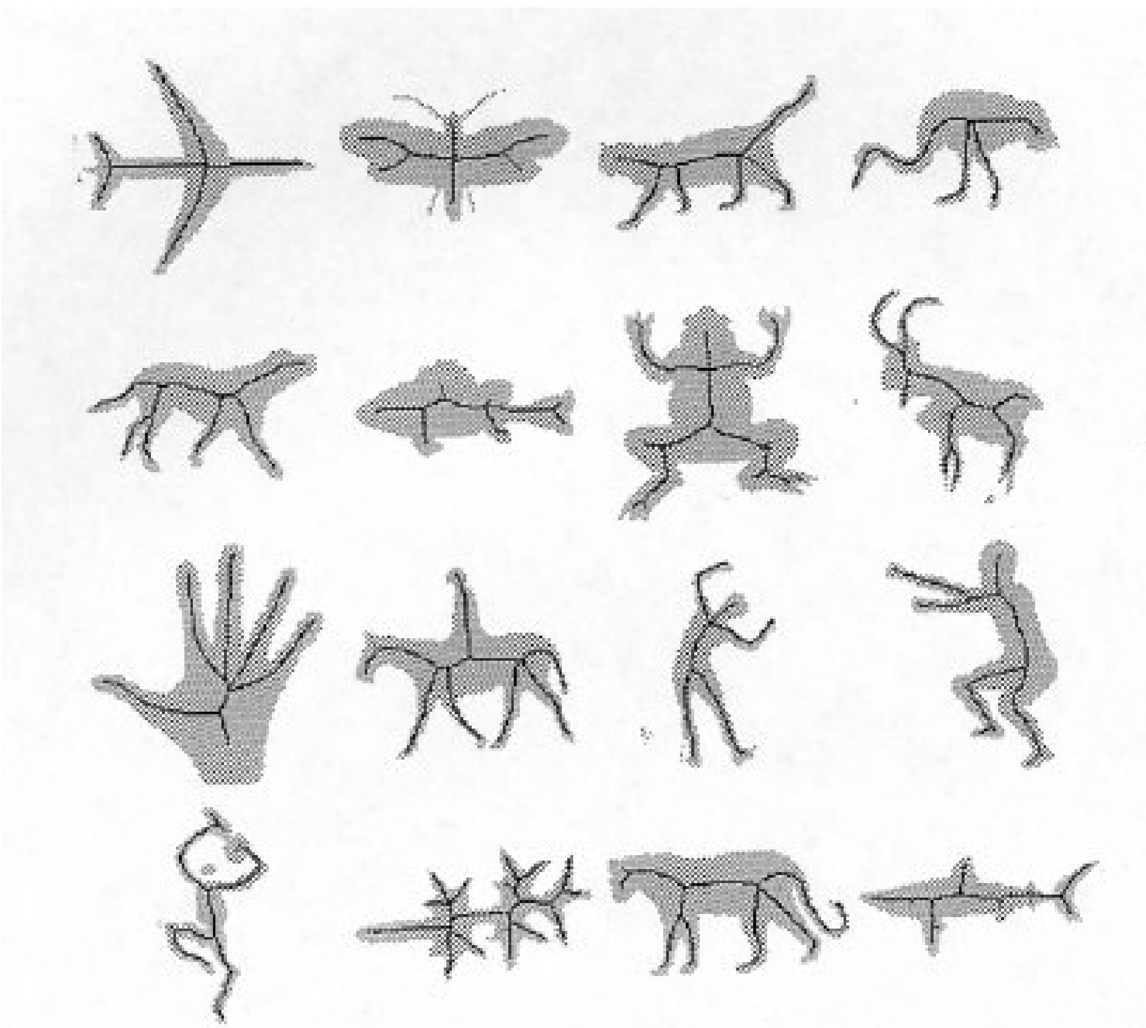
A geodesic triangle

Consider an ellipse rotated through 0, 60 and 120 degrees. These 3 ellipses form a triangle in S^2. Using the metrics with A=1.0, 0.1 and 0.01, we join them with 3 geodesics. The path in S^2 forming one of these edges is shown in the first row for the 3 metrics. The second row shows the whole triangle of shapes:



Axes: the royal road to shape description

Humans perceive shapes as having 'parts', linked in a combinatorial pattern. The axis gives this (and even bit length compression, Leonard 2004).



