Ergodic basis pursuit induces robust network reconstruction

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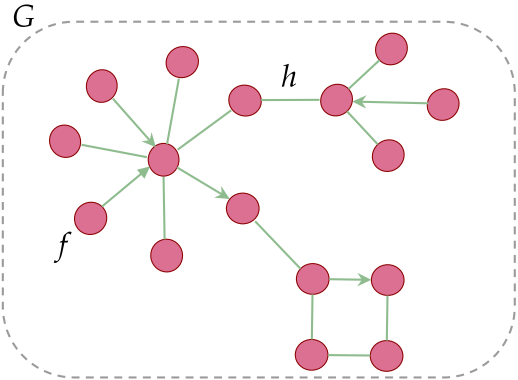
MB2022
August 28, 2022

Network dynamics is identified by a triple



The network dynamics $F : M^N \to M^N$ is given by:

$$x \in M^N \mapsto F_i(x) = f(x_i) + \alpha \sum_{j=1}^{N} A_{ij}\ h(x_j, x_i) \quad i = 1, \ldots, N.$$

Sparse representation of the network dynamics

1° assumption (f, G, h):

- The network is *sparse*.
- f and h are in the $\mathrm{span}\ \mathcal{L}_0$

$$F_i = \sum_{l=1}^{m} c_i^l \phi_l, \qquad c_i \in \mathbb{R}^m, \quad i = 1, \ldots, N,$$

where c_i is a sparse vector.

$\mathcal{L}_0$ is basis of homogeneous polynomials

Reconstruction procedure as a linear equation

$$x_i(t+1) = \sum_{l=1}^{m} c_i^l \phi_l(x(t)), \quad i = 1, \dots, N.$$

$$= \left(\phi_1(x(t)), \dots, \phi_m(x(t))\right) \begin{pmatrix} c_i^1 \\ \vdots \\ c_i^m \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} x_i(1) \\ \vdots \\ x_i(n) \end{pmatrix}}_{x_i'} = \underbrace{\begin{pmatrix} \phi_1(x(0)) & \phi_2(x(0)) & \cdots & \phi_m(x(0)) \\ \phi_1(x(1)) & \phi_2(x(1)) & \cdots & \phi_m(x(1)) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_1(x(n-1)) & \phi_2(x(n-1)) & \cdots & \phi_m(x(n-1)) \end{pmatrix}}_{\Phi_0(X)} \underbrace{\begin{pmatrix} c_i^1 \\ \vdots \\ c_i^m \end{pmatrix}}_{c_i}$$

Reconstruction procedure as a linear equation

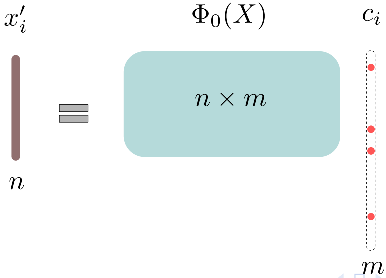
To obtain the graph structure we solve for the coefficient vector:

$$x_i' = \Phi_0(X)c_i, \quad i = 1, \ldots, N.$$

- If we have lots of data $n > m$: least square solutions

$$\underset{u \in \mathbb{R}^m}{\arg\min} \|x_i' - \Phi_0(X)u\|_2$$

- **Large networks $n < m$:** short time series

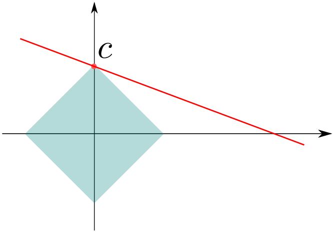


Reconstruction procedure is done separately

Network reconstruction procedure: Solve the basis pursuit:

$$\min_{u \in \mathbb{R}^m} \|u\|_1 \text{ subject to } \Phi_0(X)u = x_i'.$$

$$\Phi_0(X)u = x'$$



D. Napoletani and T. Sauer - *Reconstructing the topology of sparsely connected dynamical networks* - **Phys. Rev. E** (2008).

ℓ_2 vs. ℓ_1 minimization problem

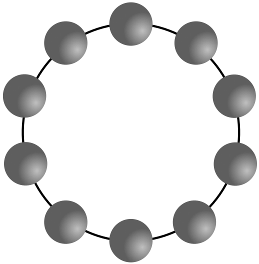
	ℓ_2 norm	ℓ_1 norm
length of time series	$n > m$	$n_0 \leq n < m$
uniqueness condition	$\Phi_0(X)$ full rank	?