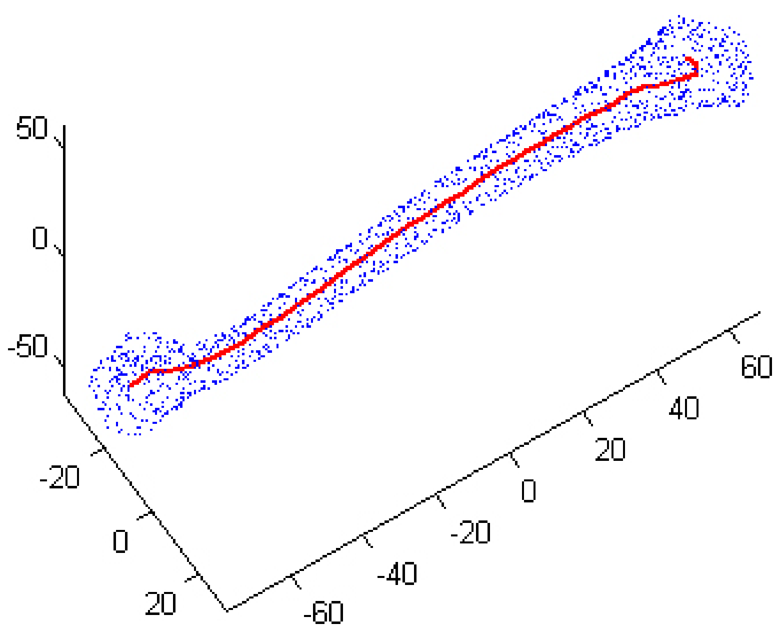
Axes in three dimensions

Axes in 3D are trickier: Yan Cao's definition:

Given a shape *S*, or even an arbitrary measure *m* with support *S*, consider the functional on potential axes:

$$E(\Gamma) = \int_S \mathrm{dist}(x,\Gamma)^p \, \mu(dx) + \alpha.\mathrm{length}(\Gamma)$$

An anatomical example:



Shape analysis via complex analysis

- 2 basic constructions from cx. analysis:

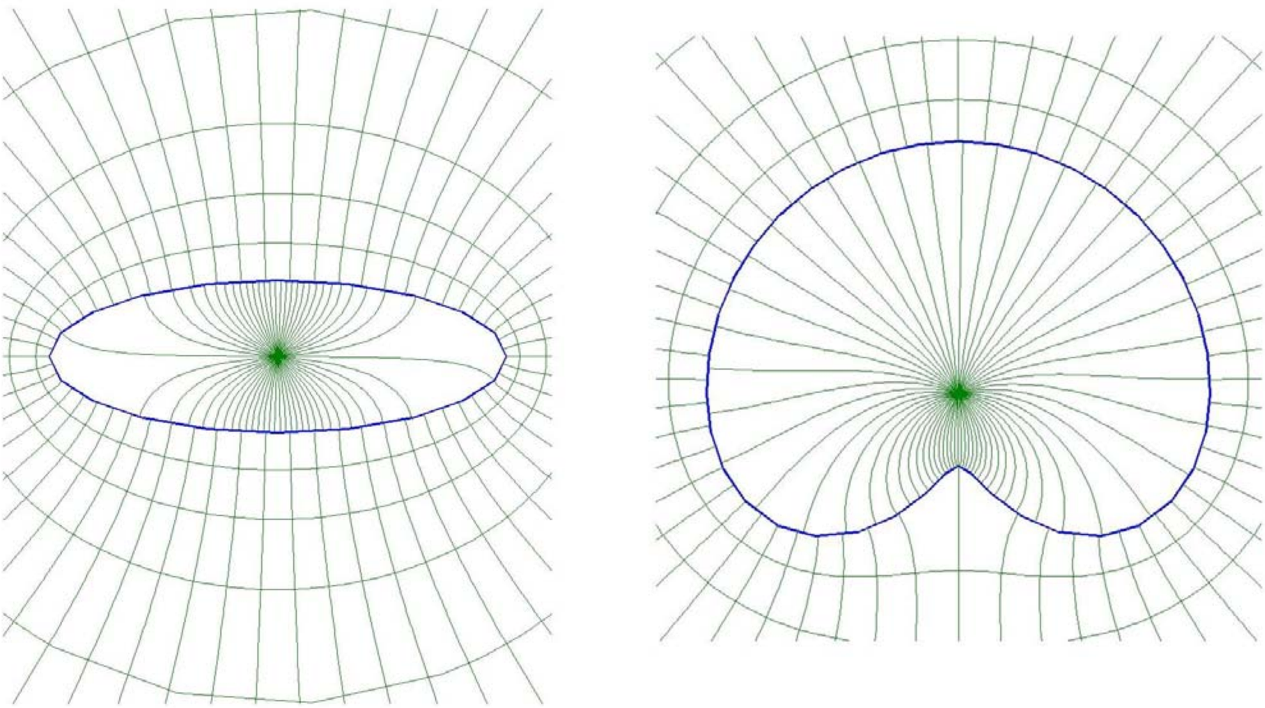
$$\forall D \subset \mathbb{C},\ \exists \phi : \Delta \xrightarrow{\approx} D,\ \text{conformal}$$
$$\text{up to } \phi \circ A,\quad A(z) = \frac{\alpha z + \beta}{\bar{\beta} z + \bar{\alpha}}\ \ (\text{called } \mathrm{SL}_2(\mathbb{R}))$$

$$\forall \text{ surfaces } S \subset \mathbb{R}^3,\ \text{diffeo. to } S^2,$$
$$\exists \phi : \widehat{\mathbb{C}} \xrightarrow{\approx} S \text{ conformal, up to}$$
$$\phi \circ A,\ A(z) = \frac{az + b}{cz + d}\ \ (\text{called } \mathrm{SL}_2(\mathbb{C}))$$

- Use the first construction:

$$\phi_0 : Int(\Delta) \xrightarrow{\approx} Int(C),$$
$$\phi_\infty : Ext(\Delta) \cup \{\infty\} \xrightarrow{\approx} Ext(C) \cup \{\infty\},$$
$$\text{with } \phi_\infty(\infty) = \infty,\ \phi'_\infty(\infty) = \text{ pos.real}$$