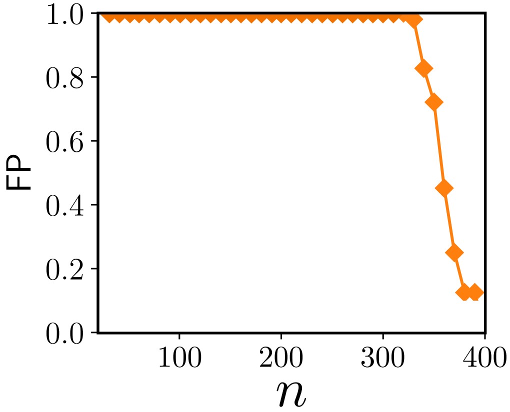
Both are unstable when the time series length is not large

Examples: ℓ_1 reconstruction

- f is the logistic map with $b = 3.98$ and $\alpha = 0.001$
- $h(x, y) = p(x) - p(y)$ where p is a cubic monimial
- G is a ring network of 10 nodes

Examples: ℓ_1 reconstruction





ℓ_1 reconstruction of logistic maps on a ring

Heuristics

How much data?

For ℓ_1 and ℓ_2 length of time series is $O(n^2)$

Determine n_0 to obtain uniqueness of solutions

Uniqueness of sparse solutions depends on Φ

We search a unique s-sparse solution via basis pursuit.

Informal statement: Uniqueness of s-sparse solutions [1]

If a set of $2s$ columns of $\Phi_0(X)$ is nearly orthonormal (RIP), then the basis pursuit has uniqueness of s-sparse solutions.

$$\delta_{2s} := \max_{\mathcal{S}\subset[m],|S|\leq 2s} \|(\Phi)^T_{\mathcal{S}}(\Phi)_{\mathcal{S}} - \mathbf{1}_{2s}\|_2 \leq \sqrt{2} - 1.$$

[1] E.J. Candes and T. Tao - *Decoding by linear programming.* - **IEEE Transactions on Information Theory, 51(12):4203–4215** (2005).

Key Idea

- Change the basis $\mathcal{L}_0$ to a new $\mathcal{L}_{new}$

$$\Phi_{new}(X) \text{ is RIP with the correct constant}$$

Challenges around the corner

- Guarantee that that solutions are also sparse in the new basis
- Obtain RIP constant for “many” trajectories

Control linear dependence

- Controlling linear dependence of pair of $\Phi_0(X)$ columns (*coherence*);
- for any two columns v_i and v_j of $\Phi_0(X)$

$$\langle v_i, v_j \rangle = \frac{1}{n} \sum_{k=0}^{n-1} \phi_i(F^k(x(0)))\phi_j(F^k(x(0)))$$

$$\to \int \phi_i \cdot \phi_j d\mu$$

Control linear dependence

Ergodicity assumption:

$$\langle v_i, v_j \rangle = \frac{1}{n} \sum_{k=0}^{n-1} \phi_i(F^k(x(0)))\phi_j(F^k(x(0)))$$

$$\to \int \underbrace{\phi_i \cdot \phi_j}_{\psi} d\mu$$

Exponential mixing systems

2° assumption: exponential decay of correlations

$$\Big| \int_{M^N} \psi \cdot (\varphi \circ F^n) d\mu - \int_{M^N} \psi d\mu \int_{M^N} \varphi d\mu \Big| \leq K(\psi, \varphi) e^{-\gamma n}, \quad n \geq 0$$

Concentration inequality result[2]:

$$\mu\Big(x(0) \in M^N \ : \ \Big| \frac{1}{n} \sum_{k=0}^{n-1} \psi \circ F^k(x(0)) \Big| \geq \varepsilon \Big) \leq 4 e^{-\theta(n, \varepsilon, \psi)}$$

[2] H. Hang and I. Steinwart - *A Bernstein-type inequality for some mixing processes and dynamical systems.* - **The Annals of Statistics** (2017).