Two examples

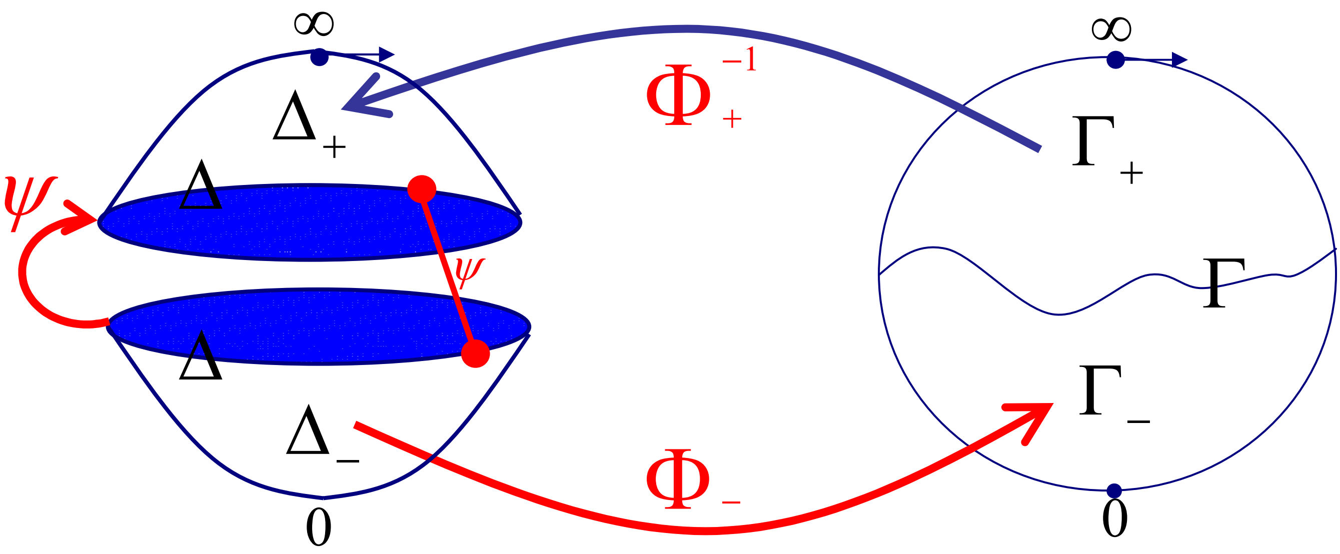


An ellipse and a kidney shaped object, with the conformal parametrization of their interiors and exteriors marked. The interior map has been chosen to carry 0 to 0, but it may take 0 to any other interior point.

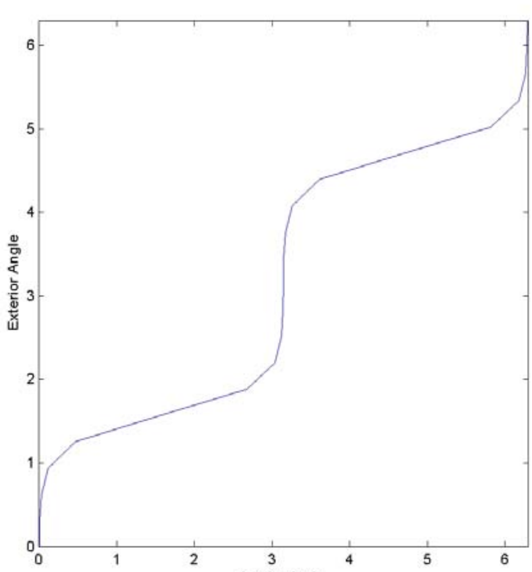
The interior and exterior parametrizations can be compared on their common boundary, defining the *fingerprint*:

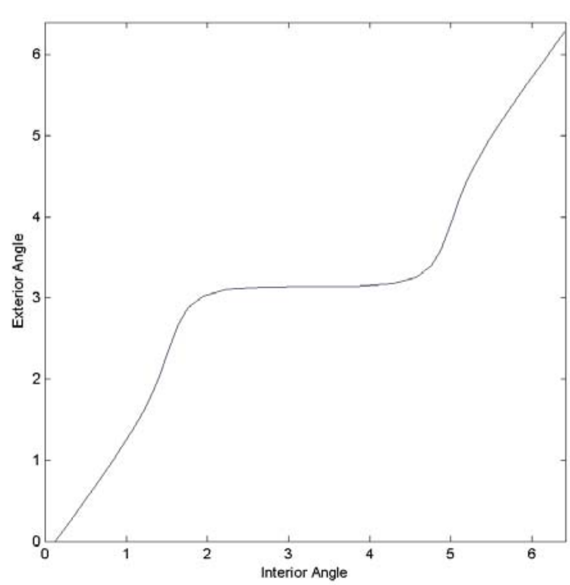
$$\psi(z) = \phi_{\infty}^{-1}(\phi_0(z)), \text{ up to } \psi \circ A, A \in SL_2(\mathbb{R})$$