Exponential mixing systems

3° **assumption: weak coupling (stochastic stability)**

There is a product measure ν close to μ

Ergodic basis pursuit has exact reconstruction

Informal statement: exact reconstruction

F has an s'-sparse representation in $\mathcal{L}_0$ and (F, μ) is an exponential mixing with $\gamma > 0$ and $d(\nu, \mu) \leq \varepsilon$. Then we construct a set of orthonormal functions w.r.t. ν,

$$\mathcal{L}_\nu = \{\varphi_i : \int_{M^N} \varphi_i \varphi_j d\nu = \delta_{ij}\}$$

such that

1. The map $\mathcal{L}_0 \mapsto \mathcal{L}_\nu$ maps s'-sparse to s-sparse representations of the dynamics
2. For large network sizes $N(\gamma)$ there is a large set of i.c. such that if

   $$n \geq K(2s-1)^2 \ln\big(m(m-1)\big),$$

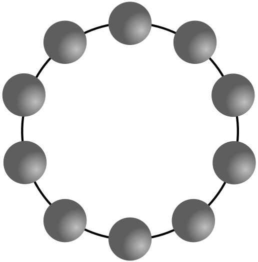
   $\Phi_\nu(X) := \Phi(\mathcal{L}_\nu, X)$ has $\delta_{2s} < \sqrt{2} - 1$.
3. The reconstruction via the ergodic basis pursuit

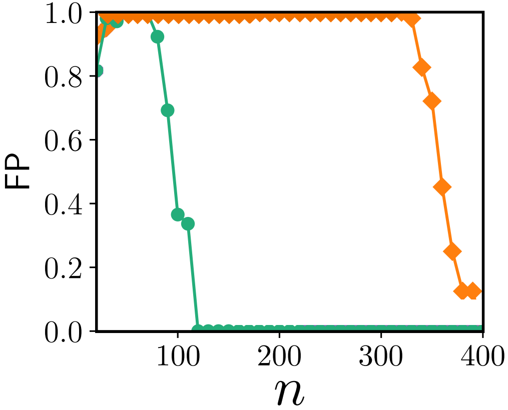
   $$\min_{u \in \mathbb{R}^m} \|u\|_1 \text{ subject to } \Phi_\nu(X)u = x'$$

   has a unique s-sparse solution.

EBP outperforms the classical basis pursuit

- using that ν is a product of Chebchev measures.





EBP (green) does not depend on the network size.

EBP outperforms the classical basis pursuit

- Length of time series

$$n_0 \geq K(2s-1)^2 \underbrace{\ln\big(m(m-1)\big)}_{\propto \ln N},$$

- Implying

$$n_0 \geq \hat{K}(2s-1)^2 \ln N,$$

- If f and h are in the basis

$$n_0 \geq \bar{K}\Delta^2 \ln N,$$

EBP outperforms the classical basis pursuit

- Estimation of the measure μ from data (weakly coupled networks)

  Kernel density estimator using all data

- Additional computational cost to orthornomalize the initial basis $\mathcal{L}_0$.

Robust reconstruction

Ergodic basis pursuit from noisy measurements

$$y(t) = x(t) + z(t),$$

where $(z_n)_{n\geq 0}$ corresponds to an i.i.d. $[-\varepsilon_0, \varepsilon_0]^N$-valued noise process for $\varepsilon_0 \in (0, 1)$ with probability measure ρ_{ε_0}.

Noise reconstruction

Informal statement: noise reconstruction

Consider the setting as before. The family $\{c^\star(\epsilon)\}_{\epsilon>0}$ given by

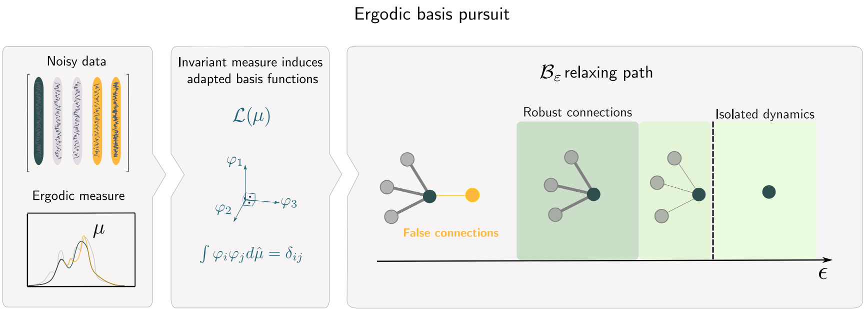
$$c^\star(\epsilon) = \underset{\tilde{u}\in\mathbb{R}^m}{\arg\min}\left\{\|\tilde{u}\|_1 \text{ subject to } \|\Phi_{\mu_0}(Y)\tilde{u} - y'\|_2 \leq \epsilon\right\}$$

satisfies for a constant $K = K(\delta_{2s}) > 0$

$$\|c^\star(\epsilon) - c\|_2 \leq K\epsilon$$

as long as $\epsilon \geq \varepsilon$.