Constructing the fingerprint



The fingerprint for the ellipse and the kidney



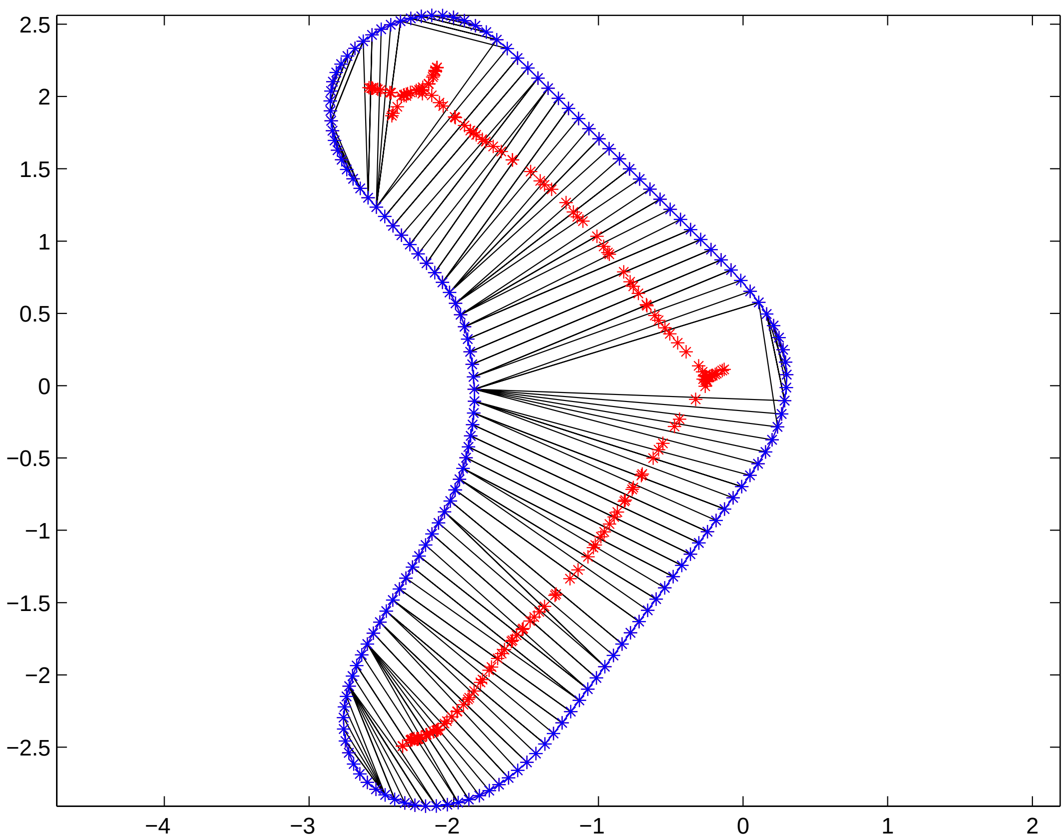


Decoding the fingerprint

- Minima of ψ correspond (roughly) to points on C nearest to $\phi_0(0)$.

$$M_A = \arg\min\left|\left(\psi \circ A\right)'\right|$$

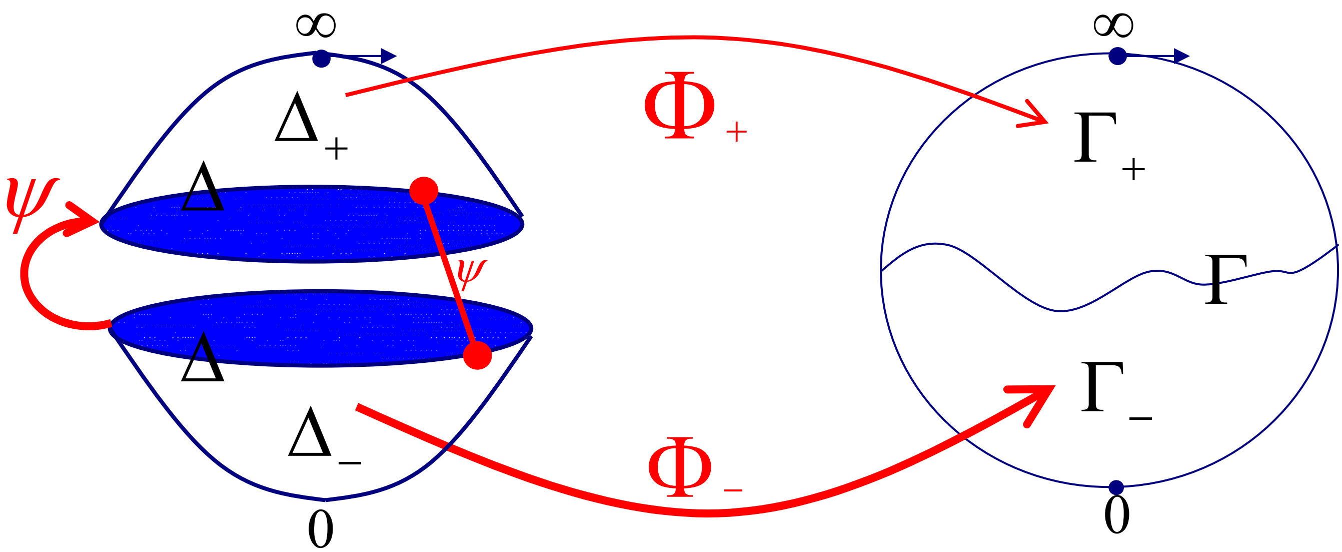
$$\text{complex axis}(C) \overset{def}{=} \left\{\phi_0(A(0)) \middle| \# M_A > 1\right\}$$



Combinatorial structure of the axis leads to a natural *cell decomposition* of S_2.

Does the fingerprint determine *C*?

- YES!! via ‘welding’. Read the diagram backwards:



- First glue on the left via ψ, then use the second basic construction to get the conformal ϕ’s, hence the image curve *C*.

$$\mathrm{Diff}(S^1)/SL_2(\mathbb{R}) \overset{\approx}{\rightleftarrows} \mathsf{S}_2/\text{transl.+scaling }(H)$$

$$\uparrow \qquad\qquad \uparrow$$

$$\mathrm{Diff}(S^1)/\text{rotations} \overset{\approx}{\rightleftarrows} \left\{\begin{array}{c}\text{curves } C\\ \text{plus basept.}\\ x_0 \in Int(C)\end{array}\right\}/H$$

S_2 is a homogeneous metric space

Welding allows the group Diff(S^1) to *act* on S_2:

1. Start with *C* and ψ.
2. Put angles θ on *C* via exterior map.
3. Cut open $\hat{\mathbb{C}}$ along *C*,
4. Reglue with twist ψ (using angles θ).
5. Find new conformal isomorphism with $\hat{\mathbb{C}}$ and thus get C^ψ.

* IMPLEMENTED BY EITAN SHARON *

There is a unique Riemannian metric on Diff(S^1) /SL_2(R), left invariant by Diff(S^1) ! – the *Weil-Petersen* metric.

Need norm on tgt. sp. to Diff(S^1) at e, 0 in SL_2(R) directions, invariant by conjugation by SL_2(R).

Tangent space at e = lie algebra of Diff(S^1), vector flds to S^1, so:

$$X = \sum a_n e^{in\theta} \partial/\partial\theta, \quad a_{-n} = \bar{a}_n$$

$$\|X\|^2 = \sum_{n \geq 2} (n^3 - n)|a_n|^2$$

A geodesic: ellipse to square

Geodesics are expected always to exist and to be unique, because sectional curvature is negative.