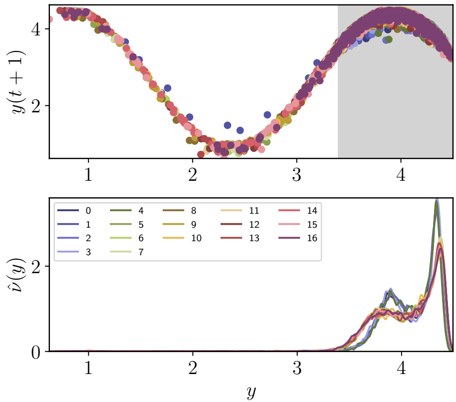
Searching robust connections



Application: optoelectronic networks (Expts from Raj Roy and Joe Hart).

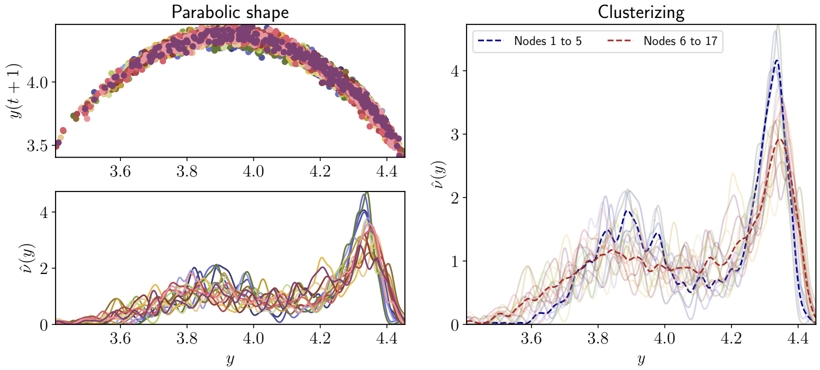
Optoelectronic experimental data and density estimation

$N = 17$ lasers diffusively coupled on a given network



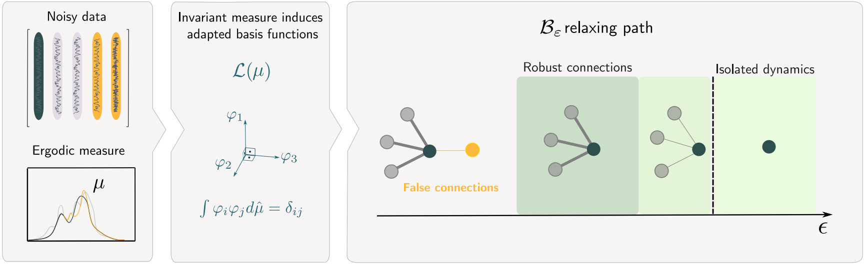
Optoelectronic experimental data and density estimation

- Discard some regions of small density
- Look for a product measure that approximates the data

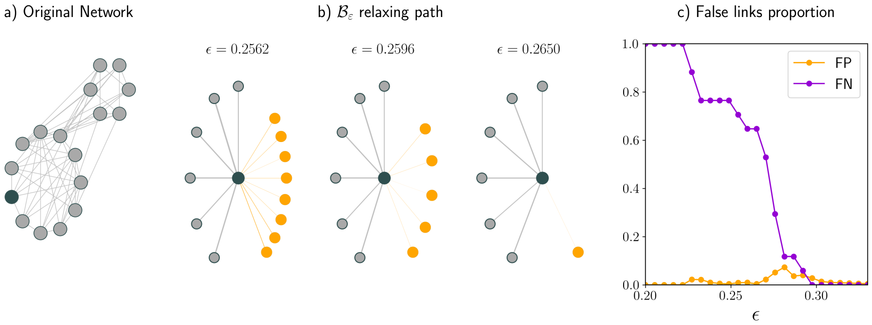


Searching robust connections (Recap)

Ergodic basis pursuit



Robust reconstruction of the optoelectronic network



For $\epsilon \approx 0.3$ excellent stable reconstruction for the whole network

Take Home

- Optimization & Ergodic Theory

  uniquess of solutions and stability

- Estimating measure from data

- Large deviation and measure approximation for networks (oportunities)

- Find a scheme for large coupling (open)